Proof Compression and $\mathcal{NP}$ vs $\mathcal{PSPACE}$

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§1. Reminder -1-

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$$\boxed{x \in L \Leftrightarrow \left(\exists u \in \{0,1\}^{p(|x|)}\right) M(x,u) = 1},$$

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- $L \subseteq \{0,1\}^*$ is in $\mathcal{PSPACE}$ if there exists a polynomial p and a TM M such that for every input $x \in \{0,1\}^*$, the total number of non-blank locations that occur during M's execution on x is at most $p(|x|)$, while $x \in L \Leftrightarrow M(x) = 1$.

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- $\mathcal{NP} = co\mathcal{NP}$ (resp. $\mathcal{NP} = \mathcal{PSPACE}$) follows from global polynomial-size provability of tautologies in classical and/or intuitionistic (resp. minimal) logic.

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- $\mathcal{NP} = co\mathcal{NP}$ (resp. $\mathcal{NP} = \mathcal{PSPACE}$) follows from global polynomial-size provability of tautologies in classical and/or intuitionistic (resp. minimal) logic.
- **Claim** [L.G.+E.H.Haeusler]: $\mathcal{NP} = \mathcal{PSPACE}$ is provable by DAG-like proof-compression techniques in Prawitz's Natural Deduction for minimal logic.

§2. The proof [L.G.]: Overview

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1. Formalize minimal propositional logic as fragment $\mathrm{LM}_{\rightarrow}$ of Hudelmaier's tree-like cutfree intuitionistic sequent calculus. For any $\mathrm{LM}_{\rightarrow}$ proof ∂ of sequent $\Rightarrow \rho$:
    1. $h(\partial)$ (= the height) is polynomial (actually linear) in $|\rho|$,
    2. $\phi(\partial)$ (= total number of formulas) and $\mu(\partial)$ (= maximal formula length) are also polynomial in $|\rho|$.
2. Show that there exists a constructive (1)+(2) preserving embedding F of $\mathrm{LM}_{\rightarrow}$ into Prawitz's tree-like natural deduction formalism $\mathrm{NM}_{\rightarrow}$ for minimal logic.
3. Elaborate polytime verifiable DAG-like deducibility in $\mathrm{NM}_{\rightarrow}$.
4. Elaborate and apply *horizontal tree-to-DAG proof compression* in $\mathrm{NM}_{\rightarrow}$. For any tree-like $\mathrm{NM}_{\rightarrow}$ input ∂, the weight of DAG-like output ∂^{C} is bounded by $h(\partial) \times \phi(\partial) \times \mu(\partial)$. Hence the weight of $(F(\partial))^{\mathrm{C}}$ for any given tree-like $\mathrm{LM}_{\rightarrow}$ proof ∂ of ρ is polynomially bounded in $|\rho|$. Since minimal logic is PSPACE-complete, conclude that $\mathcal{NP} = \mathcal{PSPACE}$.

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Recall that the validity problem for minimal propositional logic is PSPACE-complete. It will suffice to show that it is a NP problem. So consider any purely implicational formula ρ. By Hudelmaier's result, ρ is valid in the minimal logic iff there exists a tree-like $\mathrm{LM}_{\rightarrow}$ proof ∂ of ρ. Hence, by the embedding theorem and soundness and completeness of DAG-like $\mathrm{NM}_{\rightarrow}$, ρ is valid in the minimal logic iff we can "guess" a DAG-like $\mathrm{NM}_{\rightarrow}$ proof $\widehat{\partial}$ of ρ, whose weight is polynomial in $|\rho|$ (witness: $(F(\partial))^{\mathrm{C}}$). Moreover, we know that ' $\widehat{\partial}$ *is an encoded DAG-like* $\mathrm{NM}_{\rightarrow}$ *proof of* ρ ' is decidable in polynomial time with respect to $|\rho|$. Thus the existence of DAG-like $\mathrm{NM}_{\rightarrow}$ proof of ρ is verifiable in polynomial time by a non-deterministic algorithm, and hence so is the problem of ρ validity in the minimal logic, Q.E.D. □

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3. This is good, but not good enough. Because polynomial bounds on the number of subformulas fail to provide polynomial bounds on the number of sequents involved. In contrast, proof objects of natural deductions are single formulas, so there is a hope to overcome this obstacle.
4. However, full compression of natural deductions should be weakened to (say) *horizontal compression*, to save Prawitz's discharging rule(s). But this weakening still yields the result, provided that the height and the total number of formulas are polynomially bounded (via emebedding of sequent proofs).

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- The operation $\partial \hookrightarrow \partial^{\mathrm{C}}$ (called *horizontal compression*) runs by bottom-up recursion on $h(\partial)$ such that for any $n \leq h(\partial)$, the n^{th} horizontal section of ∂^{C} is obtained by merging all nodes with identical formulas occurring in the n^{th} horizontal section of ∂ (this operation is called *horizontal collapsing*).

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- Thus the horizontal compression is obtained by bottom-up iteration of the horizontal collapsing.
- The size and weigth estimates $|\partial^{\mathrm{C}}| \leq h(\partial) \times \phi(\partial)$ resp. $\|\partial^{\mathrm{C}}\| \leq h(\partial) \times \phi(\partial) \times \mu(\partial)$ are obvious, as the size of every (compressed) n^{th} horizontal section of ∂^{C} can't exceed $\phi(\partial)$.

§3. Background: Crucial technical features -1-

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- Thus every DAG-like deduction requires additional information on "legitimate" maximal deduction threads that determine the sets of open/closed assumptions. This is achieved by adding a suitable function ℓ^{G} that determines "legitimate" parents of inverse-branching nodes (regarded as "road signs" showing allowed ways from the leaves down to the root).

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- Thus every DAG-like deduction requires additional information on "legitimate" maximal deduction threads that determine the sets of open/closed assumptions. This is achieved by adding a suitable function ℓ^{G} that determines "legitimate" parents of inverse-branching nodes (regarded as "road signs" showing allowed ways from the leaves down to the root).
- A given DAG-like deduction with root formula ρ is called a proof of ρ iff all assumptions are closed.

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- Horizontal compression is supplied with corresponding ℓ^{G} compression that preserves closed assumptions (and hence provability). So if ∂ is a canonical tree-like proof of ρ then ∂^{C} is a DAG-like proof of ρ.
- DAG-like provability in question is encoded by appropriate *local proof correctness* conditions that are polytime verifiable (just as in standard tree-like case).

§4. Hudelmaier's sequent calculus $\mathrm{LM}_{\rightarrow}$ -1-

Axiom and rules of implicational minimal logic:

$$(\mathrm{M}A): \qquad \Gamma, p \Longrightarrow p$$

$$(\mathrm{M}I1 \to): \quad \frac{\Gamma, \alpha \Longrightarrow \beta}{\Gamma \Longrightarrow \alpha \to \beta} \ [(\nexists\gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]$$

$$(\mathrm{M}I2 \to): \quad \frac{\Gamma, \alpha, \beta \to \gamma \Longrightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Longrightarrow \alpha \to \beta}$$

$$(\mathrm{M}E \to P): \frac{\Gamma, p, \gamma \Longrightarrow q}{\Gamma, p, p \to \gamma \Longrightarrow q} \ [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q]$$

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$\mathrm{LM}_{\to}$ is sound and complete with respect to minimal propositional logic and tree-like deducibility. So any given formula ρ is valid in the minimal logic iff sequent $\Longrightarrow \rho$ is tree-like deducible in $\mathrm{LM}_{\to}$. Moreover:

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1. *The height of any tree-like $\mathrm{LM}_{\rightarrow}$ deduction ∂ of sequent S is linear in $|S|$. In particular if S is $\Longrightarrow \rho$, then $h(\partial) \leq 3|\rho|$.*

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2. *The foundation of any tree-like* $\mathrm{LM}_{\to}$ *deduction* ∂ *of sequent* S *is at most quadratic in* $|S|$*. In particular if* S *is* $\Longrightarrow \rho$*, then* $\phi(\partial) \leq (|\rho| + 1)^2$*, while* $|\alpha| \leq |\rho|$ *for any* α *occurring in* ∂*.*

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There exists a recursive operator F that transforms any given tree-like $\mathrm{LM}_{\rightarrow}$ deduction ∂ of $\Gamma \Longrightarrow \rho$ into a tree-like $\mathrm{NM}_{\rightarrow}$ deduction $F(\partial)$ with root-formula ρ and assumptions occurring in Γ. Moreover ∂ and $F(\partial)$ share the semi-subformula property, linearity of the height and polynomial upper bounds on the foundation. In particular if $\Gamma = \emptyset$, then $F(\partial)$ is a $\mathrm{NM}_{\rightarrow}$ proof of ρ such that:

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3. $\mu(F(\partial)) \leq 2|\rho|$.