Diophantine approximation, scalar multiplication and decidability

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- Arguably due to Skolem (1931) in *Über einige Satzfunktionen in der Arithmetik* (see Smoryński (1991)).
- Indeed, a quantifier elimination result in a suitably extended language holds.
- Rediscovered independently by Gordon (197?), Weispfenning (1999) and C. Miller (2001).
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Theorem - Gödel (1931). The theory of $(\mathbb{R}, <, +, \cdot, \mathbb{N})$ is undecidable.

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Precise question. Let α be irrational. What can be said about $(\mathbb{R}, <, +, \mathbb{N}, x \mapsto \alpha x)$ and $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$?

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Surprisingly little was know, in particular given the fact that $(\mathbb{R}, <, +, \mathbb{N})$ was (and still is) extensively used by computer scientists. Very partial results due to Weisspfenning (1999). Some results claimed by Gordon in 1970's, but the proofs were incorrect.

Connection to Diophantine Approximations/Dynamical Systems. Let $\alpha \in [0, 1]$. For $\delta \in \mathbb{R}$, we define $f_{\alpha,\delta} : \mathbb{N} \to [0, 1]$ by

$$f_{\alpha,\delta}(n) := \lfloor (n+1)\alpha + \delta \rfloor - \lfloor n\alpha + \delta \rfloor.$$

The word

$$\boldsymbol{f}_{\alpha,\delta} = f_{\alpha,\delta}(1) f_{\alpha,\delta}(2) f_{\alpha,\delta}(3) \dots$$

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When $\alpha = \frac{1}{1+\varphi}$, where φ is the golden ratio, then $\boldsymbol{f}_{\alpha,0}$ is the Fibonacci word: $0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, \dots$

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These functions are definable in $(\mathbb{R},<,+,\mathbb{N},x \mapsto \alpha x)$. Decidability yields decision procedure for questions about the Fibonacci word, like

$$\exists p \in \mathbb{N}\ p \geq 0 \wedge \exists n \in \mathbb{N} \forall i \in \mathbb{N} (i \geq n) \to (f_{\alpha,0}(i) = f_{\alpha,0}(i+p))$$

(cp. *Decision Algorithms for Fibonacci-Automatic Words* by Mousavi, Schaeffer and Shallit)

Results - H. 2015.

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Theorem. Let $S \subseteq \mathbb{R}$. Then the structure $(\mathbb{R}, <, +, \mathbb{N}, (x \mapsto \beta x)_{\beta \in S})$ defines the same sets as exactly one of the following structures:

(i) $(\mathbb{R}, <, +, \mathbb{N})$,

(ii) $(\mathbb{R}, <, +, \mathbb{N}, x \mapsto \alpha x)$, for some quadratic $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

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Consequence. When α is a quadratic real number, then the theory of $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$ is decidable.

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Is the theory of $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$ decidable for any non-quadratic α?

Prior results.

H., Tychonievich (2014). Let $\alpha, \beta \in \mathbb{R}$ be such that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$. Then $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N}, \beta\mathbb{N})$ defines multiplication on $\mathbb{R}$.

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Why? Morally, because $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N}, \beta\mathbb{N})$ allows you define the Sturmian word $f_{\alpha,\delta}$ and $f_{\beta,\delta}$ for each δ. If $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$, then you can find δ such that the pair of Sturmian words is arbitrarily complicated.

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Consider the two-sorted structure

$$\mathcal{B} := (\mathbb{N}, \mathcal{P}(\mathbb{N}), s_{\mathbb{N}}, \in),$$

where $s_{\mathbb{N}}$ is the successor function on $\mathbb{N}$ and $\in$ is the relation on $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that $\in (t, X)$ iff $t \in X$.

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Theorem - H. (2014). Let α be a quadratic irrational number. Then $(\mathbb{R}, <, +, \mathbb{N}, x \mapsto \alpha x)$ and $\mathcal{B}$ are bi-interpretable.

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$x \in \mathbb{R} \leftrightsquigarrow$ pair of a finite and an infinite sequence of $0, 1$'s:

$$b_{-n} \dots b_0 . b_1 \dots$$

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Conclusion. $\mathcal{B}$ interprets $(\mathbb{R}, <, +, \mathbb{N})$. Decidability of the theory of $(\mathbb{R}, <, +, \mathbb{N})$ follows.

Idea. How can we interpret $(\mathbb{R}, <, +, \mathbb{N}, x \mapsto \alpha x)$ in $\mathcal{B}$? Replace binary representations by Ostrowski (Zeckendorf) representations.

For simplicity, we just consider $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$.

Zeckendorf representation (1972). Let $N \in \mathbb{N}$ and F_k be the k-th Fibonacci number. Then N can be written uniquely as

$$N = \sum_{k=1}^{n} b_{k+1} F_k,$$

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$F_{-1} = 0, F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5,\ F_5 = 8, \ldots$

Zeckendorf representation of 9 is $8 + 1$ and not $5 + 3 + 1$.

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Take away: *A natural number corresponds to a finite sequence of* $0,1$*'s with no consecutive* 1*'s , and*

$<$ on $\mathbb{N} \leftrightsquigarrow$ lexicographic order on the representation

Zeckendorf representation for real numbers. Let $c \in \mathbb{R}$ be such that $-\frac{1}{\varphi} \leq c < 1 - \frac{1}{\varphi}$. Then c can be written uniquely in the form

$$c = \sum_{k=1}^{\infty} b_{k+1}(F_k \varphi - F_{k+1}),$$

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Let $b = b_1 b_2, \dots$ and $b' = b'_1 b'_2 \dots$ be two sequences of 0, 1's. We say $b \prec b'$ if there is $n \in \mathbb{N}$ minimal such that $b_n \neq b'_n$ and either

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$$< \text{ on } (-\tfrac{1}{\varphi}, 1 - \tfrac{1}{\varphi}) \leftrightsquigarrow \prec \text{ on the representations}$$

Interpretation of $(\mathbb{R}, <, +, \mathbb{N})$ in $\mathcal{B}$:

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Addition in Zeckendorf representation. Let $M, N \in \mathbb{N}$ given in Zeckendorf representation. Then the Zeckendorf representation of $M + N$ can be recognized by a finite automaton.

This is due to Frougny (1992). There is an elegant, elementary three-pass algorithm due to Ahlbach, Usatine, Frougny, Pippenger (2013).

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General case. When α is quadratic, then the theory of $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$ is decidable. Same proof works, but one has to use Ostrowski representations - a generalization of Zeckendorf representations.

Skip Ostrowski

Bemerkungen zur Theorie der Diophantischen Approximationen.

Von ALEXANDER OSTROWSKI in Hamburg.

(Aus Mitteilungen an Herrn E. HECKE.)

I.

Nach unserer letzten Unterhaltung habe ich versucht, das Problem der Abschätzung der Summen $S(x)$ mit Hilfe der Kettenbruchtheorie in Angriff zu nehmen, und bin zu sehr einfachen Resultaten gelangt. Insbesondere hat sich ein äußerst elementarer Beweis für den von den Herren HARDY und LITTLEWOOD ausgesprochenen Satz ergeben, daß $\sum_{n=1}^{x}\left(R(n\alpha)-\frac{1}{2}\right)=O(\log x)$ ist im Falle einer Irrationalität mit beschränkten Kettenbruchnennern, also insbesondere für jede reelle quadratische Irrationalität[1]).

Es sei α eine positive Irrationalzahl mit der Kettenbruchentwickelung[2])

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

Ich bilde die Reihe der Näherungsbrüche von α: $\frac{p_i}{q_i}$. Dann gilt

$$q_{i+1} = a_{i+1}q_i + q_{i-1},\quad q_0 = 1,\ q_1 = a_1. \qquad (p_i, q_i) = 1$$

Ferner gilt bekanntlich

$$\alpha - \frac{p_i}{q_i} = \frac{\vartheta_i}{q_i q_{i+1}},\ |\vartheta_i| < 1.$$

Ich bezeichne nun durch $R(\beta)$ die nicht negative Zahl $\beta - [\beta]$ und betrachte für ganze $x \geq 1$ die Summe

$$S(x) = \sum_{n=1}^{x}\left(R(n\alpha)-\frac{1}{2}\right).$$

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[1]) G. H. HARDY and J. E. LITTLEWOOD. Some Problems of Diophantine Approximation. Proceedings of the fifth int. congress of math. in Cambridge. 1913. Vol. I, p. 229. Der HARDY-LITTLEWOODsche Beweis dieses Satzes ist meines Wissens bisher noch nicht veröffentlicht worden.

[2]) Über die benutzten Sätze der Kettenbruchtheorie vergleiche man etwa O. PERRON, Die Lehre von den Kettenbrüchen, Leipzig und Berlin, 1913, besonders das zweite Kapitel.

An introduction to Diophantine approximation

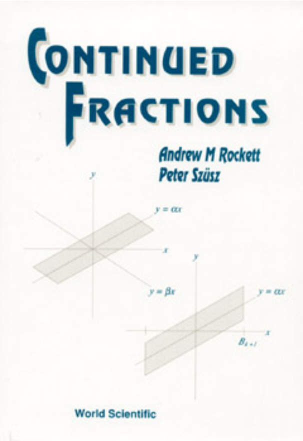
Definition A **continued fraction expansion** $[a_0; a_1, \dots, a_k, \dots]$ is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

For a real number a, we say $[a_0; a_1, \dots, a_k, \dots]$ is **continued fraction expansion of** a if $a = [a_0; a_1, \dots, a_k, \cdots]$ and $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}_{>0}$ for $i > 0$.

Fact. The continued fraction expansion of a is periodic iff a is a quadratic irrational.

$\varphi = [1; 1, \dots,]$ and $\sqrt{2} = [1; 2, \dots,]$ and $\sqrt{3} = [1; 1, 2, 1, 2, \dots]$.


CONTINUED FRACTIONS
Andrew M Rockett
Peter Szüsz
World Scientific

Definition. Let $q_{-1} := 0$, $p_{-1} := 1$ and $q_0 = 1$, $p_0 = a_0$ and for $k \geq 0$,

$$q_{k+1} := a_{k+1} \cdot q_k + q_{k-1},$$
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For φ: $q_k = F_k$ and $p_k = F_{k+1}$, so $\beta_k = F_k\varphi - F_{k+1}$.
For $\sqrt{2}$: $q_1 = 2$, $q_2 = 5$, $q_3 = 12$,..., $p_1 = 3$, $p_2 = 7$, $p_3 = 17$,...

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Definition. The k-**th difference of** a is defined as $\beta_k := q_k a - p_k$.

Fact. $\beta_k > 0$ iff $\beta_{k+1} < 0$.

Ostrowski representation (1921). Let $N \in \mathbb{N}$. Then N can be written uniquely as

$$N = \sum_{k=0}^{n} b_{k+1} q_k,$$

where $b_k \in \mathbb{N}$ such that $b_1 < a_1$, $b_k \leq a_k$ and, if $b_{k+1} = a_{k+1}$, $b_k = 0$.

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Zeckendorf representation (1972). Let $N \in \mathbb{N}$ and F_k be the k-th Fibonacci number. Then N can be written uniquely as

$$N = \sum_{k=1}^{n} b_{k+1} F_k,$$

where $b_k \in \{0, 1\}$ and if $b_{k+1} = 1$, then $b_k = 0$.

Ostrowski representation of a real number. Let $c \in \mathbb{R}$ be such that $-\frac{1}{\zeta_1} \leq c < 1 - \frac{1}{\zeta_1}$. Then c can be written uniquely in the form

$$c = \sum_{k=0}^{\infty} b_{k+1}\beta_k,$$

where $b_k \in \mathbb{N}$, $0 \leq b_1 \leq a_1 - 1$, $b_k \leq a_k$, for $k \geq 1$, and $b_k = 0$ if $b_{k+1} = a_{k+1}$, and $b_k \neq a_k$ for infinitely many odd k.

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Zeckendorf representation for real numbers. Let $c \in \mathbb{R}$ be such that $-\frac{1}{\varphi} \leq c < 1 - \frac{1}{\varphi}$. Then c can be written uniquely in the form

$$c = \sum_{k=1}^{\infty} b_{k+1}(F_k\varphi - F_{k+1}),$$

where $b_k \in \{0,1\}$ and if $b_{k+1} = 1$, then $b_k = 0$, and $b_k \neq 1$ for infinitely many odd k.

Addition in Ostrowski representation (H.-Terry (2015)). Suppose a is quadratic. Then the graph of addition of natural numbers in Ostrowski representation can be recognized by a finite automaton.

Surprisingly this was only known for some quadratic numbers (for example when $a = \varphi$). We used the ideas of Ahlbach, Usatine, Frougny, Pippenger (2013) to give an elementary three-pass algorithm.

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If α is a quadratic real number, then the theory of $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$ is decidable. What if α is not quadratic?

Observation. If the continued fraction expansion of α is non-computable, then the theory of $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$ is undecidable.

Open question I: What happens when the continued fraction expansion is nice, but not periodic? For example, $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$.

Open question II: For arbitrary α what can be said about definable sets in $(\mathbb{R}, <, +, \mathbb{N}, \alpha\mathbb{N})$?

Recall: Let $a, b \in \mathbb{R}$ be such that $1, a, b$ is $\mathbb{Q}$-linearly independent. Then $(\mathbb{R}, <, +, \mathbb{N}, a\mathbb{N}, b\mathbb{N})$ defines multiplication on $\mathbb{R}$.

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Open question III: (Cobham's theorem) Is there a set definable in both $(\mathbb{R}, <, +, \mathbb{N}, a\mathbb{N})$ and $(\mathbb{R}, <, +, \mathbb{N}, b\mathbb{N})$ that is not definable in $(\mathbb{R}, <, +, \mathbb{N})$?

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$(\mathbb{R}, <, +, \mathbb{Q}, x \mapsto \alpha x)$ definable in $(\mathbb{R}, <, +, \cdot, 2^{\mathbb{Q}}, x \mapsto x^{\alpha})$. There are non-algebraic α for which the latter structure is well-behaved.