Mathematical Practice and Human Cognition

Remarks on Quinn's "Science of Mathematics"

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Overview

- Introduction: Remark on mathematical practice
- Frank Quinn's Contributions (to a Science of Contemporary Mathematics)
- Mathematical Concepts
  - A historical example
  - Mathematical practice: Defining concept (FQ)
  - Mathematical practice: Acquisition of concepts (FQ)
  - Mathematical practice: Corroboration by math ed (cogn. sci.)
  - *Mathematical practice: convergence with phenomenology

Introduction: Remark on mathematical practice (MP)

Three meanings of MP

- MP in Math Ed and PME
- MP in "traditional" PoM: Kitcher (1984), Tymoczko (1986), Mancou (2008)
- MP in "new" PoM: PhiMSAMP (2006–2011)
  - Deliberate inclusion of insights from various disciplines

Introduction: Remark on mathematical practice (MP)

MP as a culture

- "that complex whole that includes knowledge, belief, art, morals, law, custom and any other capabilities and habits acquired by [mathematicians] as members of [their trade]." Edward Tylor, *Primitive Culture* (1871), vol. 1, p. 1

Frank Quinn: Relevant Publications

- 1992: "Theoretical Mathematics" (BAMS; jointly w/ A. Jaffe)
- 2011: "Science-of-Learning Approach to MathEd" (NAMS)
- 2012: "Revolutions in Mathematics?" (NAMS)

- 2011: *Contributions to a Science of Mathematics*

- Quinn's three periods
  - I. ??–1600: "qualitative and philosophical"
  - II. 1600–late 19th c: "quantitative and mathematical" scientific needs; elite practioner syndrome
  - III. late 19th c through Hilbert's Göttingen–?? ontologically autonomous, methodologically unique

Quinn's Third Period: Rigor

- Traumatic transition
  "the changes were forced by [the] increasingly difficulty of the mathematics and [the] ambition of the profession."
- Methodology
  Rigorous definitions along with "genuinely error-displaying methods" secure the "complete reliability" of all mathematical conclusions.
- "The slavish devotion of mathematicians to rigorous methodology is required by the subject [...] Rigor plays the same role in mathematics that agreement with the physical world plays in other sciences. Relaxing rigor is like ignoring data."

Concepts: Continuity as an example

- Period I. Philosophy and application: Leibniz' principle of continuity
- Period II. Quantitative and mathematical: ϵ-δ approach Cauchy, building on d'Alembert, Euler, Lagrange, followed by Bolzano, Dedekind, Weierstrass
- Period III. Purely mathematical: topological definition Maurice Fréchet, Frigyes Reisz (not Marcel), Felix Hausdorff, Kazimierz Kuratowksi, among others

Concepts: Continuity a a topological notion

Definition. *A topological space $\langle X, \mathcal{T} \rangle$ is a set X together with a topology $\mathcal{T}$, i. e., a family of open subsets of X, such that*

1. *$\emptyset$ and X are both open,*
2. *arbitrary unions of open sets are open,*
3. *finite intersection of open sets are open.*

Definition. *A function $f : S \to T$ between two topological spaces is continuous iff the pre-image $f^{-1}(Q)$ of every open set $Q \subset T$ is an open set $P \subset S$.*

Concepts: Quinn's question

- Increase in rigor and loss of experiential or intuitive contents result in a concentration on the mathematical substance
- Definitions are *not* simply a codification of an intuitive understanding" but "were developed and refined over long periods and with great effort," and were, in fact, "frequently a community effort."
- Quinn's question: How do human agents acquire such concepts?

Concepts: Quinn's answer

1. Sever as many ties to ordinary language as possible and limit ordinary language explanations to an absolute minimum

2. Introduce axiomatic definitions and bundle them up with a sufficient number of examples, lemmata, propositions, etc. into small cognitive packages

3. Have students practice hard with one new cognitive package at a time

4. Lather, rinse, repeat.

Concepts: How did it evolve and into what?

- Natural selection: those who did adopt another approach could no longer compete and eventually sank into oblivion

- Outcomes:
  - Core mathematics (vs mathematical sciences)
  - Empowering rank-and-file faculty (vs elite-practioner)
  - Mathematical altruism: faculty develop habits that support and nurse such practices of conceptual and methodic rigor

Quinn: Is he right?

- Soft empirical evidence
  - Quinn's own expert testimony
  - Graduate level textbooks
- Hard empirical evidence?
  - Well, 2nd part of the talk ;-)

Digression: PoMP & Quinn is right?

- Traditional PoM reduced cognitive labor to deductive proof
  - Legacy of logicism
- A PoMP may realize that such a reduction is wrong
  - Philosophy becomes richer and much more complex

Evidence from CS, MathEd, PME: Caution

- General caution: "reproducibility crisis" (Nosek 2015, Nature 2016, and representability (eg, Heinrich et al. 2010: WEIRD)
- Caution re neuroimaging: It's too early to tell

- Caution re MathEd/PME
  - Undue influence of P&P
  - No focus on advanced mathematics
  - Lack of empirical reliability: sample sizes, reproducibility
  - Lack of theoretical sophistication (eg, Anderson&Reder&Simon (2000): "Applications and Misapplications of Cognitive Psychology to Mathematics Education")

Supporting evidence from MathEd/PME (and CS)

Quinn's No 1: Sever ties to ordinary language

- From lexical decision task to priming: fact or fiction?
  (eg, Kahneman 2012 letter)
- Embodied knowledge and met-befores
  (eg, Tall 2008, 2013)
- Generic extension principle & epistemic obstacles
  (eg, Tall 1986; Cornu 1982, Sierpiǹska 1985ab)
- CS: Importance of inhibition control
  (eg, Houdé&Tzourio-Mazoyer 2003, Houdé&Borst 2015)

Supporting evidence from MathEd/PME (and CS)

Quinn's No 2: Cognitive packaging: definitions plus exercises

- Adding properties (ie, meaning) and fluidity (ie, mastery) (eg, Dreyfus 1991)
- Concept definition vs concept image (eg, Vinner 1983, 1991)
- CS: Package size matters (eg, Anderson&Lee&Fincham 2014); inhibition control (eg, Houdé et al., op cit.)

Supporting evidence from MathEd/PME (and CS)

Quinn's No 3: Practice hard!

1. Automation: load issues (eg, Thurston 1990: compressibility; Lee&Ng&Ng 2009: word problems)

2. Mathematical "Habits of Mind"
   (eg, Selden&Lim 2010; Wilkerson-Jerde&Wilensky 2009, 2011: novices vs experts)

Qualifying evidence

- Contradicting evidence
  - Different cultures: Mathematicians responding to Jaffe-Quinn
  - Tall 2013: Introduction
- Enriching evidence
  - Studies that lend support for Quinn's thesis also provide a much richer, higher-resolution picture of the cognitive processes involved

Enriching perspectives from MathEd/PME (and CS)

Quinn's No 1: Sever ties to ordinary language

- Continuity and motivation:
  conceptual-embodied – proceptual-symbolic – axiomatic formal (eg, Tall 2008, 2013)

- Continuity and generalization vs abstraction:
  $\mathbb{R}^n$ vs vector space
  (eg, Dreyfuss 1991, Dubinsky 1991, Vinner 1991)

Enriching perspectives from MathEd/PME (and CS)

Quinn's No 2: Cognitive packaging: definitions plus exercises

- Deduction vs construction: building properties of abstract objects
- Concept definition vs concept image: focus on generic or otherwise disrupting images

Enriching perspectives from MathEd/PME (and CS)

Quinn's No 3: Practice hard!

- Fluidity among images
  (eg, Dreyfus 1991; Tall 2013)
- Reification: point-wise vs. object-valued operators – focus enhancing (eg, Harel&Kaput 1991)

Some examples)

1. Concept definition vs concept image (generic images)

   - Fluidity among images
     (eg, Dreyfus 1991; Tall 2013)

   - Reification: point-wise vs. object-valued operators – focus enhancing (eg, Harel&Kaput 1991)

Some examples)

1. Conceptual entititios (reification)

   - Fluidity among images
     (eg, Dreyfus 1991; Tall 2013)

   - Reification: point-wise vs. object-valued operators – focus enhancing (eg, Harel&Kaput 1991)

Summary

1. While empirical findings lend support to Quinn's claims about the adequacy of the mathematical practice as he describes it (as a three-step program), they also suggest that a more nuanced approach is advisable.

2. Empirical findings strongly suggest that MP includes a plethora of cognitive processes that go beyond deductive proof; therefore,

   PoM is dead, long live PoMP!

Thank You!