# Tangles of Graphs and Datasets

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Datum der Disputation:

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# 1
# Introduction

Tangles are an abstract way of describing highly connected substructures, typically of graphs or datasets. The idea is that, since highly connected substructures are difficult to separate into two roughly equally sized parts, they mostly lie on one or the other side of every low-order separation; we say that they induce an orientation of the low-order separations. Tangles are a generalisation of this concept: a tangle is any such orientation of low-order separations which, in some formal sense, points towards a sufficiently large region.

Tangles were first introduced for graphs by Robertson and Seymour [1] and have since been generalised to abstract contexts [2] as well as to data-analytic and sociological applications [3].

This thesis seeks to connect the insights gained during the development of [4] with abstract tangle theory. Many of the theoretical applications required that the forbidden sets of oriented separations that define a tangle are so-called stars, while practical applications required more general forbidden subsets. This thesis fills that gap with a new framework for tangles. Central to this framework is the tangle structure tree, a unified structure that records both the tangles and certificates showing why other orientations are not tangles, in a structurally efficient way.

In chapter 2 we develop the abstract framework of tangle structure trees and prove that they exist precisely under a natural richness condition on $\mathcal{F}$.

In chapter 3 we apply this framework to recover and extend the two fundamental results about tangles: the tree-of-tangles theorem and the tangle-tree duality theorem.

In chapter 4 we apply tangles to a psychological questionnaire, clustering its questions to gain deeper insights into the questionnaire's structure, uncovering weaknesses in the current setup.

In chapter 5 we return to graph tangles and study whether every $k$-tangle in a graph is induced by a set of vertices by majority vote, and reduce this problem to graphs of bounded size.

# 2

# Tangle structure trees

We introduce a comprehensive data structure, tangle structure trees, which simultaneously displays all the $\mathcal{F}$-tangles of an abstract separation system for very general obstruction sets $\mathcal{F}$. It simultaneously also displays certificates $\sigma \in \mathcal{F}$ for any non-existence of such tangles, or for the non-extendability of low-order tangles to higher-order ones.

Our theorem can be applied to produce the structures of the classical tree-of-tangles and tangle-tree duality theorems, both for graph tangles and for their known generalizations to more general separation systems. It extends those theorems to obstruction sets $\mathcal{F}$ that need not define profiles (as they must in trees of tangles) or consist of stars of separations (as they must in tangle-tree duality).

Our existence proof for these structure trees is constructive. The construction has been implemented in open-source software available for tangle detection and further analysis.

## Contents



## 2.1 Introduction

The notion of ‘tangles’ was originally introduced by Robertson and Seymour [1] as an abstract concept of high local connectivity in graphs, one that unifies several more concrete such notions, such as highly connected subgraphs or minors. What all these notions have in common is that, given any low-order separation of the graph, any highly cohesive substructure must lie mostly on one of its two sides: since the separation has low order, it cannot split it into two roughly equal halves. A tangle remembers only how all the low-order separations are oriented in this way, each towards that highly cohesive substructure: the collection of all these oriented separations is called a *tangle*. See [5] for a precise definition and basic facts about graph tangles.

Tangles have since been generalized to more general settings than graphs. In the setting of set partitions, they offer a precise theoretical basis for 'fuzzy' real-world problems such as clustering in large datasets [3]. All these, including graphs, are special cases of so-called *abstract separation systems* [2]: an axiomatic setting that assumes only the most basic properties of graph separations. This is the setting in which tangles are most easily and comprehensively treated, and we shall use this framework also in this paper.

There are two fundamental theorems about tangles, which both have their origins in [1] and are also treated in [5]. The *tree-of-tangles* theorem exposes a tree-like structure in the graph or dataset whose tangles we consider, which 'distinguishes' the tangles in that they are shown to 'live in' different areas of this tree-like structure. The other is the *tangle-tree duality* theorem, which certifies the non-existence of a tangle by exposing another tree-like structure in the graph or dataset, one whose existence clearly precludes the existence of a tangle.[1] This is the more fundamental of the two theorems, in the sense that the other can be reduced to its abstract version (see below) but not vice versa [6] .

Both the above theorems have been generalized to $\mathcal{F}$-tangles of abstract separation systems, ways of simultaneously orienting the 'separations' of a given structure in such a way that no three of those oriented separations form an element of $\mathcal{F}$.[2] This collection $\mathcal{F}$ has to satisfy some constraints that depend on which of the two theorems we are considering.

For the (known) tree-of-tangles theorems, those which display all the $\mathcal{F}$-tangles of a separation system simultaneously, $\mathcal{F}$ has to include triples of oriented separations that are reminiscent of ultrafilters. If our separations are bipartitions of a fixed set $V$, for example, then $\mathcal{F}$ must contain all triples of the form $\{A, B, \overline{A \cap B}\}$: if $A$ and $B$ are subsets of $V$ deemed 'large', in that the $\mathcal{F}$-tangle orients the partitions $\{A, \overline{A}\}$ and $\{B, \overline{B}\}$ towards them, then the complement of $A \cap B$ cannot also be 'large'. Such $\mathcal{F}$-tangles are called *profiles*, and all existing tree-of-tangle theorems for $\mathcal{F}$-tangles assume that these are profiles [7, 8, 9, 10, 11, 12, 13, 14]. There are even a tree-of-tangles theorems that display profiles of different 'order' simultaneously. Such profiles, however, must be *robust*: they also must not contain certain triples similar to, but slightly different from, those above. More about profiles in section 2.7.[3]

For the tangle-tree duality theorems, $\mathcal{F}$ has to satisfy another constraint: its elements have to be *stars*, nested sets of oriented separations pointing towards each other [2]. The tree-like structures by which tangle-tree duality theorems certify the non-existence of $\mathcal{F}$-tangles require that $\mathcal{F}$ consist of stars: if it does not, these structures cannot be tree-like.

This state of the art leaves $\mathcal{F}$-tangles that are not profiles without any known way of displaying all those tangles simultaneously. And if $\mathcal{F}$ does not consist of stars, it leaves

[1] In a graph, this would be a tree-decomposition into parts too small for a tangle to live in.

[2] In the case of graph tangles, $\mathcal{F}$ consists of the sets of up to three oriented separations of the graph such that the union of the 'back' sides of these three separations covers the entire graph [5].

[3] In a follow-up paper [15] we show how our structure trees give rise to trees of $\mathcal{F}$-tangles whenever these are robust, even if they are not profiles.

separation systems that have no $\mathcal{F}$-tangles without any known way of organizing the elements of $\mathcal{F}$ into a data structure that displays them as easily checkable certificates for the non-existence of such tangles.

The structure trees whose existence we prove in this paper do both these things: they display all the $\mathcal{F}$-tangles of a separation system even when they are not profiles, and for those orientations of the separation system that are not $\mathcal{F}$-tangles they display certificates in $\mathcal{F}$ that show why they are not.[4]

Tangle structure trees display all this information in a single, comprehensive, data structure, which is maximally efficient in the following, structural, sense. A single tangle is most efficiently displayed by listing just those of its elements that are minimal in the poset of oriented separations that comes with a separation system. Indeed, all these are needed to determine the tangle, but any other elements can be deduced from them. Our structure trees display all the $\mathcal{F}$-tangles of a separation system $S$ simultaneously by listing only the oriented separations that are minimal in one of those tangles, including the tangles of subsystems consisting of separations of lower order. And for orientations of $S$ that are not $\mathcal{F}$-tangles it displays certificates from $\mathcal{F}$ inside those same minimal subsets of oriented separations, in a way readily accessible in the structure tree.

Our paper is organized as follows. In section 2.2 we provide the necessary background for tangles in abstract separation systems, the framework in which we shall construct our structure trees and prove their existence.

In section 2.3 we introduce basic tangle structure trees. In section 2.4 we prove their existence, and in section 2.5 we show how to make them efficient. In section 2.6 we collect all this information together to prove our main results, the '$\mathcal{F}$-tangle structure theorems'. In theorem 2.6.5 we note specifically how the main structure theorem provides certificates from $\mathcal{F}$ for the non-existence of $\mathcal{F}$-tangles if there are none.

In section 2.7, finally, we apply our results to three particular types of $\mathcal{F}$-tangles. The first two include $\mathcal{F}$-tangles in graphs: those induced by $k$-blocks, and those that are profiles. In neither of these does the defining set $\mathcal{F}$ consist of stars, so traditional tangle-tree duality fails. We derive new dichotomy theorems for these.

In our third application we treat $\mathcal{F}$-tangles that encode clusters in large datasets. These are neither profiles nor do their obstruction sets $\mathcal{F}$ consist of stars. Our structure trees display all the clusters by way of their $\mathcal{F}$-tangles, while also witnessing the absence of clusters from the other areas of the data set by displaying elements of $\mathcal{F}$.

## 2.2 Tangle basics

In this section we give precise definitions and notation for abstract tangles, largely following [2] and indicating any deviations.[5]

Tangles of graphs are ways of orienting their separations, each towards one of its two sides. Abstract tangles are designed to work in scenarios where there need not be anything to 'separate'. In order to retain our intuition from graphs, however, we continue to refer to the things of which our abstract tangles pick one of two variants

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[4] In a follow-up paper [15] we show that if $\mathcal{F}$ does consist of stars, our duality theorem implies the same tangle-tree dichotomy as the classical one.

[5] The most important difference is that, for historical reasons, the partial ordering on $\vec{S}$ used in [2] is the inverse of ours. So terms like 'large' and 'small', infima and suprema etc, are reversed.

(which they will indeed do) as ‘separations’. These are defined by noting some key properties of graph separations and making them into axioms, as follows.

A *separation system* $(\vec{S}, \le, {}^*)$ is a set $\vec{S}$, whose elements we call *oriented separations*, that comes with a partial ordering $\le$ on $\vec{S}$ and an order-reversing involution ${}^* : \vec{S} \to \vec{S}$. Thus, for any two elements[6] $\vec{r}, \vec{s}$ of $\vec{S}$ with $\vec{r} \le \vec{s}$ we have $\vec{r}^* \ge \vec{s}^*$. We write $\vec{s}^* =: \overleftarrow{s}$, and call $\overleftarrow{s}$ the *inverse* of $\vec{s}$. While we allow formally that $\vec{s} = \overleftarrow{s}$, in which case we call $\vec{s}$ and $s$ *degenerate*, this does not happen often in practice.[7]

If a separation system $\vec{U}$ happens to be a lattice, that is, if there is a supremum $\vec{r} \vee \vec{s}$ and an infimum $\vec{r} \wedge \vec{s}$ in $\vec{U}$ for every two elements $\vec{r}, \vec{s} \in \vec{U}$, we call $\vec{U}$ a *universe* of separations. It is *distributive* if it is distributive as a lattice. A separation system $\vec{S} \subseteq \vec{U}$ is *submodular* if for every two elements of $\vec{S}$ either their infimum or their supremum in $\vec{U}$ also lies in $\vec{S}$.

Very rarely we may have separations $\vec{s} \le \overleftarrow{s}$; then $\vec{s}$ is *small* and $\overleftarrow{s}$ is *large*.[8] We say that $\vec{s}$ is *trivial* (and $\overleftarrow{s}$ is *co-trivial*) in $\vec{S}$ if there exists a pair of inverse separations $\vec{r}, \overleftarrow{r} < \vec{s}$ in $\vec{S}$. Trivial separations are clearly large, so co-trivial ones are small, but the converse need not hold. See [2] for more on these technicalities if desired.

The set of *unoriented separations* in $(\vec{S}, \le, {}^*)$ is

$$S := \{\{\vec{s}, \overleftarrow{s}\} : \vec{s} \in \vec{S}\}.$$

We call the elements $\vec{s}, \overleftarrow{s}$ of $s$ its *orientations*. An *orientation of* $S$ is a set $\tau \subseteq \vec{S}$ that contains exactly one orientation of every $s \in S$. For $s \in S$ we then denote by $\tau(s)$ the unique orientation of $s$ contained in $\tau$. An orientation of a subset of $S$ is a *partial orientation* of $S$.

If $\vec{r} \ge \vec{s}$ we say that $\vec{r}$ *points towards* $s$ (and that $\overleftarrow{r}$ *points away from* $s$). We say that $\vec{r}$ *points towards* an oriented separation $\vec{s}$ whenever it points towards $s$, i.e., if $\vec{r} \ge \vec{s}$ or $\vec{r} \ge \overleftarrow{s}$, and similarly for ‘points away from’. A *star* is a set $\sigma$ of non-degenerate oriented separations that point towards each other. As is easy to check, this happens if and only if $\vec{r} \ge \overleftarrow{s}$ (and hence $\vec{s} \ge \overleftarrow{r}$) for all distinct $\vec{r}, \vec{s} \in \sigma$.

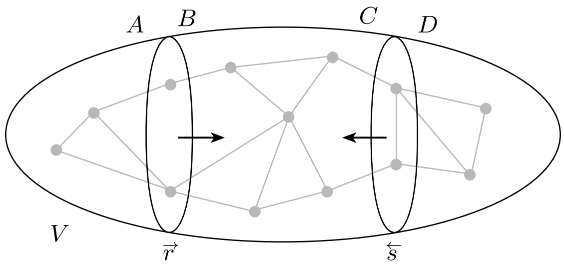


**Figure 2.1**: Nested separations $r = \{A, B\}$ and $s = \{C, D\}$ of a graph. Their orientations $\vec{r} = (A, B)$ and $\overleftarrow{s} = (D, C)$ point towards each other, since $\vec{r} \ge \vec{s}$ (as $B \supseteq D$) and $\overleftarrow{s} \ge \overleftarrow{r}$ (as $C \supseteq A$).

Two separations $s, r \in S$ are *nested* if they have orientations that are comparable under $\le$. Oriented separations are *nested* if their underlying unoriented separations are nested. A subset of $\vec{S}$ is *nested* if its elements are pairwise nested.

[6] We often denote the elements of $\vec{S}$ by letters with an arrow, in either direction, precisely in order to have a simple way to refer to their dual elements: by reversing the arrow. But the arrow directions have no meaning: an arbitrary element of $\vec{S}$ could be denoted equally as $\vec{s}$ or as $\overleftarrow{s}$.

[7] The only degenerate separation of a graph $G = (V, E)$, for example, is $\{V, V\}$.

[8] The small separations of a graph $G$ are those of the form $(V, A)$ with $A \subseteq V = V(G)$.

A subset of $\vec{S}$ is *consistent* if no pair of its elements $\vec{r}, \vec{s}$ with $r \neq s$ point away from each other. Stars are examples of consistent nested sets of oriented separations.

We shall often be interested in consistent orientations of $S$. For each of its elements $\vec{r}$, a consistent orientation of $S$ will also contain every $\vec{s} > \vec{r}$ other than, possibly, $\vec{s} = \overleftarrow{r}$. Consistent partial orientations of $S$ are easily seen to extend to consistent orientations of $S$, unless they contain a separation that is co-trivial in $\vec{S}$; see [2].

If $\sigma \subseteq \vec{S}$ is consistent, we say that $\vec{s} \in \vec{S}$ is *required* by $\sigma$ if $\vec{s} \notin \sigma$ and $\sigma \cup \{\overleftarrow{s}\}$ is inconsistent. We shall see in theorem 2.2.2 that, pathological cases aside, $\sigma \cup \{\vec{s}\}$ will then be consistent. The *closure* of $\sigma$ is

$$\lfloor \sigma \rfloor := \sigma \cup \{\vec{s} \in \vec{S} : \vec{s} \text{ is required by } \sigma\}.$$

Note that $\sigma$ requires $\vec{s} \notin \sigma$ if and only if there exists an $\vec{r} \in \sigma$ such that $r \neq s$ and $\vec{s} > \vec{r}$. Thus

$$\lfloor \sigma \rfloor = \sigma \cup \{\, \vec{s} \in \vec{S} : \exists \vec{r} \in \sigma \text{ such that } r \neq s \text{ and } \vec{s} > \vec{r}\},$$

which motivates the term of (upward) 'closure'. If $\lfloor \sigma \rfloor$ is consistent, then $\lfloor \lfloor \sigma \rfloor \rfloor$ is (defined and) easily shown to be equal to $\lfloor \sigma \rfloor$.

If $\sigma$ contains no small separations, the expression above simplifies to

$$\lfloor \sigma \rfloor = \{\, \vec{s} \in \vec{S} : \exists \vec{r} \in \sigma \text{ such that } \vec{r} \leq \vec{s}\}.$$

Indeed, any $\vec{s}$ in this latter set either lies in $\sigma$ or there is some $\vec{r} \in \sigma$ such that $\vec{r} < \vec{s}$; in that case $r \neq s$, since otherwise $\overleftarrow{s} = \vec{r} < \vec{s}$, making $\vec{r} \in \sigma$ is small.

**Lemma 2.2.1.** *Let $\sigma \subseteq \tau \subseteq \vec{S}$ be consistent sets. Then $\lfloor \sigma \rfloor \subseteq \lfloor \tau \rfloor$. If $\tau$ is an orientation of all of $S$, then $\lfloor \tau \rfloor = \tau$.*

*Proof.* If $\vec{s}$ is required by $\sigma$, then $\sigma \cup \{\overleftarrow{s}\}$ is inconsistent. Since $\sigma \subseteq \tau$, it follows that $\tau \cup \{\overleftarrow{s}\}$ is also inconsistent; hence $\vec{s}$ is either an element of or required by $\tau$. Thus every element of $\lfloor \sigma \rfloor$ lies in $\lfloor \tau \rfloor$, proving the first claim.

If $\tau$ is an orientation of all of $S$ then $\lfloor \tau \rfloor = \tau$, since for any $\vec{s} \in \lfloor \tau \rfloor \setminus \tau$ the set $\tau \cup \{\overleftarrow{s}\} = \tau$ would be inconsistent. □

Consistent orientations of $S$ can contain small separations. But examples are rare and can be counter-intuitive, so we often exclude them. Co-trivial separations cannot occur in consistent orientations of $S$. Indeed if $\overleftarrow{s}$ is co-trivial, witnessed by $r \in S$, then every orientation of $S$ will have to orient $r$ too. But it cannot do so consistently with $\overleftarrow{s}$, since both $\vec{r}$ and $\overleftarrow{r}$ are inconsistent with $\overleftarrow{s}$. Similarly, $\lfloor \{\overleftarrow{s}\} \rfloor$ is then inconsistent, since it contains $\vec{r}$ and $\overleftarrow{r}$, which are both inconsistent with $\overleftarrow{s}$.

**Lemma 2.2.2.** *Let $\tau \subseteq \vec{S}$ be consistent, and suppose that $\tau$ has no elements that are co-trivial in $\vec{S}$. Then $\lfloor \tau \rfloor$ is consistent, and $\lfloor \tau \rfloor \setminus \tau$ contains at most one orientation of any $s \in S$.*

*Proof.* If $\lfloor \tau \rfloor$ is inconsistent, it contains orientations $\vec{r}, \vec{s}$ of distinct $r, s \in S$ with $\vec{s} < \overleftarrow{r}$. Since $\tau$ is consistent, at least one of $\vec{r}, \vec{s}$ is not in $\tau$. Without loss of generality assume $\vec{r} \notin \tau$: Then $\vec{r} \in \lfloor \tau \rfloor \setminus \tau$, so $\vec{r}$ is required by $\tau$ (by definition of $\lfloor \tau \rfloor$), which means that $\tau \cup \{\overleftarrow{r}\}$ is inconsistent. Thus there exists $\vec{t} \in \tau$ with $t \neq r$ and $\overleftarrow{r} < \overleftarrow{t}$. We thus have $\vec{s} < \overleftarrow{r} < \overleftarrow{t}$, that is $\vec{s}$ and $\vec{t}$ point away from each other.

If $\vec{s} = \vec{t}$, the above inequality yields $\vec{r}, \overleftarrow{r} < \overleftarrow{t}$, which makes $\vec{t} \in \tau$ co-trivial, contradicting our assumptions. Hence $\vec{s} \neq \vec{t}$ as well as $\vec{s} \neq \overleftarrow{t}$, giving $s \neq t$. The fact that $\vec{s}$ and $\vec{t} \in \tau$ point away from each other thus implies that $\{\vec{s}, \vec{t}\}$ is inconsistent, so $\vec{s} \notin \tau$ as well as $\vec{r} \notin \tau$.

As earlier with $\vec{r}$, the fact that $\vec{s} \in \lfloor \tau \rfloor$ now implies that $\vec{r} < \overleftarrow{s} < \overleftarrow{t'}$ for some $\vec{t'} \in \tau$. The two inequalities together now give $\vec{t'} < \overleftarrow{r} < \overleftarrow{t}$, so $\vec{t'}$ and $\vec{t}$ point away from each other. As $\tau$ is consistent and contains both, this means that $t' = t$. As $\vec{t'} < \overleftarrow{t}$, we thus have $\vec{t'} = \vec{t}$. But now $\vec{r} < \overleftarrow{t'} = \overleftarrow{t}$ as well as $\overleftarrow{r} < \overleftarrow{t}$. This makes $\vec{t} \in \tau$ co-trivial in $\vec{S}$, contradicting our assumptions. This proves the first assertion.

For the second, note that $\tau$ requires any $\vec{s} \in \lfloor \tau \rfloor \setminus \tau$, by definition of $\lfloor \tau \rfloor$. This means that $\tau \cup \{\overleftarrow{s}\}$ is inconsistent, so $\overleftarrow{s} \notin \lfloor \tau \rfloor$ since $\lfloor \tau \rfloor$ is consistent. □

An *order function* on $S$ is any map $S \to \mathbb{R}$. Unless otherwise mentioned, we denote such order functions as $s \mapsto |s|$. We extend them to $\vec{S}$ by letting $|\vec{s}| := |\overleftarrow{s}| := |s|$. A separation system $(\vec{S}, \leq, ^*)$ given with an order function on $S$ is an *ordered separation system.* For every $k \in \mathbb{R}$ we let

$$\vec{S_k} := \{\vec{s} \in \vec{S} : |s| < k\};$$

this is again an ordered separation system.

We are sometimes interested in orientations of $S$ that do not have certain subsets. We typically collect those together in some set $\mathcal{F}$, whose elements we call *forbidden subsets.* Formally, if $\mathcal{F}$ is any set, we say that $\tau \subseteq \vec{S}$ *avoids* $\mathcal{F}$ if $\tau$ has no subset in $\mathcal{F}$, i.e., if no subset of $\tau$ is an element of $\mathcal{F}$.

**Definition 2.2.1.** An $\mathcal{F}$*-tangle* of $S$ is an $\mathcal{F}$-avoiding consistent orientation of $S$. The $\mathcal{F}$-tangles of the subsets $S_k$ of $S$ are the $\mathcal{F}$-tangles *in* $\vec{S}$.

When the context is clear, e.g. when $\mathcal{F}$ is some given set as above, we shall usually say 'tangle' rather than '$\mathcal{F}$-tangle' in this paper. A separation $s \in S$ *distinguishes* two tangles if they orient it differently: if one contains $\vec{s}$, the other $\overleftarrow{s}$.

## 2.3 Tangle structure trees

We adopt the graph-theoretic terminology of [5]. A *tree* is a connected acyclic graph. Given nodes $t, t'$ of a tree $T$ we write $tTt'$ for the unique $t$–$t'$ path in $T$. A *rooted tree* is a tree with a distinguished node called its *root.* Given a tree $T$ with root $r$, we define a partial order $\leq_r$ on $V(T)$ by declaring $x \leq_r y$ if $x$ lies on the path in $T$ from $r$ to $y$. Maximal elements, including the root if $|T| = 1$, are called *leaves.* Any direct successors in $<_r$ of a node of $T$ are its *children.* We write $E_v$ for the set of edges of $T$ from a node $v$ to its children.

Let $(\vec{S}, \leq, ^*)$ be a separation system, and let $\mathcal{F}$ be any set.

**Definition 2.3.1.** A *separation tree* $(T, r, \beta)$ on $\vec{S}$ consists of a rooted tree $(T, r)$ together with an edge labelling $\beta : E(T) \to \vec{S}$ such that for every non-leaf $v \in V(T)$ there exists a separation $s_v \in S$ such that $\beta$ restricts to a bijection $E_v \to \{\overleftarrow{s}, \overleftarrow{s_v}\}$ and $s_u \neq s_v$ whenever $u <_r v$.

Thus, every non-leaf node $v$ of a separation tree has either one or two children. If there is no need to refer to $r$ or $\beta$ explicitly, we usually abbreviate $(T, r, \beta)$ to $T$. For every node $v$, we write $\beta_v$ for the set $\beta\big(E(rTv)\big)$ of edge labels on the path $rTv$.

**Lemma 2.3.1.** *Let $(T, r, \beta)$ be a separation tree on $\vec{S}$. Every orientation $\tau$ of $S$ contains $\beta_\ell$ as a subset for some unique leaf $\ell \in V(T)$.*

*Proof.* Since $\beta_r = \emptyset \subseteq \tau$, there exists a node $v \in V(T)$ that is maximal with respect to $\leq_r$ subject to $\beta_v \subseteq \tau$. If $v$ is not a leaf, let $v'$ be its child with $\beta(vv') = \tau(s_v) =: \vec{s}$. As $\beta_v \subseteq \tau$, and $\vec{s} \in \tau \setminus \beta_v$ since $s_u \neq s_v$ for all $u < v$, we see that $\beta_{v'} = \beta_v \cup \{\vec{s}\} \subseteq \tau$ contradicts the maximality of $v$. Hence $v =: \ell$ is our desired leaf.

For a proof that $\ell$ is unique, let $\ell'$ be another leaf and let $v$ be the greatest common ancestor of $\ell$ and $\ell'$. Then $v$ has distinct children $w$ and $w'$ with $\beta(vw) \in \beta_\ell$ and $\beta(vw') \in \beta_{\ell'}$. Since $\beta(vw) \neq \beta(vw')$, they cannot both lie in $\tau$. As $\beta(vw) \in \beta_\ell \subseteq \tau$ we must have $\beta(vw') \notin \tau$, and hence $\beta_{\ell'} \nsubseteq \tau$. $\square$

A separation tree $T$ is *consistent* if $\beta_v \subseteq \vec{S}$ is consistent for every node $v \in T$.

**Lemma 2.3.2.** *Let $(T, r, \beta)$ be a consistent separation tree on $\vec{S}$, and let $v$ be a non-leaf node of $T$. Then $\overset{\leftrightarrows}{s}$ and $\overleftarrow{s}_v$ are $\leq$-minimal in $\beta_v \cup \{\overset{\leftrightarrows}{s}\}$ and in $\beta_v \cup \{\overleftarrow{s}_v\}$, respectively.*

*Proof.* Suppose not. Without loss of generality assume there exists $\vec{s} \in \beta_v$ with $\vec{s} < \overset{\leftrightarrows}{s}$. Let $w$ be the child of $v$ with $\beta(vw) = \overleftarrow{s}_v$. Then $\{\vec{s}, \overleftarrow{s}_v\} \subseteq \beta_w$ is inconsistent, contradicting the consistency of $(T, r, \beta)$. $\square$

**Corollary 2.3.3.** *For every consistent separation tree and non-leaf node $v$ we have $\{\overset{\leftrightarrows}{s}, \overleftarrow{s}_v\} \cap \lfloor\beta_v\rfloor = \emptyset$. That is, $s_v$ is not oriented by $\lfloor\beta_v\rfloor$.*

*Proof.* Suppose, say, that $\overset{\leftrightarrows}{s} \in \lfloor\beta_v\rfloor$. Since $s_u \neq s_v$ for all nodes $u < v$, we know that $\overset{\leftrightarrows}{s} \notin \beta_v$. Our assumption that $\overset{\leftrightarrows}{s} \in \lfloor\beta_v\rfloor$ thus means that $\overset{\leftrightarrows}{s}$ is required by $\beta_v$. Hence there exists $\vec{r} \in \beta_v$ such that $\vec{r} < \overset{\leftrightarrows}{s}$ (and $r \neq s_v$). This contradicts theorem 2.3.2. $\square$

theorem 2.3.2 and theorem 2.3.3 imply that consistent separation trees offer an efficient data structure for storing consistent orientations of $S$, in particular, for tangles. Indeed, any such orientation $\tau$ labels the tree's edges between the root and some leaf $\ell$, but this $\beta_\ell$ is only a small subset of $\tau$: elements $\vec{s}$ of $\tau$ required by $\beta_v$ for some $v < \ell$ will not appear in $\beta_\ell \setminus \beta_v$. But they can be reconstructed from $\beta_v$ by the consistency of $\tau$.

By definition of separation trees, the separations labelling their edges do not repeat along any root-to-leaf path: if $u < v$ then $s_u \neq s_v$. Hence the length of any path from the root to a leaf is at most $|S|$. Since every node has at most two children, this bounds the size of the tree:

**Corollary 2.3.4.** *Separation trees on $\vec{S}$ have at most $2^{|S|}$ leaves and fewer than $2^{|S|+1}$ nodes.* $\square$

A leaf $\ell$ of a consistent separation tree on $\vec{S}$ is an *$\mathcal{F}$-tangle leaf*, or *tangle leaf* for short, if the closure $\lfloor\beta_\ell\rfloor$ of $\beta_\ell$ is an $\mathcal{F}$-tangle of $S$. A non-leaf node $v$ is a *tangle node* if there is a tangle leaf $\ell \geq_r w$ for every successor $w$ of $v$ in $>_r$.

A leaf $\ell$ is *forbidden* (by $\mathcal{F}$) if $\beta_\ell$ contains an element of $\mathcal{F}$ as a subset. Note that tangle leaves are never forbidden.

**Definition 2.3.2.** An $\mathcal{F}$*-tangle structure tree of* $\vec{S}$ is a consistent separation tree on $\vec{S}$ in which every leaf is either a tangle leaf or forbidden, and for every non-leaf node $v$ the set $\beta_v$ has no subset in $\mathcal{F}$.

Every $\mathcal{F}$-tangle structure tree of $\vec{S}$ displays all the $\mathcal{F}$-tangles of $S$, not just some of them:

**Theorem 2.3.5.** *Let $\vec{S}$ be a separation system, let $\mathcal{F}$ be any set, and let $T$ be any $\mathcal{F}$-tangle structure tree of $\vec{S}$.*

1. *For every $\mathcal{F}$-tangle $\tau$ of $S$ there is a unique leaf $\ell$ of $T$ such that $\lfloor\beta_\ell\rfloor = \tau$.*
2. *If all the leaves of $T$ are forbidden, then $S$ has no $\mathcal{F}$-tangle.*

*Proof.* 1 By theorem 2.3.1 there is a unique leaf $\ell$ with $\beta_\ell \subseteq \tau$. This leaf is not forbidden, since $\beta_\ell \subseteq \tau$ has no subset in $\mathcal{F}$. As $T$ is a tangle structure tree, this means that $\ell$ is a tangle leaf: that $\lfloor\beta_\ell\rfloor$ is a tangle of $S$. As $\lfloor\beta_\ell\rfloor \subseteq \tau$, this tangle can only be $\tau$.

2 is immediate from 1. □

In the next section we shall determine the sets $\mathcal{F}$ for which $S$ admits a tangle structure tree whose tree-order is compatible with a given order function on $S$, in that lower-order separations label edges further down on the tree. As we shall see in theorem 2.6.3 and theorem 2.6.4, such trees display not only all the $\mathcal{F}$-tangles of the entire set $S$, as all tangle structure trees do by theorem 2.3.5, but all the $\mathcal{F}$-tangles in $\vec{S}$: the $\mathcal{F}$-tangles of the sets $S_k = \{s \in S : |s| < k\}$ with $k \in \mathbb{N}$.

## 2.4 Existence of tangle structure trees

Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be any set. We shall need two conditions on $\mathcal{F}$ to ensure that $\vec{S}$ has a tangle structure tree.

The first is that $\{\overleftarrow{s}\} \in \mathcal{F}$ for every $\vec{s} \in \vec{S}$ that is trivial in $\vec{S}$. This holds for all sets $\mathcal{F}$ of interest, and if it does we call $\mathcal{F}$ *standard for* $\vec{S}$. Intuitively, co-trivial separations $\overleftarrow{s}$ point to places in our graph or structure that are tiny – too small to house any tangle. More formally, we already saw in section 2.2 that no consistent orientation of $S$ can contain co-trivial separations. Assuming that $\mathcal{F}$ is standard for $\vec{S}$ therefore places no restrictions on the $\mathcal{F}$-tangles of $S$.

The second condition has more substance. If $T$ is any separation tree on $\vec{S}$ then, by theorem 2.3.1, every orientation $\tau$ of $S$ contains $\beta_\ell$ for a unique leaf $\ell$. If $\tau$ is consistent and $\lfloor\beta_\ell\rfloor$ orients all of $S$, then $\lfloor\beta_\ell\rfloor = \tau$ by theorem 2.2.1. If $\tau$ is not a tangle, then $\tau = \lfloor\beta_\ell\rfloor$ has a subset $\sigma$ in $\mathcal{F}$ that witnesses this. If $T$ is in fact a tangle structure tree, we know that our consistent orientation $\tau$ of $S$ is a tangle (without having to assume that $\lfloor\beta_\ell\rfloor$ orients all of $S$) unless it contains such a set $\sigma \in \mathcal{F}$ not only in $\lfloor\beta_\ell\rfloor$ but even in $\beta_\ell$: among the edge labels of $T$.

In order for this to be possible, we therefore need to make some 'richness' assumption about $\mathcal{F}$: an assumption which, in our example, ensures that $\mathcal{F}$ has enough elements to contain a subset also of $\beta_\ell$ as soon as it contains a subset of $\lfloor\beta_\ell\rfloor$.

We shall identify an essentially weakest-possible such richness condition below, in definition 2.4.1. This will need some more preparation. For motivation, readers are invited to peek at theorem 2.4.3 and the definition preceding it now. That definition clearly implies that $\mathcal{F}$ contains a subset of $\beta_\ell$ whenever it contains a subset of $\lfloor\beta_\ell\rfloor$. The notion of richness from definition 2.4.1 will still imply this, but is weaker.

A separation tree $T$ on an ordered separation system $\vec{S}$ is *ordered* if $|s_v| \le |s_w|$ whenever $v$ and $w$ are non-leaves of $T$ with $v \le w$. It is *thoroughly ordered* (in $\vec{S}$) if, for every non-leaf node $v$, the separation $s_v$ is not oriented by $\lfloor\beta_v\rfloor$[9] and has minimum order among the separations in $S$ not oriented by $\lfloor\beta_v\rfloor$.

**Lemma 2.4.1.** *Every thoroughly ordered separation tree is ordered.*

*Proof.* Let $T$ be a thoroughly ordered separation tree on $\vec{S}$. If it is not ordered, it has nodes $v, w$ with $v < w$ such that $|s_v| > |s_w|$. Since $s_w$ is not oriented by $\lfloor\beta_v\rfloor \subseteq \lfloor\beta_w\rfloor$, this contradicts the requirement that $s_v$ have minimum order among the separations not oriented by $\lfloor\beta_v\rfloor$. □

We say that $\vec{s} \in \vec{S}$ is *weakly eclipsed* by $\vec{r} \in \vec{S}$ if $\vec{r} < \vec{s}$ and $|r| \le |s|$, and *eclipsed* by $\vec{r}$ if $\vec{r} < \vec{s}$ and $|r| < |s|$. Given any set $\tau \subseteq \vec{S}$, a subset $\sigma \subseteq \tau$ is *efficient* (in $\tau$) if no element of $\sigma$ is eclipsed by any other element of $\tau$. It is *strongly efficient* if no element of $\sigma$ is weakly eclipsed by any other element of $\tau$. Note that if the order function on $S$ is injective and $\tau$ is a partial orientation of $S$, then every efficient subset of $\tau$ is strongly efficient in $\tau$.[10]

**Lemma 2.4.2.** *Let $v$ be a node of a thoroughly ordered separation tree $T$ on $\vec{S}$. Then*

1. *every strongly efficient subset of $\lfloor\beta_v\rfloor$ is contained in $\beta_v$;*
2. *if $\beta_v$ is consistent, it is efficient in any partial orientation $\tau$ of $S$ that includes $\lfloor\beta_v\rfloor$.*

*Proof.* 1 Let $\sigma$ be any strongly efficient subset of $\lfloor\beta_v\rfloor$, and let $\vec{s} \in \sigma$ be given. Suppose that $\vec{s} \notin \beta_v$. Then $\vec{s} \in \lfloor\beta_v\rfloor \setminus \beta_v$ is required by $\beta_v$, so there exists $\vec{r} \in \beta_v$ with $\vec{r} < \vec{s}$ (and $r \neq s$). As $\vec{r} \in \beta_v$, we have $r = s_u$ for some $u < v$. Choose such an $\vec{r}$ with $u$ minimal in $<_r$.

Our aim is to show that $|r| \le |s|$: then $\vec{r} \in \lfloor\beta_v\rfloor$ eclipses $\vec{s} \in \sigma$ weakly, contradicting the strong efficiency of $\sigma$ as a subset of $\lfloor\beta_v\rfloor$. Since $T$ is thoroughly ordered, we shall have $|r| \le |s|$ as desired if $\lfloor\beta_u\rfloor$ does not orient $s$: this would make $s$ a candidate for $s_u$, so $|r| > |s|$ would contradict the fact that $r = s_u$.

So let us show that neither $\vec{s}$ nor $\overleftarrow{s}$ lies in $\lfloor\beta_u\rfloor$. For $\vec{s} \notin \lfloor\beta_u\rfloor$, recall first that $\vec{s} \notin \beta_v \supseteq \beta_u$. Hence if $\vec{s} \in \lfloor\beta_u\rfloor$, there exists an $\vec{r'} \in \beta_u$ such that $\vec{r'} < \vec{s}$. This satisfies $r' = s_{u'}$ for some $u' <_r u$, so $\vec{r'} < \vec{s}$ contradicts our original choice of $\vec{r}$ given $\vec{s}$. Hence $\vec{s} \notin \lfloor\beta_u\rfloor$ as desired. Suppose now that $\overleftarrow{s} \in \lfloor\beta_u\rfloor$. As $\vec{s} > \vec{r}$, any $\vec{r'} \le \overleftarrow{s}$ in $\beta_u$ satisfies $\vec{r'} \le \overleftarrow{s} < \overleftarrow{r}$. So $r$ was already oriented by $\lfloor\beta_u\rfloor$, contradicting the fact that $r = s_u$. Hence neither $\vec{s}$ nor $\overleftarrow{s}$ lies in $\lfloor\beta_u\rfloor$, as desired.

2 Suppose not; then some $\vec{r} \in \tau$ eclipses some $\vec{s} \in \beta_v$. Let $u < v$ be such that $s = s_u$. As $|r| < |s|$ and $T$ is thoroughly ordered, we know that $\lfloor\beta_u\rfloor$ orients $r$.

Since $\tau$ is a partial orientation of $S$ containing $\vec{r}$, it cannot also contain $\overleftarrow{r}$. We thus cannot have $\overleftarrow{r} \in \lfloor\beta_u\rfloor$, since $\lfloor\beta_u\rfloor \subseteq \lfloor\beta_v\rfloor \subseteq \tau$ by theorem 2.2.1. So $\vec{r} \in \lfloor\beta_u\rfloor$. As $\vec{r} < \vec{s}$, this implies $\vec{s} \in \lfloor\beta_u\rfloor$ unless $\vec{r} = \overleftarrow{s} \in \beta_u$. Both these contradict the fact that $s = s_u$. □

[9] If $T$ is consistent, then this holds by theorem 2.3.3.

[10] We need the assumption on $\tau$, since $\vec{s}$ eclipses $\overleftarrow{s}$ weakly if $\vec{s} < \overleftarrow{s}$. So we do not want $\vec{s}, \overleftarrow{s} \in \tau$.

A typical application of theorem 2.4.2 2 is that $\beta_v$ is efficient in any tangle of $S$ that contains it. Such a tangle will be consistent and hence include $\lfloor\beta_v\rfloor$ by theorem 2.2.1.

A typical application of theorem 2.4.2 1 will be that witnesses $\sigma \in \mathcal{F}$ to the fact that some $\lfloor\beta_v\rfloor$ fails to extend to a tangle can be found not only in $\lfloor\beta_v\rfloor$ but in $\beta_v$ itself. Then our structure trees will display such witnesses $\sigma \in \mathcal{F}$ in the label sets $\beta_\ell$ of their forbidden leaves $\ell$. For this to work with the help of theorem 2.4.2, we need $\mathcal{F}$ to contain witnesses that are strongly efficient in their $\lfloor\beta_v\rfloor$.

This motivates our formal richness condition on $\mathcal{F}$:

**Definition 2.4.1.** A set $\mathcal{F}$ is *rich for* $\vec{S}$ if every consistent orientation of $S$ that has a subset in $\mathcal{F}$ also has a strongly efficient[11] subset in $\mathcal{F}$.

The assumption that $\mathcal{F}$ is rich will be central to our results. It is needed to ensure that our $\mathcal{F}$-tangle structure trees exist, and we shall see after theorem 2.4.6 that no weaker condition on $\mathcal{F}$ will ensure the same.

Although it may look a bit technical at first glance, this 'richness' requirement on $\mathcal{F}$ is quite natural, given the role of these $\mathcal{F}$ in tangle theory. For example, given a consistent orientation $\tau$ of $S$ that has a subset $\sigma \in \mathcal{F}$, we can often obtain a strongly efficient subset $\sigma' \in \mathcal{F}$ of $\tau$ simply by replacing every element $\vec{s}$ of $\sigma$ by some $\vec{s'} \leq \vec{s}$ that is minimal in $\tau$. One still has to check then that $\sigma'$ is indeed in $\mathcal{F}$. But as the idea behind those forbidden triples in $\mathcal{F}$ is that they identify areas in our graph or other structure that are 'too small to be home to a tangle', it is not unnatural for this particular $\sigma'$ to be in $\mathcal{F}$ if $\sigma$ was.

Let us cast this example in the form of a lemma. Let us call a set $\mathcal{F}$ *closed under minimization* in a subset $\tau$ of $\vec{S}$ if it contains every set $\sigma' \subseteq \tau$ obtained from some $\sigma \subseteq \tau$ in $\mathcal{F}$ by replacing every $\vec{s} \in \sigma$ with some $\vec{s'} \leq \vec{s}$.

**Lemma 2.4.3.** *If $\mathcal{F}$ is closed under minimization in every consistent orientation of $S$, then $\mathcal{F}$ is rich for $\vec{S}$.*

*Proof.* Let $\tau$ be any consistent orientation of $S$ that has a subset $\sigma$ in $\mathcal{F}$. We have to find a set $\sigma' \subseteq \tau$ in $\mathcal{F}$ that is strongly efficient in $\tau$.

Let $\sigma'$ be obtained from $\sigma$ by replacing every $\vec{s} \in \sigma$ by some $\vec{s'} \leq \vec{s}$ that is minimal in $\tau$. The set $\sigma'$ is strongly efficient in $\tau$, since any $\vec{r} < \vec{s'} \in \sigma'$ in $\tau$ contradicts the minimal choice of $\vec{s'} \leq \vec{s}$. Since $\mathcal{F}$ is closed under minimization in $\tau$, we have $\sigma' \in \mathcal{F}$ as required. □

When we apply theorem 2.4.3 later in section 2.7, we shall in fact prove that the sets $\mathcal{F}$ needed there are closed under minimization in all of $\vec{S}$. Note that this is stronger than being closed under minimization in every consistent orientation of $S$.

**Lemma 2.4.4.** *Let $\mathcal{F}$ be rich and standard for $\vec{S}$. Let $T$ be a thoroughly ordered separation tree on $\vec{S}$, and let $v \in V(T)$. Assume that $\beta_v$ is consistent and avoids $\mathcal{F}$, and that $\lfloor\beta_v\rfloor$ orients all of $S$. Then $\lfloor\beta_v\rfloor$ is a tangle of $S$.*

*Proof.* As $\beta_v$ avoids $\mathcal{F}$, which is standard, $\beta_v$ has no element that is co-trivial in $\vec{S}$. As $\beta_v$ is consistent, $\lfloor\beta_v\rfloor$ is consistent by theorem 2.2.2. Since $\lfloor\beta_v\rfloor$ orients all of $S$, it can thus only fail to be a tangle of $S$ if it has a subset $\sigma$ in $\mathcal{F}$. As $\mathcal{F}$ is rich for $\vec{S}$, we can choose $\sigma$

[11] in this orientation of $S$

to be strongly efficient as a subset of $\lfloor\beta_v\rfloor$.[12] Then even $\sigma \subseteq \beta_v$ by theorem 2.4.2 1, which contradicts our assumption that $\beta_v$ avoids $\mathcal{F}$. Hence $\lfloor\beta_v\rfloor$ is a tangle of $S$. □

**Theorem 2.4.5.** *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be a set that is rich and standard for $\vec{S}$. Then there exists a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}$.*

*Proof.* Start with the one-node tree $T_0 = \{r\}$ and let $\beta^0 = \emptyset$. We iteratively build trees $T_0 \subsetneq T_1 \subsetneq \dots$ with maps $\beta^i$ so that each $(T_i, r, \beta^i)$ is consistent and thoroughly ordered, and none of their sets $\beta^i_v$ with $v$ a non-leaf node will have a subset in $\mathcal{F}$. Note that $T_0$ has all these properties. The last of those trees will be our desired tangle structure tree.

If for some $n$ the tree $(T_n, r, \beta^n)$ is already a tangle structure tree, we are done. Otherwise $T_n$ has a leaf $v$ which is neither a tangle leaf nor forbidden. By theorem 2.4.4, $\lfloor\beta^n_v\rfloor$ does not orient all of $S$; let $s \in S$ be a separation of minimum order not oriented by $\lfloor\beta^n_v\rfloor$.

Form $T_{n+1}$ by adding two children $v_1, v_2$ at $v$. Let $\beta^{n+1}$ agree with $\beta^n$ on the edges of $T_n$, and pick orientations $\vec{s} =: \beta^{n+1}(vv_1)$ and $\overleftarrow{s} =: \beta^{n+1}(vv_2)$ of $s$; then $s = s_v$ in $T_{n+1}$. By construction, $(T_{n+1}, r, \beta^{n+1})$ is a thoroughly ordered separation tree. It is consistent, because $T_n$ was and neither orientation of $s$ lies in $\lfloor\beta^{n+1}_v\rfloor$. Its only non-leaf node $v$ that was not already a non-leaf node in $T_n$ is $v$. Since $v$ was not forbidden as a leaf of $T_n$, the set $\beta^{n+1}_v = \beta^n_v$ has no subset in $\mathcal{F}$.

This process strictly increases $|V(T_n)|$ at each step, but by theorem 2.3.4 there is an upper bound on the size of any consistent separation tree in terms of $|S|$. Hence the process terminates after finitely many steps with a thoroughly ordered tangle structure tree. □

If our order function on $S$ is injective, theorem 2.4.5 has a converse, which shows that our requirement of richness for $\mathcal{F}$ is weakest possible to ensure the existence of a tangle structure tree. This is established by our next lemma:

**Lemma 2.4.6.** *Let $\vec{S}$ be a separation system with an injective order function on $S$, and let $\mathcal{F}$ be any set. If there exists a thoroughly ordered $\mathcal{F}$-tangle structure tree $T$ of $\vec{S}$, then $\mathcal{F}$ is rich for $\vec{S}$.*

*Proof.* Let $\tau$ be any consistent orientation of $\vec{S}$ that has a subset in $\mathcal{F}$. We shall find an efficient subset $\sigma \in \mathcal{F}$ of $\tau$, which will even be strongly efficient since our order function is injective. By theorem 2.3.1, $T$ has a leaf $\ell$ with $\beta_\ell \subseteq \tau$. In particular, $\beta_\ell$ is consistent, so $\lfloor\beta_\ell\rfloor \subseteq \tau$ by theorem 2.2.1.

Since $T$ is a tangle structure tree, $\ell$ is either a tangle leaf or forbidden. If $\ell$ is a tangle leaf then $\lfloor\beta_\ell\rfloor$ is a tangle of all of $S$. But then $\lfloor\beta_\ell\rfloor = \tau$, which contradicts the fact that $\tau$ has a subset in $\mathcal{F}$.

Thus $\ell$ is forbidden, so $\beta_\ell$ has a subset $\sigma$ in $\mathcal{F}$. By theorem 2.4.2 2 this $\sigma$ is efficient in $\tau$, as desired. □

Note that if our order function on $S$ is not injective, the proof of theorem 2.4.6 still goes through as stated except for one aspect: the efficient subset $\sigma \in \mathcal{F}$ of $\tau$ it finds may not be strongly efficient in $\tau$ (as our definition of 'rich' requires).

[12] If $\lfloor\beta_v\rfloor$ contains both orientations of some $s \in S$, delete one of them to obtain a consistent orientation of $S$ as required in the definition of 'rich' for $\mathcal{F}$.

If our order function on $S$ is injective, the sets $\mathcal{F}$ for which $\vec{S}$ admits a thoroughly ordered tangle structure tree are thus precisely the rich ones:

**Theorem 2.4.7.** *Let $\vec{S}$ be a separation system with an injective order function on $S$, and let $\mathcal{F}$ be a set that is standard for $\vec{S}$. There exists a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}$ if and only if $\mathcal{F}$ is rich for $\vec{S}$. If one exists, it is unique.*

*Proof.* Existence is immediate from theorems 2.4.5 and 2.4.6. Uniqueness follows from the definition of 'thoroughly ordered' and our assumption that the order function on $S$ is injective. □

Let us illustrate the above by an example. Consider a separation system whose elements are the non-empty subsets of a 4-element set $V$, with $\leq$ defined as $\subseteq$, and $^*$ as complementation in $V$. Let $r, s$ be two crossing separations; their orientations are 2-sets. The 1-element subsets of $V$ then are precisely those of the form $\vec{r} \cap \vec{s}$ for suitable orientations $\vec{r}, \vec{s}$ of $r$ and $s$. We call the four unoriented separations which each partition $V$ into a singleton and a 3-set the *corners* of $r$ and $s$. Let $S$ be the set of $r$, $s$, and their four corners.

The set $S$ then has four *principal* consistent orientations, those that orient one corner $t$ away from $r$ and $s$ and all the other elements of $S$ towards $t$. And it has the *non-principal* consistent orientations, which orient $r$ and $s$ arbitrarily and all four corners towards $r$ and $s$. Choose an injective order function on $S$ such that $|r| < |s| < |t|$ for all corners $t$.

Let us first consider as $\mathcal{F}$ the 4-element set $\mathcal{P} = \big\{\{\vec{r}, \vec{s}, \overleftarrow{t}\} \mid \vec{t} = \vec{r} \cap \vec{s}\big\}$.[13] The unique thoroughly ordered separation tree $T$ on $\vec{S}$ then starts with orienting $r$ and $s$, as a tree $T'$ with four leaves, one for every orientation of $\{r, s\}$. For each of these leaves $v$, the next separation $s_v$ to be oriented is one of the corners $t$. If $\vec{t} = \vec{r} \cap \vec{s}$, say, we have $\vec{t} = \beta(v\ell)$ for a tangle leaf $\ell$ for the principal tangle $\lfloor \beta_\ell \rfloor = \lfloor \vec{t} \rfloor$. And $\overleftarrow{t} = \beta(v\ell')$ for a forbidden leaf $\ell'$, with $\beta_{\ell'} = \{\vec{r}, \vec{s}, \overleftarrow{t}\} \in \mathcal{P}$. As this happens at each of the four leaves $v$ of $T'$, our $T$ is a tangle structure tree of $\vec{S}$ for $\mathcal{F} = \mathcal{P}$.

Now let $\mathcal{F}$ be the set $\mathcal{R}$ of all triples in $\vec{S}$ whose complements partition $V$.[14] These are the triples of the form either $\{\vec{r}, \overleftarrow{t}, \overleftarrow{t'}\}$ with $\vec{t} = \vec{r} \cap \vec{s}$ and $\vec{t'} = \vec{r} \cap \overleftarrow{s}$, or $\{\vec{s}, \overleftarrow{t}, \overleftarrow{t'}\}$ with $\vec{t} = \vec{s} \cap \vec{r}$ and $\vec{t'} = \vec{s} \cap \overleftarrow{r}$. Our thoroughly ordered separation tree $T$ then starts with the same $T'$ as before. Every leaf $v$ of $T'$ once more sends an edge $v\ell$ to a tangle leaf $\ell$ with $\beta(v\ell) = \vec{t}$ and $\mathcal{R}$-tangle $\lfloor \beta_\ell \rfloor = \lfloor \vec{t} \rfloor$. But with $\beta_v = \{\vec{r}, \vec{s}\}$ as earlier we now have $\beta_{\ell'} = \{\vec{r}, \vec{s}, \overleftarrow{t}\} \notin \mathcal{R}$ for the other child $\ell'$ of $v$. But note that $\overleftarrow{t'} \supset \vec{s}$ for the corner $\vec{t'} = \vec{r} \cap \overleftarrow{s}$. Hence $\lfloor \beta_{\ell'} \rfloor$ is an orientation of $S$ with a subset in $\mathcal{R}$: the set $\{\vec{r}, \overleftarrow{t}, \overleftarrow{t'}\}$. Thus, $\ell'$ is neither a tangle leaf nor forbidden, and $T$ is not an $\mathcal{R}$-tangle structure tree.

As predicted by theorem 2.4.7, this difference between $\mathcal{P}$ and $\mathcal{R}$ is reflected by the fact that $\mathcal{P}$ is rich for $\vec{S}$ but $\mathcal{R}$ is not: the consistent orientation $\lfloor \beta_{\ell'} \rfloor$ of $S$ has a subset in $\mathcal{R}$, but it has no efficient subset in $\mathcal{R}$. In particular, its subset $\{\vec{r}, \overleftarrow{t}, \overleftarrow{t'}\} \in \mathcal{R}$, which prevented $T$ from being an $\mathcal{R}$-tangle structure tree, is not efficient, because $\vec{s} \subset \overleftarrow{t'}$ eclipses $\overleftarrow{t'}$.

In section 2.5 we shall prove that our tangle structure trees can be improved further, without any additional requirements on $\mathcal{F}$, by contracting 'inessential' edges: edges whose label $\vec{s}$ is neither needed in any sets $\sigma \in \mathcal{F}$ witnessing that a leaf is forbidden, nor needed to determine any tangles.

[13] We shall study these $\mathcal{P}$-tangles more closely in section 2.7.

[14] We shall study these $\mathcal{R}$-tangles more closely in [15].

Unfortunately, this contraction process can cause our structure trees to lose their property of being thoroughly ordered. We shall therefore extract from this property its essence, a slightly weaker property to be called ‘efficiency’, which is still strong enough to make the branches $rT\ell$ and their label sets $\beta_\ell$ as efficient for displaying the tangles $\lfloor \beta_\ell \rfloor$ as they are in thoroughly ordered structure trees, but weak enough to survive the contraction process we envisage for section 2.5.

Given an ordered separation system $\vec{S}$, we call a separation tree $(T, r, \beta)$ *efficient* if for every leaf $\ell$ the set $\beta_\ell$ is efficient in $\lfloor \beta_\ell \rfloor$.

**Lemma 2.4.8.** *Let $\vec{S}$ be an ordered separation system. Then every thoroughly ordered separation tree $(T, r_0, \beta)$ on $\vec{S}$ is efficient.*

*Proof.* Consider any leaf $\ell$ of $T$. If $\beta_\ell$ is not efficient in $\lfloor \beta_\ell \rfloor$ as claimed, then some $\vec{s} \in \beta_\ell$ is eclipsed by some $\vec{r} \in \lfloor \beta_\ell \rfloor$. Let $v, w < \ell$ be such that $\vec{s} = \overleftrightarrow{s}$ and $\vec{r} \geq \vec{s_w} \in \beta_\ell$.[15]

As $\vec{r}$ eclipses $\vec{s}$, we have $\vec{r} < \vec{s}$ and $|r| < |s|$; in particular, $r \neq s = s_v$. The fact that $T$ is thoroughly ordered thus implies that $r$ is already oriented by $\lfloor \beta_v \rfloor$. We cannot have $\vec{r} \in \lfloor \beta_v \rfloor$, since this would place $\overleftrightarrow{s} = \vec{s} > \vec{r}$ or $\overleftarrow{s_v}$ in $\lfloor \beta_v \rfloor$ too,[16] which would contradict the fact that $T$ is thoroughly ordered. Thus, $\overleftarrow{r} \in \lfloor \beta_v \rfloor$; let $u < v$ be such that $\overleftarrow{r} \geq \vec{s_u} \in \beta_v$.

If $u = w$, then $\vec{s_u} = \vec{s_w} \leq \vec{r} < \vec{s}$ and hence $\overleftrightarrow{s} = \vec{s} \in \lfloor \beta_v \rfloor$ or $\overleftarrow{s_v} \in \beta_v$,[16] both of which contradict the fact that $T$ is thoroughly ordered.

If $u < w$, then $\overleftarrow{s_w} \geq \overleftarrow{r} \geq \vec{s_u} \in \beta_w$. Then $\overleftarrow{s_w} \in \lfloor \beta_w \rfloor$ unless $\vec{s_w} \in \beta_w$,[16] which both contradict the fact that $T$ is thoroughly ordered.

If $w < u$, finally, then $\overleftarrow{s_u} \geq \vec{r} \geq \vec{s_w} \in \beta_u$, so $\overleftarrow{s_u} \in \lfloor \beta_u \rfloor$ unless $\vec{s_u} \in \beta_u$,[16] which both contradict the fact that $T$ is thoroughly ordered. □

## 2.5 Irreducible and efficient tangle structure trees

Let $(T, r, \beta)$ be a consistent separation tree on a separation system $\vec{S}$. Let $v$ be a node of $T$ with a child $w$.

Let us call the edge $vw$ of $T$ *necessary* for a tangle leaf $\ell \geq w$ of $T$ if $\beta(vw)$ is a $\leq$-minimal element of the tangle $\lfloor \beta_\ell \rfloor$, or equivalently, of $\beta_\ell$. We call $vw$ *necessary* for a forbidden leaf $\ell \geq w$ if every subset of $\beta_\ell$ in $\mathcal{F}$ contains $\beta(vw)$.

Let $(T, r, \beta)_{w \to v} := (T_{w \to v}, r_{w \to v}, \beta_{w \to v})$, where $T_{w \to v}$ is obtained from $T$ by contracting the edge $vw$ of $T$ and deleting any other child $w'$ of $v$ together with the subtree spanned by the nodes $u \geq_r w'$. We continue to use ‘$w$’ for the new node constructed from the edge $vw$, and thus think of $V(T_{w \to v})$ as a subset of $V(T)$. Similarly, we think of $E(T_{w \to v})$ as a subset of $E(T)$ and continue to use ‘$\beta$’ to denote its labelling $\beta_{w \to v}$. If $v = r$ we set $r_{w \to v} := w$; otherwise we keep $r_{w \to v} := r$.

Note that $T_{w \to v}$ inherits the sets $E_u$ from $T$ for its nodes $u$. In particular, leaves of $T_{w \to v}$ were also leaves of $T$, and the separations $s_u$ for non-leaf nodes $u$ of $T_{w \to v}$ are still what they were for $T$. Hence $T_{w \to v}$ is still ordered if $T$ was. However, $T_{w \to v}$ may no longer be thoroughly ordered in $\vec{S}$ even if $T$ was.

The sets $\beta_u$ remain unchanged for all $u \ngeq w$, while for all $u \geq w$ the sets $\beta_u$ in $T_{w \to v}$ arise from $\beta_u$ in $T$ by deleting $\beta(vw)$ from it. (Recall that $\beta$ was injective on $E(rTu)$, by

[15] Recall that our arrow notation for oriented separations is never fixed. We are thus free to use forward arrows to denote the orientations of $s_v$ and $s_w$ that lie in $\beta_\ell$.

[16] By the definition of $\lfloor\ \rfloor$, the assumption of $\vec{s} > \vec{r} \in \lfloor \beta_v \rfloor$ implies $\vec{s} \in \lfloor \beta_v \rfloor$ only if $\overleftarrow{s} \notin \beta_v$.

definition 2.3.1.) In particular, $T_{w\to v}$ is consistent if $T$ was. For tangle leaves $\ell$ of $T_{w\to v}$, the tangle $\lfloor\beta_\ell\rfloor$ of $S$ is the same as it was when $\beta_\ell$ was taken in $T$; its former element $\beta(vw)$ then was a non-minimal element of this tangle.

**Lemma 2.5.1.** *Let $\vec{S}$ be a separation system, and let $\mathcal{F}$ be any set. Let $(T, r_0, \beta)$ be a consistent separation tree on $\vec{S}$. Let $v$ be a node of $T$ with a successor $w$, let $\vec{s} := \beta(vw)$, and let $\ell \geq w$ be a leaf. Then $vw$ is necessary for $\ell$ if and only if $\beta_\ell \setminus \{\vec{s}\}$ has no subset in $\mathcal{F}$ and $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$ is not a tangle of $S$.*

*Proof.* Suppose first that $vw$ is necessary for $\ell$, and that $\ell$ is a tangle leaf. Then $\lfloor\beta_\ell\rfloor$ is a tangle of $S$, so $\beta_\ell \setminus \{\vec{s}\}$ has no subset in $\mathcal{F}$.

For a proof that $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$ is not a tangle of $S$, we show that it contains neither $\vec{s}$ nor $\overleftarrow{s}$. To see $\vec{s} \notin \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$, note that any $\vec{r} \in \beta_\ell \setminus \{\vec{s}\}$ with $\vec{r} \leq \vec{s}$ would satisfy $\vec{r} < \vec{s}$, since also $\vec{s} \in \beta_\ell$ and hence $r \neq s$ by definition 2.3.1. Then $\vec{s}$ would not be minimal in $\beta_\ell$, which it is since $vw$ is necessary for the tangle leaf $\ell$. And $\overleftarrow{s} \notin \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$, since $\lfloor\beta_\ell\rfloor$ is a tangle of $S$ containing $\vec{s}$. Thus, $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$ does not orient $s$, so it is not a tangle of $S$.

Suppose, second, that $vw$ is necessary for $\ell$, and that $\ell$ is a forbidden leaf. Then $\beta_\ell \setminus \{\vec{s}\}$ has no subset in $\mathcal{F}$. Suppose $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$ is a tangle of $S$. This tangle cannot contain $\vec{s}$: it would then contain the entire set $\beta_\ell$, but this has a subset in $\mathcal{F}$ since $\ell$ is a forbidden leaf. But neither can $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$ contain $\overleftarrow{s}$: then $\beta_\ell \setminus \{\vec{s}\}$ would contain some $\vec{r} \leq \overleftarrow{s}$, which would contradict the consistency of $\beta_\ell$, since $r \neq s$ by definition 2.3.1 and $\vec{s} \in \beta_\ell$ would thus point away from $\vec{r} \in \beta_\ell$.

Suppose, third, that $vw$ is not necessary for $\ell$ and that $\ell$ is a tangle leaf. Then $\lfloor\beta_\ell\rfloor$ is a tangle and $\vec{s}$ is not a minimal element of $\beta_\ell$. We complete our proof in this case by showing that $\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor = \lfloor\beta_\ell\rfloor$. If $\vec{s} \notin \beta_\ell$ this holds trivially. If $\vec{s}$ lies in $\beta_\ell$ but is not minimal in it, then there exists some $\vec{r} \in \beta_\ell$ with $\vec{r} < \vec{s}$. This implies $\vec{s} \in \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$, and hence

$$\lfloor\beta_\ell\rfloor = \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor \cup \lfloor\{\vec{s}\}\rfloor \subseteq \lfloor\lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor\rfloor = \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$$

as desired, unless $\overleftarrow{s} \in \lfloor\beta_\ell \setminus \{\vec{s}\}\rfloor$. But in that case the tangle $\lfloor\beta_\ell\rfloor$ contains both $\vec{s}$ and $\overleftarrow{s}$, which it cannot.

Suppose finally that $vw$ is not necessary for $\ell$ and that $\ell$ is a forbidden leaf. Then $\beta_\ell \setminus \{\vec{s}\}$ has a subset in $\mathcal{F}$, as desired. □

**Lemma 2.5.2.** *Let $\vec{S}$ be a separation system, and let $\mathcal{F}$ be any set. Let $(T, r, \beta)$ be a tangle structure tree of $\vec{S}$. Let $v$ be a node of $T$ with a successor $w$. Then $(T, r, \beta)_{w\to v}$ is a tangle structure tree of $\vec{S}$ if and only if $vw$ is not necessary for any leaf $\ell \geq w$ of $T$.*

*Proof.* Suppose first that $vw$ is necessary for some leaf $\ell \geq w$ of $T$. This $\ell$ is a leaf also of $T_{w\to v}$. By theorem 2.5.1, however, $\ell$ is neither a tangle leaf nor a forbidden leaf of $T_{w\to v}$. Hence $T_{w\to v}$ is not a tangle structure tree of $\vec{S}$.

Conversely, suppose that $vw$ is not necessary for any leaf $\ell \geq w$ in $T$. These $\ell$ are precisely (all) the leaves of $T_{w\to v}$. Since $\beta_\ell$ taken in $T_{w\to v}$ is obtained from $\beta_\ell$ taken in $T$ by deleting $\beta(vw)$, these $\ell$ are either forbidden or tangle leaves of $T_{w\to v}$ by theorem 2.5.1. Thus, $T_{w\to v}$ is a tangle structure tree of $\vec{S}$. □

Let us note a surprising consequence of theorem 2.5.2 and its proof. Consider a node $v$ of $T$ with children $w, w'$, and assume that $vw$ is not necessary for any leaf $\ell \geq w$ of $T$. The truth of this assertion is determined entirely by the values under $\beta$ of the subtree of $T$ that consists of the path $rTv$, the subtree spanned by all nodes $u \geq w$, and the edge $vw$. In particular, it is not affected by the values under $\beta$ of the edges of the subtree of $T$ spanned by the nodes $u \geq w'$: the subtree we delete when we form $T_{w\to v}$ from $T$.

By theorem 2.5.2, this $T_{w\to v}$ is a tangle structure tree. Every tangle of $S$ thus equals $\lfloor\beta_\ell\rfloor$ for some leaf $\ell$ of $T_{w\to v}$, which we recall is the same tangle of $S$ that was associated with $\ell$ in $T$. (In particular, $\lfloor\beta_\ell\rfloor$ is the same in $T_{w\to v}$ as in $T$.) So what about the tangles of $S$ whose associated leaf of $T$ lay in the deleted subtree, tangles that include $\beta_{w'}$ and in particular the inverse $\beta(vw')$ of $\beta(vw)$?

The surprising conclusion is that such tangles cannot exist: that if $vw$ is not necessary for any leaf $\ell \geq w$, then $\beta_{w'}$ does not extend to a tangle of $S$.[17] This is true regardless of whether $\beta_w$ extends to a tangle of $S$ or not. But in particular, $v$ cannot have been a tangle leaf of $T$; indeed, $T$ had no tangle node $v' \geq w'$:

**Corollary 2.5.3.** *Tangle structure trees $T$ and $T_{w\to v}$ as in theorem 2.5.2 have the same sets of tangle nodes.*

It is not hard to construct examples of tangle structure trees that have unnecessary edges. Indeed in many naturally occurring tangle structure trees most edges are unnecessary, and applying theorem 2.5.5 below reduces them substantially.

Let us call a node $v$ of $T$ *necessary in $T$* if for every child $w$ of $v$ there exists a leaf $\ell \geq w$ of $T$ such that $vw$ is necessary for $\ell$.

**Lemma 2.5.4.** *Let $\vec{S}$ be a separation system, and let $\mathcal{F}$ be any set. Let $(T, r, \beta)$ be a tangle structure tree of $\vec{S}$, and let $v$ be a node of $T$. Then $v$ has a child $w$ such that $(T, r, \beta)_{w\to v}$ is a tangle structure tree of $\vec{S}$ if and only if $v$ is not necessary in $T$.*

*Proof.* Suppose first that $v$ is necessary in $T$. Let $w$ be any child of $v$ in $T$. Since $v$ is necessary, there is a leaf $\ell \geq w$ such that $vw$ is necessary for $\ell$. By theorem 2.5.2, $T_{w\to v}$ is not a tangle structure tree of $\vec{S}$.

Conversely, suppose $v$ is not necessary in $T$. Then it has a child $w$ such that $vw$ is not necessary for any leaf $\ell \geq w$ in $T$. By theorem 2.5.2, $T_{w\to v}$ is a tangle structure tree of $\vec{S}$. □

A tangle structure tree $(T, r, \beta)$ is called *irreducible* if every node of $T$ is necessary. The following theorem summarizes how tangle structure trees can be contracted and pruned to irreducible trees that inherit many of their properties. The statement of the theorem assumes our notational convention that $V(T_{w\to v}) \subseteq V(T)$.

**Theorem 2.5.5.** *Let $\vec{S}$ be a separation system, and let $\mathcal{F}$ be any set. For every $\mathcal{F}$-tangle structure tree $T$ of $\vec{S}$ there exists an irreducible $\mathcal{F}$-tangle structure tree $T'$ of $\vec{S}$ obtained from $T$ by a sequence $T = T^1, \ldots, T^n = T'$ with $T^{i+1} = T^i_{w\to v}$ for suitable nodes $v, w$ of $T^i$.*

*The tree $T'$ imposes the same partial ordering on its node as $T$ does. In particular, $T'$ is ordered if $T$ is. Its leaves $\ell$ are also leaves of $T$, and their incident edges in $T$ are still edges of $T'$.*

*The tangle leaves $\ell$ of $T'$ are precisely those of $T$. Their associated tangles $\lfloor\beta_\ell\rfloor$ are the same in $T'$ as in $T$. The tangle nodes of $T'$ are precisely those of $T$. A tangle node $v$ separates two tangle leaves in $T'$ if and only if it separates them in $T$.*

[17] This does not mean that $\beta(vw')$ cannot lie in a tangle of $S$: such tangles just do not include $\beta_v$.

*Proof.* If $T^i$ is not irreducible, pick a node $v \in T^i$ that is not necessary. By theorem 2.5.4 it has a child $w$ for which $T^i_{w\to v}$ is a tangle structure tree. Let $T^{i+1} := T^i_{w\to v}$. For the second paragraph of the theorem, note that edges of the form $v\ell$, with $\ell$ a leaf, are necessary in every tangle structure tree: by theorem 2.3.3 if $\ell$ is a tangle leaf, and by definition 2.3.2 if $\ell$ is forbidden.

For the third paragraph, recall our discussion following theorem 2.5.2 and iterate theorem 2.5.3. Recall that the separation $s_v$ associated with a tangle node $v$ is the same in $T'$ as in $T$, and that $s_v$ distinguishes two tangles if and only if $v$ separates their tangle leaves in any tangle structure tree of $\vec{S}$. □

The irreducible tangle structure tree $T'$ obtained in the proof of theorem 2.5.5 is not unique. This is because we can check the nodes $v$ of $T$ for suppression in any order, and this order will affect their necessity in the tree $T^i$ in which they are considered for suppression. But it is easy to make $T'$ unique. For example, it is not hard to show that if we consider the nodes $v$ in any linear order that extends their tree order $<_r$ from $T$ – for example, level by level – the resulting tree $T'$ will be the same.

If $\mathcal{F}$ is standard, the $\mathcal{F}$-tangle structure trees we considered in section 2.4 were all efficient, because they were thoroughly ordered (theorem 2.4.8). The irreducible structure trees we obtained from them in theorem 2.5.5 may no longer be thoroughly ordered: as noted before, this property can get lost in the contraction process. But the efficiency of the original structure trees is maintained:

**Lemma 2.5.6.** *If $T'$ is obtained from an efficient tangle structure tree $T$ as in theorem 2.5.5, then $T'$ too is efficient.*

*Proof.* Consider $T^{i+1} = T^i_{w\to v}$ as in theorem 2.5.5 for $i = 1, \dots, n-1$ in turn. Assuming that $T^i$ is efficient, we have to show that so is $T^{i+1}$. Write $\beta^{i+1} := \beta^i_{w\to v}$, where $\beta^i$ is the edge labelling of $T^i$. Given any leaf $\ell$ of $T^{i+1}$, we have to show that $\beta^{i+1}_\ell$ is efficient in $\lfloor \beta^{i+1}_\ell \rfloor$. This follows from the fact that $\ell$ is also a leaf of $T^i$, the assumed efficiency of $T^i$, and the fact that $\beta^{i+1}_\ell \subseteq \beta^i_\ell$. □

## 2.6 The $\mathcal{F}$-tangle structure theorems

In this section we summarize what we have shown so far, in a few, concise, statements that we think of as the main results of this paper.

The first of these tells us when $\mathcal{F}$-tangle structure trees exist:

**Theorem 2.6.1.** *Let $\vec{S}$ be an ordered separation system. If a set $\mathcal{F}$ is standard and rich for $\vec{S}$, then $\vec{S}$ has an efficient and irreducible ordered $\mathcal{F}$-tangle structure tree.*

*Proof.* By theorem 2.4.5, $\vec{S}$ has a thoroughly ordered tangle structure tree $(T, r, \beta)$. By theorem 2.4.8 it is efficient. The desired structure tree can be obtained from $(T, r, \beta)$ by theorem 2.5.5; it is efficient by theorem 2.5.6. □

The tangle structure trees constructed for the proof of theorem 2.6.1 have a host of properties designed to help with the tangle analysis of a given separation system. It might have been natural to list these properties as part of theorem 2.6.1. However, since the latter is an existence theorem, this would have allowed us to establish these properties just for the structure tree we constructed in its proof.

We would like to stress the fact that those properties are common to all efficient and irreducible ordered tangle structure trees: these terms were designed to encode precisely those properties. Let us show this next:

**Theorem 2.6.2.** *Let $\vec{S}$ be an ordered separation system. Every efficient ordered $\mathcal{F}$-tangle structure tree $(T, r, \beta)$ for $\vec{S}$ has the following properties.*

1. *For every leaf $\ell$ of $T$, the set $\beta_\ell$ is consistent in $S$. If $\beta_\ell$ has no subset in $\mathcal{F}$, then $\lfloor\beta_\ell\rfloor$ is a tangle of $S$, and every $\mathcal{F}$-tangle of $S$ arises in this way.*
2. *For every non-leaf node $v$ of $T$, the set $\beta_v$ has no subset in $\mathcal{F}$, and $s_v$ is not oriented by $\lfloor\beta_v\rfloor$. Both its orientations $\vec{s} \in \{\overleftarrow{s}_v, \overrightarrow{s}\}$ are minimal in $\beta_v \cup \{\vec{s}\}$ in the partial ordering of $\vec{S}$. Its order $|s|$ is maximal in $\{\,|s_u| \,:\, u \leq_r v\}$.*
3. *For every orientation $\tau$ of $S$ there is a unique leaf $\ell$ of $T$ such that $\beta_\ell \subseteq \tau$. If $\tau$ is consistent, then $\lfloor\beta_\ell\rfloor \subseteq \tau$ and $\beta_\ell$ is efficient in $\tau$.*
4. *For every $\mathcal{F}$-tangle $\tau$ of $S$ there is a unique leaf $\ell$ of $T$ such that $\beta_\ell \subseteq \tau$. Then $\tau = \lfloor\beta_\ell\rfloor$, and $\beta_\ell$ is efficient in the tangle $\tau$.*
5. *For every consistent orientation $\tau$ of $S$ that is not a tangle, the set $\beta_\ell$ in* (iii) *has a subset $\sigma$ in $\mathcal{F}$. Every such $\sigma$ is efficient in $\tau$.*

*Proof.* (i) The first two statements are immediate from definition 2.3.2: tangle structure trees are consistent, and leaves that are not forbidden are tangle leaves. The third assertion is part of (iv) and will be proved there.

(ii) The first assertion is again part of definition 2.3.2. The second is theorem 2.3.3. The minimality assertion is theorem 2.3.2. The maximality assertion holds, because $(T, r, \beta)$ is ordered.

(iii) By theorem 2.3.1, $T$ has a unique leaf $\ell$ such that $\beta_\ell \subseteq \tau$. If $\tau$ is consistent, then $\lfloor\beta_\ell\rfloor \subseteq \tau$ by theorem 2.2.1. The claim now follows by theorem 2.4.2 2.

(iv) We have to show that the leaf $\ell$ from (iii) satisfies the inclusion $\lfloor\beta_\ell\rfloor \subseteq \tau$ provided there with equality. As $\tau$ has no subset in $\mathcal{F}$, the leaf $\ell$ is not forbidden. It is therefore a tangle leaf, which means that $\lfloor\beta_\ell\rfloor$ is a tangle of $S$. Since distinct tangles of $S$ cannot contain one another, this tangle can only be $\tau$.

(v) Consider the leaf $\ell$ provided for $\tau$ by (iii). If $\tau$ is not a tangle, then neither is $\lfloor\beta_\ell\rfloor \subseteq \tau$: if it was, then $\lfloor\beta_\ell\rfloor$ and $\tau$ would both be orientations of $S$, implying $\lfloor\beta_\ell\rfloor = \tau$ with a contradiction. Hence $\beta_\ell$ has a subset in $\mathcal{F}$, by (i). The efficiency claim follows from (iii), since all subsets of $\beta_\ell$ are efficient in $\tau$ if $\beta_\ell$ is. □

Let us briefly address the question of how efficient our structure trees are in displaying the $\mathcal{F}$-tangles of $S$, as well as certificates from $\mathcal{F}$ for consistent orientations of $S$ that are not $\mathcal{F}$-tangles.

There is a unique way to encode a single tangle $\tau$ efficiently: we have to know its minimal elements, but only these, since the orientations of all the other separations in $S$ are then determined by consistency of $\tau$.[18] The structure tree from theorem 2.6.1 does not achieve this for all the tangles of $S$ simultaneously. Indeed, one only needs

[18]Apply theorem 2.2.2 to the set $\sigma$ of its minimal elements to reobtain the entire tangle as $\tau = \lfloor\sigma\rfloor$.

two separations in $S$ to see that no structure tree can in general display only the minimal elements of all its tangles.[19]

However, since our structure tree $T$ in theorem 2.6.1 is irreducible, it does the next best thing. By theorem 2.6.2 (iv), the minimal elements of every tangle $\tau$ of $S$ are displayed as labels along the path $rT\ell$, where $\ell$ is the tangle leaf with $\tau = \lfloor \beta_\ell \rfloor$. Conversely, every label $\overleftrightarrow{s}$ of an edge of $T$ is minimal in *some* tangle of $S$ (with a tangle leaf $\ell > v$), or indispensable as an element of the subsets $\sigma \in \mathcal{F}$ of $\beta_\ell$ that certify that $\lfloor \beta_\ell \rfloor$ is not a tangle, for some forbidden leaf $\ell > v$.

theorem 2.6.1 is best possible in another sense too. Its characterizing condition for the existence of tangle structure trees is, essentially, that $\mathcal{F}$ must be rich. This cannot be weakened: if our order function on $S$ is injective, which in real-world applications is the rule rather than the exception, then by theorem 2.4.6 the existence of any ordered tangle structure tree (efficient and irreducible or not) implies that $\mathcal{F}$ is rich. The richness condition on $\mathcal{F}$, thus, captures precisely what we need for a tangle structure tree to exist.

If $\mathcal{F}$ is rich not only for $\vec{S}$ but also for some (or all) $\vec{S}_k$, theorem 2.6.1 applied in $\vec{S}_k$ says that there exist efficient and irreducible ordered tangle structures trees of all these $\vec{S}_k$. But we do not have to obtain these by independent applications of theorem 2.6.1, separately for each $k$: we can find them all in the original tangle structure tree $T$ for $\vec{S}$ provided by theorem 2.4.5, before we applied theorem 2.5.5 to 'reduce' it. Let us now see how.

Given an ordered separation tree $(T, r, \beta)$ on $\vec{S}$ and $k \in \mathbb{R}$, let

$$(T, r, \beta)|_k = (T|_k, r, \beta|_k)$$

be defined as follows. Since $T$ is ordered, its edges $e$ with $|\beta(e)| < k$ form a subtree of $T$ rooted at $r$, which we take as $T|_k$. As its edge labelling $\beta|_k$ we take the restriction of $\beta$ to these edges.

**Theorem 2.6.3.** *Let $(T, r, \beta)$ be a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}$. Let $k \in \mathbb{R}$, and assume that $\mathcal{F}$ is rich for $\vec{S}_k$. Then $(T, r, \beta)|_k$ is a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}_k$.*

*Proof.* It is clear from the definition of $T|_k$ that it inherits from $T$ the properties of being a thoroughly ordered and consistent separation tree on $\vec{S}_k$. Moreover, non-leaf nodes $v$ of $T|_k$ are non-leaf nodes also of $T$, so $\beta_v$ has no subset in $\mathcal{F}$.

For a proof that $T|_k$ is a tangle structure tree it remains to show that every leaf $v$ of $T|_k$ is either a tangle leaf or a forbidden leaf of $T|_k$. If it is neither, then by theorem 2.4.4 applied to the closure $\lfloor \beta_v \rfloor_k$ of $\beta_v$ in $\vec{S}_k$ does not orient all of $S_k$.

Such a node $v$ cannot be a leaf of $T$. Indeed, since $\beta_v$ has no subset in $\mathcal{F}$, it would be a tangle leaf of $T$, so $\lfloor \beta_v \rfloor$ would be a tangle of $S$, and $\lfloor \beta_v \rfloor \cap \vec{S}_k = \lfloor \beta_v \rfloor_k$ would be a tangle of $S_k$ (contrary to our assumption). So there exists an $s_v \in S$ whose orientations label the edges $e$ and $e'$ from $v$ to its children in $T$.

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[19] For example, let $S$ consist of just two nested separations, $r$ and $s$. Let $\mathcal{F} = \emptyset$, so that all three consistent orientations of $S$ are $\mathcal{F}$-tangles. The orientations $\vec{r}$ of $r$ and $\overleftarrow{s}$ of $s$ that point to each other then form a tangle in which they are both minimal, so they must both label an edge of $T$. Now if we flip either one of them, we obtain a tangle in which the other is still an element, but no longer minimal. If $|r| < |s|$, say, then $\{\vec{r}, \vec{s}\}$ would be such a tangle which our structure tree displays by using both $\vec{r}$ and $\vec{s}$ as labels of a path from the root to a tangle leaf, although $\vec{s}$ alone already determines this tangle. The tangle $\{\overleftarrow{r}, \overleftarrow{s}\}$ would be displayed by the same tree more efficiently, with $\overleftarrow{r}$ labelling an edge from the root to a leaf $\ell$, with $\beta_\ell = \{\overleftarrow{r}\}$ and $\lfloor \beta_\ell \rfloor = \{\overleftarrow{r}, \overleftarrow{s}\}$.

Since $T$ is thoroughly ordered and there exists a separation in $S_k$ not oriented by $\lfloor \beta_v \rfloor_k = \lfloor \beta_v \rfloor \cap \vec{S_k}$, we know that $|s_v| < k$. But this means that $e$ and $e'$ are edges of $T|_k$, contradicting the fact that $v$ is a leaf of $T|_k$. □

Note that the structure trees $T|_k$ found inside $T$ by theorem 2.6.3 are nested: for $i < j$ we have $T|_i \subseteq T|_j$, indeed $T|_i = (T|_j)|_i$. This hierarchy of structure trees commutes with the following well-known hierarchy of tangles. For all integers $i < j$, every tangle $\tau_j$ of $S_j$ induces a tangle $\tau_i$ of $S_i$, as $\tau_i = \tau_j \cap \vec{S_i}$. A tangle of any $S_k$ that is not induced by a tangle of any $S_{k'} \supsetneq S_k$ is *maximal* in $\vec{S}$. Every tangle in $\vec{S}$, say $\tau_k$ of order $k$, now corresponds to a tangle leaf $\ell(\tau_k)$ of $T|_k$ in that $\tau_k = \lfloor \beta_{\ell(\tau_k)} \rfloor_k$. All these $\ell(\tau_k)$ are nodes of $T$.

Conversely, if our order function on $S$ is injective, then *every* non-leaf node $v$ of $T$ is such a tangle leaf $\ell(\tau_k)$ of $T|_k$ for some tangle $\tau_k$ in $\vec{S}$, since $\beta_v$ has no subset in $\mathcal{F}$ by theorem 2.6.2 (ii): simply choose $k$ just big enough that $\beta(uv) < k$ for $u$ the parent of $v$. The positions of these nodes in $T$ then commute with the relationship of their tangles: if $\tau_j$ induces $\tau_i$ for $i \leq j$, then $\ell(\tau_i) \leq_r \ell(\tau_j)$ in the tree-order of $T$.

Thus, all the maximal tangles in $\vec{S}$ are displayed by the same structure tree:

**Corollary 2.6.4.** *Let $(T, r, \beta)$ be a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}$. Assume that the order function on $S$ is injective, and that $\mathcal{F}$ is rich for every $\vec{S_k}$.*

*Let $T'$ be obtained from $T$ by deleting any pairs $\ell, \ell'$ of forbidden leaves of $T$ that are children of the same node. Then $T'$ is a thoroughly ordered consistent separation tree on $\vec{S}$, which displays precisely the maximal $\mathcal{F}$-tangles in $\vec{S}$ as $\lfloor \beta_\ell \rfloor_k$ for leaves $\ell$ of $T'$ and suitable $k$.* □

See [15Section 6] for more details on $T'$ and applications of theorem 2.6.4.

We remark that it is not enough in theorem 2.6.3 and theorem 2.6.4 to assume that $\mathcal{F}$ is rich for $\vec{S}$, rather than for all the relevant $\vec{S_k}$. To see this, recall the example with $\mathcal{F} = \mathcal{R}$ discussed after theorem 2.4.7. Let $k \in \mathbb{R}$ be large enough that $S = S_k$ for the separation system $\vec{S}$ discussed there. The unique thoroughly ordered separation tree $T$ constructed there was not an $\mathcal{F}$-tangle structure tree, because it had leaves that were neither forbidden nor tangle leaves. Correspondingly, $\mathcal{F}$ was not rich for this $\vec{S} = \vec{S_k}$.

Now extend this separation system by adding a new separation $x$ of order $k$, making both its orientations incomparable with all the elements of $\vec{S}$, and adding them to $\mathcal{F}$ as singleton sets $\{\vec{x}\}$ and $\{\overleftarrow{x}\}$. This makes the extended $\mathcal{F}'$ rich for the extended $\vec{S'}$, because $\{\vec{x}\}$ and $\{\overleftarrow{x}\}$ are efficient and every orientation of $S'$ contains one of them. The unique $\mathcal{F}'$-tangle structure tree of $\vec{S'}$ is obtained from $T$ by adding two new leaves at every leaf of $T$, labelling their incident edges with $\vec{x}$ and $\overleftarrow{x}$. Note that all these leaves are forbidden.

Deleting them, however, as in theorem 2.6.4, returns the original $T$. Contrary to what the corollary claims, $T$ fails to display the maximal $\mathcal{F}'$-tangles in $\vec{S'}$ as $\lfloor \beta_\ell \rfloor$ for its leaves $\ell$, since these are the $\mathcal{F}$-tangles of $S$.

In the proof of theorem 2.6.3 we made use of the assumption that $T$ is thoroughly ordered. This holds for the tangle structure trees provided by theorem 2.4.5, and these are already efficient by theorem 2.4.8. But while their efficiency is maintained when we apply theorem 2.5.5 to ‘reduce’ them, as in the proof of theorem 2.6.1, the property of being thoroughly ordered is lost. For the trees $T|_k$ in theorem 2.6.3 to be tangle structure trees, however, it is essential that $T$ is thoroughly ordered.[20]

But the tangle structure trees of $\vec{S_k}$ that theorem 2.6.3 finds in $T$ can be reduced afterwards: we simply apply theorem 2.5.5 to those $T|_k$. This yields irreducible, efficient, ordered tangle structure trees for all the $\vec{S_k}$, the same that theorem 2.6.1 would give us if we applied it directly in $\vec{S_k}$.

Similarly, we can mimic the reduction process established in section 2.5 to contract the separation tree $T'$ from theorem 2.6.4, which displays all the maximal tangles in $\vec{S}$ as $\lfloor \beta_\ell \rfloor$ for suitable leaves $\ell$, to an efficient and 'irreducible' such tree. All it needs to facilitate this is that we adapt the definition of 'tangle leaves', and those of ensuing terms such as 'necessary' and 'irreducible', to refer to the maximal tangles in $\vec{S}$, rather than just the tangles of all of $S$.[21]

Finally, let us apply theorem 2.6.1 to the special case that $S$ has no $\mathcal{F}$-tangles. In this case it is desirable to be able to certify this efficiently.

In the special case that $\mathcal{F}$ consists of stars (of oriented separations; see section 2.2), such certificates are known: they are the *$S$-trees over $\mathcal{F}$* from [16], which generalize the branch-decompositions introduced by Robertson and Seymour [1] for graph tangles to $\mathcal{F}$-tangles in abstract separation systems.

Our structure trees from theorem 2.6.1 provide maximally efficient certificates for the non-existence of $\mathcal{F}$-tangles for general $\mathcal{F}$, not necessarily consisting of stars, when all their leaves are forbidden. Let us call such $\mathcal{F}$-tangle structure trees *$\mathcal{F}$-trees*.

**Theorem 2.6.5.** *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be any set that is standard and rich for $\vec{S}$. Then exactly one of the following assertions holds:*

1. *there exists an $\mathcal{F}$-tangle of $S$;*
2. *there exists an $\mathcal{F}$-tree of $\vec{S}$.*

*In the case of 2, the $\mathcal{F}$-tree can be chosen to be irreducible, efficient and ordered.*

*Proof.* By theorem 2.3.5 2, any $\mathcal{F}$-tree of $S$ precludes the existence of an $\mathcal{F}$-tangle of $\vec{S}$, so we cannot have both 1 and 2.

Let us prove that 1 or 2 holds. By theorem 2.6.1, $\vec{S}$ has an efficient and irreducible ordered $\mathcal{F}$-tangle structure tree $T$. If 1 fails, then $T$ has no tangle leaf. Then all its leaves are forbidden, so $T$ is an $\mathcal{F}$-tree. □

Let us close this section with an interesting fact exhibited by $\mathcal{F}$-trees. If $S$ has no tangle, then orientations of $S$ cannot fail to be a tangle just because they are inconsistent: even inconsistent orientations of $S$ must have a subset in $\mathcal{F}$. This is clearly not the case when $S$ does have tangles, e.g. if $\mathcal{F} = \emptyset$, as soon as it contains two nested separations (which we can orient away from each other).

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[20] In the reduction process we contracted or deleted edges whose label was 'unnecessary' for $T$ to display the $\mathcal{F}$-tangles of $S$ and any certificates in $\mathcal{F}$ of their non-existence. But such edges may have been necessary for $T|_k$ to display the tangles of $S_k$. As an extreme example, take $\mathcal{F} = \emptyset$ and assume that all the oriented separations that are $\le$-minimal in tangles of $S$ have order $k$. While $S$ may have elements of lower order that can be oriented to form lower-order tangles, those tangles will not be visible in $T$, since $\beta(E(T)) \cap \vec{S_k} = \emptyset$.

[21] The only (small) addition in substance needed here occurs early in the proof of theorem 2.5.1, where we now have to show also that $\lfloor \beta_\ell \setminus \{\vec{s}\} \rfloor$ cannot be a maximal tangle in $\vec{S}$ other than $\tau$. But this is clear, since $\lfloor \beta_\ell \setminus \{\vec{s}\} \rfloor \subseteq \tau$.

**Corollary 2.6.6.** *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be any set that is standard and rich for $\vec{S}$. If $S$ has no $\mathcal{F}$-tangle, then every orientation of $S$ has a subset in $\mathcal{F}$.*

*Proof.* Let $\tau$ be any orientation of $S$. By theorem 2.6.5, $S$ has an $\mathcal{F}$-tree $(T, r, \beta)$. Let $rT\ell$ be the maximal path in $T$ from the root such that $\beta(uv) \in \tau$ for its adjacent vertices $u < v$. Since $\ell$ is a forbidden leaf, $\mathcal{F}$ has an element $\sigma \subseteq \beta_\ell \subseteq \tau$. □

## 2.7 Applications: blocks, profiles, and cluster tangles

Graph tangles can be expressed as $\mathcal{F}$-tangles in the universe of all separations of a graph $G$: take as $\mathcal{F}$ the set $T$ of triples $\{ (A_i, B_i) \mid i = 1, 2, 3 \}$ of oriented separations such that $G[A_1] \cup G[A_2] \cup G[A_3] = G$. These are not stars. But it is not hard to show [14] that the $T$-tangles of a graph are precisely its $T^*$-tangles, where $T^*$ consists of the sets in $T$ that are stars.

Due to this history, $\mathcal{F}$-tangles have so far been studied mostly when $\mathcal{F}$ consisted of stars of separations. Such $\mathcal{F}$-tangles already allow for vast generalizations of the original tangles of graphs. However, there are contexts in which $\mathcal{F}$-tangles occur naturally and $\mathcal{F}$ does not consist of stars. Three of these are:

- blocks in graphs, more precisely: $k$-blocks for any $k \in \mathbb{N}$;
- profiles: the most comprehensive generalization of graph tangles that still admits a tree-of-tangles structure theorem;
- cluster tangles in large datasets.

We shall apply our results to all these in this section, giving a minimum of background for context in each case.

Common to all tangle contexts are two fundamental theorems about tangles in graphs, which both go back to the original paper of Robertson and Seymour [1] in which tangles were first introduced. These have become known [5] as

- the tree-of-tangles theorem; and
- the tangle-tree duality theorem.

The tree-of-tangles theorem, in its simplest form, says that all the maximal tangles in a graph can be distinguished[22] by a nested set $T$ of separations, one that can be represented as the set of separations associated with a tree-decomposition of the graph. The separations in $T$ can be chosen so that they have minimum order for any separation that distinguishes the pair of tangles they distinguish, and *canonically*, which means that $T$ is invariant under all the automorphisms of the graph [5]. These enhanced tree-of-tangle theorems have been further generalized to abstract separation systems [7, 14, 10].

The tangle-tree duality theorem, in its simplest form, says that if a graph has no tangle of some given order $k$ then it has a nested set $T$ of separations of order $< k$ that witnesses the non-existence of those tangles: any orientation $\tau$ of the graph's separations

[22] Recall that a separation *distinguishes* two tangles if they orient it differently.

of order $< k$ will in particular orient $T$, and thereby the edges of the decomposition tree associated with $T$, and following those oriented edges will take us to a node whose incident edges correspond to a set in $T^*$; thus, $\tau$ is not a tangle. The tangle-tree duality theorem, too, has been generalized to abstract separation systems [16, 14]. There are also unifications of both theorems into one [11, 12, 13].

The certificates for the non-existence of an $\mathcal{F}$-tangle offered by the tangle-tree duality theorems require that $\mathcal{F}$ consist of stars. Indeed, since $T$ is nested, the edges at any node of the decomposition tree that represents $T$ will map to a nested set of separations too, and orienting them towards that node will yield a star of separations. Hence if we look for certificates for the non-existence of $\mathcal{F}$-tangles when $\mathcal{F}$ does not consist of stars, we need a fresh start. Our theorem 2.6.5 offers such an alternative.

For our first application, let us now look at blocks. For any $k \in \mathbb{N}$, a *$k$-block* in a graph is any maximal set of at least $k$ vertices no two of which can be separated by $< k$ vertices. Every $k$-block $X$ in a graph $G$ defines an $\mathcal{B}_k$-tangle of $S_k$ for

$$\mathcal{B}_k := \left\{ \sigma \subseteq \vec{S} : \left| \bigcap \{ B \mid (A, B) \in \sigma \} \right| < k \right\},$$

where $S$ is the set of all separations of $G$: just orient every separation in $S_k$ towards $X$. Conversely, given any $\mathcal{B}_k$-tangle $\tau$ of $S_k$ in $G$, the set $\bigcap\{ B \mid (A, B) \in \tau\}$ is a $k$-block in $G$, which in turn defines $\tau$ as indicated.

One of the earliest applications of what later became $\mathcal{F}$-tangles was that finite graphs have canonical tree-decompositions that distinguish all their $k$-blocks [17]. This extended a classical decomposition theorem of Tutte for 2-blocks. But there was no tangle-tree duality theorem for blocks, since $\mathcal{B}_k$ does not consist of stars.

It was shown much later that blocks can in fact be captured by $\mathcal{F}$-tangles with $\mathcal{F}$ consisting of stars [18, 19, 14]. But these $\mathcal{F}$ have to be constructed recursively for each given graph, and are therefore highly artificial: tailor-made for the purpose, but not defining $\mathcal{F}$-tangles that can be understood from their definition. As a consequence, the certificates they yield in theory for the non-existence of blocks are of little use in practice. Our theorem 2.6.5, however, offers very tangible certificates:

**Theorem 2.7.1.** *Let $S$ be the set of separations of a graph $G$, with the usual order function [5], and let $k \in \mathbb{N}$. Then exactly one of the following assertions holds:*

1. *$G$ has a $k$-block;*
2. *there exists an $\mathcal{B}_k$-tree of $\vec{S_k}$.*

*In the case of 2, the $\mathcal{B}_k$-tree can be chosen to be irreducible, efficient, and ordered.*

*Proof.* We have already seen that $k$-blocks induce $\mathcal{B}_k$-tangles of $S_k$ and vice versa. We can thus replace 1 with the assertion that $S_k$ has a $\mathcal{B}_k$-tangle, and proceed to apply theorem 2.6.5.

Let us check the theorem's premise. The set $\mathcal{B}_k$ is standard for $\vec{S_k}$, since the all co-trivial oriented separations in $\vec{S_k}$ are of the form $(V, A)$, where $V = V(G)$ and $|A| < k$.[23] Clearly, then, $\{\{(V, A)\}\} \in \mathcal{B}_k$.

[23] In addition, the set $A$ has to lie in the separator of another separation of order $< k$.

For a proof that $\mathcal{B}_k$ is rich for $\vec{S}_k$, it suffices by theorem 2.4.3 to show that $\mathcal{B}_k$ is closed under minimization in $\vec{S}$. Given any $\sigma \in \mathcal{F}$, let $\sigma'$ be obtained from $\sigma$ by replacing every $(A, B) \in \sigma$ by some $(A', B') \leq (A, B)$ from $\vec{S}$.[24] As

$$\bigcap \left\{ B' : (A', B') \in \sigma' \right\} \subseteq \bigcap \left\{ B : (A, B) \in \sigma \right\},$$

our assumption of $\sigma \in \mathcal{B}_k$ clearly implies that $\sigma' \in \mathcal{B}_k$, as required. □

Next, profiles. Profiles of graphs were introduced by Hundertmark [20], who noticed that the above-mentioned 'tree-of-tangles' theorem for blocks [17] required in its proof much less information about blocks than is provided in their definition. All that is needed is that the orientation of $S_k$ which a $k$-block induces[25] avoids the set $\mathcal{F}$ of triples of the form $\{\vec{r}, \vec{s}, \overleftarrow{r} \vee \overleftarrow{s}\}$. In the context of graphs, this supremum is taken in the universe of all the separations of the given graph.

More generally, given any separation system $\vec{S}$ in some universe $\vec{U}$ of separations, we let

$$\mathcal{P} = \left\{ \{\vec{r}, \vec{s}, \overleftarrow{r} \vee \overleftarrow{s}\} : \vec{r}, \vec{s} \in \vec{U} \right\},$$

and refer to the $\mathcal{P}$-tangles of $S$ as its *profiles*. A profile is *regular* if none of its elements is small.

Hundertmark's discovery was seminal: since graph tangles are profiles too, the proof of the canonical tree-of-tangles theorem for blocks [17], once rewritten for profiles [7], yielded *canonical* trees of graph tangles – something that Robertson and Seymour [1] had been unable to achieve directly by their tangle methods.

So what about tangle-tree duality for profiles? It was the hope to find such a dichotomy theorem for profiles that motivated the original generalization of graph tangles to $\mathcal{F}$-tangles in abstract separation systems [16], since profiles had this form by definition. But the hope proved elusive. We did indeed find a duality theorem for $\mathcal{F}$-tangles in [16], but this once more required that $\mathcal{F}$ consisted of stars. This was good enough for tangles, since $T$-tangles coincide with $T^*$-tangles, as well as for some further duality theorems in graphs and matroids [21], but there was no tangle-tree dichotomy theorem for profiles among these.

As in the case of blocks it is possible to construct, recursively, sets $\mathcal{F}$ of stars in submodular separation systems embedded in a universe so that its profiles are precisely its $\mathcal{F}$-tangles. One can then apply [16] to obtain a tangle-tree duality theorem for profiles, encoded as such $\mathcal{F}$-tangles [19].

Alternatively, we can now obtain certificates for the non-existence of profiles directly from theorem 2.6.5.

As preparation, we need a lemma. Given a separation system $\vec{S}$ in some universe $\vec{U}$, we call a profile of $S$ *strong* if it has no subset in

$$\mathcal{P}_s = \left\{ \{\vec{r}, \vec{s}, \overleftarrow{t}\} : \vec{r}, \vec{s}, \vec{t} \in \vec{U} \text{ and } \overleftarrow{t} \leq \overleftarrow{r} \vee \overleftarrow{s} \right\}.$$

Note that strong profiles are regular: if $\vec{r} \leq \overleftarrow{r}$ then no strong profile contains $\vec{r}$, since $\vec{r} \leq \overleftarrow{r} \vee \overleftarrow{r}$ puts $\{\vec{r}\}$ in $\mathcal{P}_s$. If $\vec{S}$ is submodular and $\vec{U}$ is distributive, the converse holds too:

**Lemma 2.7.2.** *Every regular profile of a submodular separation system in a distributive universe of separations is strong.*

[24] Recall that, for graph separations, $(A', B') \leq (A, B)$ if and only if $A' \supseteq A$ and $B' \subseteq B$.

[25] orient every separation towards the side that contains the given $k$-block

*Proof.* Let $(\vec{S}, \le)$ be any submodular separation system in a distributive lattice $\vec{U}$. Suppose $S$ has a regular profile $\tau$ that is not strong. Then there exist $\vec{r}$, $\vec{s}$, $\overleftarrow{t}$ in $\tau$ such that $\overleftarrow{t} \le \overleftarrow{r} \vee \overleftarrow{s}$. Choose this triple with $\overleftarrow{t}$ as large as possible under $\le$.

Our aim will be to show that both $\vec{t} \vee \vec{r}$ and $\vec{t} \vee \vec{s}$ lie in $\vec{S}$. For if they do, they must lie in $\tau$ by the consistency of $\tau$, since $\vec{t} \vee \vec{r} \ge \vec{r} \in \tau$ and $\vec{t} \vee \vec{s} \ge \vec{s} \in \tau$ and $\tau$ is regular.[26] Then, as

$$(\vec{t} \vee \vec{s})^* \vee (\vec{t} \vee \vec{r})^* = (\overleftarrow{t} \wedge \overleftarrow{s}) \vee (\overleftarrow{t} \wedge \overleftarrow{r}) = \overleftarrow{t} \wedge (\overleftarrow{s} \vee \overleftarrow{r}) = \overleftarrow{t}$$

by distributivity, we shall have $\tau \supseteq \{\vec{t} \vee \vec{r}, \vec{t} \vee \vec{s}, \overleftarrow{t}\} \in \mathcal{P}$, which contradicts our assumption that $\tau$ is a profile.

So let us show that $\vec{t} \vee \vec{r} \in \vec{S}$; the proof of $\vec{t} \vee \vec{s} \in \vec{S}$ is analogous. By submodularity we have $\vec{t} \vee \vec{r} \in \vec{S}$ as desired unless $\vec{t} \wedge \vec{r} \in \vec{S}$, so let us assume this. Then also $\vec{S} \ni (\vec{t} \wedge \vec{r})^* = \overleftarrow{t} \vee \overleftarrow{r} \ge \overleftarrow{t} \in \tau$, so $\overleftarrow{t} \vee \overleftarrow{r} \in \tau$ by the consistency and regularity[26] of $\tau$.

If $\overleftarrow{t} \vee \overleftarrow{r} = \overleftarrow{t}$,then $\overleftarrow{t} \ge \overleftarrow{r}$ and thus $\vec{t} \le \vec{r}$, giving $\vec{t} \vee \vec{r} = \vec{r} \in \vec{S}$ as desired. But otherwise $\overleftarrow{t} \vee \overleftarrow{r} > \overleftarrow{t} \in \tau$, which contradicts the maximality of $\overleftarrow{t}$ in our choice of $\vec{r}, \vec{s}, \overleftarrow{t} \in \tau$ with $\overleftarrow{t} \le \overleftarrow{r} \vee \overleftarrow{s}$, since also $\overleftarrow{t} \vee \overleftarrow{r} \le \overleftarrow{r} \vee \overleftarrow{s}$ by definition of the supremum $\overleftarrow{t} \vee \overleftarrow{r}$. □

theorem 2.7.2 implies that the regular profiles of a submodular separation system in a distibutive universe are precisely its $\mathcal{P}_s$-tangles. Indeed, being profiles, they are $\mathcal{P}$-tangles and therefore consistent, and they avoid $\mathcal{P}_s$ since, by the lemma, they are strong, so they are $\mathcal{P}_s$-tangles. Conversely, every $\mathcal{P}_s$-tangle avoids $\mathcal{P} \subseteq \mathcal{P}_s$ and is therefore a profile, and it is strong since $\mathcal{P}_s$-tangles cannot contain small separations, as noted earlier.

**Theorem 2.7.3.** *Let $\vec{S}$ be any submodular separation system in a distibutive universe of separations. Then exactly one of the following assertions holds:*

1. *there exists a regular profile of $S$;*
2. *there exists an $\mathcal{P}_s$-tree of $\vec{S}$.*

*In the case of 2, the $\mathcal{P}_s$-tree can be chosen to be irreducible, efficient, and ordered with respect to any order function on $S$.*

*Proof.* Our aim is us apply theorem 2.6.5 to $\vec{S}$ with $\mathcal{F} = \mathcal{P}_s$, and richness referring to any order function on $S$ we may choose. As remarked after theorem 2.7.2, the regular profiles of $S$ are precisely its $\mathcal{P}_s$-tangles. It remains to check that $\mathcal{P}_s$ is standard and rich for $\vec{S}$.

For a proof that $\mathcal{P}_s$ is standard, we have to show that $\{\overleftarrow{s}\} \in \mathcal{P}_s$ whenever $\vec{s}$ is trivial in $\vec{S}$. Let $r \in S$ witness that $\vec{s}$ is trivial. Then $\vec{r}, \overleftarrow{r} < \vec{s}$, so $\overleftarrow{s} < \vec{r} < \vec{s}$. Hence $\overleftarrow{s} \le \vec{s} \vee \vec{s}$, which puts $\{\overleftarrow{s}\}$ in $\mathcal{P}_s$.

For a proof that $\mathcal{P}_s$ is rich for $\vec{S}$, it suffices by theorem 2.4.3 to show that $\mathcal{P}_s$ is closed under minimization in $\vec{S}$. Given any set $\sigma = \{\vec{r}, \vec{s}, \overleftarrow{t}\} \in \mathcal{P}_s$, consider any set $\sigma' = \{\vec{r'}, \vec{s'}, \overleftarrow{t'}\}$ with $\vec{r'} \le \vec{r}$ and $\vec{s'} \le \vec{s}$ and $\overleftarrow{t'} \le \overleftarrow{t}$. Then $\sigma' \in \mathcal{P}_s$ as desired, since $\overleftarrow{t'} \le \overleftarrow{t} \le \overleftarrow{r} \vee \overleftarrow{s} \le \overleftarrow{r'} \vee \overleftarrow{s'}$; the middle inequality comes from $\sigma \in \mathcal{P}_s$. □

Finally, let us apply our results to tangles defined by clusters in large datasets. The basic setup is that $\vec{S}$ is the set $2^V$ of all subsets of our data set $V$, the involution $^*$ given by complementation in $V$. Then $S$ is the set of bipartitions $s = \{A, B\}$ of $V$, whose

[26] Compare theorem 2.2.1 and the remark preceding it.

orientations are $\vec{s} = A$ and $\overleftarrow{s} = B$ (say). We then choose an order function on $S$ that assigns high values to separations that divide many pairs of elements of $V$ which we consider as 'close',[27] the idea being that separations of low order cannot divide many close pairs and therefore cannot cut through a cluster, only chip off a few points. This, then, implies that large clusters induce $\mathcal{F}$-tangles for $\mathcal{F}$ defined as

$$\mathcal{C}_n := \left\{ \{\vec{r}, \vec{s}, \vec{t}\} : |\vec{r} \cap \vec{s} \cap \vec{t}| < n \right\},$$

where the *agreement* value $n$ is chosen to fit the context.

Thus, $\mathcal{C}_n$-tangles in datasets are a bit like $\mathcal{B}_k$-tangles in graphs, except that $n$ is now a value separate from the order $k$ of the tangles considered. An important difference is that, unlike $k$-blocks in graphs, the clusters captured by those tangles are not normally subsets $X$ of $V$ that we can, or even wish to, specify explicitly: they can be 'fuzzy', and do not have to lie entirely on the side of the separations chosen by the tangle they induce, only mostly. Quantitatively, while the intersection of any three elements of an $\mathcal{C}_n$-tangle contains at least $n$ datapoints, the intersection of all of them will likely be empty – unlike in the case of $k$-blocks in graphs, which are equal to the intersection of *all* the sides of the graph's separations chosen by the tangle.

In other contexts than generic clustering one sometimes looks for $\mathcal{C}_n$-tangles not of the set $S$ of all the bipartitions of $V$, but of some specific, hand-designed set $S$ of bipartitions of $V$. For example, $S$ might be a questionnaire whose elements partition a population $V$ of individuals that have answered it into those that answered yes and those that answered no. In such a context, $\mathcal{C}_n$-tangles can be interpreted as 'typical' ways to answer the questions in $S$, as *mindsets* found in the population $V$ regarding the questions in $S$. See [3] for more.

In all these contexts we can now use theorem 2.6.1 to efficiently display the clusters or mindsets in $V$ as determined by $S$, as long as $\mathcal{C}_n$ is standard and rich for $\vec{S}$. If we are interested specifically in certifying that our dataset has no clusters of some desired density at all, we can similarly apply theorem 2.6.5.

The sets $\mathcal{C}_n$ are standard for all $\vec{S} \neq \emptyset$ and $n > 0$: then $\{\emptyset\} \in \mathcal{C}_n$, and only $\emptyset$ can be co-trivial in $\vec{S}$. Also, $\mathcal{C}_n$ is clearly closed under minimization in $\vec{S}$: replacing the elements $\vec{s}$ of a triple $\sigma$ with subsets $\vec{s'} \leq \vec{s}$ leads to $\bigcap \sigma' \subseteq \bigcap \sigma$, so $\sigma \in \mathcal{C}_n$ implies $\sigma' \in \mathcal{C}_n$. By theorem 2.4.3, therefore, $\mathcal{C}_n$ is rich for $\vec{S}$.

theorems 2.6.1 and 2.6.5 thus have the following instances in our context:

**Corollary 2.7.4.** *Let $\vec{S}$ be any non-empty set of subsets of a set $V$ that is closed under taking complements in $V$. Let any order function on $S$ be given, and let $n > 0$. Then $\vec{S}$ has an efficient and irreducible ordered $\mathcal{C}_n$-tangle structure tree.* □

**Corollary 2.7.5.** *Let $\vec{S}$ be any non-empty set of subsets of a set $V$ that is closed under taking complements in $V$. Let any order function on $S$ be given, and let $n > 0$. Then exactly one of the following assertions holds:*

1. *there exists a $\mathcal{C}_n$-tangle of $S$;*
2. *there exists a $\mathcal{C}_n$-tree of $\vec{S}$.*

*In case 2, the $\mathcal{C}_n$-tree can be chosen to be irreducible, efficient, and ordered.* □

[27] Choosing this order function to fit the application context allows for considerable variety in using tangles for clustering [3Chapter 9]. See [22] for a concrete example.

# 3

# Tangle structure trees II: trees of tangles and tangle-tree duality

**Tangle structure trees, introduced in [23], offer a unified data structure that displays all the tangles of a graph or data set together with certificates for the non-existence of any other tangles, either locally or overall. In this paper we apply tangle structure trees to derive new versions of the two fundamental tangle theorems: the tree-of-tangles theorem, and the tangle-tree duality theorem.**

**We extend the tree-of-tangles theorem to $\mathcal{F}$-tangles that need not be profiles. When $\mathcal{F}$ consists of stars of separations, as it does in classical tangle-tree duality theorems, we show how to convert tangle structure trees that certify the non-existence of $\mathcal{F}$-tangles into tree-decompositions that certify this in the way known from graph tangles, as $S$-trees over $\mathcal{F}$.**

## Contents



In this sequel to [23] we explore further applications of *tangle structure trees*. These are data structures, introduced in [23], which simultaneously display

- all the tangles of a given graph or abstract separation system; and
- for all its orientations that are not tangles, certificates of why they are not.

Previously, there were two separate types of theorem in tangle theory for these two purposes: *tree-of-tangle* theorems, which display all the tangles of a graph or data set in a tree-like way that 'distinguishes' different tangles by specifying a minimum-order separation that they orient differently; and *tangle-tree duality* theorems that display, also in a tree-like way, certificates for why the various orientations of the separation system are not tangles – typically in the form of small subsets of oriented separations that are *forbidden* in the given type of tangle. These two fundamental theorems, which exist in various forms and degrees of generality, are thought of as the two pillars of both graph and abstract tangle theory.

The unified way in which tangle structure trees serve both these purposes still allows us to extract, if desired, separate corollaries of the two types. In [23] we did this for the second type: we derived new tangle-tree duality-type theorems for tangles whose defining forbidden subsets were not 'stars' of separations, nested sets of typically three oriented separations pointing towards each other. All previously known tangle-tree duality theorems, in particular those for graphs, were for tangles defined by excluding such stars.

Our first main result in this paper is to extract from the tangle structure trees established in [23] a new theorem of the first type: a tree-of-tangles theorem for tangles whose set $\mathcal{F}$ of defining forbidden subsets need not form a 'profile'. All known tree-of-tangles theorems so far are for profiles, in fact, for *robust* profiles. These are still a sweeping generalization of graph tangles. But we shall prove here that we can generalize them further: we shall obtain a tree-of-tangles theorem for all $\mathcal{F}$-tangles whose set $\mathcal{F}$ of forbidden subsets of oriented separations is robust, even if not a profile.

Our second main result is about $\mathcal{F}$-tangles whose set $\mathcal{F}$ of forbidden subsets consists of stars of separations, as in all the classical tangle-tree duality theorems. We prove a general conversion theorem, which allows us to extract from our tangle structure tree for $\mathcal{F}$-tangles, of a separation system $\vec{S}$, say, an $S$-tree over $\mathcal{F}$ if $S$ admits no $\mathcal{F}$-tangles. Such an '$S$-tree over $\mathcal{F}$' is the tree structure in the most general tangle-tree duality theorems known so far for abstract separation systems (which include graphs), proved in [16]. Our conversion theorem allows us to re-prove the main result of [16] from the premise for the existence of tangle structure trees, which differs from the premise of the tangle-tree duality theorem in [16].

However we demonstrate that from our premise for the existence of tangle structure trees we can derive a weaker version of the premise in [16], the version used in practice when the duality theorem from [16] is applied in [21]. As a consequence, we shall be able in [24] to use our conversion theorem, coupled with the existence theorems for tangle structure trees in [23], to derive all the known applications of the tangle-tree duality theorem in [16] to concrete structures, such as graphs, matroids, or data sets [21].

The paper is organized as follows. In section 3.1 we give a summary of tangle basics, just what is technically needed to read this paper. More on graph tangles can be found in [5], more on abstract tangles and their applications in [2, 25, 3]. In section 3.2 we introduce tangle structure trees and summarize the main results from [23]; more on this, including motivational background, can be found there.

section 3.3 contains our conversion theorem that extracts from a tangle structure tree, of a separation system $\vec{S}$ that has no $\mathcal{F}$-tangles, an $S$-tree over $\mathcal{F}$. section 3.4 is an interlude on submodular order functions for separations, which is needed in the rest of the paper; we show how a non-injective submodular order function can be extended to an injective one without losing submodularity. The tree-of-tangles theorem for $\mathcal{F}$-tangles with arbitrary robust exclusion sets $\mathcal{F}$ is proved in section 3.5. In section 3.6 we show how to recover the essence of the tangle-tree duality theorem of [16] from our conversion theorem and our existence theorem for tangle structure trees.

## 3.1 Separation systems and their tangles

In this section we introduce the terminology that abstract tangle theory requires, following [2].[1] Readers familiar with [23] may skip this section.

Tangles of graphs are ways of orienting their separations, each towards one of its two sides. Abstract tangles are designed to work in scenarios where there need not be anything to 'separate'. In order to retain our intuition from graphs, however, we continue to refer to the things of which our abstract tangles pick one of two variants (which they will indeed do) as 'separations'. These are defined by noting some key properties of graph separations and making them into axioms, as follows.

A *separation system* $(\vec{S}, \leq, ^*)$ is a set $\vec{S}$, whose elements we call *oriented separations*, that comes with a partial ordering $\leq$ on $\vec{S}$ and an order-reversing involution $^* : \vec{S} \to \vec{S}$. Thus, for any two elements[2] $\vec{r}, \vec{s}$ of $\vec{S}$ with $\vec{r} \leq \vec{s}$ we have $\vec{r}^* \geq \vec{s}^*$. We write $\vec{s}^* =: \overleftarrow{s}$, and call $\overleftarrow{s}$ the *inverse* of $\vec{s}$. While we allow formally that $\vec{s} = \overleftarrow{s}$, in which case we call $\vec{s}$ and $s$ *degenerate*, this does not happen often in practice.[3]

If a separation system $\vec{U}$ happens to be a lattice, that is, if there is a supremum $\vec{r} \vee \vec{s}$ and an infimum $\vec{r} \wedge \vec{s}$ in $\vec{U}$ for every two elements $\vec{r}, \vec{s} \in \vec{U}$, we call $\vec{U}$ a *universe* of separations. It is *distributive* if it is distributive as a lattice. A separation system $\vec{S} \subseteq \vec{U}$ is *submodular* if for every two elements of $\vec{S}$ either their infimum or their supremum in $\vec{U}$ also lies in $\vec{S}$.

Very rarely we may have separations $\vec{s} \leq \overleftarrow{s}$; then $\vec{s}$ is *small* and $\overleftarrow{s}$ is *large*.[4] Separation systems without small elements are *regular*. We say that $\vec{s}$ is *trivial* (and $\overleftarrow{s}$ is *co-trivial*) in $\vec{S}$ if there exists a pair of inverse separations $\vec{r}, \overleftarrow{r} < \vec{s}$ in $\vec{S}$. Trivial separations are clearly large, so co-trivial ones are small, but the converse need not hold. See [2] for more on these technicalities if desired.

The set of *unoriented separations* in $(\vec{S}, \leq, ^*)$ is

$$S := \{\{\vec{s}, \overleftarrow{s}\} : \vec{s} \in \vec{S}\}.$$

We call the elements $\vec{s}, \overleftarrow{s}$ of $s$ its *orientations*. An *orientation of* $S$ is a set $\tau \subseteq \vec{S}$ that contains exactly one orientation of every $s \in S$. An orientation of a subset of $S$ is a *partial orientation* of $S$. We write $\tau(s)$ for the unique orientation of $s$ contained in $\tau$, and say that $s$ *distinguishes* two partial orientations $\tau, \tau'$ of $S$ if both are defined on $s$ and $\tau(s) \neq \tau'(s)$.

[1] There is one difference to [2]: for historical reasons, the partial ordering on $\vec{S}$ used there is the inverse of ours. So terms like 'large' and 'small', infima and suprema etc, are reversed.

[2] We often denote the elements of $\vec{S}$ by letters with an arrow, in either direction, precisely in order to have a simple way to refer to their dual elements: by reversing the arrow. But the arrow directions have no meaning: an arbitrary element of $\vec{S}$ could be denoted equally as $\vec{s}$ or as $\overleftarrow{s}$.

[3] The only degenerate separation of a graph $G = (V, E)$, for example, is $\{V, V\}$.

[4] The small separations of a graph $G$ are those of the form $(V, A)$ with $A \subseteq V = V(G)$.

If $\vec{r} \geq \vec{s}$ we say that $\vec{r}$ *points towards* $s$ (and that $\overleftarrow{r}$ *points away from* $s$). We say that $\vec{r}$ *points towards* an oriented separation $\vec{s}$ whenever it points towards $s$, i.e., if $\vec{r} \geq \vec{s}$ or $\vec{r} \geq \overleftarrow{s}$, and similarly for 'points away from'. A *star* is a set $\sigma$ of non-degenerate oriented separations that point towards each other. As is easy to check, this happens if and only if $\vec{r} \geq \overleftarrow{s}$ (and hence $\vec{s} \geq \overleftarrow{r}$) for all distinct $\vec{r}, \vec{s} \in \sigma$.

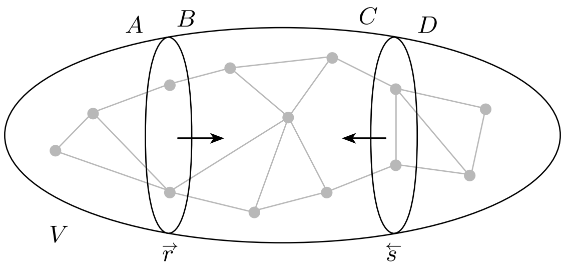


**Figure 3.1**: Nested separations $r = \{A, B\}$ and $s = \{C, D\}$ of a graph. Their orientations $\vec{r} = (A, B)$ and $\overleftarrow{s} = (D, C)$ point towards each other, since $\vec{r} \geq \vec{s}$ (as $B \supseteq D$) and $\overleftarrow{s} \geq \overleftarrow{r}$ (as $C \supseteq A$).

Two separations $r, s \in S$ are *nested* if they have orientations that are comparable under $\leq$; otherwise they *cross*. Oriented separations are *nested* if their underlying unoriented separations are nested. A subset of $\vec{S}$ is *nested* if its elements are pairwise nested.

A subset of $\vec{S}$ is *consistent* if no pair of its elements $\overleftarrow{r}, \vec{s}$ with $r \neq s$ point away from each other. Stars are examples of consistent nested sets of oriented separations.

We shall often be interested in consistent orientations of $S$. For each of its elements $\vec{r}$, a consistent orientation of $S$ will also contain every $\vec{s} > \vec{r}$ other than, possibly, $\vec{s} = \overleftarrow{r}$. Consistent partial orientations of $S$ are easily seen to extend to consistent orientations of $S$, unless they contain a separation that is co-trivial in $\vec{S}$; see [2].

If $\sigma \subseteq \vec{S}$ is consistent, we say that $\vec{s} \in \vec{S}$ is *required* by $\sigma$ if $\vec{s} \notin \sigma$ and $\sigma \cup \{\overleftarrow{s}\}$ is inconsistent. We shall see in theorem 3.1.1 that, pathological cases aside, $\sigma \cup \{\vec{s}\}$ will then be consistent. The *closure* of $\sigma$ is

$$\lfloor \sigma \rfloor := \sigma \cup \{\vec{s} \in \vec{S} : \vec{s} \text{ is required by } \sigma\}.$$

Note that $\sigma$ requires $\vec{s} \notin \sigma$ if and only if there exists an $\vec{r} \in \sigma$ such that $r \neq s$ and $\vec{r} < \vec{s}$. Thus

$$\lfloor \sigma \rfloor = \sigma \cup \{\, \vec{s} \in \vec{S} : \exists \vec{r} \in \sigma \text{ such that } r \neq s \text{ and } \vec{r} < \vec{s}\},$$

which motivates the term of (upward) 'closure'. If $\lfloor \sigma \rfloor$ is consistent, then $\lfloor \lfloor \sigma \rfloor \rfloor$ is (defined and) easily shown to be equal to $\lfloor \sigma \rfloor$.

If $\sigma$ contains no small separations, the expression above simplifies to

$$\lfloor \sigma \rfloor = \{\, \vec{s} \in \vec{S} : \exists \vec{r} \in \sigma \text{ such that } \vec{r} \leq \vec{s}\}.$$

Indeed, any $\vec{s}$ in this latter set either lies in $\sigma$ or there is some $\vec{r} \in \sigma$ such that $\vec{r} < \vec{s}$; in that case $r \neq s$, since otherwise $\overleftarrow{s} = \vec{r} < \vec{s}$, making $\vec{r} \in \sigma$ small.

**Lemma 3.1.1** ([23]). *Let $\sigma \subseteq \tau \subseteq \vec{S}$ be consistent sets. Then $\lfloor \sigma \rfloor \subseteq \lfloor \tau \rfloor$. If $\tau$ is an orientation of all of $S$, then $\lfloor \tau \rfloor = \tau$. If $\tau$ has no elements that are co-trivial in $\vec{S}$, then $\lfloor \tau \rfloor$ is consistent and $\lfloor \tau \rfloor \setminus \tau$ contains at most one orientation of any $s \in S$.*

Consistent orientations of $S$ can contain small separations. But examples are rare and can be counter-intuitive, so we often exclude them. Co-trivial separations cannot occur in consistent orientations of $S$. Indeed if $\overleftarrow{s}$ is co-trivial, witnessed by $r \in S$, then every orientation of $S$ will have to orient $r$ too. But it cannot do so consistently with $\overleftarrow{s}$, since both $\vec{r}$ and $\overleftarrow{r}$ are inconsistent with $\overleftarrow{s}$. Similarly, $\lfloor\{\overleftarrow{s}\}\rfloor$ is then inconsistent, since it contains $\vec{r}$ and $\overleftarrow{r}$, which are both inconsistent with $\overleftarrow{s}$.

An *order function* on $S$ is any map $S \to \mathbb{R}$. Unless otherwise mentioned, we denote such order functions as $s \mapsto |s|$. We extend them to $\vec{S}$ by letting $|\vec{s}| := |\overleftarrow{s}| := |s|$. A separation system $(\vec{S}, \leq, ^*)$ given with an order function on $S$ is an *ordered separation system.* For every $k \in \mathbb{R}$ we let

$$\vec{S_k} := \{\vec{s} \in \vec{S} : |s| < k\};$$

this is again an ordered separation system.

We are sometimes interested in orientations of $S$ that do not have certain subsets. We typically collect those together in some set $\mathcal{F}$, whose elements we call *forbidden subsets.* Formally, if $\mathcal{F}$ is any set, we say that $\tau \subseteq \vec{S}$ *avoids* $\mathcal{F}$ if $\tau$ has no subset in $\mathcal{F}$, i.e., if no subset of $\tau$ is an element of $\mathcal{F}$.

**Definition 3.1.1.** An $\mathcal{F}$*-tangle* of $S$ is an $\mathcal{F}$-avoiding consistent orientation of $S$. The $\mathcal{F}$-tangles of the subsets $S_k$ of $S$ are the $\mathcal{F}$-tangles *in* $\vec{S}$.

When our choice of $\mathcal{F}$ is clear from the context, or arbitrary, we shall also refer to $\mathcal{F}$-tangles simply as *tangles.*

The tangles of the sets $S_k \subseteq S$ are the *$k$-tangles of $S$*, or the *tangles of order $k$* in $\vec{S}$. When we speak of *maximal* tangles in $\vec{S}$, we refer to their partial ordering as subsets of $\vec{S}$. A maximal $\mathcal{F}$-tangle in $\vec{S}$, thus, is an $\mathcal{F}$-tangle $\tau$ of some $S_k$ for which there is no $\mathcal{F}$-tangle $\tau' \neq \tau$ of any $S_\ell$ such that $\tau = \tau' \cap \vec{S_k}$.

## 3.2 Tangle structure trees

We adopt the graph-theoretic terminology of [5]. Given a tree $T$ with a root $r$, we write $\leq_r$ for the associated partial ordering on the nodes of $T$. Maximal elements, including the root if $|T| = 1$, are called *leaves.* Any direct successors in $\leq_r$ of a node of $T$ are its *children.* We write $E_v$ for the set of edges of $T$ from a node $v$ to its children.

Let $(\vec{S}, \leq, ^*)$ be a separation system, and let $\mathcal{F}$ be any set. A *separation tree* $(T, r, \beta)$ on $\vec{S}$ consists of a rooted tree $(T, r)$ and an edge labelling $\beta : E(T) \to \vec{S}$ such that for every non-leaf $v \in V(T)$ there exists a separation $s_v \in S$ such that $\beta$ restricts to a bijection $E_v \to \{\overleftarrow{s}, \vec{s_v}\}$ and $s_u \neq s_v$ whenever $u <_r v$.

Thus, every non-leaf node $v$ of a separation tree has either one or two children. If there is no need to refer to $r$ or $\beta$ explicitly, we usually abbreviate $(T, r, \beta)$ to $T$. For every node $v$, we write $\beta_v$ for the set $\beta\big(E(rTv)\big)$ of edge labels on the path $rTv$. A separation tree is *consistent* if $\beta_v \subseteq \vec{S}$ is consistent for every node $v \in T$.

A leaf $\ell$ of a consistent separation tree on $\vec{S}$ is an *$\mathcal{F}$-tangle leaf* if the closure $\lfloor\beta_\ell\rfloor$ of $\beta_\ell$ is an $\mathcal{F}$-tangle of $S$. A non-leaf node $v$ is an *$\mathcal{F}$-tangle node* if there is an $\mathcal{F}$-tangle leaf $\ell \geq_r w$ for every (one or two) successor $w$ of $v$ in $<_r$. Tangle nodes with two children, thus, are precisely the $<_r$-infima of pairs of distinct tangle leaves.

A leaf $\ell$ is *forbidden* (by $\mathcal{F}$) if $\beta_\ell$ contains an element of $\mathcal{F}$ as a subset. Note that tangle leaves are never forbidden. An *$\mathcal{F}$-tangle structure tree of $\vec{S}$* is a consistent separation tree on $\vec{S}$ in which every leaf is either a tangle leaf or forbidden, and for every non-leaf node $v$ the set $\beta_v$ has no subset in $\mathcal{F}$.

**Theorem 3.2.1** ([23Theorem 3.7]). *Let $\vec{S}$ be a separation system, let $\mathcal{F}$ be any set, and let $T$ be an $\mathcal{F}$-tangle structure tree of $\vec{S}$. Then for every $\mathcal{F}$-tangle $\tau$ of $S$ there is a unique leaf $\ell =: \ell(\tau)$ of $T$ with $\lfloor \beta_\ell \rfloor = \tau$. In particular, if all the leaves of $T$ are forbidden, then $S$ has no $\mathcal{F}$-tangle.*

$\mathcal{F}$-tangle structure trees will be our main tool, so let us see when they exist.

Essentially, two conditions on $\mathcal{F}$ are needed to ensure that $\vec{S}$ has an $\mathcal{F}$-tangle structure tree. The first is that $\{\overleftarrow{s}\} \in \mathcal{F}$ for every $\vec{s} \in \vec{S}$ that is trivial in $\vec{S}$. This holds for all relevant $\mathcal{F}$ (see [23] for why), and if it does we call $\mathcal{F}$ *standard for $\vec{S}$*.

The second condition is the one that bites. If $T$ is any separation tree on $\vec{S}$, then every orientation $\tau$ of $S$ contains a set $\beta_\ell$ for a unique leaf $\ell$ of $T$ [23Lemma 3.2]. If $\tau$ is consistent and $\lfloor \beta_\ell \rfloor$ orients all of $S$, then $\lfloor \beta_\ell \rfloor = \tau$ by theorem 3.1.1. Hence if $\tau$ is not an $\mathcal{F}$-tangle, then $\lfloor \beta_\ell \rfloor$ has a subset $\sigma$ in $\mathcal{F}$ that witnesses this.

If $T$ is even an $\mathcal{F}$-tangle structure tree, then $\ell$ is a forbidden leaf, and so we can find such a subset $\sigma \in \mathcal{F}$ of $\tau$ not just in $\lfloor \beta_\ell \rfloor$ but even in $\beta_\ell$: among the edge labels of $T$. In order for this to be possible, we therefore need to make some 'richness' assumption about $\mathcal{F}$: an assumption which, in our example, ensures that $\mathcal{F}$ has enough elements to contain a subset also of $\beta_\ell$ as soon as it contains a subset of $\lfloor \beta_\ell \rfloor$.

The main contribution of [23] was to identify such a richness condition on $\mathcal{F}$ that is, essentially, both necessary and sufficient for $\mathcal{F}$-tangle structure trees to exist.

To motivate it, let us first state a stronger condition that is less technical, and which clearly implies that $\mathcal{F}$ contains a subset of $\beta_\ell$ whenever it contains a subset of $\lfloor \beta_\ell \rfloor$. This is that $\mathcal{F}$ is *closed under minimization* in every consistent orientation $\tau$ of $S$: that it contains every set $\sigma' \subseteq \tau$ obtained from some $\sigma \subseteq \tau$ in $\mathcal{F}$ by replacing every $\vec{s} \in \sigma$ with some $\vec{s'} \leq \vec{s}$ from $\tau$. Our richness notion in definition 3.2.1 will be weaker than this ([23Lemma 4.4]), but still strong enough to ensure the existence of $\mathcal{F}$-tangle structure trees.

A separation tree $T$ on an ordered separation system $\vec{S}$ is *ordered* if $|s_v| \leq |s_w|$ whenever $v$ and $w$ are non-leaves of $T$ with $v \leq w$. It is *thoroughly ordered* (in $\vec{S}$) if, for every non-leaf node $v$, the separation $s_v$ is not oriented by $\lfloor \beta_v \rfloor$ (i.e., $\lfloor \beta_v \rfloor$ contains neither $\overleftarrow{s}$ nor $\overleftarrow{s_v}$) and has minimum order among the separations not oriented by $\lfloor \beta_v \rfloor$. Every thoroughly ordered separation tree is ordered [23Lemma 4.1].

We say that $\vec{s} \in \vec{S}$ is *weakly eclipsed* by $\vec{r} \in \vec{S}$ if $\vec{r} < \vec{s}$ and $|r| \leq |s|$, and *eclipsed* by $\vec{r}$ if $\vec{r} < \vec{s}$ and $|r| < |s|$. Given any set $\tau \subseteq \vec{S}$, a subset $\sigma \subseteq \tau$ is *efficient* (in $\tau$) if no element of $\sigma$ is eclipsed by any other element of $\tau$. It is *strongly efficient* if no element of $\sigma$ is weakly eclipsed by any other element of $\tau$. Our separation tree $T$ is *efficient* if for every leaf $\ell$ the set $\beta_\ell$ is efficient in $\lfloor \beta_\ell \rfloor$. Thoroughly ordered separation trees are efficient [23Lemma 4.9].

**Definition 3.2.1.** A set $\mathcal{F}$ is *rich for $\vec{S}$* if every consistent orientation of $S$ that has a subset in $\mathcal{F}$ also has a strongly efficient[5] subset in $\mathcal{F}$.

[5] in this orientation of $S$

We have the following two theorems for the existence of $\mathcal{F}$-tangle structure trees.

**Theorem 3.2.2** ([23Theorems 4.6, 4.8]). *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be any set that is rich and standard for $\vec{S}$. Then there exists a thoroughly ordered $\mathcal{F}$-tangle structure tree of $\vec{S}$. It is unique if the order function on $S$ is injective.*

Note that the uniqueness part of theorem 3.2.2 does not require the existence part: an injectively ordered separation system clearly has at most one thoroughly ordered $\mathcal{F}$-tangle structure tree, regardless of what $\mathcal{F}$ is. In particular, $\mathcal{F}$ need not be standard or rich. But we know from [23Theorem 4.8] that $\mathcal{F}$, assuming it is standard, will be rich for $\vec{S}$ if the order function on $S$ is injective and an $\mathcal{F}$-tangle structure tree exists.

Thoroughly ordered structure trees can be large. But there is a way to reduce them. Given an $\mathcal{F}$-tangle structure tree $(T, r, \beta)$, consider an edge $vw$ of $T$ such that $w$ is a child of $v$. We call $vw$ *necessary* for a tangle leaf $\ell \geq w$ of $T$ if $\beta(vw)$ is a $\leq$-minimal element of the tangle $\lfloor \beta_\ell \rfloor$, or equivalently, of $\beta_\ell$. We call $vw$ *necessary* for a forbidden leaf $\ell \geq w$ if every subset of $\beta_\ell$ in $\mathcal{F}$ contains $\beta(vw)$. A node $v$ of $T$ is *necessary in $T$* if for every child $w$ of $v$ there exists a leaf $\ell \geq w$ such that $vw$ is necessary for $\ell$. If every node of $T$ is necessary, then $T$ is *irreducible*.

It is easy to make a tangle structure tree irreducible by contracting edges at unnecessary nodes and deleting certain subtrees [23Section 5]. This preserves efficiency and being ordered, but not being thoroughly ordered. Our second existence theorem for $\mathcal{F}$-tangle structure trees gains efficiency and irreducibility over that from theorem 3.2.2 at the expense of losing the thoroughness of its ordering:

**Theorem 3.2.3** ([23Theorem 6.1]). *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be any set that is rich and standard for $\vec{S}$. Then $\vec{S}$ has an efficient and irreducible ordered $\mathcal{F}$-tangle structure tree.*

See [23Theorem 6.2] for a long list of properties of these $\mathcal{F}$-tangle structure trees that make them useful for tangle analysis, both in theory and in applications.

## 3.3 Tangle-tree duality

If we apply theorem 3.2.3 to a separation system $\vec{S}$ that has no $\mathcal{F}$-tangles, it tells us that there is an $\mathcal{F}$-tangle structure tree all whose leaves are forbidden. Let us call such structure trees *$\mathcal{F}$-trees*. Conversely, if there is an $\mathcal{F}$-tree of $\vec{S}$ then $S$ has no $\mathcal{F}$-tangle, by theorem 3.2.1.

theorem 3.2.3 thus implies the following dichotomy:

**Theorem 3.3.1** ([23Theorem 6.4]). *Let $\vec{S}$ be an ordered separation system, and let $\mathcal{F}$ be standard and rich for $\vec{S}$. Then exactly one of the following assertions holds:*

1. *there exists an $\mathcal{F}$-tangle of $S$;*
2. *there exists an $\mathcal{F}$-tree of $\vec{S}$.*

*In the case of 2, the $\mathcal{F}$-tree can be chosen to be ordered, irreducible and efficient.*

Note that, unlike in all known theorems of this type, the set $\mathcal{F}$ of forbidden subsets need not consist of stars of separations (see section 3.1).

In the special case that $\mathcal{F}$ does consist of stars of separations, we shall see in theorem 3.3.5 that – as long as the $\mathcal{F}$-tree in 2 is irreducible, which the theorem says we may require – the separations in $\beta(E(T)) \subseteq \vec{S}$ will be nested. In contexts where separation systems describe genuine separations, of a set or of a graph, say, nested sets of separations are known to cut up the structure they separate in a tree-like way – for example, by a tree-decomposition of the graph. In this way, $\mathcal{F}$-trees impose a tree structure on such a set or graph, which in turn implies that its separations we are considering have no $\mathcal{F}$-tangle.

This phenomenon is one of the corner stones of classical tangle theory, known as *tangle-tree duality*. Our aim now is to derive such a dichotomy theorem also in our context, as an application of our more comprehensive tangle structure trees. Our version, theorem 3.3.6 below, will not imply the classical tangle-tree duality theorem for abstract separation systems [16], because our richness assumptions made for $\mathcal{F}$ are different. But it will offer the same dichotomy. And we shall see in [24] that it implies all known instances of the classical theorem for concrete structures, such as graphs or matroids.

To carry this out formally, let us introduce the terms in which classical tangle-tree duality is cast. Let $\vec{E}(T)$ denote the set of orientations of the edges of a tree $T$. For an oriented edge $\vec{e} = (t, t')$ of $T$ we call $t =: i(\vec{e})$ its *initial node* and $t' =: t(\vec{e})$ its *terminal node*, and we denote its inverse $(t', t)$ by $\overleftarrow{e}$. We think of $t(\vec{e})$ as the node that $\vec{e}$ *points towards*. There is a natural partial ordering $\leq$ on $\vec{E}(T)$ in which $\vec{e} > \vec{f}$ if $f \neq e$ and the unique path in $T$ from $e$ to $f$ starts at $t(\vec{e})$ and ends at $i(\vec{f})$. With the involution $^* : \vec{e} \mapsto \overleftarrow{e}$, this makes $(\vec{E}(T), \leq, ^*)$ into a separation system.

Since every two edges $e, f$ of $T$ have comparable orientations, the elements of $\vec{E}(T)$ are all nested. And for every node $t$ of $T$, the set

$$\vec{F}_t := \{ \vec{e} \in \vec{E}(T) : t(\vec{e}) = t \}$$

is a star in this separation system.

Now let $(\vec{S}, \leq, ^*)$ be any other separation system – typically, but not necessarily, consisting of separations of a set or graph. An *S-tree* $(T, \alpha)$ is a tree $T$ together with a map $\alpha : \vec{E}(T) \to \vec{S}$ that satisfies $\alpha(\overleftarrow{e}) = \alpha(\vec{e})^*$ for every $\vec{e} \in \vec{E}(T)$. It is an $S$-tree *over* a set $\mathcal{F}$ if $\alpha(\vec{F}_t) \in \mathcal{F}$ for every $t \in V(T)$.

If $\alpha$ preserves the partial ordering $\leq$ on $\vec{E}(T)$, i.e. if $\vec{e} \leq \vec{f}$ implies $\alpha(\vec{e}) \leq \alpha(\vec{f})$, and the image of $\alpha$ contains no degenerate separations, then the sets $\alpha(\vec{F}_t) \in \mathcal{F}$ will be stars in $\vec{S}$, because the $\vec{F}_t$ are stars in $\vec{E}(T)$. Conversely:

**Lemma 3.3.2** ([25]). *Let $(T, \alpha)$ be an $S$-tree over a set stars. If $\alpha$ is injective on $\vec{F}_t$ for every node $t$ of $T$, then $\vec{e} \leq \vec{f}$ implies $\alpha(\vec{e}) \leq \alpha(\vec{f})$ for all $\vec{e}, \vec{f} \in \vec{E}(T)$. In particular, $\alpha(\vec{E}(T))$ is a nested subset of $\vec{S}$.*

*Conversely, for every nested, regular, finite separation system $(\vec{S}, \leq, ^*)$ there is an $S$-tree $(T, \alpha)$ over stars such that $\alpha(\vec{E}(T)) = \vec{S}$ and $\alpha$ is injective on the stars $\vec{F}_t$.*

We remark that if $\alpha$ is not injective on some $\vec{F}_t$, then $\alpha(\vec{E}(T))$ can fail to be nested.

theorem 3.3.2 motivates the study of $S$-trees over sets $\mathcal{F}$ of stars (of separations): they make precise, in a more general way than tree-decompositions do, the 'tree-like way' in which nested sets of separations cut up any underlying structure.

The existence of an $S$-tree $(T, \alpha)$ over a set $\mathcal{F}$, not necessarily consisting of stars, precludes the existence of an $\mathcal{F}$-tangle of $S$. Indeed, given any orientation $\tau$ of $S$, the set $\alpha^{-1}(\tau)$ is an orientation of the edges of $T$. Since trees contain no cycles, this orientation will contain $\vec{F}_t$ for some $t$, so that $\alpha(\vec{F}_t) \subseteq \tau$. But as $(T, \alpha)$ is an $S$-tree over $\mathcal{F}$ we have $\alpha(\vec{F}_t) \in \mathcal{F}$, so $\tau$ is not an $\mathcal{F}$-tangle.

The converse implication, that there is an $S$-tree over $\mathcal{F}$ whenever $S$ has no $\mathcal{F}$-tangle, holds only when $\mathcal{F}$ consists of stars. But assuming this is not enough. In [16], which offers the most general theorem of this type to date, there is the further assumption that $\vec{S}$ must be *$\mathcal{F}$-separable*. This is a richness condition on $\mathcal{F}$, different from ours, which requires $\mathcal{F}$ to also contain stars which its elements 'induce on the sides' of certain separations in $S$. The precise notion of $\mathcal{F}$-separability is quite technical, but immaterial here, so we refer the reader to [16, 21] for details.

The following dichotomy between $\mathcal{F}$-tangles and $S$-trees over $\mathcal{F}$ is the classical tangle-tree duality theorem for abstract separation systems:

**Theorem 3.3.3.** *[16] Let $\vec{S}$ be a separation system inside a universe $\vec{U}$ of separations, and let $\mathcal{F} \subseteq 2^{\vec{S}}$ be a set of stars that is standard for $\vec{S}$. If $\vec{S}$ is $\mathcal{F}$-separable, then exactly one of the following assertions holds:*

1. *there exists an $\mathcal{F}$-tangle of $S$;*
2. *there exists an $S$-tree over $\mathcal{F}$.*

When $\mathcal{F}$ consists of stars, we can compare theorem 3.3.3 with what our more general theorem 3.3.1 says for such $\mathcal{F}$. Both theorems guarantee that if $S$ admits no $\mathcal{F}$-tangle then it has an easily detectable certificate in the form of a certain tree-like structure: an $S$-tree over $\mathcal{F}$ in the case of theorem 3.3.3, and an $\mathcal{F}$-tree of $\vec{S}$ in the case of our new theorem 3.3.1.

The following result shows that our theorem 3.3.1 is not only more general, because its $\mathcal{F}$ need not consist of stars, but also stronger than theorem 3.3.3, because we can obtain $S$-trees over $\mathcal{F}$ from $\mathcal{F}$-trees when $\mathcal{F}$ does consist of stars. In fact, *every* irreducible $\mathcal{F}$-tree of $\vec{S}$ then has the property that its edge labels in $\vec{S}$ are precisely the edge labels of some $S$-tree over $\mathcal{F}$:

**Theorem 3.3.4.** *Let $\vec{S}$ be a separation system without trivial elements, $\mathcal{F} \subseteq 2^{\vec{S}}$ a set of stars, and $(T, r, \beta)$ an irreducible $\mathcal{F}$-tree of $\vec{S}$. Then there exists an $S$-tree $(T', \alpha)$ over $\mathcal{F}$ and a map $\gamma : V(T') \cup \vec{E}(T') \to V(T) \cup E(T)$ such that*

1. *$\gamma$ is a bijection from the nodes of $T'$ to the leaves of $T$;*
2. *$\gamma$ is a bijection from the oriented edges of $T'$ to the edges of $T$;*
3. *for every edge $e$ of $T'$ there is a node $v_e$ of $T$ such that $\{\gamma(\vec{e}), \gamma(\overleftarrow{e})\} = E_{v_e}$, and these $v_e$ are distinct for different $e$;*
4. *for every oriented edge $\vec{e}$ of $T'$, the map $\gamma$ sends $t(\vec{e})$ to a leaf $\ell > v_e$ of $T$ such that $\gamma(\vec{e})$ is the first edge of the path $v_e T \ell$;*
5. *$\alpha = \beta \circ \gamma$.*

*In particular, $\beta(E(T)) = \alpha(\vec{E}(T'))$.*

Before we prove theorem 3.3.4, let us note a corollary for $\mathcal{F}$-trees that seems remarkable in its own right, quite independently of tangle-tree duality. If $\mathcal{F}$ consists of stars, the edges of any irreducible $\mathcal{F}$-tree map to nested separations in $\vec{S}$:

**Corollary 3.3.5.** *Let $\vec{S}$ be a separation system without trivial separations, and let $\mathcal{F} \subseteq 2^{\vec{S}}$ be a set of stars. Then for every irreducible $\mathcal{F}$-tree $(T, r, \beta)$ of $\vec{S}$ the image $\beta(E(T))$ of its edges is a nested set of separations in $\vec{S}$.*

*Proof.* Our aim is to apply theorem 3.3.2 to the $S$-tree $(T', \alpha)$ provided by theorem 3.3.4, to show that $\alpha(\vec{E}(T')) = \beta(E(T))$ is nested. To apply the lemma, we have to check that $\alpha$ is injective on the stars $\vec{F}_t$ in $\vec{E}(T')$.

Given a node $t$ of $T'$, we have $t = t(\vec{e})$ for all $\vec{e} \in \vec{F}_t$. For $\ell := \gamma(t)$ we therefore have $\gamma(\vec{F}_t) \subseteq E(rT\ell)$ by theorem 3.3.4 4. In order to show that $\alpha$ is injective on $\vec{F}_t$, it thus suffices by 5 $\alpha = \beta \circ \gamma$ and the injectivity of $\gamma$ to show that $\beta$ is injective on $E(rT\ell)$. This, however, follows from the last clause in the definition of separation trees. □

*Proof of theorem 3.3.4.* We start our construction of $T'$ by taking as $V(T')$ any set we can map to the leaves of $T$ by a bijection $\gamma$. This ensures 1.

We shall give $T'$ one edge $e$ for every non-leaf node $v =: v_e$ of $T$. To choose the ends of $e$ in $T'$, consider the children $w_1$ and $w_2$ of $v$ in $T$.[6] For each $i \in \{1, 2\}$, since $T$ is irreducible, the edge $vw_i$ is necessary for some leaf $\ell_i \geq w_i$ of $T$.[7] Let $e$ join the nodes $\gamma^{-1}(\ell_1)$ and $\gamma^{-1}(\ell_2)$ of $T'$, and let $\gamma(\vec{e}) := vw_i$ for the orientation $\vec{e}$ of $e$ for which $\gamma(t(\vec{e})) = \ell_i$. This satisfies 2–4, if we think of $T'$ as a multigraph for now, i.e., allow multiple edges.

However, since $v_e$ is the unique infimum of $\ell_1$ and $\ell_2$ in $T$, the pair $\gamma^{-1}(\{\ell_1, \ell_2\})$ of nodes in $T'$ is joined only by the edge $e$: the graph $T'$ has no parallel edges and no loops. Finally, now that $\gamma$ is fixed, let $\alpha := \beta \circ \gamma$ as in 5.

It remains to show that $T'$ is a tree, and that $(T', \alpha)$ is over $\mathcal{F}$: as $\alpha(\vec{e})^* = \alpha(\overleftarrow{e})$ by definition of $\gamma$ and $\alpha$, it will be clear that $(T', \alpha)$ is an $S$-tree once we know that $T'$ is a tree.

Let us begin by showing that $T'$ is acyclic. Consider two edges $e, f$ of $T'$ with $t(\vec{e}) = i(\vec{f})$. By 4, the edges $\gamma(\vec{e})$ and $\gamma(\overleftarrow{f})$ of $T$ at $v_e$ and $v_f$, respectively, both lie on the path $rT\ell$ for $\ell = \gamma(t(\vec{e})) = \gamma(i(\vec{f}))$. As $e \neq f$, the nodes $v_e$ and $v_f$ of $rT\ell$ are distinct, by (iii), so $v_e < v_f$ or $v_f < v_e$. As $s_u \neq s_v$ for $u < v$ in any separation tree, $\alpha(\vec{e}) = \beta(\gamma(\vec{e}))$ and $\alpha(\overleftarrow{f}) = \beta(\gamma(\overleftarrow{f}))$, compare 5, are thus orientations of distinct separations $s_{v_e} \neq s_{v_f}$ in $S$.

Since we chose $\gamma^{-1}(\ell)$ as $t(\vec{e})$ and as $t(\overleftarrow{f})$ when we picked the ends of the edges $e$ and $f$ in $T'$, the edges $\gamma(\vec{e})$ and $\gamma(\overleftarrow{f})$ of $T$ are necessary for $\ell$: their $\beta$-values $\alpha(\vec{e})$ and $\alpha(\overleftarrow{f})$ each lie in some $\sigma \in \mathcal{F}$ that is a subset of $\beta_\ell$, and every subset of $\beta_\ell$ in $\mathcal{F}$ contains both. Consider any such $\sigma \in \mathcal{F}$. Since $\sigma$ is a star, we have $\alpha(\vec{e}) \geq \alpha(\overleftarrow{f})^* = \alpha(\vec{f})$. As $s_{v_e} \neq s_{v_f}$, the above inequality is strict. The $\alpha$-values along any oriented path in $T'$ thus strictly decrease. Hence $T'$ contains no (oriented) cycles.

[6] These are distinct: $s_v$ cannot be degenerate, because $vw_1$ is necessary for a forbidden leaf $\ell \geq w_1$ of $T$, so $\beta(vw_1)$ lies in a star $\sigma \in \mathcal{F}$. Stars do not contain degenerate separations.

[7] Our proof will imply that these $\ell_i$ are unique, but for the definition it suffices to pick any.

To complete our proof that $T'$ is a tree, it suffices to show that it has one more node than edges. The first of these numbers equals the number of leaves of $T$; the second equals the number of non-leaves of $T$. This latter number is one less than the former, since $T$ is a binary tree.

To show that $(T', \alpha)$ is an $S$-tree over $\mathcal{F}$, consider any node $t$ of $T'$; we have to show that $\alpha(\vec{F}_t) \in \mathcal{F}$. As earlier, all the edges $\vec{e} \in \vec{F}_t$ are such that $\gamma(\vec{e}) \in rT\ell$ for $\ell := \gamma(t)$ is necessary for $\ell$, so $\alpha(\vec{F}_t) = \beta(\gamma(\vec{F}_t)) \subseteq \sigma \subseteq \beta_\ell$ for some $\sigma \in \mathcal{F}$. It remains to show that $\alpha(\vec{F}_t) \supseteq \sigma$.

Suppose there exists an $\vec{s} \in \sigma \setminus \alpha(\vec{F}_t)$. As $\vec{s} \in \sigma \subseteq \beta_\ell = \beta(rT\ell)$, we can find an edge $vw$ in $rT\ell$ with $v < w$ such that $\beta(vw) = \vec{s}$. Let $\vec{f} := \gamma^{-1}(vw)$. Then $\alpha(\vec{f}) = \vec{s}$, and $\gamma(\vec{f}) \in E(rT\ell)$ but $\gamma(\vec{f}) \notin \gamma(\vec{F}_t) \subseteq E(rT\ell)$. In particular, $v_f \neq v_e$ for every $\vec{e} \in \vec{F}_t$, and hence $f \neq e$, so neither $\vec{f}$ or $\overleftarrow{f}$ lies in $\vec{F}_t$. But note that $v_f$ and all these $v_e$ lie on the path $rT\ell$, so they are comparable in the tree-order on $V(T)$.

Since $T'$ is a tree, either $\vec{f}$ or $\overleftarrow{f}$ points towards $t$ in $T'$. Suppose first that $\vec{f}$ does. Then $\vec{f} \geq \vec{e}$ in $\vec{E}(T')$ for some $\vec{e} \in \vec{F}_t$. As in our acyclicity proof earlier, this implies that $\alpha(\vec{f}) \geq \alpha(\vec{e})$. As both $\vec{s} = \alpha(\vec{f})$ and $\alpha(\vec{e})$ lie in the star $\sigma$, we have $\alpha(\vec{f}) \geq \alpha(\vec{e})^*$ too. As $v_f$ and $v_e$ are distinct nodes on $rT\ell$, which implies that $s = s_{v_f} \neq s_{v_e} = \{\alpha(\vec{e}), \alpha(\vec{e})^*\}$ since $T$ is a separation tree, $\vec{s} = \alpha(\vec{f})$ is thus trivial in $\vec{S}$ witnessed by $s_{v_e}$, a contradiction to our assumptions about $\vec{S}$.

Suppose now that the edge $\overleftarrow{f}$ points towards $t$ in $T'$. Then $\overleftarrow{f} \geq \vec{e} \in \vec{F}_t$ and hence $\alpha(\vec{f})^* \geq \alpha(\vec{e})$, while $s = s_{v_f} \neq s_{v_e} = \{\alpha(\vec{e}), \alpha(\vec{e})^*\}$, as earlier. This makes $\vec{s} = \alpha(\vec{f})$ and $\alpha(\vec{e})$ inconsistent elements of $\beta_\ell$, a contradiction to the fact that $T$ is a consistent separation tree. □

Let us discuss briefly what we can say if $\vec{S}$ does have trivial elements. The proof given above stands until we have to prove that the $S$-tree $T'$ we have constructed is over $\mathcal{F}$. If $\vec{r} \in \vec{S}$ is trivial in $\vec{S}$ witnessed by $s \in S$, this can indeed fail. Indeed, assume that $\vec{s} = \alpha(\vec{e})$ with $\gamma(t(\vec{e})) = \ell$ and $\gamma(t(\overleftarrow{e})) = \ell'$. Let $v_f$ be the predecessor of $v_e$ in $T$, and assume that $\beta(v_f v_e) = \vec{r}$.

Then $v_f v_e$ is the first edge of the path in $T$ from $v_f$ to both $\ell$ and $\ell'$, so either of these would be eligible as $\gamma(t(\vec{f}))$, where $\vec{f}$ is the orientation of $f$ with $\alpha(\vec{f}) = \vec{r}$, as long as $v_f v_e$ is necessary for $\ell$ and $\ell'$, respectively. This will be the case if and only if $\vec{r}$ lies in the stars $\sigma \in \mathcal{F}$ or $\sigma' \in \mathcal{F}$ that contain $\vec{s}$ and $\overleftarrow{s}$, respectively; let us assume it lies in both. (Recall that $\beta^{-1}(\vec{s})$ and $\beta^{-1}(\overleftarrow{s})$ are necessary for $\ell$ and $\ell'$, respectively, so $\vec{s} \in \sigma$ and $\overleftarrow{s} \in \sigma'$ for some and for all stars $\sigma \subseteq \beta_\ell$ and $\sigma' \subseteq \beta_{\ell'}$ in $\mathcal{F}$.) However, when we constructed $T'$, we chose only one of $\ell$ and $\ell'$ as $\gamma(t(\vec{f}))$, say $\ell$. We then have $\vec{r} \in \sigma' \setminus \alpha(\vec{F}_{t'})$ for $t' := \gamma^{-1}(\ell')$.

However, we can easily mend this: just add a new leaf $t$ to $T'$ adjacent to $t'$, with $\alpha(tt') := \vec{r}$. We shall then need that $\{\overleftarrow{r}\} \in \mathcal{F}$, but this will be the case if we assume that $\mathcal{F}$ is standard, as we have to in order to obtain $T$ in the first place. More generally, we can add to $T'$ such new leaves at all nodes $t'$ whose $\alpha(\vec{F}_{t'})$ fails to contain a trivial separation $\vec{r}$ whose triviality is witnessed by some $s$ with $\vec{s} \in \alpha(\vec{F}_{t'})$. The modified tree will no longer satisfy the detailed clauses in theorem 3.3.4, but it will be an $S$-tree over $\mathcal{F}$ all whose non-trivial labels $\alpha(\vec{s})$ are obtained as in our proof.

Note that the 'tree-likeness' in $S$-trees over $\mathcal{F}$ and in $\mathcal{F}$-trees, respectively, refers to very different structures: while $S$-trees over $\mathcal{F}$ in theorem 3.3.3 exhibit tree-likeness in the (object divided by the) separation system $\vec{S}$ itself – for example, they define tree-decompositions with parts specified by $\mathcal{F}$ if $S$ consists of separations of a graph – the $\mathcal{F}$-trees in theorem 3.3.1 are 'tree-like' only in the formal sense of a decision-tree for how to orient the elements of $S$. It is all the more remarkable, therefore, that the structuring trees found in $S$ by theorem 3.3.3 can be recovered from the $\mathcal{F}$-trees in theorem 3.3.1, as shown in theorem 3.3.4. We thus have a new tangle-tree duality theorem: one that offers the same dichotomy as theorem 3.3.3, but with a different premise.

To state this in optimal form, we need one more definition. Given any set $\mathcal{F} \subseteq 2^{\vec{S}}$ and any order function on $S$, let $\mathcal{F}_{\text{eff}}$ denote the set of all sets $\sigma \in \mathcal{F}$ that are efficient in their closure $\lfloor \sigma \rfloor$ in $\vec{S}$ (see section 3.2). Note that this definition makes sense independently of any tangle structure trees.

**Corollary 3.3.6.** *Let $\vec{S}$ be an ordered separation system without trivial separations. Let $\mathcal{F} \subseteq 2^{\vec{S}}$ be a rich set of stars. Then exactly one of the following assertions holds:*

1. *there exists an $\mathcal{F}$-tangle of $S$;*
2. *there exists an $S$-tree over $\mathcal{F}$.*

*In the case of 2, the $S$-tree can be chosen to be over $\mathcal{F}_{\text{eff}}$.*

*Proof.* As indicated after theorem 3.3.2, the assertions 1 and 2 cannot both hold. Let us assume that 1 fails, and show 2. By theorem 3.3.1, there is an $\mathcal{F}$-tree $T$ of $\vec{S}$, which may be chosen irreducible, efficient and ordered. If $T$ is efficient, it will be an $\mathcal{F}_{\text{eff}}$-tree, since any $\sigma \subseteq \beta_\ell$ from $\mathcal{F}$ is then efficient in $\lfloor \beta_\ell \rfloor$ and therefore also efficient in $\lfloor \sigma \rfloor \subseteq \lfloor \beta_\ell \rfloor$. Now 2 and the final statement of the corollary follow from theorem 3.3.4. □

While theorem 3.3.3 and theorem 3.3.6 proclaim the same tangle-tree dichotomy, the requirements they make on $\mathcal{F}$ in their premise are different: $\mathcal{F}$-separability of $\vec{S}$ and richness of $\mathcal{F}$, respectively. In order to derive theorem 3.3.3 formally from theorem 3.3.6 (in the case that $\mathcal{F}$ consists of stars) we would need to show that if $\vec{S}$ is $\mathcal{F}$-separable then $\mathcal{F}$ is rich for $\vec{S}$. The condition of $\mathcal{F}$-separability, however, was tailor-made for the proof of theorem 3.3.3. It therefore seems unlikely that we can formally derive from it that $\mathcal{F}$ is rich. However, in section 3.6 we shall derive this from a condition that is only slightly stronger than $\mathcal{F}$-separability. This stronger condition, in fact, holds and is used in all the known applications of theorem 3.3.3, rather than $\mathcal{F}$-separability itself. This will be explored in [24].

## 3.4 Submodular order functions

Submodular order functions are a key ingredient to much of tangle theory. We shall need them in this paper too, both for our tree-of-tangles theorems in section 3.5 and to retrieve the essence of the tangle-tree duality theorem from [16] in section 3.6.

In both these applications we shall need tangle structure trees that are not just ordered but thoroughly ordered. These exist reliably only when our order function is injective. If it is not – which often happens in theoretical applications[8] – we have

[8]Order functions used for real-world tangle applications, such as in clustering, tend to be injective. But the standard order function on graph separations, for example, is not.

to tweak it slightly to make it so. We shall prove in this section that we can do this while preserving its submodularity.

We say that an order function $o'$ on a set $S$ *refines* another order function $o$ on $S$ if

$$o(r) < o(s) \Rightarrow o'(r) < o'(s) \quad (\forall r, s \in S).$$

Thus, $o'$ refines $o$ if and only if $o'(r) \leq o'(s)$ implies $o(r) \leq o(s)$. But we do not have the same implication for '<', since $o'$ need not be constant on the subsets of $S$ whose elements have a common value under $o$.

Let $\vec{S}$ be a separation system with an order function $o$, and let $\mathcal{F}$ be any set. Recall that the $\mathcal{F}$-tangles of the subsets

$$S_k = \{ s \in S \mid o(s) < k \}$$

of $S$ are the $\mathcal{F}$-tangles *in* $\vec{S}$ with respect to $o$.

**Lemma 3.4.1.** *If $o'$ refines $o$, then the tangles in $\vec{S}$ with respect to $o$ are also tangles in $\vec{S}$ with respect to $o'$.*

*Proof.* If $o'$ refines $o$ then, for every $k \in \mathbb{R}$ and with $S_k$ defined with respect to $o$ as above, the range $o'(S_k)$ in $\mathbb{R}$ is an initial segment of $o'(S)$. For every $k$ choose $k' \in \mathbb{R}$ larger than every element of $o'(S_k)$ but smaller than every element of $o'(S) \setminus o'(S_k)$. Then $S_k$ can be rewritten as $\{ s \in S \mid o'(s) < k' \}$, which puts the $\mathcal{F}$-tangles of $S_k$ in tangle$(\vec{S}, \mathcal{F}, o')$. □

Let us consider a separation system $\vec{U}$ that is a universe: one whose partial ordering $\leq$ makes it into a lattice. Any two $\vec{r}, \vec{s} \in \vec{U}$ thus have a supremum $\vec{r} \vee \vec{s}$ and an infimum $\vec{r} \wedge \vec{s}$ in $\vec{U}$. Let us call a function $u: \vec{U} \to \mathbb{R}$ *submodular* if

$$u(\vec{r} \vee \vec{s}) + u(\vec{r} \wedge \vec{s}) \leq u(\vec{r}) + u(\vec{s})$$

for all $\vec{r}, \vec{s} \in \vec{U}$, and *structurally submodular* if

$$u(\vec{r} \vee \vec{s}) \leq u(\vec{r}) \quad \text{or} \quad u(\vec{r} \wedge \vec{s}) \leq u(\vec{s}) \tag{†}$$

for all $\vec{r}, \vec{s} \in \vec{U}$. Note that submodular functions on $\vec{U}$ are also structurally submodular, but not conversely. Unlike submodularity, the structural submodularity of a real function on $\vec{U}$ is preserved when we compose it with an order isomorphism on $\mathbb{R}$, but it is not preserved by summing functions.

Let us call an order function $|\ |$ on the set $U$ of unoriented separations *submodular* if the function $\vec{s} \mapsto |s|$ it induces on $\vec{U}$ is submodular, and similarly for 'structurally submodular'. As earlier, submodularity of order functions on $U$ is preserved under summing (but not under composition with order isomorphisms), while structural submodularity of order functions on $U$ is preserved under composition with order isomorphisms (but not under summing).

Note also that, given any $k \in \mathbb{R}$, the separation system $\vec{U_k} = \{\vec{s} \in \vec{U} : |s| < k\}$ is submodular, as defined at the start of section 3.1, as soon as $|\ |$ is structurally submodular on $\vec{U}$. Conversely, if $\vec{S} \subseteq \vec{U}$ is submodular, one can define a structurally submodular order function on $U$ so that $\vec{S} = \vec{U_k}$. But there need not exist a submodular order function on $U$ with this property; see [26Example 5.11].

It is not hard to refine a submodular order function to make it injective. Ideally, we would like to do this while preserving its submodularity. We shall prove a little less than this, which is still good enough for our use of submodularity: that we can refine it to an order function that is still structurally submodular.

**Lemma 3.4.2.** *Let $\vec{U}$ be a separation universe with a submodular function $u$. Then $u' : \vec{s} \mapsto u(\overleftarrow{s})$ and $w : s \mapsto u(\vec{s}) + u(\overleftarrow{s})$ are also submodular order functions on $U$.*

*Proof.* Since $u$ is submodular, we also have

$$u'(\vec{s} \vee \vec{r}) + u'(\vec{s} \wedge \vec{r}) = u(\overleftarrow{s} \wedge \overleftarrow{r}) + u(\overleftarrow{s} \vee \overleftarrow{r}) \leq u(\overleftarrow{s}) + u(\overleftarrow{r}) = u'(\vec{s}) + u'(\vec{r}),$$

so $u'$ is submodular. The sum of two submodular functions is clearly submodular, hence so is also $w = u + u'$. □

**Lemma 3.4.3.** *Let $\vec{U}$ be a separation system and $\vec{t} \in \vec{U}$. The function $\mathbf{1}_{\vec{t}} : \vec{U} \to \{0, 1\}$ given by*

$$\mathbf{1}_{\vec{t}}(\vec{s}) = \begin{cases} 0 & \text{if } \vec{s} \leq \vec{t}, \\ 1 & \text{otherwise.} \end{cases}$$

*is submodular.*

*Proof.* Let $\vec{r}, \vec{s} \in \vec{U}$. We have to show that

$$\mathbf{1}_{\vec{t}}(\vec{r} \wedge \vec{s}) + \mathbf{1}_{\vec{t}}(\vec{r} \vee \vec{s}) \leq \mathbf{1}_{\vec{t}}(\vec{r}) + \mathbf{1}_{\vec{t}}(\vec{s}).$$

If the right-hand side of the inequality is 2, the inequality holds trivially. So let us assume that $\mathbf{1}_{\vec{t}}(\vec{r}) = 0$. Then $\vec{r} \wedge \vec{s} \leq \vec{r} \leq \vec{t}$, so $\mathbf{1}_{\vec{t}}(\vec{r} \wedge \vec{s}) = 0$. Hence the left-hand side of the inequality is at most 1. The inequality thus holds unless the right-hand side is zero, i.e., unless also $\mathbf{1}_{\vec{t}}(\vec{s}) = 0$. But then also $\vec{s} \leq \vec{t}$ and thus $\vec{r} \vee \vec{s} \leq \vec{t}$, so $\mathbf{1}_{\vec{t}}(\vec{r} \vee \vec{s}) = 0$, making the left-hand side zero too. □

It may help to think of the sums in our next lemma as $n$-ary integer expansions.

**Lemma 3.4.4.** *Let $\vec{U}$ be a separation universe, $\iota : \vec{U} \to \{0, \dots, |\vec{U}| - 1\}$ a bijection, and $n \in \mathbb{N}$. Then*

$$\gamma_n^\iota(\vec{s}) := \sum_{\vec{t} \in \vec{U},\, \vec{t} \ngeq \vec{s}} n^{\iota(\vec{t})} = \sum_{\vec{t} \in \vec{U}} n^{\iota(\vec{t})} \mathbf{1}_{\vec{t}}(\vec{s}).$$

*In particular $\gamma_n^\iota$ is submodular.*

*Proof.* The first part is immediate, as the additional summands in the second sum are all zero.

The submodularity of $\gamma_n^\iota$ then follows from theorem 3.4.3 and the fact that sums of submodular functions are again submodular.

□

**Lemma 3.4.5.** *Let $\vec{U}$ be a separation universe with a submodular order function $o$. Then there exists an injective submodular order function $o'$ on $U$ that refines $o$.*

*Proof.* We shall construct a small injective and submodular perturbation $\delta : U \to \mathbb{R}$, so that $o' := o + \delta$ is injective and refines $o$.

First choose $\varepsilon > 0$ as follows. Let $\Delta := \{|o(s) - o(r)| : s, r \in U,\ o(s) \neq o(r)\}$. If $\Delta \neq \emptyset$ set $\varepsilon := \min \Delta > 0$; otherwise set $\varepsilon := 1$. In either case $\varepsilon > 0$.

Fix an arbitrary bijection $\iota : \vec{U} \to \{0, \dots, |\vec{U}| - 1\}$ and set $\gamma(s) := \gamma_3^\iota(\vec{s}) + \gamma_3^\iota(\overleftarrow{s})$. By theorem 3.4.2 and theorem 3.4.4 the function $\gamma : U \to \mathbb{R}$ is submodular. We claim $\gamma$ is injective. Write $\gamma(s)$ in base 3:

$$\gamma(s) = \sum_{k=0}^{|\vec{U}|-1} a_k(s)\, 3^k, \qquad a_k(s) \in \{0, 1, 2\}.$$

By construction we have $a_k(s) \leq 1$ if and only if the separation $\vec{t} = \iota^{-1}(k)$ points towards $s$. If $\gamma(s) = \gamma(r)$ then their base-3 expansions coincide, hence every $\vec{t} \in \vec{U}$ points towards $s$ if and only if it points towards $r$. From this we will now deduce $s = r$, showing that $\gamma$ is injective.

Let $\vec{s_0}$ be any orientation of $s$. Suppose that, for some $n \in \mathbb{N}$, we have defined an orientation $\vec{s_n}$ of $s$. Then since $\vec{t} := \vec{s_n}$ points towards $s$, it also points towards $r$. So there exists an orientation $\vec{r_n}$ of $r$ with $\vec{s_n} \geq \vec{r_n}$. Analogously, given any orientation $\vec{r_n}$ of $r$ we can find an orientation $\vec{s}_{n+1}$ of $s$ with $\vec{r_n} \geq \vec{s}_{n+1}$. For any orientation $\vec{s_0}$ of $s$ we can thus construct an infinite chain $\vec{s_0} \geq \vec{r_0} \geq \vec{s_1} \geq \ldots$ of, alternately, orientations of $s$ and $r$. As $r$ and $s$ have at most two orientations each, at most two of these inequalities can be strict. Hence for $m$ large enough we have $\vec{s_m} = \vec{r_m}$, and therefore $s = r$ as desired. This completes our proof that $\gamma$ is injective.

Note that $0 \leq \gamma(s) < 3^{|\vec{U}|}$ for every $s$. Define

$$\delta(s) := \left(\tfrac{1}{2}\varepsilon/3^{|\vec{U}|}\right) \gamma(s).$$

Then $0 \leq \delta(s) < \varepsilon/2$ for all $s$. Note that $\delta$ is injective and submodular, because $\gamma$ is and $\frac{1}{2}\varepsilon/3^{|\vec{U}|} > 0$.

Set $o' := o + \delta$. As a sum of submodular functions, $o'$ is submodular. To see that $o'$ is injective, suppose $o'(r) = o'(s)$. Then

$$|o(r) - o(s)| = |\delta(r) - \delta(s)| \leq |\delta(r)| + |\delta(s)| < \varepsilon.$$

By the choice of $\varepsilon$ we must have $o(s) = o(r)$, and hence $\delta(s) = \delta(r)$. Since $\delta$ is injective this gives $s = r$. Finally, $o'$ refines $o$, because whenever $o(r) > o(s)$ we have

$$o'(r) - o'(s) = o(r) - o(s) + \delta(r) - \delta(s) \geq \varepsilon - |\delta(r)| - |\delta(s)| > 0\,,$$

so $o'(r) > o'(s)$. □

An *enumeration* of a set $S$ is any bijection between $S$ and $\{1, \ldots, |S|\}$.

**Lemma 3.4.6.** *Let $\vec{U}$ be a separation universe with a submodular order function $o$. Then there exists a structurally submodular enumeration of $U$ that refines $o$.*

*Proof.* Compose the injective submodular refinement $o'$ of $o$ provided by theorem 3.4.5 with an order isomorphism $\varphi$ from its range to $\{1, \ldots, |U|\}$. Since $o'$, being submodular, is also structurally submodular, and structural submodularity is preserved under composition with order isomorphisms, the enumeration $\varphi \circ o'$ of $U$ is structurally submodular, and like $o'$ it refines $o$. □

## 3.5 Trees of tangles

In section 3.3 we derived one of the two fundamental types of theorem about tangles from our tangle structure trees: a so-called tangle-tree duality theorem. In this section we shall do the same for the second pillar of tangle theory: its so-called tree-of-tangles theorems.

Let $\vec{S}$ be an ordered separation system. Recall from section 3.1 that a separation *distinguishes* two tangles in $\vec{S}$ if they orient it differently. Let us say that it distinguishes them *optimally*[9] if it distinguishes them but no separation in $S$ of lower order does. A nested set $N \subseteq S$ is a *tree*[10] *of the tangles of* $S$ if every two $\mathcal{F}$-tangles of $S$ (for some given $\mathcal{F}$) are distinguished optimally by a separation in $N$. Likewise, $N$ is a *tree of the tangles in* $\vec{S}$ if it is nested and every two maximal $\mathcal{F}$-tangles in $\vec{S}$ are distinguished optimally by a separation in $N$.

**Definition 3.5.1.** Let $\vec{S}$ be a separation system contained in a universe $\vec{U}$. An orientation $\tau$ of $S$ is *robust* in $\vec{U}$ if for no $s \in U$ there is a triple $\{\vec{r}, \overleftarrow{r} \vee \vec{s}, \overleftarrow{r} \vee \overleftarrow{s}\}$ in $\tau$ with $|\overleftarrow{r} \vee \vec{s}|, |\overleftarrow{r} \vee \overleftarrow{s}| < |r|$.[11]

These triples will be illustrated in fig. 3.2. Note that while $\vec{r}$ can lie in such a triple in $\tau$ only if it lies in $\vec{S}$, the separation $s$ need not lie in $S$: it just gives rise to that triple in $\vec{S}$ together with $\vec{r} \in \vec{S}$.

We shall find in every separation system $\vec{S}$ that lives in some universe with an injective submodular order function a tree of all the robust $\mathcal{F}$-tangles of $S$, as well as a more comprehensive tree of all the robust $\mathcal{F}$-tangles in $\vec{S}$. Here, $\mathcal{F}$ has to be standard and rich, as usual, but need not satisfy any further requirements. But to ease our terminology, we shall encode the robustness assumption in $\mathcal{F}$, by assuming that $\mathcal{F}$ contains all the triples from definition 3.5.1. We can then simply speak of (trees of) $\mathcal{F}$-tangles rather than of robust $\mathcal{F}$-tangles.

The most comprehensive tree-of-tangles theorems known so far are for $\mathcal{F}$-tangles that are not only robust but also *profiles*: consistent orientations of a separation system $\vec{S}$ that contain no triple of the form $\{\vec{r}, \vec{s}, \overleftarrow{r} \vee \overleftarrow{s}\}$, where the supremum is once more taken in some fixed universe of separations containing $\vec{S}$ [7].

Tangles of graphs are robust profiles [5].[12] So the tree-of-tangles theorems for robust profiles of abstract separation systems from [7] generalize the tree-of-tangles theorem for graphs of Robertson and Seymour [1], and our results below will further generalize those from [7] to $\mathcal{F}$-tangles that are robust but need not be profiles.

There is another advance in [7] over [1]: the trees of tangles found in [7] are *canonical*. This means, roughly, that the automorphisms of the given graph or separation system $\vec{S}$ leave the tree of tangles $N \subseteq S$ invariant. This is usually proved by observing that the construction of $N$ depends only on invariants of the graph or separation system rather than, say, of an enumeration of $\vec{S}$ assumed for the sake of the construction, as was the case for the first trees of tangles for graphs in [1]. As a consequence, such a canonical tree of tangles can then be computed by an algorithm that yields the same result regardless of the order in which the elements of the graph or separation system are presented to it.

Our tree-of-tangles theorems for all robust $\mathcal{F}$-tangles, theorems 3.5.1 and 3.5.8 below, requires that the order function given on $S$ is injective. This assumption makes

[9]The usual terminology is to say ‘efficiently’ rather than ‘optimally’. We use the latter only in this paper, to avoid confusion with our use of the term ‘efficient’ for subsets of $\vec{S}$.

[10]Recall that nested sets of separations of structures such as sets or graphs divide them in a ‘tree-like’ way [25]; hence the ‘tree’ in the name.

[11]If this definition appears technical, think of orientations of bipartitions of a set as designating the side they point to as ‘big’. Robust orientations of set partitions are then like a finite analogue to ultrafilters: big subsets never split into two small subsets.

[12]Indeed, all tangles of abstract separation systems, as defined in [14], are profiles (see there).If the ambient universe of their separation system is distributive, they are easily seen to be robust. Graph tangles are examples of such abstract tangles in a distributive universe.

canonicity, as defined above, trivial: unless $|S| = 1$, the identity will be the only automorphism of $\vec{S}$ as an *ordered* separation system.

As we saw in section 3.4, we can make a given submodular order function injective without losing its submodularity. But this conversion is not canonical: it depends on choices not determined by the invariants of the separation system considered.[13] If we ignore the order function on our separation system and refer in our notion of canonicity only to the unordered system, as a poset with an order-reversion involution, then the tree of tangles we shall obtain in this section will not be canonical: it will depend on the choice of (injective) order function on $S$, because the (thoroughly ordered) tangle structure tree from which we construct it depends on it.

The following definition allows us to encode the robustness of our tangles in $\mathcal{F}$. Let $\vec{S}$ be a separation system in a universe $\vec{U}$ of separations. Let

$$\mathcal{R}(\vec{U}) := \big\{\{\vec{r}, \overleftarrow{r} \vee \vec{s}, \overleftarrow{r} \vee \overleftarrow{s}\} \subseteq \vec{U} \,:\, r, s \in U \text{ and } |\overleftarrow{r} \vee \vec{s}|, |\overleftarrow{r} \vee \overleftarrow{s}| < |r|\big\}.$$

These triples are easily identified in fig. 3.2, for example.

Here is our basic tree-of-tangles theorem for robust $\mathcal{F}$-tangles with arbitrary $\mathcal{F}$, or equivalently, for all $\mathcal{F}$-tangles with arbitrary $\mathcal{F}$ that include $\mathcal{R}(\vec{U})$:

**Theorem 3.5.1.** *Let $\vec{U}$ be a universe of separations with an injective and structurally submodular order function. Let $k \in \mathbb{R}$ and $\vec{S} := \vec{U}_k$. Let $\mathcal{F}$ be any set that is rich and standard for $\vec{S}$ and includes $\mathcal{R}(\vec{U})$. Let $(T, r, \beta)$ be the unique thoroughly ordered $\mathcal{F}$-tangle structure tree for $\vec{S}$ (cf. theorem 3.2.2). Let $N$ be the set of all separations $s_v \in S$ with $v$ a tangle node of $T$.*

*Then $N$ is a tree of the tangles of $S$; in particular, it is nested. It consists of all the separations in $S$ that distinguish two $\mathcal{F}$-tangles of $S$ optimally.*

*Proof.* We shall build up our proof of theorem 3.5.1 as a sequence of a few lemmas. Let $\vec{U}$, $k$, $\vec{S}$, $\mathcal{F}$, $(T, r, \beta)$ and $N$ be as stated in the theorem, with an injective and structurally submodular order function on $U$.

The *corners* of two separations $r, s \in S$ are the four separations in $U$ that have orientations of the form $\vec{r} \wedge \vec{s}$. fig. 3.2 shows a corner of two bipartitions of a set, where $\vec{r} \wedge \vec{s}$ is the intersection of the subsets to which $\vec{r}$ and $\vec{s}$ point.

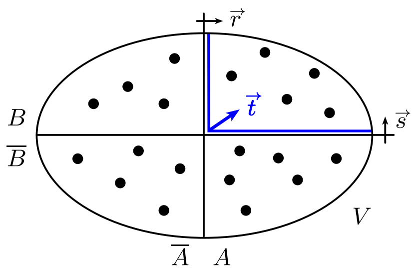


**Figure 3.2:** A corner $\vec{t} = \vec{r} \wedge \vec{s}$ of two bipartitions of a set $V$.

Two corners $t$ and $t'$ of $r$ and $s$ lie *on the same side of* $r$ if these separations have orientations $\vec{r} \geq \vec{t}, \vec{t}'$. Corners on the same side of one of two crossing separations are *adjacent*; non-adjacent corners are *opposite*. Since our order function on $U$ is structurally submodular, all four ordered pairs $(t, t')$ of opposite corners satisfy $|t| \leq |r|$ or $|t'| \leq |s|$.

[13] In our case, this was the choice of the bijection $\iota$ in the proof of theorem 3.4.5.

In fig. 3.2, the triples $\{\vec r, \overleftarrow{r} \vee \vec s, \overleftarrow{r} \vee \overleftarrow{s}\}$ from the definition of $\mathcal{R}(\vec U)$ consist of $\vec r$ and the two corners on the right-hand side of $r$ both oriented towards $r$, the last element of the triple being $\overleftarrow{t} = \overleftarrow{r} \vee \overleftarrow{s}$.

**Lemma 3.5.2.** *Any two crossing separations $r, s \in S$ have two adjacent corners that either lie on the same side of $r$ and have order $< |r|$ or lie on the same side of $s$ and have order $< |s|$.*

*Proof.* Let $|r| \le |s|$. If all four corners have order $< |s|$ we are done; let $t_1$ be a corner of order $\ge |s|$. As $r$ and $s$ cross we have $r \neq s \neq t_1$, so in fact $|r| < |s| < |t_1|$.

By structural submodularity, the corner $t_2$ opposite $t_1$ has order $\le |r|$, and hence order $< |r|$, since $t_2 \neq r$ as earlier. If the corner $t_3$ on the same side of $r$ as $t_2$ also has order $< |r|$ we are done, so we assume not. Then $|t_3| > |r|$, and its opposite corner $t_4$ satisfies $|t_4| < |s|$. Since $t_4$ lies on the same side of $s$ as $t_2$, and $|t_2| < |r| < |s|$, this completes the proof. □

**Lemma 3.5.3.** (i) *If $s \in S$ distinguishes two tangles $\tau, \tau'$ of $S$ optimally then $s = s_v$, where $v$ is the tangle node of $T$ that is the infimum in $T$ of the tangle leaves $\ell = \ell(\tau)$ and $\ell' = \ell(\tau')$.*[14]

(ii) *Conversely, if the tangle node $v$ of $T$ is the infimum in $T$ of two tangle leaves $\ell = \ell(\tau)$ and $\ell' = \ell(\tau')$, then $s_v$ distinguishes $\tau$ and $\tau'$ optimally.*

*Proof.* (i) By definition of $v$, the tangles $\tau = \lfloor \beta_\ell \rfloor$ and $\tau' = \lfloor \beta_{\ell'} \rfloor$ both include $\lfloor \beta_v \rfloor$, but they differ on $s_v$. Since $s$ distinguishes $\tau$ and $\tau'$ optimally, we thus have $|s| \le |s_v|$, while $s$ is not oriented by $\lfloor \beta_v \rfloor$. As $T$ is thoroughly ordered, this implies $|s| \ge |s_v|$, and hence $s = s_v$ since $|\ |$ is injective.

(ii) The separation $s_v$ distinguishes $\tau$ from $\tau'$ by definition of $v$. Since $T$ is thoroughly ordered, any $s \in S$ with $|s| < |s_v|$ has an orientation $\vec s$ in $\lfloor \beta_v \rfloor$, which is a subset of both $\tau = \lfloor \beta_\ell \rfloor$ and $\tau' = \lfloor \beta_{\ell'} \rfloor$. Hence $\tau(s) = \tau'(s) = \vec s$, so $s$ does not distinguish $\tau$ from $\tau'$. □

**Lemma 3.5.4.** *$N$ consists of all the separations in $S$ that distinguish two tangles of $S$ optimally.*

*Proof.* Immediate from theorem 3.5.3 and the definition of $N$. □

Let us call a node $v$ of $T$ *critical* if $s_v$ has an orientation $\overleftrightarrow{s}$ that is either co-trivial in $\vec S$ or such that $\lfloor \beta_v \rfloor \cup \{\overleftrightarrow{s}\}$ contains a triple from $\mathcal{R}(\vec U)$.

**Lemma 3.5.5.** *Tangle nodes of $T$ are never critical.*

*Proof.* Suppose $T$ has a critical tangle node $v$. Let $\overleftrightarrow{s}$ be as in the definition of its criticality. Let $w$ be the successor of $v$ with $\beta(vw) = \overleftrightarrow{s}$, and let $\ell \ge w$ be a tangle leaf of $T$.

Then $\overleftrightarrow{s}$ lies in the $\mathcal{F}$-tangle $\lfloor \beta_\ell \rfloor$. As $\mathcal{F}$ is standard, $\overleftrightarrow{s}$ cannot be co-trivial. Hence $\lfloor \beta_v \rfloor \cup \{\overleftrightarrow{s}\}$ contains a triple from $\mathcal{R}(\vec U) \subseteq \mathcal{F}$. As $\lfloor \beta_v \rfloor \cup \{\overleftrightarrow{s}\} \subseteq \lfloor \beta_\ell \rfloor$, this means that $\ell$ is not a tangle leaf, contrary to its definition. □

**Lemma 3.5.6.** *Let $u, v$ be non-leaf nodes of $T$. If $s_u$ and $s_v$ cross, then either $u$ or $v$ is critical.*

*Proof.* By theorem 3.5.2, we can find $r \in \{s_u, s_v\}$ such that two corners $t, t'$ of $s_u$ and $s_v$ lie on the same side of $r$ and have order $< |r|$. Let $\overleftarrow{r}$ orient $r$ away from these corners. Then these have orientations $\overleftarrow{t}, \overleftarrow{t'} \le \overleftarrow{r}$. Note that $t, t' \in S$, since $|t|, |t'| < |r|$ and $r \in S = U_k$, as well as $\{\vec r, \overleftarrow{t}, \overleftarrow{t'}\} \in \mathcal{R}(\vec U)$.

[14] See theorem 3.2.1 for the definition of $\ell(\tau)$ etc.

Let $r = s_v$, say, and let $w$ be the child of $v$ for which $\beta(vw) = \overleftarrow{r}$. If $\overleftarrow{r}$ is co-trivial in $\vec{S}$, then $v$ is critical as desired, so we assume that $\overleftarrow{r}$ is not co-trivial. Since $T$ is an $\mathcal{F}$-tangle structure tree and $v$ is not a leaf, the set $\beta_v$ has no subset in $\mathcal{F}$. As $\mathcal{F}$ is standard, this means that $\beta_v$, and hence also $\beta_w = \beta_v \cup \{\overleftarrow{r}\}$, has no co-trival elements.

Since $t$ and $t'$ have order $< |r|$ but $s_v = r$, our assumption that $T$ is thoroughly ordered implies that $t$ and $t'$ were ineligible for selection as $s_v$: they must have orientations in $\lfloor \beta_v \rfloor \subseteq \lfloor \beta_w \rfloor$. Now $\lfloor \beta_w \rfloor$ is consistent by theorem 3.1.1, because $\beta_w$ is consistent by definition of $T$ and has no co-trivial elements. But the only orientations of $t$ and $t'$ that are consistent with $\overleftarrow{r} \in \beta_w$ are $\overleftarrow{t}$ and $\overleftarrow{t'}$, since $\vec{t}, \vec{t'} \leq \vec{r}$. Thus, $\overleftarrow{t}, \overleftarrow{t'} \in \lfloor \beta_v \rfloor$.

This makes $\{\vec{r}, \overleftarrow{t}, \overleftarrow{t'}\} \in \mathcal{R}(\vec{U})$ a witness of the fact that $v$ is critical. □

**Lemma 3.5.7.** *N is a tree of the tangles of S. In particular, N is nested.*

*Proof.* $N$ is nested by theorems 3.5.5 and 3.5.6. By theorem 3.5.4, this makes it a tree of the tangles of $S$. □

This completes our proof of theorem 3.5.1. □

The proof of theorem 3.5.1 starts from the premise that we have a thoroughly ordered tangle structure tree for our given separation system. theorem 3.2.2 provides this when $\mathcal{F}$ is standard and rich. Both these properties are required in the premise of the theorem; but the richness of $\mathcal{F}$ is never used in the proof. However it follows from the existence of the structure tree that the proof does need, combined with the injectivity of the order function on $S$ [23Theorem 4.8]. So we may as well require it.

The tangle structure tree in which we find our tree of tangles does not, however, have to be thoroughly ordered: we could work in any tree obtained from the unique thoroughly ordered structure tree provided by theorem 3.2.2 by reducing it as in the proof of theorem 3.2.3 given in [23]. This is because tangle nodes are never deleted in the reduction process, remain tangle nodes, and their associated separations $s_v$ remain optimal distinguishers of any two tangles such that $v$ is the infimum in $T$ of their tangle leaves. See [23Section 5] for more analysis.

theorem 3.5.1 is our basic tree-of-tangles theorem for robust $\mathcal{F}$-tangles. It applies only to the tangles of a fixed separation system $\vec{S} = \vec{U}_k$ inside some universe $\vec{U}$. The classical tree-of-tangles theorem for profiles [7], however, is more comprehensive (in its more restricted context): it finds a canonical tree of tangles for all the (maximal) tangles in $\vec{S}$, regardless of their order.[15]

We can extend theorem 3.5.1 to yield such a more general tree of tangles in $\vec{S}$, too. Our definition of tangle structure trees in this context remains the same as before, except that the notion of tangle leaves is extended to capture all the tangles in $\vec{S}$:

**Definition 3.5.2.** An *$\mathcal{F}$-tangle structure tree in $\vec{S}$* is a consistent separation tree $T$ on $\vec{S}$ in which every leaf is either a tangle leaf or forbidden, and for every non-leaf $v$ the set $\beta_v$ has no subset in $\mathcal{F}$.

Here, a leaf $\ell$ of $T$ is a *tangle leaf* if $\lfloor \beta_\ell \rfloor$ is a maximal $\mathcal{F}$-tangle in $\vec{S}$. It is *forbidden* if $\beta_\ell$ has a subset in $\mathcal{F}$.

Definitions that involve the term 'tangle leaf', such as that of *necessary* edges or nodes of $T$, adapt accordingly: they now refer to 'tangle leaves' as defined above.

[15] Tangles of $S_k$ are often called 'tangles in $\vec{S}$ of order $k$'.

Here, then, is our more comprehensive tree-of-tangles theorem. Its tree of tangles displays more tangles than theorem 3.5.1 does, namely, all the maximal tangles in $\vec{S}$ rather than just the tangles of $S$ (which are also maximal tangles in $\vec{S}$). However, its premise is also stronger. So neither theorem implies the other.

**Theorem 3.5.8.** *Let $\vec{S}$ be a universe of separations with an injective and structurally submodular order function. Let $\mathcal{F}$ be any set that includes $\mathcal{R}(\vec{S})$ and is rich and standard for $\vec{S}_k$ for every $k \in \mathbb{R}$. Then there exists a tree of the tangles in $\vec{S}$.*

*Proof.* We start by observing that the thoroughly ordered $\mathcal{F}$-tangle structure trees $(T_k, r_0, \beta^k)$ provided by theorem 3.5.1 for the separation systems $\vec{S}_k \subseteq \vec{S}$ are unique by theorem 3.2.2. And they are naturally nested: growing from the same root $r_0$, they satisfy $T_i \subseteq T_j$ and $\beta^i \subseteq \beta^j$ for all $i < j$.

Let $n$ be large enough that $S = S_n$, and let $T := T_n$. Then every non-leaf node $v$ of $T$ is a leaf of some $T_k$: just choose $k$ big enough that $v \in T_k$, but small enough that $|s_v| \geq k$. As $v$ is a non-leaf node of $T$, the fact that $(T_n, r_0, \beta^n)$ is an $\mathcal{F}$-tangle structure tree implies that $\beta^n_v \supseteq \beta^k_v$ has no subset in $\mathcal{F}$. Thus, $v$ is a tangle leaf of $T_k$ in the usual sense of section 3.2.

Let $T'$ be obtained from $T$ by deleting any pairs $\ell, \ell'$ of forbidden leaves of $T$ that are children of the same node, and let $\beta := \beta^n \restriction E(T')$. Then every leaf of $T'$ is either a forbidden leaf of $T$, or a tangle leaf $\ell$ of $T_k$ for some $k$. In the latter case, its associated tangle $\tau$ of $S_k$ is a maximal tangle in $\vec{S}$: it cannot extend to a tangle $\tau' \supsetneq \tau$ of any $S_{k'}$. Indeed, this $\tau'$ would be associated with a tangle leaf $\ell' >_r \ell$ of $T_{k'}$, so $\tau'$ would include either $\beta_\ell \cup \{\vec{s}_\ell\}$ or $\beta_\ell \cup \{\overleftarrow{s}_\ell\}$. This cannot happen for any $\mathcal{F}$-tangle $\tau'$, as both these sets have a subset in $\mathcal{F}$ by the choice of $\ell$.

Thus, every leaf $\ell$ of $T'$ is either forbidden (and also a leaf of $T$), or it is a tangle leaf in our new sense that $\lfloor \beta_\ell \rfloor$ is a maximal $\mathcal{F}$-tangle in $\vec{S}$. Conversely, for every maximal tangle $\tau$ in $\vec{S}$ there is a leaf $\ell$ of $T'$ such that $\lfloor \beta_\ell \rfloor = \tau$. But, unlike in $T$, for every non-leaf node $v$ of $T'$ there exists a tangle leaf $\ell > v$ of $T'$.

Let $V$ be the set of all non-leaf nodes of $T'$ that are not the parent of a forbidden leaf of $T$. This $V$ is precisely the set of all tangle nodes of $T'$. Let $N = \{ s_v \mid v \in V\}$. We shall prove that $N$ is our desired tree of tangles: that it is nested, and that it consists of precisely the separations in $S$ that distinguish two maximal $\mathcal{F}$-tangles in $\vec{S}$ optimally.

Let us show first that every two maximal tangles in $\vec{S}$, say a tangle $\tau_i$ of $S_i$ and a tangle $\tau_j$ of $S_j$ with $i \leq j$, are distinguished optimally by some separation in $N$. Let us apply theorem 3.5.1 with $k := i$ to the tangles $\tau_i$ and $\tau_j \cap \vec{S}_i \neq \tau_i$ of $S_i$. By the proof of the theorem, these tangles are distinguished optimally by $s_v$ for $v$ the infimum of their tangle leaves in $T_i$. This $v$ lies in $V$, and $s_v$ also distinguishes $\tau_i$ and $\tau_j$ optimally as tangles in $\vec{S}$, i.e., as partial orientations of $S$.

Conversely, let us show that every $s_v \in N$ distinguishes some such pair $\tau_i, \tau_j$ of tangles in $\vec{S}$ optimally. As $v$ lies in $V$, it has two children in $T'$, neither of which is a forbidden leaf of $T$. Since all forbidden leaves of $T'$ are also forbidden leaves of $T$, the definition of $T'$ implies that $v$ is the infimum of some tangle leaves $\ell_i$ and $\ell_j$ of $T'$ associated with tangles $\tau_i$ of $S_i$ and $\tau_j$ of $S_j$. By our earlier arguments, these are maximal tangles in $\vec{S}$, and $s_v$ distinguishes them optimally.

To see that $N$ is nested, consider distinct nodes $u, v \in V$ in $T$. Suppose $s_u$ and $s_v$ cross. By theorem 3.5.2, we can find $r \in \{s_u, s_v\}$ such that two corners $t, t'$ of $s_u$ and $s_v$ lie on the same side of $r$ and have order $< |r|$. Let us assume that $r = s_v$. As earlier in our proof,

$v$ is a tangle node in some $T_i$. But now $v$ is critical in $T_i$, by the proof of theorem 3.5.6. (Note that this proof no longer considers $u$ after identifying $v$ by theorem 3.5.2, exactly as we did here. This is important, since in our context $u$ may not be a node of $T_i$.) This contradicts theorem 3.5.5 applied to $T_i$.[16] □

Let us summarize the details from our proof of theorem 3.5.8:

**Corollary 3.5.9.** *Let $\vec{S}$ be a universe of separations with an injective and structurally submodular order function. Let $\mathcal{F}$ be any set that includes $\mathcal{R}(\vec{S})$ and is rich and standard for $\vec{S}_k$ for every $k \in \mathbb{R}$. Let $T'$ be obtained from the unique thoroughly ordered $\mathcal{F}$-tangle structure tree $(T, r, \beta)$ for $\vec{S}$ by deleting any pairs $\ell, \ell'$ of forbidden leaves of $T$ that are children of the same node. Let $V$ be the set of tangle nodes of $T'$.*

*Then $N = \{ s_v \mid v \in V\}$ is a tree of the tangles in $\vec{S}$. It is nested and consists of all the separations in $S$ that distinguish two maximal $\mathcal{F}$-tangles in $\vec{S}$ optimally.*

## 3.6 Zooming in: how to make consistent subsets more efficient

When we compared our new tangle-tree duality results from section 3.3 with the classical such theorems, such as theorem 3.3.3 [16], we observed two things. One was that our dichotomy theorem 3.3.1 is more general than that from [16] since, unlike there, our sets $\mathcal{F}$ of forbidden subsets need not be stars of separations. Due to this greater generality, however, our witnesses for the non-existence of tangles, though 'tree-like' in the sense that they are $\mathcal{F}$-trees, do not impose a tree structure on whatever our separation system separates (e.g., a graph or dataset), as theorem 3.3.3 from [16] does.

For the case that $\mathcal{F}$ does consist of stars we went on to prove that our results imply a tangle-tree duality theorem just like theorem 3.3.3, albeit from a different premise. At the end of section 3.3 we remarked that, since our premise (that $\mathcal{F}$ is rich) differs so fundamentally from the premise in [16] (that $\vec{S}$ is $\mathcal{F}$-separable), it is unlikely that our results could imply those of [16] formally.

Our aim in this section is to show that our premise (and hence, tangle-tree duality) does, however, follow from a slight strengthening of $\mathcal{F}$-separability, one that is actually verified and used when the duality theorem of [16] is applied [21]. As a result, we shall be able to deduce all the applications of [16] to concrete structures of interest [21] from our results. This will be done in [24].

Assuming a structurally submodular injective order function on $\vec{S}$, we shall prove that $\mathcal{F}$ is rich as soon as it is *closed under shifting*. We can then apply our theorem 3.3.6 to obtain the same tangle-tree dichotomy as offered by theorem 3.3.3, which also makes this assumption about $\mathcal{F}$. In fact, our notion of 'closed under shifting' will be weaker than that in [16], making our result stronger. While [16] makes no formal submodularity assumption, it assumes a consequence of it (separability of $\vec{S}$), which in practice is always established by assuming submodularity [21Lemma 3.4].

Our strategy for proving, from assumptions similar to those made in [16], that some given $\mathcal{F}$ is rich, is as follows. Recall that $\mathcal{F}$ is *rich* if every consistent orientation $\tau$

[16] We cannot apply the two lemmas to $T'$ directly because $T'$, unlike $T$ and $T_i$, need not be a tangle structure tree of any $\vec{S'}$ with $S' \subseteq S$.

of $S$ that has a subset in $\mathcal{F}$ also has an efficient such subset: one that is 'maximally focused' in the sense that none of its elements is eclipsed by another separation in $\tau$. Our strategy will be to start from an arbitrary set $\sigma \in \mathcal{F}$, and then to make it more focused by iteratively 'zooming in' until it is efficient.

For intuition, think of any consistent orientation $\tau$ of $\vec{S}$ as pointing towards some place in a dataset $V$ that $S$ may be separating. If $\tau$ is a tangle, this could be a cluster; if not, it could be the (possibly empty) interior $\bigcap \sigma$ of a star $\sigma \in \mathcal{F}$. For two oriented separations $\vec{r}, \vec{s}$ in $\tau$, think of $\vec{s} > \vec{r}$ as expressing that $\vec{s}$ points towards $r$ and, beyond it, to whatever place $\vec{r}$ points to in $V$. Thus, while $\vec{r}$ and $\vec{s}$ point to the same place, $\vec{r}$ is closer to it. Hence if $\vec{r}$ eclipses $\vec{s}$, then replacing $\vec{s}$ with $\vec{r}$ in some subset $\sigma$ of $\tau$ – e.g., one in $\mathcal{F}$, or a set of elements whose closure is a tangle of $S$ – makes $\sigma$ more focused on the place to which $\tau$ points.

Our strategy for making a given set $\sigma \in \mathcal{F}$ more efficient will thus be to replace, iteratively, an element of $\sigma$ with another element of $\tau$ that eclipses it. The rest of $\sigma$ will either be left unchanged (at the start of this section) or 'shifted' to one side of the eclipsing separation, as in [16] (later in this section).

Let us call a collection $\mathcal{F}$ of subsets of $\vec{S}$ *closed under eclipsing* in $\tau \subseteq \vec{S}$ if any set $\sigma'$ obtained from a subset $\sigma \in \mathcal{F}$ of $\tau$ by replacing some $\vec{s} \in \sigma$ with a separation $\vec{r} \in \tau$ that weakly eclipses $\vec{s}$ will also lie in $\mathcal{F}$. Note that $\sigma'$ will be strictly smaller than $\sigma$ in the natural extension of our partial ordering $\le$ on $\vec{S}$ to $2^{\vec{S}}$, in which $\sigma \geq \sigma'$ means that there exists a map $f : \sigma \to \sigma'$ such that $\vec{s} \geq f(\vec{s})$ for all $\vec{s} \in \sigma$.

**Lemma 3.6.1.** *If $\vec{S}$ is an ordered separation system and $\mathcal{F} \subseteq 2^{\vec{S}}$ is closed under eclipsing in every consistent orientation of $S$, then $\mathcal{F}$ is rich for $\vec{S}$.*

*Proof.* To prove that $\mathcal{F}$ is rich, let $\tau$ be any consistent orientation of $S$ that has a subset in $\mathcal{F}$. We claim that every set $\sigma$ that is minimal in $\mathcal{F} \cap 2^{\tau} \neq \emptyset$ with respect to our partial ordering on $2^{\vec{S}}$ is strongly efficient in $\tau$.

Indeed, suppose $\sigma$ is not strongly efficient in $\tau$. Then there exists an $\vec{r} \in \tau$ that weakly eclipses some $\vec{s} \in \sigma$. Let $\sigma' := (\sigma \setminus \{\vec{s}\}) \cup \{\vec{r}\}$. Then $\sigma > \sigma' \in \mathcal{F}$, since $\mathcal{F}$ is closed under eclipsing in $\tau$. This contradicts the minimality of $\sigma$, since $\sigma' \subseteq \tau$. □

We now introduce our second way of making a set $\sigma \in \mathcal{F}$ more focused, which is essentially the shifting operation from [21, 16]. Let $\vec{s} \in \vec{S}$ be non-trivial in $\vec{S}$ and non-degenerate. Write $S_{\le \vec{s}}$ for the set of all $t \in S$ to which $\vec{s}$ points, those with an orientation $\vec{t} \le \vec{s}$. Given any $\vec{r} \le \vec{s}$ in $\vec{S}$, define the $\vec{S}_{\le \vec{s}} \to \vec{U}$ map[17]

$$f\!\downarrow^{\vec{s}}_{\vec{r}}(\vec{t}) := \begin{cases} \vec{t} \wedge \vec{r} & \text{if } \overleftarrow{s} \neq \vec{t} \le \vec{s}; \\ (\overleftarrow{t} \wedge \vec{r})^* & \text{if } \overleftarrow{s} \neq \overleftarrow{t} \le \vec{s}. \end{cases}$$

It is easy to see that such *shifting maps* preserve the partial ordering between their arguments; in particular, they map stars to stars [16]. We say that a separation $\vec{r} \le \vec{s}$ in $\vec{S}$ *emulates* $\vec{s}$ in $\vec{S}$ if our shifting map has its image inside $\vec{S}$, i.e., if $\vec{t} \wedge \vec{r} \in \vec{S}$ for every $\vec{t} \in \vec{S}$ with $\overleftarrow{s} \neq \vec{t} \le \vec{s}$.

Let us show that shifting stars can reduce their *order*, the sum of the orders of their elements under a given order function on our separation system.

[17]Note that, as $\vec{s}$ is non-trivial, $\vec{t} \le \vec{s}$ and $\overleftarrow{t} \le \vec{s}$ cannot both hold unless $s = t$, in which case $\vec{s} \wedge \vec{r} = \vec{r}$ (rather than its inverse) is chosen as the image of $\vec{s}$, and $\overleftarrow{r}$ as the image of $\overleftarrow{s}$.

**Lemma 3.6.2.** *Let $\vec{U}$ be a universe of separations with a structurally submodular order function, and let $S = U_\ell$ for some $\ell \in \mathbb{R}$. Let $\tau$ be a consistent orientation of $S$. Assume that some element $\vec{s}$ of a star $\sigma \subseteq \tau$ is non-trivial in $\vec{S}$ and eclipsed by some $\vec{r} \in \tau$. Then $\vec{r}$ can be chosen so that it emulates $\vec{s}$ in $\vec{S}$ and $\sigma' := f\downarrow^{\vec{s}}_{\vec{r}}(\sigma)$ is a star in $\tau$ of lower order than $\sigma$.*

*Proof.* Choose $\vec{r}$ with $|r|$ minimum, and subject to this maximal under the partial ordering on $\vec{S}$. Let us show first that $\vec{r}$ emulates $\vec{s}$ in $\vec{S}$.

We have to show that $\vec{r} \wedge \vec{t} \in \vec{S}$ for any $\vec{t} \in \vec{S}$ with $\overleftarrow{s} \neq \vec{t} \leq \vec{s}$, i.e., that $|\vec{r} \wedge \vec{t}| < \ell$. By the structural submodularity of our order function we have $|\vec{r} \wedge \vec{t}| \leq |t| < \ell$ as desired if $|\vec{r} \vee \vec{t}| > |r|$, so let us assume that $|\vec{r} \vee \vec{t}| \leq |r|$. Then $\vec{r} \vee \vec{t} \in \vec{S}$; let us show that, in fact, $\vec{r} \vee \vec{t} \in \tau$.

As $\vec{r} \vee \vec{t} \geq \vec{r} \in \tau$, the consistency of $\tau$ implies $\vec{r} \vee \vec{t} \in \tau$ as desired, unless the unoriented separation underlying $\vec{r} \vee \vec{t}$ is $r$. If $\vec{r} \vee \vec{t} = \vec{r}$, we are done since $\vec{r} \in \tau$, so let us assume that $\overleftarrow{r} = \vec{r} \vee \vec{t} \geq \vec{r}$. Then $\vec{s} \geq \vec{r} \vee \vec{t} = \overleftarrow{r}$, since $\vec{s} \geq \vec{r}$ as well as $\vec{s} \geq \vec{t}$ by assumption, which makes $\vec{s}$ trivial witnessed by $r$, contrary to assumption, unless $s = r$. But if $\vec{s} = \vec{r}$ we have $\vec{r} \vee \vec{t} = \vec{s} \vee \vec{t} = \vec{s} \in \tau$, while if $\vec{s} = \overleftarrow{r}$ we have $\vec{r} \vee \vec{t} = \overleftarrow{r} = \vec{s} \in \tau$. This completes our proof that $\vec{r} \vee \vec{t} \in \tau$.

As noted, we have $\vec{s} \geq \vec{r} \vee \vec{t}$, as well as $|\vec{r} \vee \vec{t}| \leq |r| < |s|$, since $\vec{r}$ eclipses $\vec{s}$. So $\vec{r} \vee \vec{t}$ eclipses $\vec{s}$, and was thus a candidate for $\vec{r}$. As $|\vec{r} \vee \vec{t}| \leq |r|$, this contradicts the choice of $\vec{r}$ unless $\vec{r} \vee \vec{t} = \vec{r}$. But in that case $\vec{r} \geq \vec{t}$, giving $\vec{r} \wedge \vec{t} = \vec{t} \in \vec{S}$ as desired. This completes our proof that $\vec{r}$ emulates $\vec{s}$.

As shifting preserves the partial ordering on $\vec{S}$, we know that $\sigma'$ is a star. Let us show that it has lower order than $\sigma$. Since $f\downarrow^{\vec{s}}_{\vec{r}}(\vec{s}) = \vec{r}$ and $|r| < |s|$, it suffices to show that $\left|f\downarrow^{\vec{s}}_{\vec{r}}(\vec{t})\right| = |\vec{r} \wedge \vec{t}| \leq |t|$ for any other elements $\overleftarrow{t}$ of $\sigma$. But this follows from structural submodularity and the choice of $\vec{r}$, as earlier, unless $\vec{r} \wedge \vec{t} = \vec{t}$. But in that case we also have $|\vec{r} \wedge \vec{t}| = |t|$ as desired.

It remains to show that $\sigma' \subseteq \tau$. We have $f\downarrow^{\vec{s}}_{\vec{r}}(\vec{s}) = \vec{r} \in \tau$ by explicit assumption about $\vec{r}$. For any $\overleftarrow{t} \in \sigma$ other than $\vec{s}$, we have $\vec{s} \geq \vec{t}$, since $\sigma$ is a star, and hence $f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{t}) = (\vec{t} \wedge \vec{r})^* = \overleftarrow{t} \vee \overleftarrow{r} \geq \overleftarrow{t} \in \tau$ by the definition of $f\downarrow^{\vec{s}}_{\vec{r}}$. This implies $f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{t}) \in \tau$ by the consistency of $\tau$ unless $f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{t}) = \vec{t}$, so let us show that this cannot be the case.

If it is, then we have shown that $\vec{t} = f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{t}) \geq \overleftarrow{t}$. Then $\vec{s} \geq \vec{t} \geq \overleftarrow{t}$, which contradicts our assumption that $\vec{s}$ is non-trivial unless $s = t$. But in that case we have $\vec{s} = \vec{t}$, since $\vec{s} \neq \overleftarrow{t}$ as these are distinct elements of $\sigma$. But if $\vec{s} = \vec{t}$ then $\overleftarrow{r} = f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{s}) = f\downarrow^{\vec{s}}_{\vec{r}}(\overleftarrow{t}) = \vec{t} = \vec{s}$, which contradicts the fact that $|r| < |s|$. □

Call a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars *closed under shifting*[18] in $\tau \subseteq \vec{S}$ if for every $\sigma \in \mathcal{F} \cap 2^{\tau}$ and every $\vec{r} \in \tau$ that weakly eclipses and emulates some non-trivial $\vec{s} \in \sigma$ in $\vec{S}$ we have $f\downarrow^{\vec{s}}_{\vec{r}}(\sigma) \in \mathcal{F}$.

**Lemma 3.6.3.** *Let $\vec{U}$ be a universe of separations with a structurally submodular injective order function, and let $S = U_\ell$ for some $\ell \in \mathbb{R}$. Let $\mathcal{F}$ be a set of stars of non-trivial separations in $\vec{S}$ that is closed under shifting in $\vec{S}$.*[18] *Then $\mathcal{F}$ is rich for $\vec{S}$.*

[18] Our definition of 'closed under shifting' is weaker than that in [21, 16], making our subsequent results stronger. We can even strengthen them further: as is easily checked, we can weaken in both theorems 3.6.3 and 3.6.4 the assumption that $\mathcal{F}$ is closed under shifting in $\vec{S}$ to assuming that $\mathcal{F}$ is closed under shifting in every consistent orientation of $S$. We omitted this for simplicity.

*Proof.* To prove that $\mathcal{F}$ is rich, consider any consistent orientation $\tau$ of $S$ that includes some $\sigma \in \mathcal{F}$. Choose such a star $\sigma$ in $\tau$ of minimum order. We shall show that $\sigma$ is strongly efficient in $\tau$.

If not, there exists an $\vec{r} \in \tau$ that weakly eclipses some $\vec{s} \in \sigma$. Since our order function is injective, any such $\vec{r}$ in fact eclipses $\vec{s}$. By theorem 3.6.2, we can choose $\vec{r}$ so that it emulates $\vec{s}$ in $\vec{S}$ and $\sigma' := f\downarrow^{\vec{s}}_{\vec{r}}(\sigma)$ is a star in $\tau$ of lower order than $\sigma$. As $\mathcal{F}$ is closed under shifting we have $\sigma' \in \mathcal{F}$, which contradicts our choice of $\sigma$. □

**Theorem 3.6.4.** *Let $\vec{U}$ be a universe of separations with either a submodular or an injective structurally submodular order function. Let $S = U_\ell$ for some $\ell \in \mathbb{R}$. Assume that $\vec{S}$ has no trivial elements. Let $\mathcal{F}$ be a set of stars of separations in $\vec{S}$ that is closed under shifting in $\vec{S}$.*[18] *Then exactly one of the following assertions holds:*

1. *there exists an $\mathcal{F}$-tangle of $S$;*
2. *there exists an $S$-tree over $\mathcal{F}$.*

*Proof.* Recall from the proof of theorem 3.4.1 that if $S$ has the form of $S = U_\ell$ with respect to a given order function $o$, it also has this form with respect to any refinement of $o$. By theorem 3.4.6 we may therefore assume that our order function on $U$ is injective and structurally submodular. By theorem 3.6.3, $\mathcal{F}$ is rich for $\vec{S}$. The theorem now follows from theorem 3.3.6. □

While theorem 3.6.4 is probably to be the most widely applicable tangle-tree duality theorem to date, it may appear restrictive that $S$ must have the form of $U_\ell$, and that we need a submodular order function on $U$. The original tangle-tree duality theorem from [16], theorem 3.3.3, makes no such assumptions. However, it makes another assumption, one not made in theorem 3.6.4: that $\vec{S}$ is $\mathcal{F}$-separable.

This is not a coincidence: $\mathcal{F}$-separability is essentially a technical summary of exactly those consequences of our two more natural assumptions, submodularity and $S = U_\ell$, that are technically needed in the proof of theorem 3.3.3.

Finally, the premise of theorem 3.6.4 differs from that of theorem 3.3.3 in that it requires $\vec{S}$ not to have trivial elements. We need this for our conversion of $\mathcal{F}$-trees into $S$-trees over $\mathcal{F}$ in theorem 3.3.4. But it entails no loss of generality: the trivial separations of any separation system lie in all consistent orientations, so we can just delete them from our given system and add them back later if desired [2, 25].

## 3.7 Applications

In the previous sections we were able to show both pillars of traditional tangle theorems, the tangle-tree duality theorem theorem 3.3.6, and the tree-of-tangles theorem theorem 3.5.8.

In the last section we already showed that sets of stars which are closed under shifting are *rich*. It turns out that even in theoretical applications where the stars are not closed under shifting, the forbidden sets considered are rich, though with the caveat that the order functions have to be made injective.

The applications we are going to cover in Section 3.7.1 are feature systems and their cousins: the advanced feature systems and set separation systems. These cover most separation systems relevant for practical applications and are prominently featured in [3].

In the following subsections we revisit the applications from [27], where the tangle-tree duality theorem is used to derive further duality theorems.

### 3.7.1 Set separation systems, feature systems and advanced feature systems

Let $V$ be a finite set. We denote by $\mathbb{R}^V$ the set of all real-valued functions $f : V \to \mathbb{R}$. We define

$$f \le g :\iff f(v) \le g(v) \text{ for every } v \in V,$$

$$f^* := -f.$$

Then any finite set of functions $\vec{S}$ which is closed under negation forms a separation system $(\vec{S}, \le, ^*)$. Such a separation system is called an *advanced feature system on* $V$. Its elements are called *(advanced) features* and the sets $\{f, -f\}$ for $f \in \vec{S}$ are called *(advanced) potential features.* The supremum $\vee$ and infimum $\wedge$ on advanced features are defined using the pointwise maximum and minimum respectively.

A *feature* is a subset $\emptyset \subsetneq A \subsetneq V$. We identify the feature $A$ with the function

$$f_A(v) := \begin{cases} 1 & \text{if } v \in A, \\ -1 & \text{if } v \notin A. \end{cases}$$

It is easy to check that $f_A \le f_B$ if and only if $A \subseteq B$ and $f_A^* = f_{V \setminus A} = f_{A^C}$. In this way the advanced features induce a $\le$ relation and involution on the features. If $\vec{S}$ is a set of features closed under complements, then $(\vec{S}, \subseteq, ^C)$ is a *feature system.* Due to our previous observations, feature systems can also be interpreted as advanced feature systems. While not technically a universe, we often treat the features of $V$ as if they were part of the larger universe, which includes $V$ and $\emptyset$. This does not change the mathematical substance of the later arguments.

A *set separation* of $V$ is a set $\{A, B\}$ where $A \cup B = V$. An *oriented set separation* is a tuple $(A, B)$ where $\{A, B\}$ is a set separation. We identify an oriented set separation $(A, B)$ with the advanced feature $f_{A,B} := f_B - f_A$ (extending $f_A$ to $A = V$ and $A = \emptyset$ as well). It is easy to see that for set separations $\{A, B\}$ and $\{C, D\}$ of $V$ we have $f_{A,B} \le f_{C,D}$ if and only if $A \supseteq C$ and $B \subseteq D$ and $f_{A,B}^* = f_{B,A}$. Separation systems consisting of set separations with the previously noted operations are called *set separation systems.* As with feature systems, set separation systems can also be interpreted as advanced feature systems.

There are three types of forbidden subsets usually considered for advanced feature systems $\vec{S}$ on a set $V$. Let $k, n \in \mathbb{N}$.

$$\begin{aligned} \mathcal{A}_k^{\le n}(\vec{S}) &:= \{\sigma \subseteq \vec{S} : |\sigma| \le n \wedge |\{v \in V : f(v) > 0 \;\forall f \in \sigma\}| < k\}, \\ \mathcal{B}_k^{\le n}(\vec{S}) &:= \{\sigma \subseteq \vec{S} : |\sigma| \le n \wedge |\{v \in V : f(v) \ge 0 \;\forall f \in \sigma\}| < k\}, \\ \mathcal{C}_k^{\le n}(\vec{S}) &:= \{\sigma \subseteq \vec{S} : |\sigma| \le n \wedge |\{v \in V : \sum_{f \in \sigma} f(v) > 0\}| < k\} \end{aligned}$$

Let us now discuss each of these classes of forbidden subsets. We start with $\mathcal{A}_k^{\le n}$. In the special case that $\vec{S}$ is a feature system, it can easily be checked that $\mathcal{A}_k^{\le n}$ are the $\sigma \subseteq \vec{S}$ for which $|\sigma| \le n$ and $|\bigcap_{B \in \sigma} B| < k$. If $\vec{S}$ is a set separation system then

$\mathcal{A}_k^{\leq n}$ are the $\sigma \subseteq \vec{S}$ for which $|\sigma| \leq n$ and $\left|\bigcap_{(A,B)\in\sigma}(B \setminus A)\right| < k$. This class of forbidden subsets appears, for example, in [3].

We will now show several useful properties of $\mathcal{A}_k^{\leq n}$ which make them suitable for practical applications.

**Lemma 3.7.1.** *Let $\vec{S}$ be an ordered advanced feature system. Then $\mathcal{A}_k^{\leq n}(\vec{S})$ is closed under eclipsing and standard.*

*If $\vec{U}$ is an advanced submodular feature universe and $\vec{S} = \vec{U}_\ell$ for some $\ell \in \mathbb{R}$ then*

$$\mathcal{A}_k^{\leq n*} := \{\sigma \in \mathcal{A}_k^{\leq n} : \sigma \text{ is a star}\}$$

*is closed under shifting and standard. If $n \geq 3$ then $\mathcal{R}(\vec{S}) \subseteq \mathcal{A}_k^{\leq n*}$.*

*Proof.* We start by showing that $\mathcal{A}_k^{\leq n}$ is closed under eclipsing. Let $\sigma \in \mathcal{A}_k^{\leq n}$ and let $g \in \vec{S}$ weakly eclipse some $f \in \sigma$. Let $\sigma' := \sigma \setminus \{f\} \cup \{g\}$ and $v \in V$. We show that if $h'(v) > 0$ for every $h' \in \sigma'$ then also $h(v) > 0$ for every $h \in \sigma$, thereby showing $\sigma' \in \mathcal{A}_k^{\leq n}$. Since $g(v) > 0$ and $f > g$ we have $f(v) > 0$, the other $h \in \sigma$ are also contained in $\sigma'$ and therefore $h(v) > 0$. Hence we have shown that $\mathcal{A}_k^{\leq n}$ is closed under eclipsing.

Next we show that $\mathcal{A}_k^{\leq n*}$ is standard. If $h$ is co-trivial, then $h(v) \leq 0$ for every $v \in V$. So $\{h\} \in \mathcal{A}_k^{\leq n*}$. That $\mathcal{A}_k^{\leq n}$ is standard follows since $\mathcal{A}_k^{\leq n*}$ is standard and $\mathcal{A}_k^{\leq n*} \subseteq \mathcal{A}_k^{\leq n}$.

To show that $\mathcal{R}(\vec{S}) \subseteq \mathcal{A}_k^{\leq n*}$ for $n \geq 3$ consider $f, g \in \vec{S}$ and let $v \in V$. We have to show that one of $f(v)$, $(-f \vee g)(v)$ and $(-f \vee -g)(v)$ is at most 0. If $f(v) \leq 0$ we are done. So assume $f(v) > 0$. We have $g(v) \leq 0$ or $-g(v) \leq 0$. Assume without loss of generality that $g(v) \leq 0$. Then $(-f \vee g)(v) \leq 0$.

Finally, we show that $\mathcal{A}_k^{\leq n*}$ is closed under shifting. Let $\sigma \in \mathcal{A}_k^{\leq n*}$ and let $b \in \vec{S}$ weakly eclipse some $a \in \sigma$. We have to show that $\sigma' := f\downarrow_b^a(\sigma) \in \mathcal{A}_k^{\leq n*}$. Let $v \in V$ and $g(v) > 0$ for every $g \in \sigma'$. To show $\sigma' \in \mathcal{A}_k^{\leq n*}$ we have to show $h(v) > 0$ for every $h \in \sigma$. Since $b(v) > 0$ and $a > b$ we obtain $a(v) > 0$. Every other element $h \in \sigma' \setminus \{b\}$ is of the form $h = -b \vee g$ for some $g \in \sigma \setminus \{a\}$. If $h(v) > 0$ then since $b(v) > 0$ we must have $g(v) > 0$. □

Next let us consider the forbidden subsets $\mathcal{B}_k^{\leq n}$. We have $\mathcal{B}_k^{\leq n} \subseteq \mathcal{A}_k^{\leq n}$. If $\vec{S}$ is a feature system then $\mathcal{B}_k^{\leq n} = \mathcal{A}_k^{\leq n}$. The difference can be seen if $\vec{S}$ is a set separation system, then $\mathcal{B}_k^{\leq n}$ are the $\sigma \subseteq \vec{S}$ for which $|\sigma| \leq n$ and $\left|\bigcap_{(A,B)\in\sigma} B\right| < k$. This class of forbidden subsets is normally used in the context of graphs as we will see in the next subsection.

**Lemma 3.7.2.** *Let $\vec{S}$ be an ordered advanced feature system. Then $\mathcal{B}_k^{\leq n}$ is closed under eclipsing.*

*Proof.* The proof of $\mathcal{B}_k^{\leq n}$ being closed under eclipsing follows by the same argument as in theorem 3.7.1. □

Finally, we consider $\mathcal{C}_k^{\leq n}$. These forbidden subsets, introduced in [3], are only considered on advanced features, since their definition takes advantage of the additional information contained in advanced features, compared to features or set separations.

**Lemma 3.7.3.** *Let $\vec{S}$ be an ordered advanced feature system. Then $\mathcal{C}_k^{\leq n}$ is closed under eclipsing and standard.*

*Proof.* We show that $\mathcal{C}_k^{\leq n}$ is closed under eclipsing. Let $\sigma \in \mathcal{C}_k^{\leq n}$ and let $g \in \vec{S}$ weakly eclipse some $f \in \sigma$. Let $\sigma' := \sigma \setminus \{f\} \cup \{g\}$ and $v \in V$. We show that if $\sum_{h \in \sigma'} h(v) > 0$ then $\sum_{h \in \sigma} h(v) > 0$.

$$\sum_{h \in \sigma'} h(v) = g(v) - f(v) + \sum_{h \in \sigma} h(v) \leq \sum_{h \in \sigma} h(v).$$

Hence we have shown that $\mathcal{C}_k^{\leq n}$ is closed under eclipsing. If $h$ is co-trivial, then $h(v) \leq 0$ for every $v \in V$, so $\sum_{g \in \{h\}} g(v) > 0$ never holds. Hence $\{h\} \in \mathcal{C}_k^{\leq n}$, and $\mathcal{C}_k^{\leq n}$ is standard. □

Let us collect which families of forbidden sets and stars are rich in the following corollary.

**Corollary 3.7.4.** *Let $\vec{S}$ be an ordered advanced feature system and $k, n \in \mathbb{N}$ then $\mathcal{A}_k^{\leq n}$, $\mathcal{B}_k^{\leq n}$ and $\mathcal{C}_k^{\leq n}$ are rich for $\vec{S}$.*

*Let $\vec{U}$ be a universe of advanced features with an injective structurally submodular order function. Let $S := U_\ell$ for some $\ell \in \mathbb{R}$. Assume that $\vec{S}$ has no trivial elements. Then $\mathcal{A}_k^{\leq n*}$ is rich for $\vec{S}$.*

*Proof.* The first part follows from theorem 3.6.1, theorem 3.7.1, theorem 3.7.2 and theorem 3.7.3. The second part follows from theorem 3.6.3 and theorem 3.7.1. □

When applying advanced features as suggested in [3], we often define our order function empirically, using real-world data. Because real-world data is inherently noisy, our order functions are often injective, even without making them injective using theorem 3.4.6.

Another consequence of the previous lemmas is a tangle-tree duality result for advanced features and a tree-of-tangles result.

**Corollary 3.7.5.** *Let $\vec{U}$ be a universe of advanced features with either a submodular or an injective structurally submodular order function. Let $S := U_\ell$ for some $\ell \in \mathbb{R}$. Assume that $\vec{S}$ has no trivial elements. Let $n, k \in \mathbb{N}$. Exactly one of the following assertions holds:*

1. *there exists an $\mathcal{A}_k^{\leq n*}$-tangle of $S$;*
2. *there exists an $S$-tree over $\mathcal{A}_k^{\leq n*}$.*

*Furthermore, if $n \geq 3$, then there exists a canonical tree of $\mathcal{A}_k^{\leq n*}$-tangles in $\vec{U}$.*

*Proof.* By theorem 3.7.1 the stars $\mathcal{A}_k^{\leq n*}$ are closed under shifting in $\vec{S}$, and so the duality follows from theorem 3.6.4.

If $n \geq 3$, then theorem 3.7.1 gives $\mathcal{A}_k^{\leq n*}$ standard and $\mathcal{R}(\vec{U}) \subseteq \mathcal{A}_k^{\leq n*}$. Then the tree-of-tangles result follows from theorem 3.4.5, theorem 3.7.4 and theorem 3.5.8. □

### 3.7.2 Duality in Graphs

Let $G = (V, E)$ be a graph. A *graph separation* of $G$ is a set separation $\{A, B\}$ of $V$ such that there does not exist an edge $e \in E$ from $B \setminus A$ to $A \setminus B$. It is a well-known result that the separations of any graph $G$ form a universe of set separations. The order of a graph separation is defined as $\|A, B\| := |A \cap B|$, which is submodular. In this part we let $\vec{S}$ denote the set of all graph separations of $G$. A graph tangle is a $\mathcal{T}$-tangle of $S_k$ with

$$\mathcal{T} := \{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq \vec{S} : G[A_1] \cup G[A_2] \cup G[A_3] = G\}.$$

The set $\mathcal{T}^*$ is the set of all elements of $\mathcal{T}$ which are stars. The fact that $\mathcal{T}^*$ is closed under shifting is shown in the proof of [27Theorem 4.1]. Note that while our definition of closed under shifting is different from the one in [27], our definition is weaker and thus follows from the old definition.

**Theorem 3.7.6** (Tangle-tree duality theorem for graphs [27Theorem 4.1] ). *For every $k > 0$, every graph $G$ satisfies exactly one of the following assertions:*

1. *$G$ has a $\mathcal{T}^*$-tangle of $S_k$.*
2. *$G$ has an $S_k$-tree over $\mathcal{T}^*$.*

*Proof.* We may assume that $\vec{S}_k$ has no trivial elements since, by the discussion after theorem 3.6.4, deleting the trivial separations does not change the tangles. The theorem then follows from theorem 3.6.4 and the fact that $\mathcal{T}^*$ is closed under shifting. □

Diestel and Oum used the following set of stars in [27] to derive the duality between brambles and tree-width in graphs using tangle duality.

$$\mathcal{B}_k^* := \bigcup_{n\in\mathbb{N}} \mathcal{B}_k^{\le n*}.$$

More precisely, they showed that the existence of a bramble of order $k$ in a graph $G$ is equivalent to the existence of an $\mathcal{B}_k^*$-tangle of $S_k$ [27Lemma 6.4] and that a graph $G$ has tree-width at least $k-1$ if and only if $G$ has no $S_k$-tree over $\mathcal{B}_k^*$ [27Lemma 6.3]. They also showed that $\mathcal{B}_k^*$ is closed under shifting in $S_k$ [27Lemma 6.1]. We can therefore use our new duality theorem to re-obtain the following result.

**Theorem 3.7.7** (Tangle-bramble-treewidth duality theorem for graphs [27Theorem 6.5]). *The following assertions are equivalent for all finite graphs $G$ and $k > 0$:*

1. *$G$ has a bramble of order at least $k$.*
2. *$G$ has an $\mathcal{B}_k^*$-tangle of $S_k$.*
3. *$G$ has no $S_k$-tree over $\mathcal{B}_k^*$.*
4. *$G$ has tree-width at least $k-1$.*

*Proof.* (i) ⇔ (ii) follows from [27Lemma 6.4] and (iii) ⇔ (iv) follows from [27Lemma 6.3]. (ii) ⇔ (iii) follows from [27Lemma 6.1] and theorem 3.6.4. □

Another duality that Diestel and Oum proved in [27] using tangle duality is the duality between blockages and path-width in graphs. They showed that the existence of a blockage of order $k-1$ in a graph $G$ is equivalent to the existence of an $\mathcal{B}_k^{\le 2*}$-tangle of $S_k$ [27Theorem 7.2] and that $G$ has path-width at least $k-1$ if and only if $G$ has no $S_k$-tree over $\mathcal{B}_k^{\le 2*}$ [27Lemma 6.3]. We therefore re-obtain the following result.

**Theorem 3.7.8** (Tangle-blockage-pathwidth duality theorem for graphs [27Theorem 7.2]). *The following assertions are equivalent for $G \neq \emptyset$ and $k > 0$:*

1. *$G$ has a blockage of order $k-1$.*
2. *$G$ has an $\mathcal{B}_k^{\le 2*}$-tangle of $S_k$.*

3. *G has no $S_k$-tree over $\mathcal{B}_k^{\leq 2*}$.*

4. *G has path-width at least $k-1$.*

*Proof.* (i) ⇔ (ii) follows from [27Theorem 7.2] and (iii) ⇔ (iv) follows from [27Lemma 6.3]. (ii) ⇔ (iii) follows from [27Lemma 6.1] and theorem 3.6.4. □

### 3.7.3 Tangle duality for tree-width in matroids

Let $M = (E, I)$ be a matroid with rank function $r$. The rank function has the following properties:

(R1) $r(\emptyset) = 0$ and $r(X) \geq 0$ for every $X \subseteq E$,

(R2) $r$ is a submodular set function,

(R3) for every $X \subseteq E$ and every $a \in E$ we have $r(X) \leq r(X \cup \{a\}) \leq r(X) + 1$.

Since features of $E$ are subsets of $E$ we can interpret $r$ as a function on the features of $E$ as well.

The function $\lambda(\vec{s}) := r(\vec{s}) + r(\overleftarrow{s}) - r(E)$ is called the *connectivity function* of $M$. The connectivity function can be used to define an order function on the features $\vec{S}(E)$ of $E$. This function is known to be non-negative, submodular and symmetric.

For $\sigma \subseteq \vec{S}(E)$ define

$$\langle\sigma\rangle := \sum_{\vec{s}\in\sigma}(r(\vec{s}) - r(E)) = \sum_{\vec{s}\in\sigma}(\lambda(\vec{s}) - r(\overleftarrow{s})).$$

For $k \in \mathbb{N}$ define the sets $\mathcal{F}_k := \{\sigma \subseteq \vec{S} : \sigma \text{ is a star and } \langle\sigma\rangle < k\}$. We will now show that this set of stars is rich. We start by proving an addition to theorem 3.6.2.

**Lemma 3.7.9.** *Let $\vec{U}$ be a universe of separations with a structurally submodular order function, and let $S = U_\ell$ for some $\ell \in \mathbb{R}$. Let $\tau$ be a consistent orientation of $S$. Assume that some element $\vec{s}$ of a star $\sigma \subseteq \tau$ is non-trivial in $\vec{S}$ and weakly eclipsed by some $\vec{r} \in \tau$. If $|r| \leq |r'|$ for all $\vec{r}'$ weakly eclipsing $\vec{s}$ with $\vec{r}' > \vec{r}$, then $\vec{r}$ emulates $\vec{s}$ in $\vec{S}$ and $\sigma' := f\downarrow^{\vec{s}}_{\vec{r}}(\sigma)$ is a star in $\tau$ of order not greater than $\sigma$.*

*Proof.* That $\vec{r}$ emulates $\vec{s}$ in $\vec{S}$ and $\sigma' \subseteq \tau$ follow analogously to the proof of theorem 3.6.2.

As shifting preserves the partial ordering on $\vec{S}$, we know that $\sigma'$ is a star. Let us show that it has order not greater than $\sigma$. Since $f\downarrow^{\vec{s}}_{\vec{r}}(\vec{s}) = \vec{r}$ and $|r| \leq |s|$, it suffices to show that $\left|f\downarrow^{\vec{s}}_{\vec{r}}(\vec{t})\right| = |\vec{r} \wedge \vec{t}| \leq |t|$ for any other elements $\overleftarrow{t}$ of $\sigma$. But this follows from submodularity and the fact that $|\vec{r} \vee \vec{t}| \geq |r|$, which holds analogously to theorem 3.6.2. □

**Lemma 3.7.10.** *Let $k, \ell \in \mathbb{N}$. The set $\mathcal{F}_k$ is rich for $\vec{S}_\ell$ for all injective and structurally submodular order functions $|\,|$ refining $\lambda$.*

*Proof.* Let us define $\Phi(\sigma) := -\sum_{\vec{s}\in\sigma} r(\overleftarrow{s})$, $\Xi(\sigma) := \sum_{\vec{s}\in\sigma} \lambda(\vec{s})$ and $\Delta(\sigma) := \sum_{\vec{s}\in\sigma} |s|$. Then $\langle\sigma\rangle = \Phi(\sigma) + \Xi(\sigma)$. Let $\tau$ be a consistent orientation of $S_\ell$ which contains an element $\sigma$ of $\mathcal{F}_k$ as a subset. Choose $\sigma$ such that $\Delta(\sigma)$ is minimal for all $\sigma \in \mathcal{F}_k$ contained in $\tau$. Suppose some $\vec{s} \in \sigma$ is weakly eclipsed by some $\vec{r} \in \tau$. Choose $\vec{r}$ of minimal order such that it weakly eclipses $\vec{s}$ and let $\sigma' := f\downarrow^{\vec{s}}_{\vec{r}}(\sigma)$. Since the order function is injective, $\vec{r}$ in fact eclipses $\vec{s}$. Using theorem 3.6.2 and its proof we obtain $\Delta(\sigma') < \Delta(\sigma)$. Then using theorem 3.7.9 we obtain that $\sigma'$ is a star in $\tau$ with $\Xi(\sigma') \leq \Xi(\sigma)$. Moreover, the contribution of $\vec{s}$ to $\Phi$ does not increase because $\overleftarrow{r} > \overleftarrow{s}$ and hence $-r(\overleftarrow{r}) \leq -r(\overleftarrow{s})$. For any other $\overleftarrow{t} \in \sigma$, the shifted element is $\vec{r} \wedge \vec{t}$, whose inverse is $\overleftarrow{r} \vee \overleftarrow{t} \geq \overleftarrow{t}$, so again $-r(\overleftarrow{r} \vee \overleftarrow{t}) \leq -r(\overleftarrow{t})$. Therefore $\Phi(\sigma') \leq \Phi(\sigma)$. Thus $\langle\sigma'\rangle \leq \langle\sigma\rangle$ and $\sigma' \in \mathcal{F}_k$. Thus we have a contradiction. □

As stated in [27], a *tree-decomposition* of $M$ is a pair $(T, \tau)$, where $T$ is a tree and $\tau : E \to V(T)$ is any map. Let $t$ be a node of $T$, and let $T_1, \dots, T_d$ be the components of $T - t$. Then the *width* of $t$ is the number

$$\sum_{i=1}^{d} r(E \setminus F_i) - (d-1)r(E),$$

where $F_i = \tau^{-1}(V(T_i))$. The *width* of $(T, \tau)$ is the maximum width of the nodes of $T$. The *tree-width* of $M$ is the minimum width over all tree-decompositions of $M$.

Diestel and Oum were able to show in [27Lemma 8.4] that $M$ has an $S_k$-tree $(T, \alpha)$ over $\mathcal{F}_k$ if and only if it admits a tree-decomposition $(T, \tau)$ of width $< k$.

**Theorem 3.7.11** (Tangle-treewidth duality theorem for matroids [27Theorem 8.5]). *Let $M$ be a matroid, and let $k > 0$ be an integer. Then the following statements are equivalent:*

1. *$M$ has tree-width at least $k$.*
2. *$M$ has no $S_k$-tree over $\mathcal{F}_k$.*
3. *$M$ has an $\mathcal{F}_k$-tangle of $S_k$.*

*Proof.* By [27Lemma 8.4], assertions (i) and (ii) are equivalent. Since features are never trivial, the separations in $\vec{S}_k$ are never trivial either. Viewing $\lambda$ as an order function on the ambient separation universe of subsets of $E$, theorem 3.4.6 yields an injective and structurally submodular order function $|\,|$ refining $\lambda$ such that the separation system $S_k$ defined using $\lambda$ is identical to the separation system $S_{k'}$ defined using $|\,|$ for some $k' \in \mathbb{N}$. By theorem 3.7.10, the set $\mathcal{F}_k$ is rich for $\vec{S}_{k'} = \vec{S}_k$. Hence assertions (ii) and (iii) are equivalent by theorem 3.3.6. □

### 3.7.4 Weakly submodular partition functions

A *partition* of a finite set $E$ is a set of disjoint non-empty subsets of $E$ whose union is $E$. Denote the set of all partitions of $E$ by $\mathcal{P}(E)$. A *partition function* is a map $\Psi : \mathcal{P}(E) \to \mathbb{R} \cup \{\infty\}$.

Let $\vec{S}$ be the feature system of $E$ and let, for the rest of this section, $\mathcal{F} \subseteq 2^{\vec{S}}$ be the set of stars with empty intersection. There is a bijection between the stars $\sigma \in \mathcal{F}$ and the corresponding partitions $\{\overleftarrow{s}\}_{\vec{s} \in \sigma}$. Indeed, if $\sigma$ is a star with empty intersection, then the sets $\overleftarrow{s}$ with $\vec{s} \in \sigma$ are pairwise disjoint and cover $E$. So we can consider $\Psi$ to be a function $\Psi : \mathcal{F} \to \mathbb{R} \cup \{\infty\}$.

Using these observations, we obtain the following restatement of the definition of a *weakly submodular partition function* $\Psi$. It is equivalent to the definition stated in [27]. We say that $\Psi$ is a weakly submodular partition function if one of the following conditions holds for every pair of stars $\sigma, \varrho$ and every choice $\vec{r} \in \sigma$, $\vec{s} \in \varrho$:

1. there exists a $\vec{t}$ such that $\vec{r} \geq \vec{t} \geq \vec{r} \wedge \overleftarrow{s}$ and $\Psi(\sigma) > \Psi(f \downarrow^{\vec{r}}_{\vec{t}} (\sigma))$,
2. $\Psi(\varrho) \geq \Psi(f \downarrow^{\vec{s}}_{\vec{s} \wedge \overleftarrow{r}} (\varrho))$.

The partition function $\Psi$ induces an order function on $\vec{S}$ by $|s| := \Psi(\{\vec{s}, \overleftarrow{s}\})$.

**Lemma 3.7.12.** *Let $\sigma \in \mathcal{F}$, $\vec{s} \in \sigma$ and $\vec{r} \in \vec{S}$. Then one of the following statements is true:*

*(W1)* $\Psi(\sigma) \geq \Psi(f \downarrow^{\vec{s}}_{\vec{s} \wedge \overleftarrow{r}} (\sigma))$,

*(W2) there exists a $\vec{r} \geq \vec{t} \geq \vec{r} \wedge \overleftarrow{s}$ such that $|t| < |r|$.*

*Furthermore, one of the following statements is true:*

*(O1)* $|\vec{s} \wedge \overleftarrow{r}| \leq |\vec{s}|$,

*(O2) there exists a $\vec{r} \geq \vec{t} \geq \vec{r} \wedge \overleftarrow{s}$ such that $|t| < |r|$.*

*We call an order function* $|\;|$ weakly submodular *if, for every pair $\vec{r}, \vec{s}$, at least one of the statements* (O1) *and* (O2) *holds.*

*Proof.* Follows directly from the definition. □

Let $\vec{T} \subseteq \vec{S}$ be those features of cardinality 1. Define

$$\mathcal{F}^1_k := \{\{\vec{s}\} \subseteq \vec{T} : |s| < k\}, \quad \mathcal{F}^2_k := \{\sigma \in \mathcal{F} : \Psi(\sigma) < k\}, \quad \mathcal{F}_k := \mathcal{F}^2_k \cup \mathcal{F}^1_k.$$

A feature of cardinality 1 cannot be weakly eclipsed, because any strictly smaller feature would have to be empty, so $\mathcal{F}^1_k$ is rich.

The order function that maps every separation of $\vec{S}$ to 0 is submodular. Applying theorem 3.4.5 to this order function and scaling the resulting injective submodular perturbation if necessary, we obtain an order function $\delta$ on $\vec{S}$ such that $\psi := |\;| + \delta$ refines $|\;|$ and is injective. Note that $\psi$ itself is not submodular, but it is weakly submodular.

**Lemma 3.7.13.** *$\mathcal{F}^2_k$ is rich for $\vec{S}_\ell$ for every $k, \ell \in \mathbb{R}$ for the order function $\psi$.*

*Proof.* Let us define

$$\Xi(\sigma) := \sum_{\vec{s} \in \sigma} \psi(s)$$

Let us further define

$$\Phi_\varepsilon(\sigma) := \Psi(\sigma) + \varepsilon \cdot \Xi(\sigma),$$

and for $k' \in \mathbb{R}$ let

$$\mathcal{G}^\varepsilon_{k'} := \{\sigma \in \mathcal{F} : \Phi_\varepsilon(\sigma) < k'\}.$$

There exist $\varepsilon, k' \in \mathbb{R}$ such that $\mathcal{G}^\varepsilon_{k'} = \mathcal{F}^2_k$. So we only have to show that $\mathcal{G}^\varepsilon_{k'}$ is rich for $\vec{S}_\ell$.

Let $\tau$ be a consistent orientation of $\vec{S}_\ell$ that contains some $\sigma \in \mathcal{G}^\varepsilon_{k'}$. Choose $\sigma \in \mathcal{G}^\varepsilon_{k'} \cap 2^\tau$ such that $\Xi(\sigma)$ is minimal. Suppose there exists an $\vec{r} \in \tau$ which weakly eclipses some $\vec{s} \in \sigma$. We choose $\vec{r}$ such that $\psi(r)$ is minimal. Then, because $\psi$ is injective, $\psi(r) < \psi(s)$. Let $\sigma' := f \downarrow^{\vec{s}}_{\vec{r}} (\sigma)$.

By the minimality of $\psi(\vec{r})$, and hence of $|\vec{r}|$ because $\psi$ refines $|\;|$, the separation $\vec{r}$ has minimal $|\;|$-order among those weakly eclipsing $\vec{s}$. Therefore (W2) cannot hold when applying theorem 3.7.12 to the order function $|\ |$. So (W1) must hold, and hence $\Psi(\sigma) \geq \Psi(\sigma')$.

For every separation $\overleftarrow{t} \in \sigma \setminus \{\vec{s}\}$ it holds that $\psi(\vec{t} \wedge \vec{r}) \leq \psi(\vec{t})$. Otherwise there would exist an $\vec{r'}$ with $\vec{r} < \vec{r'} \leq \vec{r} \vee \vec{t}$ and $\psi(\vec{r'}) < \psi(\vec{r})$. This implies that $\Xi(\sigma') < \Xi(\sigma)$.

In conclusion, $\sigma' \subseteq \tau$ and $\sigma' \in \mathcal{G}^{\varepsilon}_{k'}$, contradicting the minimality of $\Xi(\sigma)$. This contradiction completes the proof. □

**Theorem 3.7.14.** *Let $\Psi$ be a weakly submodular partition function on a finite set $E$, and let $k, \ell \in \mathbb{R}$. Then $\mathcal{F}_k$ is rich for $\vec{S}_\ell$ for $\psi$.*

*Proof.* Since $\mathcal{F}^1_k$ is rich for $\vec{S}_\ell$ and, by theorem 3.7.13, we also know that $\mathcal{F}^2_k$ is rich for $\vec{S}_\ell$, their union $\mathcal{F}_k$ is also rich for $\vec{S}_\ell$. □

This allows us to obtain the following duality result.

**Corollary 3.7.15.** *The following assertions are equivalent for all weakly submodular partition functions $\Psi$ of a finite set $E$ and $k > 0$:*

1. *There exists an $\mathcal{F}_k$-tangle of $S_k$.*
2. *There exists no $S_k$-tree over $\mathcal{F}_k$.*

*Proof.* Choose $k' \in \mathbb{R}$ such that the separation system $S_k$ defined with respect to the order function $|\ |$ is identical to the separation system $S_{k'}$ defined with respect to the order function $\psi$. By theorem 3.7.14, the set $\mathcal{F}_k$ is rich for this separation system with respect to $\psi$. The equivalence now follows from theorem 3.3.6. □

A *k-bramble* for a weakly submodular partition function $\Psi$ of $E$, as stated in [27], is a non-empty set of intersecting subsets of $E$ that contains an element from every partition $P$ of $E$ with $\Psi(P) < k$. It is *non-principal* if it contains no singleton set $\{e\}$. If all the stars in $\mathcal{F}_k$ are contained in $\vec{S}_k$ then $\Psi$ is called *monotone*.

Diestel and Oum deduced in [27] the following theorem from the corresponding tangle-tree duality statement.

**Theorem 3.7.16** ([27] Theorem 10.2). *The following assertions are equivalent for all monotone weakly submodular partition functions $\Psi$ of a finite set $E$ and $k > 0$:*

1. *There exists a non-principal $k$-bramble for $\Psi$.*
2. *$S_k$ has an $\mathcal{F}_k$-tangle.*
3. *There exists no $S_k$-tree over $\mathcal{F}_k$.*

# 4

# Traits and tangles: An analysis of the Big Five paradigm by tangle-based clustering

**Using the recently developed mathematical theory of tangles, we re-assess the mathematical foundations for applications of the five factor model in personality tests by a new, mathematically rigorous, quantitative method. Our findings broadly confirm the validity of current tests, but also show that more detailed information can be extracted from existing data.**

**We found that the big five traits appear at different levels of scrutiny. Some already emerge at a coarse resolution of our tools at which others cannot yet be discerned, while at a resolution where these *can* be discerned, and distinguished, some of the former traits are no longer visible but have split into more refined traits or disintegrated altogether.**

**We also identified traits other than the five targeted in those tests. These include more general traits combining two or more of the big five, as well as more specific traits refining some of them.**

**All our analysis is structural and quantitative, and thus rigorous in explicitly defined mathematical terms. Since tangles, once computed, can be described concisely in terms of very few explicit statements referring only to the test questions used, our findings are also directly open to interpretation by experts in psychology.**

**Tangle analysis can be applied similarly to other topics in psychology. Our paper is intended to serve as a first indication of what may be possible.**

## Contents

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This paper can be read at two levels.

- At the methodological level it showcases a new mathematical clustering method suited particularly to the social sciences [4, 3]: indirect clustering by *tangles*.
- At the content level it offers additional, non-statistical, soundness validation (which we shall argue is needed) and new structural analysis of the Five Factor Model in personality testing.

Tangles have their origin in structural graph theory. Although they do offer a new, indirect, approach to generic clustering, which we apply here, their particular strength lies in direct applications to problems typical for the social sciences, as outlined in [3].

## 4.1 Introduction

The 'big five' paradigm, also known as the *Five Factor Model* (FFM) [28], holds that the essence of an individual's personality can be captured by evaluating it in terms of the five 'OCEAN' traits: *o*penness to experience, *c*onscientiousness, *e*xtraversion, *a*greeableness and *n*euroticism. These five traits have been advanced as interpretations of the five principal factors originally found by Tupes and Christal [29] in data of Cattell [30] and Fiske [31], and also of five factors found by others in different data; see [32]. In a typical personality assessment based on this paradigm, such as [33, 34, 35], the person to be evaluated has to answer, for each of these five traits, a set of questions designed to test to what degree an individual has the trait in question.

There are a number of fundamental assumptions underlying this approach, which are the topic of this paper.

The first of these stems from the fact that the above five traits are not themselves the factors, in the technical factor-analytical sense, but are interpretations of such factors. In personality tests such as [35], however, they are targeted directly. The multitude of traits studied, e.g., in [30] and [31], has thus been compressed into five factors; these were interpreted; and their interpretations were fanned out again into the many aspects displayed in [36] and assessed in [35].

Compression followed by de-compression always bears the risk of information loss or shift. Hence the following question arises:

(1) To what extent are current personality tests that are designed to target the traits named as OCEAN related, in a reproducible quantitative sense, to the original Five Factors from which they derive their justification?

We believe that settling this question requires data of a kind we have been unable to find in the literature; see our discussion of this problem in section 4.1.1.

Alternatively, one can interpret the reference which these tests make to the original Five Factors as inspiration rather than justification, and seek to validate these tests directly. In particular, do the several questions into which each of the five OCEAN traits is fanned out really test the same thing? For example, we might, in our own use of the term ‘conscientiousness’, count both orderliness and a sense of responsibility for others as aspects of this ‘trait’. But do these two phenomena really occur together in individuals more often than not?

Note that this does not follow even if we assume that ‘conscientiousness’ is a suitable name for one of the five principal factors found in those original studies. What this assumption and the definition of factor analysis tell us is that several of the traits originally tested for are highly correlated with *some* aspect of conscientiousness. It does not tell us that all its aspects we might think of when designing tests such as [35] must be highly correlated with each other.

Ideally, we would like to start with a positive answer to the following question:

(2) Are the five OCEAN traits extensionally verifiable *as traits*, in some objective sense that does not depend on our own, culturally determined, notions associated with those five words?

In other words, can we ground our interpretative notions of the five groups of personality aspects named as OCEAN in quantitatively verifiable data that confirms them as genuine traits, whether or not these correspond exactly to the original Five Factors as stipulated in (1)?

Note that no experiment can be expected to prove assertions such as ‘these ten questions are well suited to test for openness’ (say). This is because the only way to define a comprehensive trait such as openness rigorously at the data level is by what mathematicians would call a *class*: to name some explicitly chosen set of ‘markers’ known to be highly correlated with each other, and decree that whatever they have in common (a postulated entity thought to ‘explain’ their high correlation) shall be called 'openness'. But then this set of markers could be tested for directly – which would make its justification tautological and offer no added value.

However, one *can* develop criteria for the following generalisation of (2), which can also be applied directly to any concrete personality test:

(3) To what extent does a given set of questions form markers of *some* trait?

Such criteria are suggested in section 4.1.2. Their validation involves clustering.

Buchanan, Johnson and Goldberg [37] offer evidence of a positive answer to (3) for a test similar to [35]. This evidence is based on factor analysis run on the answers they obtained for questions designed to test for the OCEAN traits. While factor analysis can, in principle, be used for clustering, this is not its brief or even its forte. We shall argue in section 4.1.3 that clustering by factor analysis is too narrowly defined to meet our criteria for (3) from section 4.1.2. If one accepts our criticism, a consequence will be that the results from [37] do not sufficiently support a positive answer to (3).

Using a new clustering method developed specifically for 'fuzzy' datasets as are common in the social sciences, clustering by *tangles* [3], we offer here an analysis more in line with our criteria for (3) of the test results obtained in [38, 39] for the IPIP questions [36] developed to test the FFM markers of Goldberg [34]. We give a preview of this in section 4.2.

Our analysis broadly confirms that the 50 questions asked in [36] correspond to five traits, in a sense that meets our criteria from section 4.1.2. Interestingly, though, these traits are not visible at the same level of 'resolution' of our clustering tool of tangles: some already emerge at a coarse resolution at which others cannot yet be distinguished, while at a resolution where these *can* be distinguished some of the former traits are no longer visible as clusters, but have disintegrated into single questions. *Our analysis, thus, finds more structure* than just five clusters: structure between the five expected traits, but also between them and their subtraits and more comprehensive traits.

We have found this structure to be robust across different, indeed disjoint, sets of participants whose answers to the questions in [36] we evaluated. But our results are limited, so far, to the data published in [38, 39]. It would be interesting to see if they can be replicated with results obtained in other personality tests developed for the 'big five' paradigm.

Tangles have their origin in structural discrete mathematics, in particular in graph theory. The use of tangles in data analysis, including for the social sciences, is new and treated in depth in [3]. An easy-to-read introduction can be found in [40], and we outline what we need here in section 4.3. Tangle software is freely available via [4].

### 4.1.1 From factors to traits: information loss or shift through interpretation in pursuit of challenge (1)

The FFM has its origin in the fact that several studies found five main factors in personality tests each testing for some larger set of traits, different sets for these original studies. Formally, this precludes any notion that these are 'the same' five factors. However, they are said [32] to have similar *interpretations*: those summarised, e.g., as 'OCEAN'. Moreover, only this shift from the data level to that of interpretation enables us to talk about (five) *traits* at all, rather than about intuitively inaccessible mathematical entities constructed to formally 'explain' (linearly combine to) various other traits.

When FFM-based personality tests such as [35] are developed, however, these interpretations necessarily take on a life of their own: the questions designed to test for the 'big five traits' target whatever our intuitive ideas for these traits encompass. In particular, they do not target the entirely abstract five factors that Tupes and Christal [29] found in Cattell's test for 35 other traits.

While this is entirely legitimate, and in practical terms unavoidable, it does involve a double jump: one from the original factors to their interpretations, and another from those interpretations to the inventories designed to test them. Any claim that these inventories test for the original Five Factors, therefore, is a hypothesis that needs validation.

In the case of Tupes and Christal's five factors this would require a study in which, for example, the 50-question inventory from Goldberg [34] was tested alongside Cattell's original 35 traits. The set of 10 questions targeting a particular one of the five traits, $t$ say, would then have to be shown to be highly correlated to one of the principal five factors found for Cattell's 35 traits in *this* study, which in turn would have to be shown to be similar in terms of the Cattell questions to the factor found by Tupes and Christal which $t$ interprets.

However, this is not what happened. Instead, Goldberg [34] subjected the returns for his 50 questions to another run of factor analysis. As expected, he found five factors: one for each set of 10 questions targeting a trait, and highly correlated to these 10 questions. This does not amount to a positive answer to our question (1) – only, at best, to a positive answer to our question (2) or (3).

We look into this further in section 4.1.3. We shall argue that factor analysis, while being the well-established tool of choice for *finding* hitherto untargeted fundamental traits in terms of which others can be explained, is not the most powerful tool available for evaluating whether inventories of questions designed to test for these traits live up to their brief. Our aim in this paper is to offer an additional tool for this purpose.

### 4.1.2 What is a trait? Two extensional criteria for (2) and (3)

Our question (2) asks whether the five FFM traits are 'extensionally verifiable': whether the groupings of personality aspects implicit in our semantic notion of each of the five terms defining these traits are borne out in reality.

For example, there may be reasons for us to subsume certain distinct personality aspects in our notion of agreeableness that do not, in fact, occur more often together in an individual than other aspects. In such a case 'agreeableness', while being a convenient term for us to refer to such a *combination* of personality aspects, would not name a 'trait' – at least none fundamental enough to serve as one of five pillars on which to build comprehensive personality assessments.

The work reported in this paper is based on the assumption that there is a consensus that a *trait* should satisfy the two formal requirements outlined below.In section 4.1.3 we describe a method which, we shall argue, can validate these two criteria better than what appears to be the current standard method.

Consider a term $t$ in our language for which we wish to decide whether or not we should call it a 'trait'. To decide this, we might devise a questionnaire $Q_t$ each of whose questions targets one aspect of $t$ included in our informal notion for $t$, so that all these aspects are represented in $Q_t$.

Our first requirement on $t$ for qualifying as a trait is what we might call *extensional cohesion*: that the answers received from any sufficiently diverse population for any such set $Q_t$ of questions are highly correlated,[1] statistically or in some other relevant sense.[2] Only if they are can the answers to the questions in $Q_t$ given by a concrete person be treated as measuring the same thing – the 'trait' $t$. However, such cohesion alone is not enough to warrant this.

Our second requirement on $t$ is what we might call *extensional completeness*. This is defined only when $Q_t$ is one of several sets of questions in some comprehensive questionnaire $Q$, such as the 'big five' inventories in [34] and [36]. It stipulates that the answers received for $Q_t$ shall *not* be highly correlated to those received for any set $Q' \subseteq Q \setminus Q_t$, such as a similar set $Q_{t'}$ devised to test for a trait $t'$ other than $t$. For if they were, and both $Q_t$ and $Q_{t'}$ (say) had already been shown to be extensionally cohesive, then $t$ and $t'$ would merely be two manifestations of some more comprehensive trait, testable by $Q_t \cup Q_{t'}$ but perhaps not yet understood in interpretation terms. This would raise legitimate concerns about any view of $t$ and $t'$ as different personality traits: it might still be convenient for us, but would not be borne out extensionally as envisaged in (2).

In this paper we report on our analysis of the data in [38, 39], where $Q$ is the set of questions from [36]. This is composed of five subsets $Q_t$, one for each of the five terms $t$ in OCEAN. We found that each of these $Q_t$ is both extensionally cohesive and extensionally complete in $Q$ at *some* level of 'resolution' of our tools (see section 4.2.2). But we also found that this does not happen at the same level: there is no level of resolution for our tangle-based validation tool at which all the five sets $Q_t$ simultaneously satisfy the above two requirements on traits.

### 4.1.3 Traits as clusters: a generic paradigm for (3)

Our two requirements from section 4.1.2 for sets of variables, in our case questions, to qualify for being regarded as capturing some common 'trait' as envisaged in (3) are best described in clustering terms.

Let us view the set $Q$ of all the questions included in a given FFM test as a 'space',[3] each question $q \in Q$ being a point in this space. If we interpret high correlation between two questions as close proximity in this space, the extensional cohesion and completeness required of each of the five subsets $Q_t$ of $Q$, one for each trait $t$, translate to the requirement that these five sets $Q_t$ *should form distinguishable clusters in the space* $Q$. For example, a subset of $Q$ found to be extensionally incohesive could not be a cluster: it might be a union of several clusters not hanging together, or even of proper subsets of such clusters. An extensionally cohesive but incomplete subset of $Q$ might form only a partial cluster, or there might be no clusters in $Q$ at all.

Buchanan, Johnson and Goldberg [37] and Goldberg [34], offer validations of their FFM test inventories, which are similar to those we investigated [38, 39]. These can

[1]The correlation may be positive or negative. Since positive and negative correlations between two questions can be converted into each other by inverting one of these questions, and positive correlation is transitive if it is high enough, we can make all correlations in a group of pairwise highly correlated questions positive simply by rephrasing some of the questions.

[2]Tangles in datasets are defined in terms of set partitions rather than pairs of data points. This enables us to apply more comprehensive notions of correlation than statistical correlation of pairs of questions, such as that of 'mutual information' from discrete entropy.

[3]A metric space, for example – we do not need a vector space. Formally, all we need is a set with a positive real-valued function on its pairs.

be described in clustering terms, too. In their studies, as well as in [38] and [39], there is an inventory $Q$ consisting of five sets $Q_t$ of ten questions, one for each of the five terms $t$ in OCEAN. In order to validate their test, they ran factor analysis on the returns they received. They found five main factors, one factor $f_t$ for each of the five sets $Q_t$, and note that the questions in $Q_t$ are highly correlated to $f_t$ but not to any other $f_{t'}$. They interpret this as validation of both their test and, presumably, of the FFM itself in the sense of our question (1).

This should then imply a positive answer also to our weaker question (3), and indeed to an extent it does. Expressed in clustering terms, their validation shows that the answers received for their tests form clusters around the five factors they found, viewed as virtual centre points: clusters with respect to correlation interpreted as proximity, as earlier.

However, this confirmation of (3) is not the strongest possible which their data might offer. This is because factor analysis, when used for clustering as here, produces clusters subject to constraints that are not needed: it usually requires the centre points of the clusters to be orthogonal when viewed as vectors in the appropriate inner product space, and the centre points of clusters are chosen so as to maximise proximity not just to the points in their corresponding cluster, but to *all* the data points (subject to being orthogonal). This is mitigated somewhat by an application of *varimax* to the five factors originally found, but remains an unnecessary constraint from a pure clustering perspective.

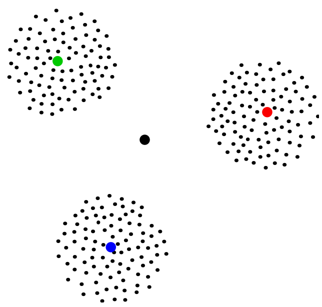

**Figure 4.1:** Factor analysis, used for clustering of variables with respect to correlation as proximity, might find the black dot as a cluster centre.

fig. 4.1 shows an extreme example to illustrate the latter point. The small dots indicate the variables to be clustered. Two dots are close in the figure if they are highly correlated as variables. The central black dot indicates a combined variable that factor analysis might find: it is computed so as to maximise its average correlation to *all* the variables to be clustered, so the black dot is placed so as to minimise its average distance from the small dots. Standard clustering would instead find the three coloured dots as cluster centres: they each maximise the average correlation only to the group of variables in their respective cluster.

More generally, recasting our two criteria from section 4.1.2 in clustering terms indicates that statistical correlation – even without the two unnecessary constraints just mentioned – is not the only possible basis for their quantification: it is one way to measure the similarity of variables (questions), but not the only one. Even more generally, there may be – and indeed are – ways of defining and detecting clusters that are not just based on the similarity of *pairs* of points.

Tangles can detect, and quantify, clusters defined in more general ways than by similarity of pairs of points, let alone just by statistical correlation. Moreover, they unify all standard clustering methods by extracting something like their common structural essence.

### 4.1.4 Testing for our two criteria: from clustering to tangles

In section 4.1.3 we argued that generic clustering may offer a way of validating FFM tests that is not subject to some constraints inherent in using factor analysis for this purpose. In this section we make the case that, among the generic clustering methods available, tangle-based clustering may be a particularly good choice to detect clusters meeting our criteria from section 4.1.2 in particularly 'fuzzy' environments such as personality data.

Clustering, traditionally, seeks to divide a set of data points into subsets of points that are particularly tightly knit together. Sometimes, in *soft clustering*, points are allowed to spread their 'membership' over more than one cluster. But there is no generally accepted definition of 'cluster'. This appears to be unavoidable, because the task to decide for every single data point which, if any, cluster it should be assigned to, is simply too much to ask of real-world data.

Tangles are designed to overcome this problem. They deliberately refrain from attempting to define a cluster as a set of points. Instead, they merely point to roughly where in the data space clusters can be found, and say something about their relative structure – such as whether some large but loose cluster includes one or more smaller dense ones. The gain from ignoring the precise (but error-prone and partly random) information about individual points given in the data, which yields only fuzzy information about cluster-like sets of such points, is that the coarser information retained is precise, robust, and can be examined mathematically. This is what tangles do: *tangles are a precise way of capturing intrinsically fuzzy data* [3]. This applies in particular when the data in question consists of answers to an FFM questionnaire.

The precise but coarser information from the data of [38, 39] that is encoded in tangles will be enough to answer our question (3): we do not need to come up with concrete subsets of $Q$, possibly but not necessarily the sets $Q_t$, to prove the extensional cohesion and completeness of the five corresponding traits. And moreover, we obtain additional information on how these traits are related: how they emerge from more comprehensive traits, how they split into subtraits, and all this with quantitative information about the level of precision, or 'resolution',[4] at which this happens.

## 4.2 Data, objectives, and setup

Our aim was to validate, and possibly refine, the Five Factor Model (FFM) [28] by analysing the returns received in [38, 39] for the IPIP questions [36] developed to test the FFM markers developed by Goldberg [34]. These questions were designed specifically to test for the five traits commonly abbreviated as OCEAN, ten questions for each of these traits. We applied tangle-based clustering to the set of these questions based only on the answers received, that is, ignoring the information of which trait each question was designed to test.

[4]This will be encoded as the *order* of the corresponding tangles.

### 4.2.1 Clusters and traits: the expected outcome

The anticipated result was that, at the most basic level, we would recover five clusters in this set of questions: the clusters corresponding to, in traditional clustering terms, the five subsets of the questions designed to test for the five traits.

However, tangle-based clustering can reveal more detailed information than just detecting a certain number of clusters. It can relate these to each other in structural ways which, when we interpret them here, may reveal how those five traits are related: how some of them can be distinguished less clearly than others, and hence form more comprehensive traits at a coarser level, and perhaps split into more detailed traits at a finer level. These levels come on an absolute scale, the *order* of the tangles found, so that these developments can be compared across the various traits and subtraits as they emerge when the data is analysed at increasing 'resolution'. In particular, the five target traits, if recovered, would have their place in any such evolutionary hierarchy of traits that may be factually observable in [38] and [39].

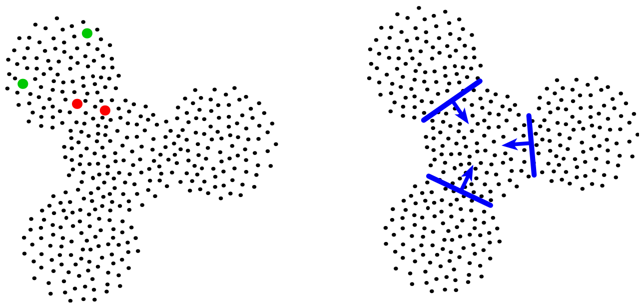

**Figure 4.2:** Four clusters and three bottlenecks

### 4.2.2 Clustering by tangles

Consider a dataset as depicted in fig. 4.2. Intuitively, there are four clusters: a central one, and three around it. However as soon as we try to come up with a rigorous explicit definition of 'cluster' that underpins this intuition, we get into trouble. For example, if we say that a *cluster* is a set of pairwise close points, or a maximal such set, we can easily find pairs of points that invalidate this attempt: points that are close to each other but which lie in different point clouds that form our intuitive clusters, such as the two red dots in the picture, or points that lie in the same intuitive cluster but far from each other, such as the two green dots.

Clustering by tangles sidesteps these issues by taking a radically different, indirect approach. On the right in the picture we can see some natural ways to split the dataset in two, splits that do not cut right through any of the visible clusters but skirt around them. We call such natural partitions the *bottlenecks* of the dataset. For every bottleneck, every visible cluster lies mostly on one of its two sides: otherwise it would not be a bottleneck. In this way, each cluster *orients* every bottleneck towards the side that contains most of it. The figure shows this for the central cluster by little arrows on the three bottlenecks.

This information, the simultaneous orientation of each bottleneck towards one of its sides, is called a *tangle*; it is all that we wish to remember about each of those four clusters. Note that this information is robust in a fuzzy environment such as fig. 4.2: the way in which each of the four clusters orients those three bottlenecks will not change if we draw the bottlenecks slightly differently or move a few of our data points.

Note, however, that we are still far from a general definition of tangles that can replace the more traditional attempts at defining clusters. This is because our definition of the four tangles in the figure depended on a pre-conceived, if intuitive, traditional notion of cluster: we defined 'bottlenecks' as splits of our dataset that leave its clusters largely intact, and we defined 'tangles' as simultaneous orientations of all the bottlenecks towards some cluster.

The crucial starting observation in applied tangle theory is that one can define both 'bottlenecks' and 'tangles' axiomatically, without reference to clusters. One can then either extract point clusters from those tangles – or choose not to, if the information about the dataset sought from determining its clusters can be derived from the tangles directly, as will be the case in our application here.

Although tangles contain less detailed information than traditional point clusters, they offer additional insight. One is that they come at different levels: if we allow only few, particularly narrow, bottlenecks we get few tangles that hang together loosely. If we orient wider bottlenecks too, we get more and denser tangles. These will *refine* the earlier loose ones in that they orient the same bottlenecks as those do, but some wider ones in addition.

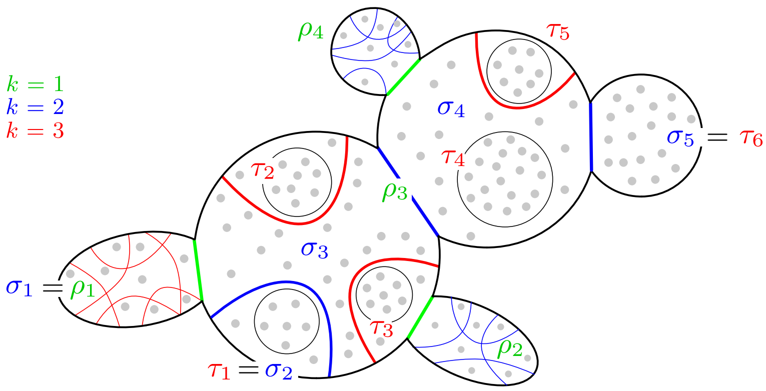


**Figure 4.3**: A hierarchy of tangles of increasing order $k = 1, 2, 3$

fig. 4.3, from [3], shows tangles of increasing *order*, or bottleneck width. There are four tangles, $\rho_1, \ldots, \rho_4$, of order $k = 1$. Adding slightly wider bottlenecks to find the tangles of order 2, we see that the tangle $\rho_1$ of order 1 remains unchanged, the tangles $\rho_2$ and $\rho_4$ disintegrate, and the tangle $\rho_3$ splits into the four tangles $\sigma_2, \ldots, \sigma_5$. When we allow even wider bottlenecks for tangles of order 3, we find that the tangle $\sigma_1 = \rho_1$ disintegrates, the tangles $\sigma_2$ and $\sigma_5$ remain unchanged, and the tangles $\sigma_3$ and $\sigma_4$ of order 2 each split into two tangles of order 3, into $\tau_2, \tau_3$ and $\tau_4, \tau_5$ respectively.

The lowest order at which a certain tangle first appears, where it arises as one of several tangles of that order which refine the same tangle of lower order, is known as its *complexity*; the highest order up to which this tangle remains unchanged is its *cohesion*, and the difference between the two (plus 1) is its *visibility*. In the example of fig. 4.3, the tangle $\sigma_1$ of order 2 has complexity 1, because it equals (is the unique extension to an order-two tangle of) the tangle $\rho_1$ of order 1 which refines no tangle of lower order still; it has cohesion 2, because it is still a tangle of order 2 but no longer one of order 3; and so it has visibility $2 = (2 - 1) + 1$. The tangle $\sigma_4$ of order 2 has complexity 2, because it is not the only order-two tangle that refines the tangle $\rho_3$ of order 1 (as $\sigma_2$, $\sigma_3$, $\sigma_5$ do too); it also has cohesion only 2, since $\tau_4$ and $\tau_5$ are two tangles of order 3 that both refine it; and so it has visibility $(2 - 2) + 1 = 1$. See [3Ch.7.4, 14.6] for details.

Another advantage over traditional clustering is that the latter often requires us to specify in advance how many clusters we seek to detect. Tangles reflect whatever clusters the dataset contains. We can set a number of parameters, such as the tangles' required agreement value, which will influence this in general. But once that is done, the number of tangles of any given order found will depend on the data only, not on any choice of ours.

Most importantly, tangles are robust when the data is inherently fuzzy, such as survey data in psychology, sociology, or economics.

### 4.2.3 Tangles and traits

When the points in our dataset of section 4.2.2 are questions about personality, narrow bottlenecks will divide questions whose answer sets are only loosely correlated, in some well-defined sense of our choice.[5] As the width of the bottlenecks considered increases, these bottlenecks (and hence their tangles) can distinguish also between more highly correlated questions. A tangle will 'disintegrate' when the bottlenecks allowed become so wide that they cease to be true 'bottlenecks' but dissect the set of questions into singleton subsets (as indicated in fig. 4.3): when a question is considered as 'highly correlated' only to itself.

In our context it will be possible, in principle,[6] to interpret each of the tangles of our set $Q$ of 50 questions found in the data of [38, 39] as related to one particular personality aspect. These different aspects will be more fundamentally different when their corresponding tangles have low order, because they will be distinguished by splits that cannot separate highly correlated questions. They will be more specific when their corresponding tangles have higher order.

We can then formalise traits as equivalence classes of tangles of different order, where a tangle $\tau$ of order $k$ is *equivalent* to a tangle $\tau'$ of order $k' \geq k$ if, essentially, $\tau'$ is the unique extension of $\tau$ to a tangle of order $k'$.[7] A *trait*, then, formalised as such an equivalence class of tangles, can be thought of as a single tangle that 'is born' at some order $k$ and 'dies' after some maximum order $k' \geq k$.

In section 4.2.2 we called the minimum and the maximum $k$ for which 'a given tangle' is a $k$-tangle its 'complexity' and 'cohesion'. We can now define this more precisely: the *complexity* and the *cohesion* of a trait – an equivalence class of tangles – are the smallest and the largest $k$ for which this trait (class) contains a tangle of order $k$. As before, the *visibility* of a trait is the difference between its cohesion and its complexity plus 1.

Inasmuch as (equivalence classes of) tangles of partitions of a set $Q$ of questions formalise the informal notion of clusters in $Q$, our formal notions of complexity and cohesion for traits formalise what in section 4.1.2 we called the desired external completeness and cohesion of clusters in $Q$. They do so in a precise, quantitative, way which can be computed from the data of answer sets received for $Q$. And they give

[5] As mentioned earlier, the similarity measure between different questions we shall use will mostly be that of mutual information in the sense of information theory, not statistical correlation.

[6] Such interpretations need not be readily expressible in traditional terms. Indeed, one of the main advantages of tangles is that they enable us to detect hitherto unknown patterns: patterns we do not have to first guess and then confirm, the standard process in the social sciences. But tangles, once found, are immediately accessible to interpretation by experts; see below.

[7] The precise definition is a little narrower: a tangle $\tau$ of order $k$ is *equivalent* to a tangle $\tau'$ of order $k' \geq k$ if every tangle of order $\ell$ with $k \leq \ell \leq k'$ that induces $\tau$ is in turn induced by $\tau'$; see [3Ch.14.6].

rise to the new formal notion of visibility, which may be an interesting new parameter for traits found in personality tests.

## 4.3 Methods 1: Tangle basics

For this section let $Q$ be an arbitrary finite set.

### 4.3.1 Partitions

**Definition 4.3.1.** (partition and sides)
Let $A, B \subseteq Q$ be two subsets of $Q$. If $A$ and $B$ are disjoint and $A \cup B = Q$, then $\{A, B\}$ is called a *partition of* $Q$. The sets $A$ and $B$ are called the *sides* of the partition.

We denote partitions either as a set, like $\{A, B\}$, or by single letters such as $r$ or $s$. We denote the two sides of a partition $s$ by $\vec{s}$ and $\overleftarrow{s}$. The directions of these arrows are not fixed in advance. For example, if $s = \{A, B\}$ then $A$ could be denoted by either $\overleftarrow{s}$ or $\vec{s}$. But once we have chosen $A$ to be denoted as $\vec{s}$, say, the other side $B$ of this partition will implicitly have been denoted as $\overleftarrow{s}$.

Given a set $S$ of partitions of $Q$, we denote the set of all their sides as $\vec{S}$. Note that for every $\vec{s} \in \vec{S}$ also $\overleftarrow{s}$ lies in $\vec{S}$, since it is also a side of $s \in S$. If $s$ is a partition of $Q$ and $q \in Q$, we write $q(s)$ for the side of $s$ that contains $q$.

Two sets are commonly called *nested* if one contains the other. We call two partitions *nested* if they have nested sides.

### 4.3.2 Order functions and similarity functions

**Definition 4.3.2.** (order of partition)
Given a set $S$ of partitions of $Q$ and an $S \to \mathbb{R}$ function $s \mapsto |s|$ that we have called an 'order function', we call $|s|$ the *order* of $s$, for any $s \in S$.

All order functions considered in this paper will take non-negative values. They will also all be injective: not by definition, but because we shall tweak the data so as to make them injective. This makes our computations easier, because it spares us having to make arbitrary choices, but it has no material consequences.

Our aim in choosing order functions will be to assign low order to particularly natural partitions of $Q$, partitions we called 'bottlenecks' in section 4.2.

One way of defining order functions without reference to any informal notion of cluster is to assign low order to partitions that split few pairs of points we regard as 'similar'. Functions $Q^2 \to \mathbb{R}$ intended to express this are called *similarity functions* on $Q$. One simple choice of an order function on the partitions of $Q$, then, is to sum up the similarities of the pairs of points split by a partition:

**Definition 4.3.3.** (cut weight order)
Let $w : Q^2 \to \mathbb{R}$ be a similarity function on $Q$, and let $\{A, B\}$ be a partition of $Q$. The *cut weight* of $\{A, B\}$ is defined as

$$\operatorname{cut}(A, B) := \sum_{a \in A,\, b \in B} w(a, b).$$

While the cut weight order function is natural, it tends to assign higher order to more balanced partitions, because these split more pairs. For example, if the sides $A, B$ of a partition of $Q$ are about equally large, then the number of pairs $(a, b)$ whose weights $w(a, b)$ are summed in the definition of *cut* is quadratic in $|Q|$, about $(|Q|/2)^2$, while if the partition is unbalanced in that it divides only some $k$ points of $Q$ from the rest, this number is linear in $|Q|$, about $k \cdot |Q|$. Thus, no matter what the weights $w(a, b)$ are, the value of $cut(A, B)$ tends to be larger when $\{A, B\}$ is more balanced.

This is a problem, since we do not want our particularly natural, and hence low-order, partitions to be cluttered up by unbalanced but unnatural partitions. We therefore try to correct the bias of the order function *cut* towards unbalanced partitions. For example, we often take as the order of $\{A, B\}$ not the sum of the weights $w(a, b)$, as above, but something more like their average:

**Definition 4.3.4.** (ratio cut weight order)
Given a similarity function on $Q$, the *ratio cut weight* of a partition $\{A, B\}$ of $Q$ is defined as

$$\mathrm{Rcut}\,(A, B) := \frac{\mathrm{cut}(A, B)}{|A||B|}|Q| = \mathrm{cut}\,(A, B)\left(\frac{1}{|A|} + \frac{1}{|B|}\right).$$

There are many other commonly used order functions; see [3]. Our order function of choice will be that of ratio cut weight, based primarily on a similarity function called ‘mutual information’ borrowed from information theory. But we tested also the ‘cosine’ similarity function. Both these are defined in section 4.4.2.

### 4.3.3 Tangles: basic notion, types, and hierarchy

Let $S$ be a set of partitions of $Q$. The few graph-theoretic terms used in this section are explained in [41].

**Definition 4.3.5.** (orientation of set of partitions)
A set $\sigma \subseteq \vec{S}$ which, for each $s \in S$, contains exactly one side of $s$ is called an *orientation* of $S$. The side of $s$ that lies in $\sigma$ is denoted by $\sigma(s)$.

fig. 4.2 indicates an orientation, by arrows, of the set of three ‘bottleneck’ partitions shown in blue. As in that example, orientations of partitions can be defined by subsets of $Q$ that lie mostly on the same side of any such partition. We say that an orientation $\sigma$ of $S$ is *guided* by any subset $X$ of $Q$ that satisfies

$$|X \cap \vec{s}| > |X \cap \overleftarrow{s}|$$

for every $\vec{s} \in \sigma$. An orientation $\sigma$ of $S$ can have more than one guiding set, and it can also have none.

**Example 4.3.1.** *For every $q \in Q$ there is a unique orientation of $S$ that is guided by $\{q\}$. We call it the* principal *orientation of $S$ defined by $q$.*

As explained in section 4.2.2, the idea behind tangles is that while clusters, conceived of as subsets $X$ of $Q$, are notoriously difficult to pin down precisely, they will – whatever their precise definition if any – guide the same orientations of the ‘bottleneck’ partitions of $Q$ if these are narrow enough. For example, if more than two thirds of $X$ lie on the same side of any $s \in S$, the orientation $\sigma$ of $S$ guided by $X$ will not change if we modify $X$ a little, perhaps by using a competing definition of point cluster, or because our data is not perfectly reliable.

In this example, the intersection of any three sets in $\sigma$ contains an element of $X$, since each of them misses less than a third of $X$. This motivates the following formal definition of tangles:

**Definition 4.3.6.** (tangle)
An orientation $\tau$ of $S$ is a *tangle* of $S$ if $\vec{r} \cap \vec{s} \cap \vec{t} \neq \emptyset$ for all triples $\vec{r}, \vec{s}, \vec{t}$ of elements of $\tau$.

Note that this definition, while motivated by the properties of a subset $X$ of $Q$ that we might think of as a point cluster, no longer depends on such an informal notion of 'cluster': the point it requires to lie in the intersection of any three[8] elements of a tangle can be any point in $Q$ and does not have to come from any previously defined 'cluster' $X$.

Due to this difference, a set of partitions of a dataset $Q$ can have tangles that are not guided by any obvious subsets of $Q$: tangles are more general than orientations of set partitions guided by 'clusters' in that set, however defined.

In order to reduce the number of tangles of $S$, we often require that its triples contain not just one but some specified number of elements of $Q$:

**Definition 4.3.7.** (agreement value)
The *agreement value* $a(\tau)$ of a tangle $\tau$ of $S$ is the minimum number of elements of $Q$ that lie in the intersection of any three sets in $\tau$.

Thus, if every three sides of partitions of $S$ that lie in a given tangle $\tau$ have at least $n$ elements of $Q$ in common, the agreement value of $\tau$ is at least $n$.

Recall that every $q \in Q$ defines a map $s \mapsto q(s)$ from $S$ to $\vec{S}$, just as every tangle $\tau$ of $S$ defines an $S \to \vec{S}$ map $s \mapsto \tau(s)$. We think of $q$ as being similar to $\tau$ if these two functions agree on many $s \in S$:

**Definition 4.3.8.** (point-tangle similarity)
Given a tangle $\tau$ of $S$ and any $q \in Q$, we call the number

$$\sigma(q, \tau) := \left|\{\, s \in S \,:\, q(s) = \tau(s) \,\}\right|$$

the *similarity* between $q$ and $\tau$.

**Definition 4.3.9.** ($S_k$; $k$-tangles of $S$; tangles in $\vec{S}$; order of a tangle; $T(S)$)
Let $s \mapsto |s| \in \mathbb{R}$ be an order function on $S$. For every integer $k$ we write

$$S_k := \{s \in S \,:\, |s| < k\}$$

and call the tangles of $S_k$ the *$k$-tangles of $S$* or the *tangles of order $k$ in $\vec{S}$*. We write $T(S)$ for the set of all tangles in $\vec{S}$, irrespective of their order.

Note that the $k$-tangles of $S$ are not tangles of $S$ (unless $k$ is very large), since they only orient its subset $S_k$. For $k < k'$, every $k'$-tangle $\tau'$ of $S$ defines (or *induces*, or *refines*) a $k$-tangle of $S$, the tangle $\tau := \tau' \cap \vec{S_k}$ of $S_k$. Note that $\tau \subseteq \tau'$ in this case. Conversely, given any two tangles $\tau \subseteq \tau'$ in $\vec{S}$, there will be integers $k \leq k'$ such that $\tau'$ is a $k'$-tangle of $S$ and $\tau = \tau' \cap \vec{S_k}$ a $k$-tangle.

[8] There is some magic in this number, which is discussed at length in [3Section 7.3].

**Definition 4.3.10.** (tangle equivalence)
A $k$-tangle $\tau$ of $S$ is called *equivalent* to a $k'$-tangle $\tau'$ of $S$ with $k \leq k'$ if every $\ell$-tangle of $S$ with $k \leq \ell \leq k'$ that induces $\tau$ is in turn induced by $\tau'$.

It is not hard to prove, but also not entirely obvious, that this is indeed an equivalence relation on the set of tangles in $\vec{S}$. Note that refinement is well defined on equivalence classes: if a tangle $\tau$ refines a tangle $\sigma$ not equivalent to $\tau$, then any tangle $\tau'$ equivalent to $\tau$ refines any tangle $\sigma'$ equivalent to $\sigma$.

**Definition 4.3.11.** (complexity, cohesion, visibility)
The *complexity* and the *cohesion* of an equivalence class of tangles in $\vec{S}$ are the smallest and the largest order of a tangle in that class, respectively. The *visibility* of the class is its cohesion minus its complexity plus 1.

**Definition 4.3.12.** (tangle search tree)
The *tangle search tree* of $S$ is the graph with node set $T(S)$ in which two nodes $\rho, \tau$ are adjacent if $\rho \subsetneq \tau$ and there is no tangle $\sigma$ in $\vec{S}$ such that $\rho \subsetneq \sigma \subsetneq \tau$. We often write $T(S)$ also for this tree, not just for its set of nodes.

The tangle search tree is indeed a tree. Since $S_k = \emptyset$ for sufficiently small $k$, and $\emptyset$ is a $k$-tangle for this $k$, the empty set $\emptyset$ is a node of $T(S)$, which we take as its root. If our order function is injective, as we shall make it throughout, then every child of a tangle in the tangle search tree orients exactly one more partition in $S$ than its parent, so every node has at most two children. The tangle search tree has a non-root level for every integer $k$ that occurs as an order of a partition in $S$. If these are all the integers between 1 and some $K$, the tangles at level $k = 1, \dots, K$ are the $(k+1)$-tangles of $S$, those that orient every partition in $S$ of order up to $k$.

Note that two tangles in $T(S)$ are equivalent if and only if the path that links them in the tangle search tree is vertical[9] and has no inner vertices at which the tree branches: in graph-theoretical terms, its inner vertices all have degree 2. Contracting the maximal such paths in the tree $T(S)$ to single edges between their ends, and deleting the root $\emptyset$ if it has only one child, turns $T(S)$ into a *binary* tree $\hat{T}(S)$, one whose non-leaf nodes[10] all have exactly two children. These nodes correspond to the equivalence classes of tangles defined earlier.

**Definition 4.3.13.** (tree of traits)
In contexts when the equivalence classes of tangles in $\vec{S}$ (other than possibly $\{\emptyset\}$) are called 'traits', the tree on $\hat{T}(S)$ will be called the *tree of traits.* We denote this tree by $\hat{T}(S)$ too.

For every pair of tangles in $\vec{S}$ of which neither induces (i.e., is a superset of) the other there exists an $s \in S$ which these two tangles orient differently. We say that $s$ *distinguishes* these two tangles. If $s$ has lowest order among all the partitions in $S$ that distinguish the two tangles, it distinguishes them *efficiently*.

**Definition 4.3.14.** (efficient distinguisher of incomparable tangles)
If the order function on $S$ is injective, then any $s \in S$ that distinguishes two tangles in $\vec{S}$ efficiently is unique. We call it their *efficient distinguisher.*

[9]Formally: ascending or descending in the tree order on $T(S)$ associated with the root $\emptyset$.
[10]The root of a tree never counts as a leaf [41].

## 4.4 Methods 2: Tangle particulars for our analysis

Our aim in this paper is to determine, and analyse, tangles of partitions of the set $Q$ of questions used in the FFM test in [38, 39]. We shall always denote as $I$ the set of individuals on whose answers to these questions we base our tangles. This set $I$ will vary as we compute and test our tangles, as will the set $S$ of partitions of $Q$ whose tangles we compute. But $Q$ will always be the same. Thus, we shall study the tangles in $\vec{S}$ for a suitable set $S$ of partitions of $Q$, based on the answers to $Q$ given by various sets $I$ of people.

Before we describe more specifically how we chose the parameters giving rise to these tangles, let us formalise the notion of 'trait' as envisaged in section 4.2.3, now rigorously on the basis of definition 4.3.10:

**Definition 4.4.1.** (tangle trait)
A *(tangle) trait* is an equivalence class of tangles in $\vec{S}$ other than $\{\emptyset\}$.

Note that $\{\emptyset\}$ may or may not form an equivalence class in $T(S)$: it does so if and only if the empty tangle $\emptyset$ has two descendents in the tangle search tree.

In order for the term of 'tangles in $\vec{S}$' to be defined, we had to choose an order function on $S$. We used the order function of 'ratio cut weight' introduced in section 4.3.2. This order function requires us to first choose a similarity function on $Q$. We used two different options for this, in order to test our results for stability. These similarity functions are defined in section 4.4.2.

Whichever similarity function on $Q$ we worked with, a key element of our work was that these similarities should be extensionally determined: they should be computable from how the individuals in $I$ answered those questions, regardless of any interpretation, or intention for why those questions were included in $Q$, such as to test for a particular trait. Moreover, our aim was that every individual in $I$ should carry the same weight in determining the similarities between questions in $Q$. Since different individuals have different habits in how they answer *any* question, such as their different levels of assertiveness or positivity, our aim of giving them equal weight required us to factor out those individual habits. We describe how we did that in section 4.4.1. Note that such normalisation must be done with care, since the personal habits we normalise are themselves personality traits like those that $Q$ intends to measure. We also normalised the answers for each particular question to give them mean zero, so that questions suggesting traits generally seen as positive were treated no differently from other questions.

In section 4.4.3 we explain how we chose $S$. In section 4.4.4 we present our tangle search algorithm, slightly simplified; the precise version is described in [4].

### 4.4.1 Normalising the answers received for $Q$

In our data, the answers solicited by the questions in $Q$ were on scale from 1 to 5, with 1 indicating strong disagreement and 5 strong agreement. This makes $Q$ into a set of functions $q: I \to \mathbb{R}$, where $q(i) \in \{1, \dots, 5\}$ records the answer given to $q$ by $i$. Dually, every $i \in I$ defines a function $i: Q \to \mathbb{R}$ that sends each question $q \in Q$ to its answer $i(q)$ given by $i$. Thus, $i(q) = q(i)$ for all $i \in I$ and $q \in Q$ whenever some $I$ has been fixed.

Some individuals tend to agree more than others with questions as they are phrased. This can be seen formally by the fact that the functions $i: Q \to \mathbb{R}$ have different means $\mu_i$ for different $i \in I$. To compensate for this we normalised, for every $i \in I$ separately, their answers to give them mean 0, by subtracting $\mu_i$ from their given answers.

Next, we compensated for different levels of assertiveness amongst different people. The assertiveness of person $i \in I$ can be measured by the standard deviation $\sigma_i$ of the function $i: Q \to \mathbb{R}$. We normalised these by dividing, for each $i \in I$ separately, their answers (after normalising their mean to 0) by $\sigma_i$, to give them all the standard deviation of 1.

In a third step, we did some normalisation between the questions $q : I \to \mathbb{R}$. Some questions have more people agree with them than other questions, perhaps because they are testing for a trait that is generally seen as positive. We compensated for this by subtracting, separately for each question $q \in Q$, from the answers received for $q$ the median of all these answers.

After this last normalisation, every $q \in Q$ had equally many answers $i(q) \leq 0$ as answers $i(q) \geq 0$, counted over all $i \in I$. However, this last normalisation between the questions may have undone some of our earlier normalisation between individuals. We therefore repeated all three steps cyclically until this process converged to functions that satisfied all three of our normalisation goals: to functions $i: Q \to \mathbb{R}$ and $q: I \to \mathbb{R}$ with $i(q) = q(i)$ such that every $i$ has mean 0 and standard deviation 1, and every $q$ has median 0. On the data we investigated, this convergence happened quickly, and only a few rounds of iteration were needed.

From now on we shall use our notation of $q: I \to \mathbb{R}$ for $q \in Q$, and $i: Q \to \mathbb{R}$ for $i \in I$ to refer to these normalised functions computed from their original versions in the data of [38, 39].

### 4.4.2 Similarity functions

Our first similarity function is based on the notion of discrete entropy in information theory, defined below. More background, examples, and a discussion of how this captures similarites can be found in [3Ch. 9.5].

**Definition 4.4.2.** (entropy and mutual information)
The *(discrete) entropy* of a function $f : I \to X$, where $X$ is any finite set, is the number

$$I(f) := \sum_{x \in X} \left(|f^{-1}(x)|/|I|\right) \cdot \log\left(|I|/|f^{-1}(x)|\right).$$

The *product* $(f \times g): I \to X \times Y$ of two functions $f : I \to X$ and $g : I \to Y$ is defined by setting $(f \times g)(i) := (f(i), g(i))$ for all $i \in I$. The *mutual information* of $f : I \to X$ and $g : I \to Y$, then, is defined as

$$\mathbb{I}(f, g) := I(f) + I(g) - I(f \times g).$$

**Definition 4.4.3.** (entropy similarity)
The *entropy similarity* of two questions $p, q \in Q$ is their mutual information $\mathbb{I}(p, q)$ as $I \to \mathbb{R}$ functions.

If we view the set $\mathbb{R}^I$ of all $I \to \mathbb{R}$ functions as a copy of the vector space $\mathbb{R}^n$ with $n = |I|$, we can compare entropy similarity to other similarity functions between pairs of points of $\mathbb{R}^n$. We shall do this with the following similarity measure on $\mathbb{R}^n$, which is based on the cosine of the angle between two vectors:

**Definition 4.4.4.** (cosine similarity)
The *cosine similarity* of two points $p, q \in \mathbb{R}^n$ is defined as

$$c(p,q) := \frac{p \cdot q}{\|p\| \, \|q\|}.$$

Here $\cdot$ denotes the dot product in $\mathbb{R}^n$, while $\|p\|$ and $\|q\|$ denote the lengths of $p$ and $q$, respectively. The *absolute cosine similarity* of $p$ and $q$ is defined as

$$ac(p,q) := |c(p,q)|.$$

### 4.4.3 Choosing partitions

As $|Q| = 50$, there are about $2^{50}$ partitions of $Q$, so we could not include them all in our set $S$ of partitions whose tangles we computed. This set $S$ should contain a diverse selection of low-order partitions, so that our results approximated the tangles of the set of all partitions of $Q$.

We constructed such a set $S$ in three steps, described below. Performing these steps entailed two choices: of one of two matrices, $L$ or $J$ (see below); and of one of the two similarity measures from section 4.4.2. As our order function $s \mapsto |s|$ we always used 'ratio cut weight' from definition 4.3.4.

In the first step we computed an orthonormal set of eigenvectors of a certain matrix, which we then converted into partitions of $Q$. This yielded a set of at most $|Q| = 50$ partitions, typically fewer. We did this for one of two $m \times m$ matrices, $L$ and $J$, where $m := |Q| = 50$. Let the rows and columns of these matrices be indexed not by numbers but by the elements of $Q$ directly.

The matrix $L$ is commonly known as the 'combinatorial Laplacian' of the complete graph on $Q$ with 'edge weights' given by a similarity function $\sigma$ on $Q$. In row $p$ and column $q$ the matrix $L$ has the entry $-\sigma(p,q)$ if $p \neq q$, and entry $\sum_{q' \in Q \setminus \{q\}} \sigma(q,q')$ if $p = q$. See [3Ch. 9.2, 10.3] for more about $L$ and the role of its eigenvectors. The similarity functions we used are those from section 4.4.2.

$L$ has $m$ real eigenvalues $0 = \lambda_0 \leq \cdots \leq \lambda_{m-1}$, counted with multiplicities. If $Q$ had a partition $\{A, B\}$ such that $\sigma(a,b) = 0$ for all $a \in A$ and $b \in B$, we would add it to $S$ straight away and start again with $Q$ replaced by $A$ and, separately, by $B$. This did not in fact happen. The first eigenvalue $\lambda_0 = 0$ of $L$ therefore always had multiplicity 1 (as is well known), so $0 < \lambda_1 \leq \cdots \leq \lambda_{m-1}$. We then picked eigenvectors $v_1, v_2, \ldots$ for the eigenvalues $\lambda_1, \lambda_2, \ldots$ one by one, all orthogonal to both the eigenvector $\mathbb{1}$ for $\lambda_0$ and to each other.

The matrix $J$ has entries $-(p \cdot q)$ in row $p$ and column $q$, for all $p, q \in Q$, where '$\cdot$' once more denotes the dot product in $\mathbb{R}^n$, as in definition 4.4.4. The matrix $J$ has $m$ real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m \leq 0$, counted with multiplicities, whose corresponding orthogonal eigenvectors $v_1, v_2, \ldots$ are the *principal components* of the 'variables' in $Q$ for the data given by $I \subseteq \mathbb{R}^Q$. Note, however, that we are using these principal components not as discussed, and criticised, in section 4.1.3 (see fig. 4.1), but in the dual (and opposite) role: rather than proposing them as centre points of clusters in $Q$, we use them to define natural partitions of $Q$.

The eigenvectors $v_1, v_2, \ldots$ of either $L$ or $J$ were then turned into partitions of $Q$. Given $v_i : Q \to \mathbb{R}$, we chose the partition

$$s_i := \big\{ \{q \in Q : v_i(q) < 0\}, \{q \in Q : v_i(q) \geq 0\} \big\}.$$

Let us call the set $S$ of these partitions $S^0$.

In the second step we iteratively enlarged our current set $S$ of partitions by including, for all the pairs of partitions $\{A, B\}, \{C, D\}$ already in $S$, also their four *corners* in $\vec{S}$. These are the sets

$$A \cap C,\ B \cap C,\ A \cap D,\ B \cap D.$$

The partitions of $Q$ which these sets form with their complements in $Q$ are the four *corner partitions* of $\{A, B\}$ and $\{C, D\}$. Tangle algorithms, see [4], work better when corners of partitions in $S$ are also in $\vec{S}$ if they have low order. Let us call the resulting set $S$ of partitions $S^1 \supseteq S^0$.

We then performed the third step on the partitions in $S^1$. For each $s \in S^1$ we tried to find a single $q \in Q$ such that moving $q$ across the partition $s$ would reduce the order of $s$. If we were able to find such elements $q$, we moved the $q$ across $s$ for which this decreased the order of $s$ the most, adding the modified partition to $S$. We then repeated this with the modified partition, iteratively until no such $q$ could be found. We added all these modified partitions to $S^1$. We took the resulting set of partitions of $Q$ as the set $S$ whose tangles we would finally compute.

Let us take a moment to motivate the above three steps; more background can be found in [3]. Partitions obtained from orthonormal eigenvectors of $L$ and $J$ are particularly useful starting partitions for our construction: they are more balanced than other partitions of low order, in that they divide $Q$ into sides of roughly equal size, and among such balanced partitions they have particularly low order.

Adding corners helps with weeding out 'fake' tangles: tangles of only some of the partitions of $Q$ that do not extend to tangles of all the partitions – those we are trying to approximate.

Partitions whose orders are local minima with respect to moving single elements across include our target partitions, the efficient distinguishers of tangles of agreement value at least 2 of *all* the partitions of $Q$. Since tangles of agreement 2 cannot orient two partitions differently if these differ by a single $q \in Q$ (ie, both towards the moving $q$ or both away from it), the way in which our tangles orient the local minima partitions determines by how they orient our earlier set $S^1$. Thus, only tangles of $S^1$ that extend to tangles of the increased final set $S$ were considered further. This weeds out tangles of $S^1$ that satisfy the tangle axioms 'for the wrong reason' that $S^1$ did not include enough low-order partitions to put the tangle axioms to a sufficiently rigorous test.

### 4.4.4 Finding the tangles

Having constructed one of the sets $S$ of partitions of $Q$ described in section 4.4.3, we constructed all the tangles in $\vec{S}$ of agreement value at least $a = 2$, as follows.

We began by sorting the partitions in $S$ by order, as $|s_0| < |s_1| < \dots$ say. Then for $k = 0, 1, \dots$ in turn we built the set $T_k$ of all tangles of $\vec{S_k}$, where

$$S_k := \{s_i \mid i < k\}.$$

Note that this coincides with our definition 4.3.9 of $S_k$ if $|s_i| = i$ for all $i < k$, which we shall assume to simplify our exposition.[11] Note that $T_0 = \{\emptyset\}$, since the empty tangle is the unique tangle of $S_0 = \emptyset$.

[11] The actual value of an order never matters for tangles: all that matters is the linear ordering which the order function used imposes on $S$.

Having constructed $T_k$ for some $k \geq 0$, we let $T_{k+1}$ consist of all the tangles of the form $\tau \cup \{\vec{s}_k\}$ or $\tau \cup \{\overleftarrow{s}_k\}$ with $\tau \in T_k$. For any given $\tau \in T_k$, sometimes both these extensions of $\tau$ were tangles, or just one of them, or neither: this depended on whether $\vec{s}_k$ or $\overleftarrow{s}_k$ formed a triple with two elements of $\tau$ whose intersection consisted of fewer than $a$ elements of $Q$, in which case adding it to $\tau$ would not yield a tangle of $S_{k+1}$ of agreement value at least $a$.

This process ended only once the last partition in $S$, its element of largest order, was examined. Then $T := T_1 \cup T_2 \cup \ldots$ was our set of all tangles in $\vec{S}$.

## 4.5 Results

In this section we present our findings. In section 4.5.1 we specify the datasets in which we looked for tangles. We state which parameters we used to compute these tangles (see sections 4.4.2 and 4.4.3), and note how many traits they yielded (definition 4.4.1). We finally define what we mean when we use the 'big five' labels of $O, C, E, A, N$ to name some of these traits.

In section 4.5.2 we present the trees of traits we found (definitions 4.3.12 and 4.3.13), analyse their structure, and indicate what this means from an interpretation point of view.

Interpretation of our results is addressed more comprehensively in section 4.5.3. While we have to leave interpretation as such to experts in psychology, we say how exactly such experts can read off the facts they need from our structural analysis in section 4.5.2 and supplementary data listed in the Appendix. One of the fortes of tangle clustering is that it condenses the most crucial information into a very small set of facts that can be stated explicitly at the interpretation level. This is done in the Appendix, in section 4.7.3.

In section 4.5.4 we list, for each of the traits found, which of the 50 questions in $Q$ represent that trait best. In particular, we do this for the $OCEAN$ traits, but also for the other traits we found. This also has a direct impact on interpretation, which is discussed in section 4.5.4 too.

Finally, we address the question of how reliable our results are. This has two aspects, which we call 'robustness of methods' and 'stability of findings'. By *robustness* of our methods we mean invariance of our findings under small changes of the input, while we keep those methods unchanged. By *stability* we mean the converse: the invariance of our findings under small changes of the methods, performed on the same input. These are discussed in sections 4.5.5 and 4.5.6, respectively.

### 4.5.1 The tangle traits we found

We computed, separately for the two datasets in [38] and [39], the tangles in $\vec{S}$ of agreement at least 2. We focus here on reporting our findings for the larger of those two datasets, the data from [39]. We also report more briefly what tangles we found in [38], and discuss the differences (section 4.5.2) and their significance. Both studies were conducted with the same questionnaire $Q$ of 50 questions. When computing their tangles we used identical parameters, specified below.

Let us remark right away that there will indeed be differences between our findings for the two studies of [38] and [39]. This motivated us to investigate to what extent these differences were due to substantial differences in the data, or perhaps to a lack of robustness of our methods. To answer this we re-applied our algorithm, with the same parameters as before, to disjoint subsets of the participants in the larger study of [39]. As we shall see in section 4.5.5, the tangles we found on these subsets were very similar to those we found on the entire dataset of [39]. Hence they all differed, and in similar ways, from the tangles we found in [38].

We have thus been able not only to find reproducible tangle-based traits in the studies of [38] and [39]. We can also say with some confidence which aspects of the relationship between those traits, in each of the two datasets, are stable across those two studies (which were four years apart and conducted with different target sets of participants), and which were specific to one study or the other.

The tangles analysed below are based on mutual information as the similarity function (definition 4.4.3), and on its Laplacian $L$ as the matrix (section 4.4.3). In section 4.5.6 we shall compare these tangles with the tangles we found for the cosine similarity function (definition 4.4.4), again with the matrix $L$. Since these two similarity functions measure different things, we expected them to return different tangles, which was indeed the case. As both mutual information and cosine similarity are accessible to interpretation, the (small but interesting) difference between the tangles found can also be interpreted. We must leave this to the experts in psychology, but would like to point out this possibility.

We further ran tests to see whether our choice of the matrix $L$ rather than $J$ affected the tangles we found. Any difference here would be harder to interpret. However, we found no such difference; see section 4.5.6.

In each of the datasets from [38] and [39] we found several thousand tangles. In the case of [38] these combined to eleven traits, in the case of [39] to fifteen (see definition 4.4.1). We denoted some of these traits by the letters $O, C, E, A, N$, according to the following rule.

Let us first denote by the letters $O, C, E, A, N$ the five subsets of $Q$ designed to test for the trait indicated by the respective letter. Consider any one of these subsets, $O$ say. If it so happened that $O$, but none of the other four sets $C, E, A, N \subseteq Q$, guides the lowest-order tangle in some trait $T$, and if no trait refined by $T$ has this property too, we denoted $T$ as $O$. Similarly, we used each of the letters $C, E, A, N$ to denote the set of questions in $Q$ it indicates, and then by implication also the trait corresponding to that set, as defined above for $O$.

The tree of the fifteen traits we found in [39] is depicted in fig. 4.4. The tree of the eleven traits we found in [38] is depicted in fig. 4.5.

### 4.5.2 Structure: how different traits are related

The trees of traits, shown in fig. 4.4 for [39] and in fig. 4.5 for [38], display a hierarchy between those traits. This has two aspects. The first aspect is the structure of the tree: this tells us how the traits are related to each other, i.e., which of them refines which. The second aspect is at which level the various traits occur: this tells us their complexity, cohesion, and visibility (definition 4.3.11).

Let us look at the first of these aspects. Traits sitting above other traits in the figure can be viewed as a common generalisation of those. The trait $T_8$ in fig. 4.4, for example, is shown above the traits $A$ and $E$, so it is something like a common generalisation of the two. Indeed, the partitions in $S$ oriented by the tangles in $T_8$ are oriented in the same way by the tangles in $A$ and by the tangles in $E$. Those latter tangles, however,

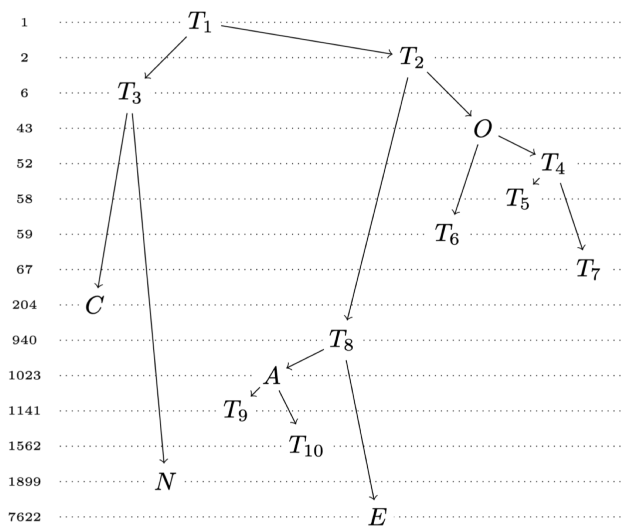


**Figure 4.4:** The tree of traits found in the larger study of [39]

may disagree on their orientation of partitions whose order is too large for them to be oriented by the tangles in $T_8$. In particular, they disagree on their unique efficient distinguisher, the partition $s$ of lowest order which they orient differently. Since $A$ and $E$ both refine $T_8$, the order of $s$ is at least that of the largest tangle in $T_8$.

By contrast, the trait $T_2$ is an even broader common generalisation of $A$ and $E$, one that also encompasses $O$ – but neither $C$ nor $N$. The traits $A$ and $E$ are distinguished from $O$ by the same efficient distinguisher $s$, whose order is at least that of the largest tangle in $T_2$. This $s$ is also the efficient distinguisher of $T_8$ and $O$, but it does not distinguish $A$ from $E$.

Let us now turn to the second aspect of fig. 4.4, the levels at which the various traits are shown. The vertical scale in the figure indicates order: of the tangles representing the traits depicted, and of the partitions of $Q$ in $S$ that distinguish them.[12]

The trait $T_6$, for example, is entered in the diagram at level 59. This means that the largest order of a tangle in this trait is 59. The lowest order of a tangle in $T_6$ can also be read out of the picture: it is the lowest order at which the highest-order tangle in the parent of $T_6$, trait $O$, splits into tangles belonging to its two children, the traits $T_4$ and $T_6$. As $O$ is entered at level 43, this order of the efficient distinguisher between the tangles in $T_4$ and those in $T_6$ is at least 43.[13] If it is exactly 43, the trait $T_6$ has complexity 43, cohesion 59, and visibility $59 - 43 + 1 = 17$. Its visibility is indicated roughly by the length of the line that joins $T_6$ to $O$ in the figure. This is higher than the visibility of the trait $T_5$, say, but lower than that of the traits $C$ and $N$.

A particularly interesting aspect of the vertical distribution of the traits in fig. 4.4 is that some of the $OCEAN$ traits are born only when others have already died: there is no order of separations in $S$ at which they overlap. Traits $A$ and $E$, for example, are born just below level 940, when their parent trait $T_8$ splits. At this time the traits $C$

[12] But note that fig. 4.4 is not drawn to scale. The lengths of the lines, which we shall see indicate the visibility of traits, thus have to be compared with caution: lines further up span a smaller vertical difference than their lengths appear to indicate.

[13] More precisely, it is the lowest order of a separation $s$ not oriented by the largest tangle in $O$. Since that tangle has order 43, it orients precisely the separations in $S$ that have order less than 43. So $s$ is the element of $S$ of the smallest order that is at least 43.

and $O$ are already dead. Trait $E$ then outlasts trait $A$ by a considerable margin, as does trait $N$, which was born even before $A$ and $E$.

*In summary, while the 'big five traits' targeted by Q do show up as tangle traits, they do so at very different times, or levels of 'resolution' of the lens – order – through which we view our tangles and the traits to which they combine.*

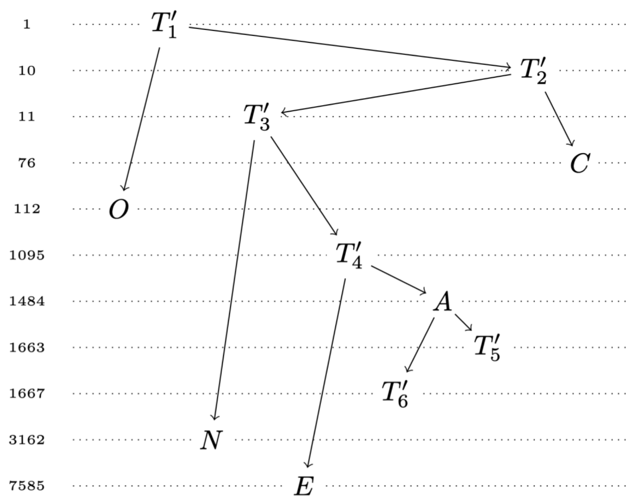


**Figure 4.5:** The tree of traits found in the smaller study of [38]

Let us now turn to the smaller study of [38]. Its tree of traits is depicted in fig. 4.5. For any comparison with the tree of traits for the larger study [39], note that while the respective traits $O, C, E, A, N$ were named according to the same rule, applied independently to the two datasets, the enumerations $T_1, T_2, \ldots$ and $T'_1, T'_2, \ldots$ of the other traits are arbitrary and not related.

The most important result on which the two studies agree is that both feature traits that merit being called $O, C, E, A, N$ according to our rule in section 4.5.1: their lowest-order tangle is guided by the subset of $Q$ specified by the letter that names them, but by no other such subset, and this applies to no smaller trait.

One structural similarity between [38] and [39] is that the traits $A$ and $E$ are direct siblings: in both studies they have a common parent (the trait $T_8$ in [38] and the trait $T'_4$ in [38]). And in both datasets the trait $A$ then splits into two more traits, while $E$ does not have distinct refinements.

A closer look also reveals some interesting differences between [38] and [39].

A particularly striking feature of [38] is that the $OCEAN$ traits are, successively, split off one line of refining non-$OCEAN$ traits, the traits $T'_1 > T'_2 > T'_3 > T'_4$ (where $>$ indicates refinement as shown in the figure). In [39], by contrast, the the $OCEAN$ traits form two groups, $C, N$ versus $A, E, O$, where $A$ and $E$ form a subgroup of the latter group $A, E, O$. These two major groups are in fact separated by *two* levels of more general traits: by the trait $T_1$ (which the two groups refine in different ways), but also by its two children, the traits $T_2$ and $T_3$. Given the fact that $T_2$ and $T_3$ are themselves traits, this seems quite remarkable – and potentially interesting from an interpretation point of view. While we limit ourselves to structural comparisons here, we shall see in section 4.5.3 that it is possible to use the information from the Appendix to write down how these traits orient their distinguishers. This provides experts in psychology with direct interpretable information.

In [39], the traits $C$ and $N$ have a direct common generalisation, the trait $T_3$. In [38], they also have a common generalisation that does not also generalise any of the other $OCEAN$ traits, the trait $T_2'$. This, however, is not a direct generalisation just of the two traits $C$ and $N$: there is an intermediate trait $T_3'$, which is already a generalisation of $N$ (though not of $C$), and the common generalisation $T_2'$ of $C$ and $N$ also generalises $T_3'$. The study of [38] thus sees more structure than [39] in the relationship between the traits $C$ and $N$.

Another interesting difference between [38] and [39] is that, in [38], the trait $O$ does not further refine, while in [39] it splits into traits $T_6$ and $T_4$, which in turn splits into $T_5$ and $T_7$. Thus, [39] reveals substantial structure inside the trait $O$, which [38] did not see.

### 4.5.3 Interpretation

Tangles, unlike clusters found by some more traditional clustering algorithms, are in principle open to interpretation: every tangle is a set of oriented partitions of $Q$, and these are known explicitly. An expert psychologist can draw interpretative conclusions from the way our tangle traits orient these partitions.

In our particular case, we can even say something interpretative about the tangle traits we found without any expert knowledge of psychology. This is because we were able to relate some of these traits, $O$, $C$, $E$, $A$ and $N$, to five groups of questions in $Q$, as indicated in section 4.5.1. This already yields interpretative information for these five tangle traits themselves, if one considers these five sets of questions as understood at the interpretation level. And this is a reasonable assumption: recall that these five sets of questions were *designed* to flesh out some *interpretation* of the Five Factors, see section 4.1.1 and the Introduction. Our five tangle traits named $O, C, E, A, N$ therefore inherit those interpretations in the way made precise in section 4.5.1.

So what about interpreting the other ten of our fifteen tangle traits in [39], say? Our structural tangle analysis in section 4.5.2 yields that some of those other ten traits are common generalisations of specific subsets of the $OCEAN$ traits, and thereby inherit their interpretations. But even those traits that are not common generalisations of $OCEAN$ traits can be interpreted succinctly, as follows.

Each of the 15 traits in [39] is determined uniquely by which of the efficient tangle distinguishers it orients, and how. While there are seven efficient distinguishers in total, one for every branching node of the tree of traits (fig. 4.4), the tangles in a given trait orient only those distinguishers whose order is smaller than their complexity. But different traits are distinguished by at least one of these distinguishers, so each trait is identified uniquely by how it orients the efficient distinguishers that it does orient.

And every oriented distinguisher is directly accessible to interpretation: it is one concrete subset of $Q$. For the efficient trait distinguishers, these subsets are listed explicitly in the Appendix, in section 4.7.3. From these lists one can combine, for each of our 15 traits $T$ other than the all-encompassing $T_1$, how the tangles in $T$ orient the distinguishers $s$ which they orient at all, those of order less than the complexity of $T$. Conversely, given any two traits $T$ and $T'$, fig. 4.4 tells us which $s \in S$ is their unique efficient distinguisher, and its list in section 4.7.3 then tells us how $T$ and $T'$ orient $s$ towards its opposite sides and which concrete subsets of $Q$ those sides are.

A more direct, and simpler, route to interpreting our tangle traits is to identify, for each given trait, the questions in $Q$ that represent it best – in a precise, formal sense. We discuss this next.

### 4.5.4 Which questions represent the traits best?

Recall that our questionnaire $Q$ was designed for the purpose of testing the five OCEAN traits, originally in the pre-formal sense of 'trait' discussed and criticised in section 4.1. In section 4.5.1 we gave a formal definition of when we denote any of our tangle traits (which, in contrast, are a formal concept) by those $OCEAN$ labels. For these five traits, but also for all the other six (respectively, ten) traits we found in [38] and [39], our tangle framework allows us to determine explicitly which of the 50 questions in $Q$ are best suited to test for that particular trait. This works as follows.

Every tangle in $\vec{S}$, by definition, orients some of the partitions of $Q$ towards one of their two sides. Every element of $Q$ does that too: we can think of it as orienting every partition of $Q$ towards the side that contains it. We can therefore compare the questions $q$ in $Q$ directly with the tangles $\tau$ in $\vec{S}$, simply by counting the number $\sigma(q, \tau)$ of partitions they orient in the same way (definition 4.3.8).

In order to assess how well a question $q$ represents an entire trait $T$, now, rather than a given tangle, it would be natural to simply use $\sigma(q, \tau)$ for some $\tau \in T$. But which $\tau \in T$ should we choose for this comparison? Fortunately, it does not matter: as long as we compare $q$ and $\tau$ only on partitions $s \in S$ that really matter, the result will be the same.

Indeed, given $q \in Q$ and a tangle trait $T$, write $k^-$ for the complexity of $T$ and $k^+$ for its cohesion. Then the tangles in $T$ agree on all the partitions in

$$S_T := \{ s \in S : k^- \leq |s| + 1 \leq k^+ \}$$

which they orient, and they disagree, with any tangle that can be distinguished from them at all, on all partitions in $S_T$ that such a tangle also orients. The partitions in $S_T$, therefore, are those on which the trait $T$ comes into its own: those on which its tangles differ from tangles in all traits that are essentially different from $T$.

We now define $T(s) := \tau(s)$ for all $s \in S_T$, where $\tau$ is any tangle in $T$ that orients $s$. (We do not have to specify which $\tau$, since they all agree on $s$.) Then

$$\sigma(q, T) := \left|\{ s \in S_T : q(s) = T(s) \}\right|$$

measures what we want: the similarity between $q$ and the tangles in $T$ on the partitions that reflect $T$ best.

section 4.7.2 lists, for each of the traits found in [38] and [39], the three questions $q$ in $Q$ that represent this trait $T$ best as measured by $\sigma(q, T)$.[14] These tables contain a wealth of information that is directly accessible to interpretation. Not only are the individual traits each represented by three concrete questions; it is also instructive to compare how the structural relationship between the traits, as expressed by the trees of traits shown in figs. 4.4 and 4.5, compares with the sets of questions that represent them.

For example, fig. 4.4 shows that the trait $O$ in [39] splits into the subtraits $T_4$ and $T_6$. These are represented in section 4.7.2 by disjoint sets of questions: $T_4$ by $O_4, O_5, O_{10}$ and $T_6$ by $O_1, O_7, O_8$, which helps us distinguish them also at the interpretation level. The subtrait $T_4$ then splits again, into $T_5$ and $T_7$. But these are both represented best by $O_5, O_6, O_{10}$, the same set of questions: their difference as subtraits of $T_4$ is more subtle than what their representing questions indicate. But this difference is captured succinctly by one explicitly known partition: their efficient distinguisher. This is listed in section 4.7.3, associated with $T_4$.

[14] The questions themselves are listed in section 4.7.1.

We must leave any further analysis to the experts in psychology, contenting ourselves with pointing out how the information on this can be based is made explicit in our tangle data.

### 4.5.5 Robustness of our methods

Recall that by *robustness* of our methods we mean the invariance of our findings under small changes to the input, while we keep those methods unchanged. To test this, we repeated our tangle analysis of the large study [39] described earlier in this section on ten subsets of its participants.

We first divided its entire set of about a million participants into 5 disjoint subsets of 200.000 participants and analysed their tangles. We then chose five further random subsets of about 10.000 participants each, and again analysed their tangles.

The trees of traits we found in the five large subsets were very similar to that of the entire dataset (fig. 4.4). Two of them were in fact identical to it. Another two were identical to it except that trait $A$ lacked the two children it had in fig. 4.4, denoted there as $T_9$ and $T_{10}$. The fifth tree of tangles is shown in fig. 4.6. It differs from the other four in that the two long branches hanging off $T_2$ and $T_3$ in fig. 4.4 are swapped. So there is now a joint trait generalising $C, N, O$ but not $A, E$, whereas in fig. 4.4 there was one generalising $A, E, O$ but not $C, N$. The $OCEAN$ traits themselves are independent, as before.

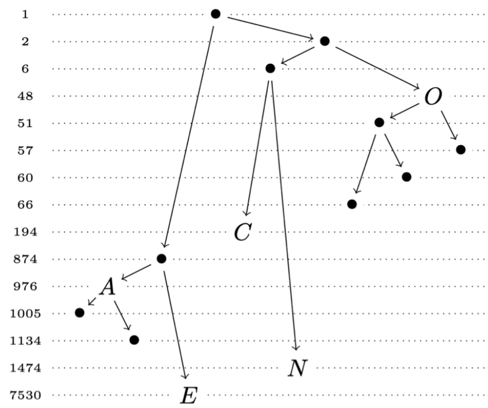


**Figure 4.6:** The only different tree of traits on a large subset of [39]

The trees of traits we found on the five smaller subsets of [39] are depicted in fig. 4.7. They show a little more variation – not unexpected for smaller datasets. The first is again identical to that of fig. 4.4. The second is identical to it except that $A$ has no children, as earlier. The other three are variants of the tree in fig. 4.6.[15]

Thus, the traits we found in the ten subsets of [39], as well as their tree structure, are all very similar to those for the entire dataset (fig. 4.4). In each of them there are five traits that merit being denoted as $O, C, E, A, N$ by our rule in section 4.5.1. Their relative structure is also the same as before: traits $A$ and $E$ combine to a more comprehensive trait not generalising any other $OCEAN$ traits. And so do $C$ and $N$, except in the last tree of fig. 4.7. In six of the ten trees, $O$ combines with the supertrait of $A$ and $E$ to a

[15]Note that for the tree structure it is irrelevant which two branches emanating from a node is drawn left and which right. The branch containing $C$, $N$ and $O$ in last tree in fig. 4.7 is in fact identical to that in the tree before, except that the parent trait of $C$ and $N$ has split into two traits.

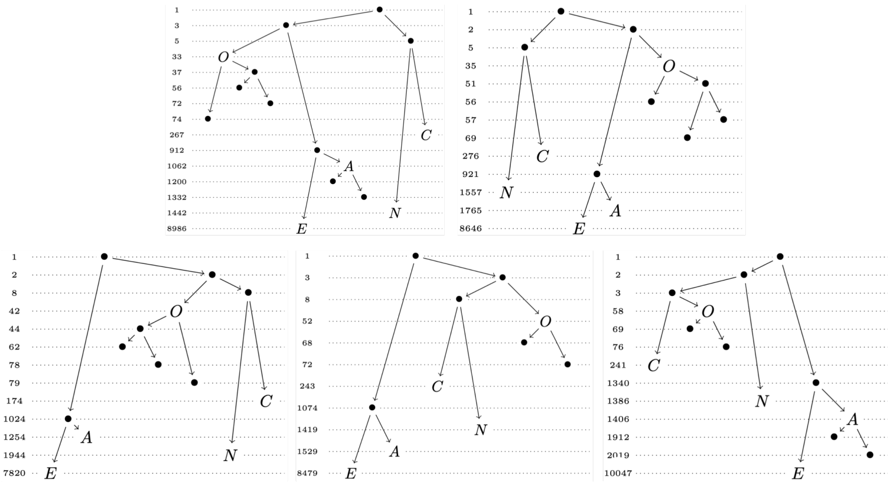


**Figure 4.7**: The trees of traits for the five small subsets of [39]

trait not generalising $C$ or $N$, as in fig. 4.4, while in the other four it combines with $C$ and $N$ to a trait not generalising $A$ or $E$.

Overall, our tangle analysis of the ten subsets of [39] yields essentially the same results as that of the entire dataset. This means that our tangle analysis is robust in the sense defined earlier: the essential aspects of its results are invariant under changes of data that are not due to other factors, such as a different time when the study was conducted, or a different sample of participants.

If we take this as confirmation of the robustness our methods also with respect to our analysis of the data from [38], the implication is that the difference our analysis in section 4.5.2 found between the studies of [38] and [39] reflects genuine difference in the data. This, in turn, may have interesting implications for their interpretation, since the differences between [38] and [39], as well as what they have in common (such as being based on the same questionnaire $Q$), are known explicitly.

### 4.5.6 Stability

Recall that by *stability* of our findings we mean their invariance under small changes of our methods, performed on the same input.

In our methods we initially made some reasonable choices that we kept fixed throughout our tangle analysis. Those included the use of the ratio cut weight order function (based, however, on a similarity function which we did vary), to require our tangles to have a minimum agreement of 2, and the various normalisations described in section 4.4.1.

However we tested variation of two parameters: of our choice of the similarity function used by the ratio cut weight order function (see section 4.4.2), and of our choice of the matrix ($L$ or $J$, see section 4.4.3) whose eigenvectors provided the initial set of partitions of $Q$ from which we then built our final set $S$ of partitions.

We found that replacing $L$ with $J$ made no structural difference to the traits we found and their relationships. The only difference lay in the complexity and cohesion of the traits, but with very little resulting difference for their visibilities. So our findings were perfectly stable in this respect.

This is not unremarkable, and we interpret it as an indication of reliability for the tangle method in clustering in general. Its fundamental idea is to ignore just the right amount of detail in a fuzzy environment to identify the ‘true’ clusters and their mutual relationship. Both these turned out to be identical in our tangle analysis even when its input data, the starting set $S$ of partitions of $Q$, was very different as we obtained it from the eigenvectors of the matrices $L$ and $J$, respectively.

Let us now report briefly what we found when we based the ratio cut weight order, which we used throughout, on cosine similarity (definition 4.4.4) rather than on entropy similarity (definition 4.4.3). The trees of traits we found in [38] and [39], respectively, are shown in fig. 4.8.

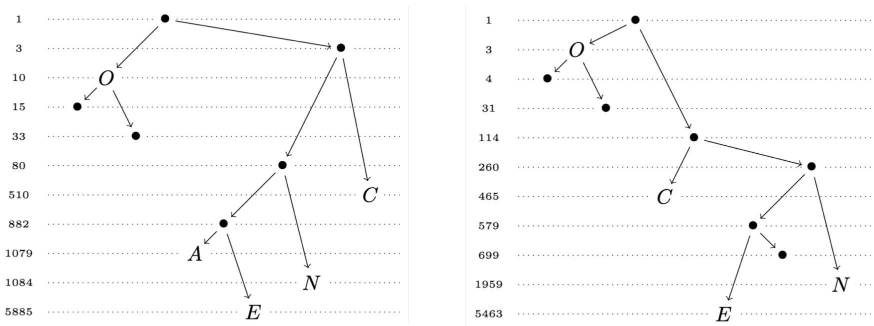


**Figure 4.8**: The trees of traits computed with cosine similarity. The tree for the large study [39] is shown on the left, that for the small study [38] on the right.

Once more, we used the labels of $O, C, E, A, N$ for traits that corresponded to the subsets of $Q$ bearing these names, according to the rule from section 4.5.1. This time, no trait found in the smaller study of [38] could be named as $A$. But it had traits corresponding to $O, C, E$ and $N$, while in the larger study of [39] all five $OCEAN$ traits occurred.

Unlike for the traits based on entropy similarity, the trees of traits were now strikingly similar for the smaller and the larger study, [38] and [39]. The only structural difference between the two is that the sibling of $E$ is $A$ in [39], while in [38] it is an unmarked trait. Both $A$ and this unmarked trait have low visibility in their respective tree. More detailed analysis shows that the unmarked trait is in fact quite close to meeting our standards for being named as $A$, whereas trait $A$ in [39] meets these only just.

An explicit look at the partitions of $Q$ we collected into $S$ using cosine rather than entropy similarity indicates that they are rougher: the entropy-based low-order partitions conform more readily to what, from an interpretation point of view, we would expect to be relevant partitions of $Q$. But this roughness is smoothed again by the tangles they form, and even more so by the traits to which these combine: we get a similar number of traits as before, the expected $OCEAN$ traits are among them, and there are further traits that refine or generalise these $OCEAN$ traits, thereby endowing them with additional structure.

Finally, there is a striking structural similarity between the two trees of traits based on cosine similarity (fig. 4.8) with the tree of traits based on entropy similarity (fig. 4.5) for the smaller study [38]. Indeed, in structural terms the tree of traits shown on the left in fig. 4.8 (which is based on the *larger* study [39]) differs from that of fig. 4.5 (for the *smaller* study [38]) only in that $O$ splits into two subtraits in the former but not in the latter, while $A$ splits into two subtraits in the latter but not in the former. Otherwise, the two trees are structurally identical.[16] The similarity between these two trees seems remarkable because they are based on different studies, [38] and [39], which in our earlier analysis we found to exhibit more distinct trait structures (which, moreover, we found to be robust, see section 4.5.5).

We cannot explain this phenomenon. The fact that the trees of traits are more similar for [38] and [39] under cosine similarity (fig. 4.8) than they are under entropy similarity (figs. 4.4 and 4.5) can be interpreted as an indication that tangle analysis based on cosine similarity is a blunter tool than tangle analysis based on entropy similarity. But the fact that the more sensitive tool yields, for one of these studies [38], results very similar to those returned by the blunter tool for both studies, while it returns reliably different results for the other study [39], is harder to interpret. It might be seen as an indication of an anomaly in the design of [39], which analysis based on cosine similarity is (perhaps beneficially) too blunt to pick up, or it may be an artefact that indicates nothing significant.

## 4.6 Conclusion

The problem this paper aimed to address was to bridge the gap between the mathematical notion of a *factor*, in the factor analysis performed on personality data from Cattell [30] and Fiske [31] by Tupes and Christal [29], and interpretations of these ('big five') factors on which questionnaires for personality assessment are currently based. The gap stems from the fact that factors, by definition, are abstract linear combinations of answers obtained for the studies analysed, while questionnaires testing for such factors are formulated based not on those linear combinations of the original questions but on new questions designed to test for an *interpretation* of those factors, conceived of as a 'trait'.

To do this, we developed quantitative criteria for groups of questions to merit being thought of as testing for any common trait at all. This required us to provide a formal definition of 'trait', against which such questionnaires could then be tested.

We first proposed, in general clustering terms, two criteria for a group of questions to achieve this: as 'extensional cohesion' and 'extensional completeness'. Our use of the novel clustering method of *tangle analysis* then enabled us to propose a rigorous notion of 'trait', on which we based our further investigations: that of an equivalence class of tangles under some generic equivalence relation from tangle theory, whose origin lies in the tangle analysis of visual data. This formal notion of (tangle) trait satisfies our earlier requirements of extensional cohesion and completion; indeed these are built in to the very notion of a tangle and of tangle equivalence. But it goes further, in that it identifies the structural essence of clusters in a fuzzy environment, while filtering out unnecessarily detailed but less essential data as 'noise'.

We analysed the data returned by the separate studies of [38] and [39], both for the same questionnaire $Q$ designed to test for the 'big five' OCEAN traits. Our analysis confirms that the five groups of questions in $Q$ designed to test for those five 'traits' do indeed each test for something that satisfies our formal notion of a tangle trait, with some reservations in the case of trait $A$ ('agreeableness'). But in addition to the five $OCEAN$ traits we also found some others, both common generalisations and refinements of $OCEAN$ traits: six in the smaller study of [38], and ten in the larger study of [39].

[16] In mathematical terms: once we delete the two pairs of children named above, there exists a graph isomorphism between the two trees that fixes their $OCEAN$ traits, i.e., which maps corresponding $OCEAN$ traits to each other.

These traits, including the five *OCEAN* traits, are not equally cohesive and complete, and our analysis quantifies to what extent they are. *They are also interrelated in ways that were not, to the best of our knowledge, known before.* These relationships are exhibited by *trees of traits* (section 4.5.2).

For each of the five *OCEAN* traits, but also for the other traits we found, our analysis can identify the questions in $Q$ that represent this trait best, in a rigorous, quantitative sense. In section 4.7.2 we list the top three questions for each trait. This is one aspect in which the tangle traits we found are directly interpretable – unlike the factors found in larger studies by factor analysis.

Further interpretability of the traits we found can be read off explicitly from their *efficient distinguishers*: five in [38], and seven in [39]. Each of these is one concrete partition of $Q$ into two sets of questions. These are listed explicitly in section 4.7.3, and are thus directly accessible to expert interpretation.

Finally, we tested the robustness of our methods (section 4.5.5) for the larger study of [39], as well as the stability of our findings under changes to our tangle parameters (section 4.5.6). We found that both the traits found and their relationships as expressed in the trees of traits were essentially invariant under changes of input data taken disjointly and independently from the same study. We found that our results were perfectly stable under changes to the parameters $L$ or $J$ that determined the partitions of $Q$ for our tangles, and largely stable when we replaced the entropy-based similarity measure we used with the cruder measure of cosine similarity. Both these are state-of-the-art similarity measures in information theory and data science.

In summary, we found that the method of tangle-based clustering can be used reliably to find previously unknown traits in existing data obtained for FFM-based personality tests, and to relate known traits to each other, all in a rigorously defined quantitative sense.

## 4.7 Appendix

### 4.7.1 The questionnaire $Q$

Here is a list of the 50 questions in the questionnaire $Q$ used in [38] and [39]:

**Openness**

$O_1$ I have a rich vocabulary.

$O_2$ I have difficulty understanding abstract ideas.

$O_3$ I have a vivid imagination.

$O_4$ I am not interested in abstract ideas.

$O_5$ I have excellent ideas.

$O_6$ I do not have a good imagination.

$O_7$ I am quick to understand things.

$O_8$ I use difficult words.

$O_9$ I spend time reflecting on things.

$O_{10}$ I am full of ideas.

**Conscientiousness**

$C_1$ I am always prepared.

$C_2$ I leave my belongings around.

$C_3$ I pay attention to details.

$C_4$ I make a mess of things.

$C_5$ I get chores done right away.

$C_6$ I often forget to put things back in their proper place.

$C_7$ I like order.

$C_8$ I shirk my duties.

$C_9$ I follow a schedule.

$C_{10}$ I am exacting in my work.

**Extraversion**

$E_1$ I am the life of the party.

$E_2$ I don't talk a lot.

$E_3$ I feel comfortable around people.

$E_4$ I keep in the background.

$E_5$ I start conversations.

$E_6$ I have little to say.

$E_7$ I talk to a lot of different people at parties.

$E_8$ I don't like to draw attention to myself.

$E_9$ I don't mind being the center of attention.

$E_{10}$ I am quiet around strangers.

**Agreeableness**

$A_1$ I feel little concern for others.

$A_2$ I am interested in people.

$A_3$ I insult people.

$A_4$ I sympathize with others' feelings.

$A_5$ I am not interested in other people's problems.

$A_6$ I have a soft heart.

$A_7$ I am not really interested in others.

$A_8$ I take time out for others.

$A_9$ I feel others' emotions.

$A_{10}$ I make people feel at ease.

**Neuroticism**

$N_1$ I get stressed out easily.

$N_2$ I am relaxed most of the time.

$N_3$ I worry about things.

$N_4$ I seldom feel blue.

$N_5$ I am easily disturbed.

$N_6$ I get upset easily.

$N_7$ I change my mood a lot.

$N_8$ I have frequent mood swings.

$N_9$ I get irritated easily.

$N_{10}$ I often feel blue.

### 4.7.2 All traits, represented by typical questions

In section 4.5.4 we saw that tangles in $\vec{S}$, and by implication also tangle traits $T$, can be compared directly with individual questions $q$ from $Q$ (see section 4.7.1), by counting how many of the partitions in $S$ they orient in the same direction. We said that when this number is high the question $q$ *represents* the trait $T$ well.

The following two tables list the top three representatives from $Q$, in order (best first), for each of the traits we found in [38] and [39]:

| | | | | |
|---|---|---|---|---|
| $\boldsymbol{O}$: $O_{10}, O_6, O_1$ | $\boldsymbol{C}$: $C_5, C_6, C_1$ | $\boldsymbol{E}$: $E_7, E_5, E_2$ | $\boldsymbol{A}$: $A_9, A_1, A_8$ | $\boldsymbol{N}$: $N_6, N_8, N_9$ |
| $\boldsymbol{T_2}$: $O_{10}, C_5, N_3$ | $\boldsymbol{T_3}$: $C_5, N_9, N_8$ | $\boldsymbol{T_4}$: $O_{10}, O_5, O_4$ | $\boldsymbol{T_5}$: $O_{10}, O_6, O_5$ | $\boldsymbol{T_6}$: $O_8, O_1, O_7$ |
| $\boldsymbol{T_7}$: $O_{10}, O_5, O_6$ | $\boldsymbol{T_8}$: $A_5, A_7, E_2$ | $\boldsymbol{T_9}$: $A_5, A_7, A_4$ | $\boldsymbol{T_{10}}$: $A_2, A_7, A_5$ | |

The questions best representing the traits in [39]

| | | | | |
|---|---|---|---|---|
| $\boldsymbol{O}$: $O_2, O_1, O_8$ | $\boldsymbol{C}$: $C_5, C_6, C_9$ | $\boldsymbol{E}$: $E_7, E_5, E_1$ | $\boldsymbol{A}$: $A_5, A_2, A_4$ | $\boldsymbol{N}$: $N_6, N_9, N_8$ |
| $\boldsymbol{T_2'}$: $A_5, A_6, A_2$ | $\boldsymbol{T_3'}$: $O_{10}, C_5, N_3$ | $\boldsymbol{T_4'}$: $E_1, E_2, E_3$ | $\boldsymbol{T_5'}$: $A_5, A_4, A_8$ | $\boldsymbol{T_6'}$: $A_2, A_7, A_5$ |

The questions best representing the traits in [38]

### 4.7.3 All traits, represented by the tangle-distinguishers

In section 4.5.3 we noted that traits are described most succinctly by the small collection of partitions in $S$ that distinguish them efficiently from other traits. In this appendix we list these distinguishers explicitly, and then list for every trait how it orients them.

The efficient distinguisher $s$ of two traits $T_1$ and $T_2$ is naturally associated with their unique common ancestor $T$ in the tree of traits that displays them. Indeed, by definition $s$ is the partition of $Q$ in $S$ of lowest order that $T_1$ and $T_2$ orient differently. So they orient all partitions of lower order than $|s|$ in the same way. The set of these oriented partitions is the largest-order tangle in $T$, and $T$ has cohesion $|s|$.

In the tables below we denote this partition $s$ of $Q$ as $s(T)$. Note that $s$ is the efficient distinguisher not only for $T_1$ and $T_2$, but for all pairs of distinct traits that both refine $T$ but no tangle of higher cohesion. The two sides of a partition $s = s(T)$ are stated, as subsets of $Q$, in the left and right half of the box below $s$.

First, the efficient distinguishers from [39], one associated with each branching node of the tree of traits in fig. 4.4:

| $s(T_1)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A3 | A1 A2 A4 A5 A6 A7 A8 A9 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

| $s(T_2)$ | |
|---|---|
| E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 | |
| | N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 |
| A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 | |
| | C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

| $s(T_3)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| | N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 |
| | A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 | |

| $s(O)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A3 | A1 A2 A4 A5 A6 A7 A8 A9 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| O1 O8 | O2 O3 O4 O5 O6 O7 O9 O10 |

| $s(T_4)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| | A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| O1 O2 O4 O7 O8 O9 | O3 O5 O6 O10 |

| $s(T_8)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 | |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

| $s(A)$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A1 A3 A4 A5 A6 A8 A9 | A2 A7 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

The 14 traits in [39] other than $T_1$ orient these distinguishers as follows. Left and right arrows correspond to the sides of $s(T)$ shown left and right above.

$T_2$: $\vec{s}(T_1)$
$T_3$: $\overleftarrow{s}(T_1)$ $\vec{s}(T_2)$
$O$: $\vec{s}(T_1)$ $\vec{s}(T_2)$ $\overleftarrow{s}(T_3)$
$T_4$: $\vec{s}(T_1)$ $\vec{s}(T_2)$ $\vec{s}(O)$ $\overleftarrow{s}(T_3)$
$T_5$: $\vec{s}(T_1)$ $\vec{s}(T_2)$ $\vec{s}(O)$ $\overleftarrow{s}(T_4)$ $\overleftarrow{s}(T_3)$
$T_6$: $\vec{s}(T_1)$ $\vec{s}(T_2)$ $\overleftarrow{s}(O)$ $\overleftarrow{s}(T_4)$ $\overleftarrow{s}(T_3)$
$T_7$: $\vec{s}(T_1)$ $\vec{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\overleftarrow{s}(T_3)$
$C$: $\overleftarrow{s}(T_1)$ $\vec{s}(T_2)$ $\overleftarrow{s}(O)$ $\overleftarrow{s}(T_4)$ $\overleftarrow{s}(T_3)$
$T_8$: $\vec{s}(T_1)$ $\overleftarrow{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\vec{s}(T_3)$
$A$: $\vec{s}(T_1)$ $\overleftarrow{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\overleftarrow{s}(T_8)$ $\vec{s}(T_3)$
$T_9$: $\vec{s}(T_1)$ $\overleftarrow{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\overleftarrow{s}(T_8)$ $\overleftarrow{s}(A)$ $\vec{s}(T_3)$
$T_{10}$: $\vec{s}(T_1)$ $\overleftarrow{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\overleftarrow{s}(T_8)$ $\vec{s}(A)$ $\vec{s}(T_3)$
$N$: $\overleftarrow{s}(T_1)$ $\vec{s}(T_2)$ $\overleftarrow{s}(O)$ $\overleftarrow{s}(T_4)$ $\overleftarrow{s}(T_8)$ $\overleftarrow{s}(A)$ $\vec{s}(T_3)$
$E$: $\vec{s}(T_1)$ $\overleftarrow{s}(T_2)$ $\vec{s}(O)$ $\vec{s}(T_4)$ $\vec{s}(T_8)$ $\vec{s}(A)$ $\vec{s}(T_3)$

Next, the efficient distinguishers from [38], this time associated with the branching nodes of the tree of traits in fig. 4.5:

| $s(T_1')$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| | N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 |
| | A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 |
| | C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 |
| O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 | |

| $s(T_2')$ | |
|---|---|
| E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 | |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 | |
| | C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

| $s(T_3')$ | |
|---|---|
| | E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| | A1 A2 A3 A4 A5 A6 A7 A8 A9 A10 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 | |

| $s(T_4')$ | |
|---|---|
| E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 | |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A10 | A1 A2 A3 A4 A5 A6 A7 A8 A9 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

| $s(A)$ | |
|---|---|
| E1 E2 E3 E4 E5 E6 E7 E8 E9 E10 | |
| N1 N2 N3 N4 N5 N6 N7 N8 N9 N10 | |
| A2 A7 A10 | A1 A3 A4 A5 A6 A8 A9 |
| C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 | |
| | O1 O2 O3 O4 O5 O6 O7 O8 O9 O10 |

The 10 traits in [38] other than $T_1$ orient these distinguishers as follows. Left and right arrows correspond to the sides of $s(T)$ shown left and right above.

$T_2'$: $\vec{s}(T_1')$
$T_3'$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$
$C$: $\vec{s}(T_1')$ $\vec{s}(T_2')$ $\overleftarrow{s}(T_3')$
$O$: $\overleftarrow{s}(T_1')$ $\vec{s}(T_2')$ $\overleftarrow{s}(T_3')$
$T_4'$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\vec{s}(T_3')$
$A$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\vec{s}(T_3')$ $\vec{s}(T_4')$
$T_5'$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\vec{s}(T_3')$ $\vec{s}(T_4')$ $\vec{s}(A)$
$T_6'$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\vec{s}(T_3')$ $\vec{s}(T_4')$ $\overleftarrow{s}(A)$
$N$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\overleftarrow{s}(T_3')$ $\overleftarrow{s}(T_4')$ $\overleftarrow{s}(A)$
$E$: $\vec{s}(T_1')$ $\overleftarrow{s}(T_2')$ $\vec{s}(T_3')$ $\overleftarrow{s}(T_4')$ $\overleftarrow{s}(A)$

# 5

# On vertex sets inducing tangles

**Diestel, Hundertmark and Lemanczyk asked whether every $k$-tangle in a graph is induced by a set of vertices by majority vote. We reduce their question to graphs whose size is bounded by a function in $k$. Additionally, we show that if for any fixed $k$ this problem has a positive answer, then every $k$-tangle is induced by a vertex set whose size is bounded in $k$. More generally, we prove for all $k$ that every $k$-tangle in a graph $G$ is induced by a weight function $V(G) \to \mathbb{N}$ whose total weight is bounded in $k$. As the key step of our proofs, we show that any given $k$-tangle in a graph $G$ is the lift of a $k$-tangle in some topological minor of $G$ whose size is bounded in $k$.**

## Contents

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## 5.1 Introduction

### 5.1.1 Vertex sets inducing tangles

Tangles are an abstract notion of 'clusters' in graphs that originates in the theory of graph minors developed by Robertson and Seymour [42]. They allow for a unified treatment of various concrete highly cohesive substructures in graphs, such as large clique or grid minors. Tangles describe these clusters in a graph indirectly. Instead of describing what the cluster is composed of, they describe its position, in that they orient the low-order separations of the graph towards it. Intuitively, a concrete cluster orients all low-order separations by majority vote, that is the cluster orients such a separation $\{A, B\}$ towards its side, $A$ or $B$, which contains most of the cluster. Such a side exists; otherwise, the cluster would be separated by few vertices, which contradicts its high cohesion.

The orientations of the low-order separations induced by concrete clusters are 'consistent', in that they all point to the cluster. Robertson and Seymour's key innovation was to distil from this an abstract notion of 'consistency' which leads to the notion of tangle [1]: Formally, a *$k$-tangle* $\tau$ in a graph $G$ is an orientation of the separations of $G$ of order $< k$ such that there do not exist three separations $(A_i, B_i) \in \tau$ such that the union of the small sides $G[A_i]$ covers the whole graph. We refer to $k$ as the *order* of $\tau$, and we will denote a tangle of unspecified order as simply a *tangle in $G$*.

While every concrete cluster induces a tangle by majority vote, is the converse also true in that all tangles stem from concrete clusters in this way? Without a precise definition of concrete cluster, it seems difficult to answer this question. However, the question remains interesting if we consider arbitrary vertex sets instead of concrete clusters: Is every tangle at least induced by the majority vote of some set of vertices? Diestel, Hundertmark and Lemanczyk [7]*Section 7 formalised this problem as follows. A set $X$ of vertices of a graph $G$ *induces* a tangle $\tau$ in $G$ if for every separation $(A, B) \in \tau$ we have $|X \cap A| < |X \cap B|$. For example, the vertex set of a complete subgraph induces a tangle in this way.

**Problem 5.1.1.** *[7] Is every tangle in a graph $G$ induced by some set $X \subseteq V(G)$?*

In what follows we will often consider theorem 5.1.1 for all tangles of some fixed order $k \in \mathbb{N}$ and then say for short: theorem 5.1.1 for $k$.

We remark that theorem 5.1.1 is already answered in the affirmative for $k \leq 3$: Such sets $X$ inducing $k$-tangles exist for $k \leq 2$ due to the well-known correspondence of these tangles to components and blocks, respectively (cf. [1]*(2.6)). For $k = 3$, Elbracht

[43]*Theorem 1.2 proved theorem 5.1.1 directly. Independently, Grohe [44]*Theorem 4.8 proved a direct correspondence between the 3-tangles and the 'proper triconnected components' of a graph, which yields another proof of theorem 5.1.1 for $k = 3$. Similarly, it should be possible to derive theorem 5.1.1 for $k = 4$ from the recent characterisation of 4-tangles in terms of 'internal 4-connectedness' by Carmesin and Kurkofka [45]*Theorem 1. Besides these results for small $k$, Diestel, Elbracht and Jacobs [46]*Theorem 12 showed that theorem 5.1.1 is true for every $k$-tangle $\tau$ in a graph $G$ which *extends* to a $2k$-tangle $\tau'$ in $G$, that is, $\tau \subseteq \tau'$.

For general $k$, Elbracht, Kneip and Teegen [47]*Theorem 2 made substantial progress towards theorem 5.1.1 by proving a relaxed weighted version. A *weight function* $w: V(G) \to \mathbb{N}$ on the vertex set $V(G)$ of a graph $G$ *induces* a tangle $\tau$ in $G$ if $w(A) < w(B)$ for every $(A, B) \in \tau$. Note that a set $X \subseteq V(G)$ induces a tangle $\tau$ if and only if its indicator function $\mathbb{1}_X$ on $V(G)$ induces $\tau$.

**Theorem 5.1.2.** *[47] Every tangle in a graph $G$ is induced by some weight function on $V(G)$.*

Contrary to their positive result, Elbracht, Kneip and Teegen [47]*Theorem 10 explicitly construct an example which shows that not only theorem 5.1.1, but also theorem 5.1.2 fails for tangles in general discrete contexts, such as matroids or data sets (see e.g [48, 21, 49, 7]). However, no such example is known for tangles in graphs. Thus, theorem 5.1.1 is open for all $k \geq 4$.

### 5.1.2 Our contributions to theorem 5.1.1

In this paper, we reduce theorem 5.1.1 for every $k$ to graphs whose size is bounded by a function in $k$:

**Theorem 1.** *For every integer $k \geq 1$, there exists $M = M(k) \in O(3^{k^{k^5}})$ such that for every $k$-tangle $\tau$ in a graph $G$, there exists a $k$-tangle $\tau'$ in a connected topological minor $G'$ of $G$ with fewer than $M$ edges such that if a weight function $w'$ on $V(G')$ induces the tangle $\tau'$, then the weight function $w$ on $V(G)$ which extends $w'$ by zero[1] induces the tangle $\tau$. In particular, a set of vertices which induces $\tau'$ also induces $\tau$.*

As an immediate corollary of Theorem 1, we obtain that one may verify the validity of theorem 5.1.1 for any fixed $k$ computationally in explicitly bounded, though impractically long, time by checking theorem 5.1.1 for every $k$-tangle in every connected graph with fewer than $M$ edges.

**Corollary 2.** *For $k \geq 1$, there exists $M = M(k) \in O(3^{k^{k^5}})$ such that theorem 5.1.1 holds for $k$ if it holds for all $k$-tangles in connected graphs $G$ with fewer than $M$ edges.*

Our second corollary of Theorem 1 asserts that whenever theorem 5.1.1 is true for some fixed $k$, then every $k$-tangle is induced by a set of vertices of size bounded in $k$. This extends the known fact that one may choose the set $X$ in theorem 5.1.1 that induces a given $k$-tangle with $k \leq 2$ to be of size at most $k$; this follows from the characterisation of 1- and 2-tangles by components and blocks, respectively. More generally, we prove

[1]Given two sets $X' \subseteq X$, a function $w: X \to \mathbb{N}$ *extends* a function $w': X' \to \mathbb{N}$ *by zero* if $w$ restricted to $X'$ is $w'$ and $w$ restricted to $X \setminus X'$ is 0.

that for every $k$-tangle $\tau$ in a graph $G$ one may choose a weight function that induces $\tau$, which exists by theorem 5.1.2, in such way that its *total weight* $w(V(G)) = \sum_{v \in V(G)} f(v)$, which it distributes on $V(G)$, is bounded in $k$.

**Corollary 3.** *For every integer $k \geq 1$, there exists $K = K(k)$ such that for every $k$-tangle $\tau$ in a graph $G$ there exists a weight function $V(G) \to \mathbb{N}$ which induces $\tau$ and whose total weight $w(V(G))$ is bounded by $K$. In particular, the support of $w$ has size $\leq K$.*

*Moreover, if theorem 5.1.1 holds for $k$, then every $k$-tangle in a graph is induced by a set of at most $M(k)$ vertices, where $M(k)$ is given by Theorem 1.*

We derive Theorem 3 from Theorem 1 by first fixing a weight function for every $k$-tangle in a connected graph with fewer than $M$ edges (such weight functions exist by theorem 5.1.2) and then taking $K$ as the maximum of the total weights of these finitely many fixed weight functions. The moreover-part follows immediately from Theorem 1 by fixing each such weight function to be an indicator function of some inducing set given by the assumed positive answer to theorem 5.1.1 (see section 5.8 for details).

### 5.1.3 An inductive proof method for tangles

Our proof of Theorem 1 is based on the following theorem which allows inductive proofs for statements about tangles in graphs. We expect that this inductive proof method will be of independent interest.

**Theorem 4.** *For every integer $k \geq 1$ there is some $M(k) \in O(3^{k^{k^5}})$ such that the following holds: Let $\tau$ be a $k$-tangle in a graph $G$. Then there exists a sequence $G_0, \dots, G_m$ of graphs and $k$-tangles $\tau_i$ in $G_i$ for every $i \in \{0, \dots, m\}$ such that*

- *$G_0 = G$, $\tau_0 = \tau$;*
- *$G_i$ is obtained from $G_{i-1}$ by deleting an edge, suppressing a vertex, or taking a proper component;*
- *the $k$-tangle $\tau_{i-1}$ in $G_{i-1}$ survives as the $k$-tangle $\tau_i$ in $G_i$ for every $i \in [m]$;*
- *$G_m$ is connected and has fewer than $M(k)$ edges.*

Before we proceed to explain the crucial term 'survive' in this theorem, we remark that it is necessary to allow the suppression of a vertex in Theorem 4. Indeed, there are connected graphs $G$ with arbitrarily many edges which have a $k$-tangle such that every graph obtained from $G$ by deleting an edge has no $k$-tangles at all (example 5.3.1).

What does it mean that the $k$-tangle $\tau$ 'survives' as a $k$-tangle $\tau'$ in $G'$ obtained from $G$ by deleting an edge, suppressing a vertex, or taking a proper component? Let us here consider the first case that $G' = G - e$ is obtained from $G$ by deleting an edge $e$ of $G$. Then every separation of $G$ is also a separation of its subgraph $G'$. But in general $G'$ admits more separations than $G$; namely those separations $\{A, B\}$ of $G'$ which have an endpoint of $e$ in each of the two sets $A \setminus B$ and $B \setminus A$. A $k$-tangle $\tau$ in $G$ *extends* to a $k$-tangle $\tau'$ in $G'$ if $\tau \subseteq \tau'$. Then we also say that the $k$-tangle $\tau$ *survives* as the $k$-tangle $\tau'$ in $G'$. We remark that such an extension $\tau'$ of $\tau$ may or may not exist in $G'$; if it exists, it need not be unique. If such a $\tau'$ exists and is unique, then we also say that $\tau$ *induces* $\tau'$.

For the cases that $G'$ is a component of $G$ or obtained from $G$ by suppressing a vertex of $G$, we refer the reader to section 5.3.2 for the details. For readers familiar with the fact that a tangle of order $k \geq 3$ in a minor of a graph $G$ 'lifts' to a $k$-tangle in $G$ (cf. [1]*(6.1) and [45]*Lemma 2.1), we remark that 'surviving' is the reverse notion.

### 5.1.4 Overview of the proof of Theorem 4

It suffices to describe one step of the construction of the above sequence, that is to find $G_i$ and the $k$-tangle $\tau_i$ in it given $\tau_{i-1}$ and $G_{i-1}$. We reduce the argument to several cases. Let $\tau$ be a $k$-tangle in a graph $G$. The following is an immediate consequence of the well-known correspondence of 1- and 2-tangles in $G$ to the components and blocks of $G$, respectively:

1. if $k = 1$, then $\tau$ extends to some $k$-tangle in $G - e$ for every edge $e$ of $G$ (theorem 5.3.1), and
2. if $k = 2$, then $\tau$ extends to a $k$-tangle in $G - e$ for some edge $e$ of $G$ (theorem 5.3.2).

A rather simple analysis will yield that

3. if $G$ is disconnected, then $\tau$ induces to a $k$-tangle in a unique component of $G$ (theorem 5.3.3),
4. if $k \geq 3$ and $e$ is the unique edge incident to a vertex of degree 1, then $\tau$ induces a $k$-tangle in $G - e$ (theorem 5.3.4), and
5. if $k \geq 3$, then $\tau$ induces a $k$-tangle in every graph obtained from $G$ by suppressing any vertex of degree 2 (theorem 5.3.5).

We remark that in each of the above 1 to 5, the tangle $\tau$ survives as a $k$-tangle in a strictly smaller graph (section 5.3). It remains to consider the case that $\tau$ is a tangle of order $k \geq 3$ in a connected graph with minimum degree $\geq 3$. Recall that a suppressed vertex always has degree 2. So as soon as we restrict to connected graphs of minimum degree at least 3, the statement requires that we find an edge $e$ to delete.

The proof hinges on an argument that this will always be possible: by carefully picking the edge $e$ of $G$, we may always extend $\tau$ in $G' = G - e$ as long as $G$ is sufficiently large by a function in the tangle's order $k$.

**Theorem 5.** *For every integer $k \geq 3$, there is some $M = M(k) \in O(3^{k^{k^5}})$ such that the following holds:*

*For every $k$-tangle in a connected graph $G$ with minimum degree $\geq 3$ and at least $M$ edges, there is an edge $e$ of $G$ such that $\tau$ extends to a $k$-tangle in $G - e$.*

Together 1 to 5 along with Theorem 5 complete the proof, since in any given case one of them will ensure that the $k$-tangle $\tau$ in a sufficiently large graph survives in some smaller graph.

### 5.1.5 Proof sketch of Theorem 5

Our task is to find a suitable edge $e$ of $G$ such that $\tau$ extends to some $k$-tangle $\tau'$ in $G' = G - e$. For this, we consider two cases. First, we assume that the graph $G$ contains a tangle $\tilde{\tau}$ of order $> k$ (section 5.4). If $\tau \subseteq \tilde{\tau}$, then we will observe that for every edge $e$ of $G$ the $k$-tangle $\tau$ extends to the $k$-tangle $\tau'$ in $G'$ which is essentially the restriction of $\tilde{\tau}$ to the separations of order $< k$ of $G'$ (theorem 5.4.3). Else we may find an edge $e$ far away from $\tau$ and close to $\tilde{\tau}$ such that the high order of $\tilde{\tau}$ enables us to define a suitable extension $\tau'$ of $\tau$ in $G' = G - e$ (theorem 5.4.4).

Second, we assume that the graph $G$ contains no tangle of order $> k$ (section 5.7). We remark that this case requires significantly more effort than the first. In the analysis of this case we aim to decompose $G$ in such a way that we can control the separations which arise from the deletion of an edge $e$ in a suitable location (section 5.6). To obtain the desired decomposition, we start with the tree-decomposition obtained from the tangle-tree duality theorem [50]*Theorem 12.5.1 due to the absence of high-order tangles. As $G$ is sufficiently large, the decomposition tree contains a very long path whose structure we may regularise to obtain a *rainbow-cloud decomposition* of $G$ (theorem 5.5.1).

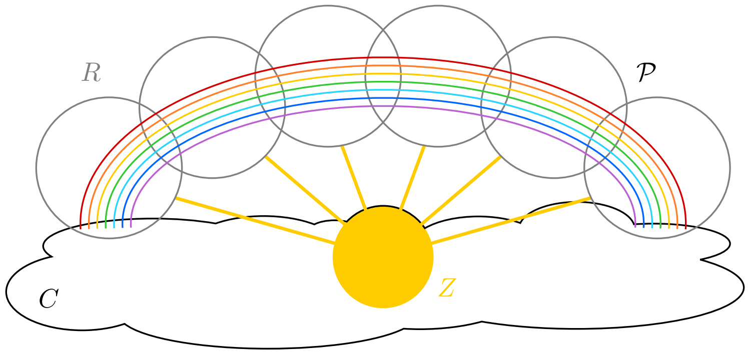


**Figure 5.1:** Schematic drawing of a rainbow-cloud decomposition

Roughly speaking (see also fig. 5.1), a rainbow-cloud decomposition of $G$ consists of a long *rainbow*, an induced subgraph $R$ of $G$ with a very regular and linear connectivity structure, which has many connections to the *sun* $Z \subsetneq V(G)$ and the remaining graph is gathered in a *cloud*, another induced subgraph $C$ with $G = G[Z \cup V(R)] \cup C$ (see section 5.5 for the precise definition). We show that if we choose the edge $e$ deep inside the rainbow $R$ and far away from $\tau$, then the connectivity structure of the long rainbow $R$ ensures that the new separations which arise from the deletion of $e$ may be oriented in such a way that $\tau$ extends to a $k$-tangle $\tau'$ in the graph $G - e$ (theorem 5.7.1).

### 5.1.6 How this paper is organised

We recall the relevant terminology concerning tangles in section 5.2. We then introduce in section 5.3 the notion of 'survives' and prove Theorem 4 in several special cases: the case $k \leq 2$, the case $\delta(G) \leq 2$, and when the graph $G$ is disconnected. In section 5.4 we then prove Theorem 5, the remaining case of Theorem 4 when $G$ contains a tangle of order $k+1$. In section 5.5, we introduce the definition of 'rainbow-cloud decompositions' and prove that such a decomposition exists in the absence of $(k+1)$-tangles in $G$, assuming that $G$ is sufficiently large. Building on several lemmas which we prove in section 5.6 about the interaction of rainbow-cloud decompositions and separations, we complete the proof of Theorem 5 in section 5.7. In section 5.8 we collect all the individual cases to prove Theorem 4, derive Theorem 1, and deduce Theorem 2 and Theorem 3.

## 5.2 Preliminaries

In this section, we recall the relevant terms and definitions concerning tangles in graphs. For general graph-theoretic concepts and notation, we follow [50]. In particular, we denote the set $\{1, \dots, n\}$ with $n \in \mathbb{N}$ by $[n]$. A path is *trivial* if it has length 0. We remark that all graphs in this paper are finite.

### 5.2.1 Separations of sets

While we will only work with separations of graphs in this paper, we formally introduce the notion of a separation on a set to transfer separations from one graph to another.

Given an arbitrary set $V$, an *(unoriented) separation* of $V$ is an unordered pair $\{A, B\}$ of subsets $A, B$ of $V$ such that $A \cup B = V$. The sets $A$ and $B$ are the *sides* of the separation $\{A, B\}$, and $A \setminus B$ and $B \setminus A$ are its *strict sides*. The separation $\{A, B\}$ *separates* every two sets $X \subseteq A$ and $Y \subseteq B$. The *order* of the separation, denoted $|A, B|$, is the cardinality of its *separator* $A \cap B$. Every separation $\{A, B\}$ of $V$ has two *orientations*, $(A, B)$ and $(B, A)$. These orientations of $\{A, B\}$ are *oriented separations* of $V$, that is, ordered pairs of subsets of $V$ whose union equals $V$. We refer to $A$ as the *small* side of an oriented separation $(A, B)$ and to $B$ as its *big* side. Given a set $V$, we write $U(V)$ for the set of all (unoriented) separations of $V$, and let $S_k(V) := \{\{A, B\} \in U(V) : |A \cap B| < k\}$. The set of all oriented separations of $V$ is denoted by $\vec{U}(V)$ and the set of all oriented separations of $V$ of order less than $k$ by $\vec{S}_k(V)$. We will often also refer to 'oriented separations' simply as 'separations' if the meaning is clear from the context. Similarly, we will transfer definitions for separations verbatim to oriented separations and vice-versa if the transfer is unambiguous.

The oriented separations of $V$ have a natural partial order:

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

With this, the set $\vec{U}(V)$ is a lattice with *infimum* $(A, B) \wedge (C, D) := (A \cap C, B \cup D)$ and *supremum* $(A, B) \vee (C, D) := (A \cup C, B \cap D)$. Moreover, $\vec{U}(V)$ is *distributive*, that is, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in \vec{U}(V)$. It is well-known [51]*Chapter IX Corollary 1 that a lattice is distributive if and only if every element of the lattice is uniquely determined by its infimum and supremum with any given element.

If two unoriented separations of a set $V$ have orientations which are comparable, then they are called *nested*; if they are not nested, then the two separations *cross*.

By double counting, we obtain that the order of the supremum and infimum of $\{A, B\}$ and $\{C, D\}$ sum up to

$$|(A, B) \wedge (C, D)| + |(A, B) \vee (C, D)| = |A, B| + |C, D|.$$

The inequality '$\leq$' implied by the equality above implies that $|\cdot|$ is a *submodular* function on the lattice $(\vec{U}(V), \leq)$. In particular, this implies that for two separations $\{A, B\}, \{C, D\}$ of order $< k$, at least one of $\{A \cap C, B \cup D\}$ and $\{A \cup C, B \cap D\}$ has also order $< k$.

### 5.2.2 Tangles and separations of graphs

Let $G = (V, E)$ be a graph. A *separation $\{A, B\}$ of $G$* is a separation of its vertex set $V$ such that no edge of $G$ joins $A \setminus B$ and $B \setminus A$ (equivalently: $G = G[A] \cup G[B]$). We write $U(G)$

for the set of all unoriented separations of $G$ and $S_k(G) := \{\{A,B\} \in U(G) : |A \cap B| < k\}$; as above, we then define $\vec{U}(G)$ and $\vec{S}_k(G)$ as the respective sets of all oriented separations. Note that $U(G) \subseteq U(V)$ by definition. Moreover, $(\vec{U}(G), \leq)$ is a sublattice of $(\vec{U}(V), \leq)$, as it is straight forward to check that given two separations $(A,B)$, $(C,D)$ of $G$, the ordered pairs $(A \cap C, B \cup D)$ and $(A \cup C, B \cap D)$ are again separations of $G$.

Now let $k \geq 1$ be an integer. An *orientation* of $S_k(G)$ is a set $O \subseteq \vec{S}_k(G)$ that contains precisely one orientation of every separation $\{A,B\} \in S_k(G)$. If an orientation $O$ of $S_k(G)$ contains $(A,B)$, we say that $O$ *orients* $\{A,B\}$ *as* $(A,B)$. A *$k$-tangle* or *tangle of order $k$* in $G$ is an orientation $\tau$ of $S_k(G)$ which contains no element of

$$\mathcal{T} := \mathcal{T}(G) := \{\{(A_1,B_1),(A_2,B_2),(A_3,B_3)\} \subseteq \vec{U}(G) : G[A_1] \cup G[A_2] \cup G[A_3] = G\}.$$

as a subset. We say the $k$-tangle *avoids* $\mathcal{T}$. The elements of $\mathcal{T}$ are called *forbidden triples*. A forbidden triple of $\mathcal{T}$ such that two of the $(A_i,B_i)$ are equal is a *forbidden tuple*. Tangles of unspecified order are referred to as *tangles in $G$*. A $k$-tangle $\tau$ in $G$ *extends* to a $k^*$-tangle $\tau^*$ in $G$ for $k^* \geq k$ if $\tau \subseteq \tau^*$. We say that a separation $\{A,B\}$ of $G$ *distinguishes* two tangles $\tau, \tau'$ in $G$ if $\tau$ and $\tau'$ contain distinct orientations of $\{A,B\}$.

It is well-known [1]*(2.6) that tangles of order 1 and 2 are in a one-to-one correspondence to the components and blocks[2], respectively:

**Proposition 5.2.1.** *For a tangle $\tau$ in a graph $G$, let $X_\tau := \bigcap_{(A,B)\in\tau} B$. Then the map $\tau \mapsto G[X_\tau]$ is a bijection between the tangles of order 1 and the components of $G$ as well as between the tangles of order 2 and the set of all blocks of $G$.* □

From its definition, tangles have several properties that we will frequently make use of. First, every tangle $\tau$ is *consistent*: there are no two separations $(A,B),(C,D) \in \tau$ such that $(D,C) \leq (A,B)$. Secondly, if $\tau$ has order $k$, then $(X, V(G)) \in \tau$ for all $X \subseteq V(G)$ of size $< k$; this property is sometimes called the *regularity* of $\tau$. Thirdly, $\tau$ has the *profile property*: if $(A,B),(C,D) \in \tau$ and their supremum $(A \cup C, B \cap D)$ again has order $< k$, then it is also in the tangle $\tau$.

Finally, let us make two observations about the interplay of tangles with the partial order $\leq$ on oriented separations. First, if an orientation $\tau$ of $S_k(G)$ is not a tangle and thus contains a forbidden triple, then $\tau$ also contains a forbidden triple whose elements are $\leq$-maximal in $\tau$. Secondly, if $\tau$ is a tangle in $G$, then every $\leq$-maximal separation $(A,B) \in \tau$ satisfies that $G[B \setminus A]$ is connected.

### 5.2.3 Topological minors

Let $G$ be a graph. By *suppressing* a degree-2 vertex $v$ of $G$, we obtain the graph $G' := G - v + uw$ where $u, w$ are the two neighbours of $v$ in $G$. As this operation is only defined for vertices of degree 2, we often just say that the graph $G'$ is obtained from $G$ by suppressing a vertex. A *topological minor* $G'$ of $G$ is a graph obtained from $G$ by a sequence of deleting edges, suppressing vertices and deleting vertices. Equivalently, $G'$ is obtained from $G$ by a sequence of suppressing vertices in a subgraph of $G$. We remark that if a topological minor $G'$ of $G$ is connected, we may obtain $G'$ also by a sequence of deleting edges, suppressing vertices and passing to proper components of the current graph.

[2]Recall that a *block* of a graph is a $\subsetneq$-maximal connected subgraph $H$ such that $H - v$ is connected for every vertex $v \in H$. In particular, as the empty graph is not considered to be connected, every block contains some edge.

### 5.2.4 Lifts of tangles

It is well-known that tangles in minors lift to tangles in the host graph (see [1]*(6.1) or [45]*Lemma 2.1). Here, we introduce the notion only for topological minors.

Let $\tau'$ be a $k$-tangle in a subgraph $G'$ of $G$. Then the *lift* $\tau$ of $\tau'$ to $G$ is the set consisting of precisely those $(A, B) \in \vec{S}_k(G)$ whose restriction $(A \cap V(G'), B \cap V(G'))$ is in $\tau'$. It is immediate that $\tau$ is a $k$-tangle in $G$, since $G'$ is a subgraph of $G$ and thus every forbidden triple in $\tau$ restricts to a forbidden triple in $\tau'$ by definition.

Now let $\tau'$ be a tangle of order $k \geq 3$ in a graph $G'$ obtained from $G$ by suppressing a vertex $v$. Denote the two neighbours of $v$ in $G$ by $u_1, u_2$. We define the *lift* $\tau$ of $\tau'$ to $G$, as follows.

Let $(A, B) \in \vec{S}_k(G)$ be arbitrary. We denote by $(A', B')$ the pair $(A \setminus \{v\}, B \setminus \{v\})$ and by $C'_i$ the set $C' \cup \{u_i\}$ for $C' = A', B'$ and $i = 1, 2$.

1. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$, then $(A', B')$ is a separation of $G'$, and we include $(A, B)$ in $\tau$ if $(A', B') \in \tau'$.

2. If, for $\{i, j\} = \{1, 2\}$, $u_i \in A \setminus B$ and $u_j \in B \setminus A$, then $(A', B'_i)$ and $(A'_j, B')$ are separations of $G'$, and we include $(A, B)$ in $\tau$ if at least one of $(A', B'_i)$ and $(A'_j, B')$ is in $\tau'$.

Note that in 2 either both $(A', B'_i), (A'_j, B')$ are in $\tau'$ or both their inverses are in $\tau'$, which implies that $\tau$ does not contain both orientations of a separation, and is hence an orientation of $S_k(G)$. Indeed, we otherwise have $(A', B'_i), (B', A'_j) \in \tau$, as we cannot have $(B'_i, A), (A'_j, B') \in \tau$ because of $(A', B'_i) \leq (A'_j, B')$ and the consistency of the tangle $\tau'$. But we have $(G'[A'] \cup G'[B']) + u_1u_2 = G'$, and thus $(A', B'_i)$, $(B', A'_j)$ and $(\{u_1, u_2\}, V(G'))$ would form a forbidden triple in $\tau'$, since $(\{u_1, u_2\}, V(G')) \in \tau$ due to the regularity of the $k$-tangle $\tau$ with $k \geq 3$. From this it also follows that a forbidden triple in $\tau$ would correspond to a forbidden triple in $\tau'$, by replacing each $(A, B)$ in the forbidden triple by $(A'_1, B')$ or $(A'_2, B')$ if necessary. Thus, the lift $\tau$ of the tangle $\tau'$ of $G'$ of order $k \geq 3$ to $G$ as defined above is indeed a $k$-tangle in $G$.

Given a $k$-tangle $\tau'$ in a topological minor $G'$ of a graph $G$, we now obtain the *lift* $\tau$ of $\tau'$ to $G$ by iteratively considering the lifts along the sequence of edge deletions, vertex suppressions and vertex deletions from which $G'$ originated from $G$.

## 5.3 Definition of 'survive' and special cases of Theorem 4

In this section we define what it means for a tangle $\tau$ in a graph $G$ to 'survive' in a topological minor $G'$ of $G$. This notion can be seen as a converse to lifting a tangle. But let us emphasise that while a tangle of order at least 3 in $G'$ always lifts to $G$, a tangle in $G$ need not survive in $G'$. We first define 'survive' for subgraphs $G'$ of $G$ in section 5.3.1 and then for graphs $G'$ obtained from $G$ by suppressing a single vertex of degree 2 in section 5.3.2. Alongside these definitions, we prove several lemmas which all deal with special cases of Theorem 4. Finally, we provide example 5.3.1 which demonstrates that suppressing vertices of degree 2 needs to be allowed in Theorem 4.

### 5.3.1 Extending and inducing tangles in subgraphs

Recall that for every subgraph $G'$ of a graph $G$ on the same vertex set, a separation of $G$ is also a separation of $G'$, i.e. $U(G) \subseteq U(G')$ and $S_k(G) \subseteq S_k(G')$ for all integers $k \geq 1$. We say that an orientation $\tau$ of $S_k(G)$ for some integer $k \geq 1$ *extends* to an orientation $\tau'$ of $S_k(G')$ if $\tau'$ orients every separation in $S_k(G)$ in the same way as $\tau$ (equivalently: $\tau \subseteq \tau'$). In this situation, we also sometimes say that the $k$-tangle $\tau$ in $G$ *survives* as the $k$-tangle $\tau'$ in $G'$. If $\tau$ is a $k$-tangle in $G$ and there exists precisely one $k$-tangle $\tau'$ in $G'$ to which $\tau$ extends, then we also say that $\tau$ *induces* the $k$-tangle $\tau'$. In this paper, $G'$ will often either be a component of $G$ or arise from $G$ by the deletion of a single edge $e \in G$. We remark that if a $k$-tangle $\tau$ in $G$ extends to a $k$-tangle $\tau'$ in a subgraph $G'$ of $G$, then $\tau$ is obviously the lift of $\tau'$ to $G$.

**Lemma 5.3.1.** *Let $\tau$ be a 1-tangle in a graph $G$. Then $\tau$ extends to a 1-tangle $\tau'$ in $G - e$ for every edge $e \in G$.*

*Proof.* Let $e$ be an arbitrary edge of $G$, and let $G' := G - e$. By theorem 5.2.1, the set $X_\tau := \bigcap_{(A,B)\in\tau} B$ is a component of $G$ corresponding to the 1-tangle $\tau$. Let $X'$ be a component of $G'$ whose vertex set is contained in $X_\tau$. By theorem 5.2.1 the component $X'$ of $G'$ corresponds to a 1-tangle $\tau'$ in $G'$ with $X_{\tau'} = \bigcap_{(A,B)\in\tau'} B = V(X')$. Since every vertex of $G$ lies on precisely one side of every separation of $G$ of order 0, a separation $(A, B)$ of $G$ of order 0 is contained in $\tau'$ if and only if $X_{\tau'} \subseteq B$. So for every $(A, B) \in \tau$, we have $X_{\tau'} \subseteq X_\tau \subseteq B$ and thus $(A, B) \in \tau'$. Hence, $\tau$ extends to $\tau'$. □

We remark that in the proof of theorem 5.3.1 the deletion of $e$ may give rise to (at most) two components $X'$ in $G[X_\tau]$. Thus, $\tau$ does not necessarily extend to a unique 1-tangle in $G'$.

**Lemma 5.3.2.** *Let $\tau$ be a 2-tangle in a graph $G$ with at least 2 edges. Then $\tau$ extends to a 2-tangle $\tau'$ in $G - e$ for some edge $e \in G$.*

*Proof.* By theorem 5.2.1, $G[X_\tau]$ is a block of $G$; in particular, it contains an edge $f$. Let $e \neq f$ be any other edge in $G$, and consider $G' := G - e$. Let $X'$ be the vertex set of the block of $G'$ containing $f$, and note that $X' \subseteq X_\tau$ by the definition of $G'$. By theorem 5.2.1, the block $X'$ corresponds to a 2-tangle $\tau'$ of $G'$; in particular, $f \in G'[X'] = G'[X_{\tau'}]$. Since every edge in $G$ lies on precisely one side of every separation of $G$ of order $\leq 1$, a separation $(A, B)$ of $G$ of order $\leq 1$ is contained in $\tau'$ if and only if $f \in G[B]$. For every $(A, B) \in \tau$, we have $f \in G[X_\tau] \subseteq G[B]$ and hence $(A, B) \in \tau'$. Thus, $\tau$ extends to $\tau'$. □

Similarly, as in the proof of theorem 5.3.1 also in the proof of theorem 5.3.2 the block $X_\tau$ may split into two (or even more) blocks in $G'$. Thus, $\tau$ does not necessarily extend to a unique 2-tangle in $G'$.

**Lemma 5.3.3.** *Let $k \geq 1$ be an integer, and let $G$ be a graph with a $k$-tangle $\tau$. Then there exists a (unique) component $G'$ of $G$ such that $\tau$ extends to a $k$-tangle $\tau'$ in $G'$. Moreover, $\tau$ induces the $k$-tangle $\tau'$ in $G'$.*

*Proof.* Let $\tau_1$ be the 1-tangle in $G$ with $\tau_1 \subseteq \tau$. By theorem 5.2.1, $G[X]$ with $X := X_{\tau_1} = \bigcap_{(A,B)\in\tau_1} B$ is a component of $G$. We claim that $G' := G[X]$ is as desired.

We define an orientation $\tau'$ of $S_k(G')$ based on $\tau$. For any separation $\{A', B'\}$ of order $< k$ of $G'$, we consider the separation $\{A, B\}$ of $G$ defined by $A := A' \cup (V(G) \setminus X)$

and $B := B'$. This again has order $< k$ and is hence oriented by $\tau$. So if $(A, B) \in \tau$, then we put $(A', B') \in \tau'$, and if $(B, A) \in \tau$, we put $(B', A') \in \tau'$.

We now show that $\tau$ extends to $\tau'$. It suffices to check that if we included $(A', B')$ or $(B', A')$ in $\tau$ because of $(A, B)$ or $(B, A)$ in $\tau$, respectively, then there exists no $(C, D) \in \tau$ such that $(C \cap V(X), D \cap V(X)) = (B', A')$ or $= (A', B')$, respectively. Suppose for a contradiction that there is such a $(C, D) \in \tau$. The definition of $X := X_{\tau_1}$ ensures that the separation $((V(G) \setminus X), X)$ of order 0 is in $\tau_1 \subseteq \tau$. Thus, the forbidden triple

$$\{(A, B), (C, D)), ((V(G) \setminus X), X)\} \text{ or } \{(B, A), (C, D)), ((V(G) \setminus X), X)\})$$

is contained in the tangle $\tau$, respectively, which is a contradiction. Thus, $\tau$ extends to $\tau'$.

It remains to show that the orientation $\tau'$ of $S_k(G')$ is even a tangle in $G'$. Suppose for a contradiction that there exists a forbidden triple $\{(A'_i, B'_i) : i \in [3]\}$ in $\tau'$. As above, $(A_i, B_i) := (A'_i \cup (V(G) \setminus X), B'_i)$ is a separation of $G$ of order $< k$ and by construction contained in $\tau$ for all $i \in [3]$. Since $G'$ is a component of $G$, it is immediate that $\{(A_i, B_i) : i \in [3]\}$ forms a forbidden triple in $\tau$, which is a contradiction.

We remark that the proof immediately ensures that $G'$ is the unique component such that $\tau$ extends to a $k$-tangle in it, and also $\tau'$ is the unique such $k$-tangle. □

**Lemma 5.3.4.** *Let $\tau$ be a tangle in $G$ of order $k \geq 3$. Suppose that $G$ has a vertex $v$ of degree 1 and let $e$ be the unique edge incident to $v$. Then $\tau$ induces a $k$-tangle $\tau'$ in $G - e$.*

*Proof.* Let $\tau'$ be the subset of $\vec{S}_k(G')$ which contains $(A, B) \in \tau'$ if $(A, B) \in \tau$, $(A \setminus \{v\}, B \cup \{v\}) \in \tau$ or $(A \cup \{v\}, B \setminus \{v\}) \in \tau$. The regularity of the tangle $\tau$ of order $\geq 3$ ensures that $(e, V(G)) \in \tau$. Thus, $\tau'$ is an orientation of $S_k(G')$, as any violation would form together with $(e, V(G))$ a forbidden triple in $\tau$.

Suppose for a contradiction that $\{(A'_i, B'_i) : i \in [3]\}$ is a forbidden triple in $\tau'$. Since $G' = \bigcup_{i \in [3]} G'[A'_i]$, some $A'_j$ contains the other endvertex $u \neq v$ of $e$. Thus, $(A_j, B_j) := (A'_j \cup \{v\}, B'_j \setminus \{v\})$ is a separation of $G$ and thus in $\tau$, as $\tau'$ is an orientation. Let $(A_i, B_i) \in \tau$ which witnesses that $(A'_i, B'_i) \in \tau'$ for $i \in [3] \setminus \{j\}$. Now the $(A_i, B_i)$ form a forbidden triple in $\tau$, which is a contradiction. □

### 5.3.2 Inducing tangles in graphs obtained by vertex suppression

Recall that for a given vertex $v \in V(G)$ of degree 2, the graph obtained from $G$ by *suppressing the vertex $v$* is $G' := G - v + uw$ where $u, w$ are the two neighbours of $v$ in $G$. Let $\{A, B\}$ be a separation of $G'$. If $u, w \in A$, then $\{A \cup \{v\}, B\}$ is a separation of $G$, and analogously if $u, w \in B$, then $\{A, B \cup \{v\}\}$ is a separation of $G$. In particular, they have the same order as $\{A, B\}$. We remark that at least one of $u, w \in A$ and $u, w \in B$ holds, as $uw$ is an edge of $G'$.

We say that a $k$-tangle $\tau$ of $S_k(G)$ for some integer $k \geq 1$ *induces* the subset $\tau' \subseteq \vec{S}_k(G')$ consisting of those $(A, B) \in \vec{S}_k(G')$ such that at least one of $(A \cup \{v\}, B)$ and $(A, B \cup \{v\})$ is in $\tau$. In particular, if $\{A \cup \{v\}, B\}$ is a separation of $G$, then one of its orientations is contained in $\tau$, and thus at least one of $(A, B)$ or $(B, A)$ is in $\tau'$; but $\tau'$ might contain both if also $\{A, B \cup \{v\}\}$ is a separation of $G$. theorem 5.3.5 below ensures that if $k \geq 3$, then not only the latter does not happen, but $\tau'$ is even a $k$-tangle. In this situation, we also sometimes say that the $k$-tangle $\tau$ in $G$ *survives* as the $k$-tangle $\tau'$ in the graph $G'$ obtained from $G$ by suppressing a vertex. We remark that if a $k$-tangle $\tau$ in $G$ extends to a $k$-tangle $\tau'$ in the graph $G'$ obtained from $G$ by suppressing a vertex, then $\tau$ is obviously the lift of $\tau'$ to $G$.

**Lemma 5.3.5.** *Let $G$ be a graph, and let $G' = G - v + uw$ be the graph obtained by suppressing a vertex $v$ of degree 2, where its two neighbours in $G$ are $u, w$. Then for a given $k$-tangle $\tau$ with $k \geq 3$ in $G$ the set $\tau' \subseteq \vec{S}_k(G')$ induced by $\tau$ is a $k$-tangle in $G'$.*

*Proof.* We claim that $(A \cup \{v\}, B) \in \tau$ if and only if $(A, B \cup \{v\}) \in \tau$, if both $\{A \cup \{v\}, B\}$ and $\{A, B \cup \{v\}\}$ are separations of $G$. Suppose for a contradiction that this is not the case. The consistency of $\tau$ ensures that $(A, B \cup \{v\}) \in \tau$ and $(B, A \cup \{v\}) \in \tau$. Since $k \geq 3$ and every tangle is regular, $(\{u, v, w\}, V(G) \setminus \{v\}) \in \tau$. Now $\{(A, B \cup \{v\}), (B, A \cup \{v\}), (\{u, v, w\}, V(G) \setminus \{v\})\}$ forms a forbidden triple in the tangle $\tau$, which is a contradiction.

The claim above shows that $\tau'$ is an orientation of $S_k(G')$. It remains to show that it is a $k$-tangle in $G'$. Suppose that $\{(A'_i, B'_i) : i \in [3]\}$ is a forbidden triple in $\tau'$. Then there is $j \in [3]$ with $u, w \in A'_j$, since $uw$ is an edge in $G'$. Hence, $\{A'_j \cup \{v\}, B'_j\}$ is a separation of $G$, and thus $(A_j, B_j) := (A'_j \cup \{v\}, B'_j) \in \tau$ by definition of $\tau'$. For every $i \in [3] \setminus \{i\}$, we let $(A_i, B_i) \in \tau$ be the separation which witnesses that $(A'_i, B'_i) \in \tau'$. Then $\{(A_i, B_i) : i \in [3]\}$ is a forbidden triple in $\tau$, as $\{(A'_i, B'_i) : i \in [3]\}$ is a forbidden triple in $G'$ and $u, v, w \in A_j$. This is a contradiction to the $\tau$ being a tangle in $G$. □

### 5.3.3 Suppressing vertices of degree 2 in Theorem 4

We conclude this section with an example that shows that there are connected graphs $G$ with arbitrarily many edges which have a $k$-tangle such that every graph obtained from $G$ by deleting an edge has no $k$-tangles at all. In particular, in Theorem 4 it is necessary to allow the suppression of a vertex.

**Example 5.3.1.** *For every $k, M \in \mathbb{N}$ with $k \geq 3$, there exists a connected graph $G$ with at least $M$ edges and which has a $k$-tangle such that, for every edge $e$ of $G$, the graph $G - e$ does not have a $k$-tangle.*

*Proof.* Let $G'$ be some connected graph which has a $k$-tangle but which is such that, for every edge $e$ of $G'$, the graph $G' - e$ does not have any $k$-tangles. Such graphs exist: Take any graph $H$ that has a $k$-tangle, and let $H_0 := H \supset H_1 \supset \cdots \supset H_n$ be a maximal sequence such that $H_{i+1}$ is obtained from $H_i$ by either deleting an edge or taking a proper component of $H_i$, and such that $H_n$ still has a $k$-tangle. Set $G' := H_n$. By the maximal choice of the sequence $(H_i)_{i \in [n]}$ and because of theorem 5.3.3, $G'$ is connected and no graph obtained from $G'$ by deleting an edge has a $k$-tangle.

Since edgeless graphs have no tangles of order $\geq 2$, the graph $G'$ contains an edge $uv$. Let $G$ be obtained from $G'$ replacing $uv$ by a $u$–$v$ path of length at least $M + 1$. Then $G$ has at least $M$ edges, and it has a $k$-tangle as every tangle of order $k \geq 3$ in $G'$ lifts to a $k$-tangle in $G$. But $G - e$ has no $k$-tangle for every edge $e$ of $G$, since any such tangle would induce a $k$-tangle in $G' - e'$ by theorems 5.3.4 and 5.3.5 where $e' := e$ if $e \in E(G)$ or $e' := uv$ otherwise. □

## 5.4 Proof of Theorem 5 if $G$ has a higher-order tangle

In this section we prove Theorem 5 for graphs which contain a tangle of order $> k$.

**Theorem 5.4.1.** *Let $\tau$ be a tangle in $G$ of order $k \geq 2$. Suppose further that there exists a $(k+1)$-tangle $\tau^*$ in $G$. Then there is an edge $e \in E(G)$ such that $\tau$ extends to some $k$-tangle $\tau'$ in $G - e$.*

We distinguish two cases: First, we show that if $\tau$ itself extends to a $(k+1)$-tangle in $G$, then $\tau$ extends to a $k$-tangle in $G-e$ for every edge $e \in E(G)$ (theorem 5.4.3). Otherwise, there exists a $(k+1)$-tangle $\tilde{\tau}$ in $G$ which does not extend $\tau$. Here, $\tau$ does not extend to a $k$-tangle in $G-e$ for every edge $e \in G$, but only for some such edges $e$ and we will need some care to find them (theorem 5.4.4).

For both cases, we will make use of the following observation.

**Lemma 5.4.2.** *Let $k \geq 3$ be an integer, and let $G$ be a graph with a $k$-tangle $\tau$. Further, let $e$ be an edge of $G$ and let $\{A, B\}$ be a separation of $G-e$ with $e \in E_G(A \setminus B, B \setminus A)$. If $|A \cap B| < k-1$, then $(A \cup e, B) \in \tau$ if and only if $(A, B \cup e) \in \tau$.*

*Proof.* The assumptions immediately imply that $\{A \cup e, B\}$ and $\{A, B \cup e\}$ are separations of $G$ of order $< k$, and so $\tau$ orients both of them. The consistency of $\tau$ together with $(A, B \cup e) \leq (A \cup e, B)$ yields the forwards implication. The backwards implication follows from the fact that $\{(A, B \cup e), (B, A \cup e), (e, V(G))\}$ is a forbidden triple but $(e, V(G)) \in \tau$, since $\tau$ is a $k$-tangle with $k \geq 3$ and $V(G)$ cannot be the small side of any separation in $\tau$. □

We start with the case in which $\tau$ extends to a $(k+1)$-tangle in $G$:

**Lemma 5.4.3.** *Let $k \geq 2$ be an integer, and let $G$ be a graph with a $k$-tangle $\tau$. If $\tau$ extends to a $(k+1)$-tangle $\tilde{\tau}$ in $G$, then $\tau$ extends to a $k$-tangle $\tau'$ in $G-e$ for every edge $e \in E(G)$.*

*Proof.* Consider $G' := G - e$. We define an orientation $\tau'$ of $S_k(G')$ as follows: If $\{A, B\} \in S_k(G')$ is also a separation of $G$, then we put $(A, B) \in \tau'$ if and only if $(A, B) \in \tau$. Otherwise, the edge $e$ has one endvertex in $A \setminus B$ and the other one in $B \setminus A$. So both $\{A \cup e, B\}$ and $\{A, B \cup e\}$ are separations of $G$ of order $|A, B| + 1 \leq k$, and these two separations are oriented by the $(k+1)$-tangle $\tilde{\tau}$, and we have $(A \cup e, B) \in \tau$ if and only if $(A, B \cup e) \in \tilde{\tau}$ by theorem 5.4.2. Then we put $(A, B) \in \tau'$ if and only if $(A \cup e, B) \in \tilde{\tau}$ (equivalently: $(A, B \cup e)$). The first part of the definition guarantees that $\tau$ extends to $\tau'$.

It remains to show that $\tau'$ is a $k$-tangle in $G'$. If there exists a forbidden triple $\{(A_i, B_i) : i \in [3]\}$ in $G'$ for $\tau'$, then we obtain a forbidden triple for $\tilde{\tau}$ in $G$ by replacing those $(A_i, B_i)$ that are not already separations of $G$ with $(A_i \cup e, B_i)$. The arising triple is then by the definition of $\tau'$ a forbidden triple in $\tilde{\tau}$. This contradicts that $\tilde{\tau}$ is a tangle in $G$ by assumption. □

We remark that one can also show a vertex-version of theorem 5.4.3 along the same lines: if a $k$-tangle $\tau$ in $G$ extends to a $(k+1)$-tangle in $G$, then $\tau$ extends to a $k$-tangle in $G' := G - v$ for every vertex $v \in V(G)$.

Now we turn to the case that $G$ has a $(k+1)$-tangle $\tilde{\tau}$ which does not extend $\tau$. Let us briefly describe our proof strategy: First, we observe that there exists a separation $(B, A) \in \tilde{\tau}$ which is $\leq$-maximal in $\tilde{\tau} \cap \vec{S}_k(G)$ and distinguishes $\tau$ and $\tilde{\tau}$. We will then delete an arbitrary edge $e$ on the side of $\{A, B\}$ which is small with respect to $\tau$, i.e. $e \in G[A \setminus B]$. To define the desired $k$-tangle $\tau'$ of $G'$ to which $\tau$ shall extend, we first orient all separations of $G'$ that are 'forced' by $\tau$ in that they have an orientation which was either already in $\tau$ or which must be in $\tau'$ to achieve the desired consistency of $\tau'$. The remaining separations are then oriented according to $\tilde{\tau}$; for this, we draw on the fact that $\tilde{\tau}$ has order $k+1$ and hence naturally defines an orientation of all the separations in $S_k(G')$ using theorem 5.4.2. While $\tau$ extends to this orientation $\tau'$ by construction, the main part of the proof is devoted to show that $\tau'$ is indeed a tangle.

Intuitively speaking, the construction of $\tau'$ ensures that neither the separations forced by $\tau$ nor those oriented according to $\tilde{\tau}$ contain a forbidden triple. The maximal choice of $(B, A)$ together with submodularity arguments then ensures that there is also no forbidden triple consisting of both kinds of separations: it allows us to transfer any such forbidden triple in $\tau'$ either into one in $\tau$ or into one in $\tilde{\tau}$.

**Lemma 5.4.4.** *Let $k \geq 2$ be an integer, and let $G$ be a graph with a $k$-tangle $\tau$. If there exists a $(k+1)$-tangle $\tilde{\tau}$ in $G$ with $\tau \nsubseteq \tilde{\tau}$, then there exists an edge $e \in E(G)$ such that $\tau$ extends to a $k$-tangle $\tau'$ in $G - e$.*

*Proof.* We first find an edge $e$ of $G$ that we afterwards prove to be as desired. For this, let $(B, A) \in \tilde{\tau}$ be a separation of $G$ which distinguishes $\tau$ and $\tilde{\tau}$ and is $\leq$-maximal in $\tilde{\tau}$ among all such distinguishing separations. Note that $(B, A)$ is even maximal in $\tilde{\tau} \cap \vec{S}_k$: any separation $(C, D) \in \tilde{\tau} \cap \vec{S}_k$ with $(B, A) < (C, D)$ would also distinguish $\tau$ and $\tilde{\tau}$ since $(D, C) \in \tau$ by the consistency of $\tau$. We then choose an arbitrary edge $e$ in $G[A \setminus B]$. Let us first show its existence.

There exists a vertex $v \in A \setminus B$, since otherwise $(V(G), A) = (B, A) \in \tilde{\tau}$, contradicting that $\tilde{\tau}$ is a tangle. If the vertex $v$ has a neighbour in $A \setminus B$, then the edge joining them is as desired. So suppose for a contradiction that all neighbours of $v$ are in $A \cap B$. We can then find a forbidden triple in $\tilde{\tau}$: First, we can move $v$ from $A \setminus B$ to $B \setminus A$ to obtain a new separation $\{A \setminus \{v\}, B \cup \{v\}\}$ of $G$, which has the same order as $\{A, B\}$. Thus, $\tilde{\tau}$ contains an orientation of it, and we must have $(A \setminus \{v\}, B \cup \{v\}) \in \tilde{\tau}$ due to the maximality of $(B, A)$ in $\tilde{\tau} \cap \vec{S}_k(G)$. Secondly, since $|A \cap B| < k$ and $\tilde{\tau}$ is a $(k+1)$-tangle in $G$, we have $((A \cap B) \cup \{v\}, V(G)) \in \tilde{\tau}$. Hence, $\{(B, A), (A \setminus \{v\}, B \cup \{v\}), ((A \cap B) \cup \{v\}, V(G))\}$ is contained in the tangle $\tilde{\tau}$, but it is also a forbidden triple, which is a contradiction. All in all, $v$ has a neighbour in $A \setminus B$; in particular, $G[A \setminus B]$ contains an edge.

From now on, we prove that the chosen edge $e \in G[A \setminus B]$ is as desired. For this, we consider $G' := G - e$ and construct an orientation $\tau'$ of $S_k(G')$ to which $\tau$ extends. It then remains to show that $\tau'$ is a tangle in $G'$.

For the construction of $\tau'$, note that $\tau'$ has to contain not only $\tau$, but also all orientations of separations of $S_k(G')$ that are 'forced' by the request that $\tau'$ shall again be a tangle and hence especially consistent. More formally, we say that $\tau$ *forces* an orientation of a separation $\{C, D\}$ of $G'$ if there exists a separation $(E, F) \in \tau$ such that $(C, D) \leq (E, F)$ or $(D, C) \leq (E, F)$. In particular, $\tau$ forces an orientation of every separation in $S_k(G')$ which is also a separation of $G$ and the separations in $\vec{S}_k(G')$ which are maximal among all those forced by $\tau$ are separations of $G$. Note that the consistency of $\tau$ ensures that at most one orientation of a separation in $S_k(G')$ is forced by $\tau$.

We now define the orientation $\tau'$ of $S_k(G')$. If $\tau$ forces an orientation of a separation $\{C, D\} \in S_k(G')$, then we put the respective orientation in $\tau'$. Otherwise, $\{C, D\}$ is especially not a separation of $G$, so $e$ has one endvertex in $C \setminus D$ and the other one in $D \setminus C$. Then $\{C \cup e, D\}$ and $\{C, D \cup e\}$ are separations of $G$ of order at most $k$, as $\{C, D\}$ has order less than $k$. Thus, both these separations are oriented by the $(k+1)$-tangle $\tilde{\tau}$ in $G$, and by theorem 5.4.2, we have $(C \cup e, D) \in \tilde{\tau}$ if and only if $(C, D \cup e) \in \tilde{\tau}$. Now if $(C \cup e, D) \in \tilde{\tau}$ (equivalently: $(C, D \cup e) \in \tilde{\tau}$), then we put $(C, D) \in \tau'$, and if $(D \cup e, C) \in \tilde{\tau}$ (equivalently: $(D, C \cup e) \in \tilde{\tau}$), then we put $(D, C) \in \tau'$. This definition of $\tau'$ ensures that $\tau'$ indeed is an orientation of $S_k(G')$ and also that $\tau$ extends to $\tau'$.

Thus, it remains to show that $\tau'$ is indeed a tangle. Suppose for a contradiction that there is a forbidden triple $\{(C_i, D_i) : i \in [3]\}$ in $\tau'$. Without loss of generality, we may assume that all the $(C_i, D_i)$ are $\leq$-maximal in $\tau'$. We now aim to use $\{(C_i, D_i) : i \in [3]\}$ together with the construction of $\tau'$ to find a forbidden triple in $G$ which is contained in either $\tau$ or $\tilde{\tau}$. This then yields a contradiction since both $\tau$ and $\tilde{\tau}$ are tangles in $G$. Towards this, we first give a condition on the $(C_i, D_i)$ which allows us to find a forbidden triple in $\tilde{\tau}$ and prove afterwards that if this condition does not hold, then we can find a forbidden triple in $\tau$.

First, suppose that each $\{C_i, D_i\}$ either crosses $\{A, B\}$ or satisfies $(C_i, D_i) \leq (B, A)$. In this case, we aim to find a forbidden triple in $\tilde{\tau}$. Towards this, the following lemma shows that $(C_i, D_i) \in \tilde{\tau}$ if $\{C_i, D_i\}$ crosses $\{A, B\}$ and is also a separation of $G$.

**Sublemma 5.4.5.** *Let $\{C, D\}$ be a separation of $G'$ that is also a separation of $G$ and whose orientation $(C, D) \in \tau'$ is $\leq$-maximal in $\tau'$. If $\{C, D\}$ crosses $\{A, B\}$, then $(C, D) \in \tilde{\tau}$.*

*Proof.* Assume that the infimum $(A \cap C, B \cup D)$ of $(A, B)$ and $(C, D)$ has order less than $k$. By the maximality of $(B, A)$ in $\tilde{\tau} \cap \vec{S}_k$, we then have $(A \cap C, B \cup D) \in \tilde{\tau}$. Since $\{(B, A), (A \cap C, B \cup D), (D, C)\}$ is a forbidden triple in $G$, this then implies $(C, D) \in \tilde{\tau}$, as desired.

It remains to prove that $\{A \cap C, B \cup D\}$ has order less than $k$. By submodularity, it suffices to show that $\{A \cup C, B \cap D\}$ has order at least $k$. Suppose for a contradiction that it has order less than $k$. Then $\tau$ contains an orientation of $\{A \cup C, B \cap D\}$. Since $\tau$ extends to $\tau'$, we have $(C, D)$ in $\tau$. On the one hand, as $(C, D)$ is also $\leq$-maximal in $\tau$, we must have that its supremum $(A \cup C, B \cap D)$ with $(A, B)$ is not in $\tau$. On the other hand, the profile property of $\tau$ ensures that $(B \cap D, A \cup C) \notin \tau$, as $(A, B) \in \tau$. This is a contradiction. ∎

In this first case, where each $\{C_i, D_i\}$ either crosses $\{A, B\}$ or satisfies $(C_i, D_i) \leq (B, A)$, we can use theorem 5.4.5 to obtain a forbidden triple $\{(C_i', D_i') : i \in [3]\}$ in $\tilde{\tau}$ as follows: For $i \in [3]$, assume first that $\{C_i, D_i\}$ is also a separation of $G$. If $(C_i, D_i) \leq (B, A)$, then $(C_i', D_i') := (C_i, D_i) \in \tilde{\tau}$ since $(B, A) \in \tilde{\tau}$ and $\tilde{\tau}$ is consistent. Otherwise, $\{C_i, D_i\}$ crosses $\{A, B\}$ and then $(C_i', D_i') := (C_i, D_i) \in \tilde{\tau}$ by theorem 5.4.5. Secondly, if $\{C_i, D_i\}$ is not a separation of $G$, then the maximality of $(C_i, D_i)$ in $\tau'$ implies that $(C_i, D_i)$ cannot be forced by $\tau$. Thus, we have $(C_i', D_i') := (C_i \cup e, D_i) \in \tilde{\tau}$ by construction. Now $\{(C_i', D_i') : i \in [3]\}$ is a forbidden triple in the tangle $\tilde{\tau}$ in $G$, which is a contradiction.

So we assume that some $\{C_i, D_i\}$, say $\{C_1, D_1\}$, neither crosses $\{A, B\}$ nor satisfies $(C_i, D_i) \leq (B, A)$. In this case, we aim to find a forbidden triple in $\tau$. We claim that $(A, B) \leq (C_1, D_1)$ and that this yields $(C_1, D_1) \in \tau$: By our assumptions, $\{C_1, D_1\}$ is nested with $\{A, B\}$, but we do not have $(C_1, D_1) \leq (B, A)$ (equivalently: $(A, B) \leq (D_1, C_1)$). If $(D_1, C_1) \leq (A, B)$, then $(D_1, C_1) \in \tau'$ is forced by $(A, B) \in \tau$ but $\tau'$ is an orientation which already contains $(C_1, D_1)$, which is a contradiction. Furthermore, we cannot have $(C_1, D_1) < (A, B)$, since $(C_1, D_1)$ is maximal in $\tau'$. Thus, $(A, B) \leq (C_1, D_1)$; in particular, $e \in G[A \setminus B] \subseteq G[C_1 \setminus D_1]$. Therefore, $(C_1, D_1)$ is not only a separation of $G - e = G'$, but also one of $G$. Since $\tau$ extends to $\tau'$ and $(C_1, D_1) \in \tau'$, we obtain $(C_1, D_1) \in \tau$.

As shown, we have $e \in G[C_1]$. So if $(C_2, D_2)$ and $(C_3, D_3)$ are separations of $G$, then they are not only in $\tau'$ but also in $\tau$ as $\tau$ extends to $\tau'$ which yields our desired forbidden triple in $\tau$. So suppose that $(C_i, D_i)$ with $i \in \{2, 3\}$ is not a separation of $G$. We claim that $\{C_i, D_i\}$ crosses $\{A, B\}$. Indeed, if $\{C_i, D_i\}$ had an orientation that is greater than $(A, B)$, then $\{C_i, D_i\}$ would be a separation of $G$, as the deleted edge $e$ is contained in $G[A \setminus B]$. If $(C_i, D_i) < (A, B)$, then $(C_i, D_i)$ would in contradiction to its choice not be maximal in $\tau'$, since $(A, B)$ is also in $\tau'$, If $(D_i, C_i) \leq (A, B)$, then $(D_i, C_i) \in \tau'$ is forced by $(A, B) \in \tau$, which is a contradiction to $(C_i, D_i) \in \tau'$. So $\{C_i, D_i\}$ cannot be nested with $\{A, B\}$, that is, they cross. But then the following lemma shows that the infimum of $(C_i, D_i)$ and $(B, A)$ is in $\tau$.

**Sublemma 5.4.6.** *Let $\{C, D\}$ be a separation of $G'$ that is not a separation of $G$ and whose orientation $(C, D) \in \tau'$ is maximal in $\tau'$. If $\{C, D\}$ crosses $\{A, B\}$, then $(B \cap C, A \cup D) \in \tau$.*

*Proof.* Since $e \in G[A \setminus B]$, $\{B \cap C, A \cup D\}$ is a separation of $G$. Assume that it has order less than $k$. Then $\tau$ contains an orientation of $\{B \cap C, A \cup D\}$, and this orientation must not be $(A \cup D, B \cap C)$ as $\tau$ would otherwise force the orientation $(D, C)$ of $\{C, D\}$ to be in the orientation $\tau'$ which already contains $(C, D)$.

It remains to show that $\{B \cap C, A \cup D\}$ indeed has order less than $k$; suppose for a contradiction otherwise. Since both $\{A, B\}$ and $\{C, D\}$ have order at most $k - 1$, this implies that $\{B \cup C, A \cap D\}$ has order less than $k - 1$ by submodularity. The edge $e$ is in $G[A \setminus B]$ by its choice. Additionally, it has one endvertex in $C \setminus D$ and the other one in $D \setminus C$ because $\{C, D\}$ is not a separation of $G$ by assumption. Therefore, the order of $\{B \cup C \cup e, A \cap D\}$ increases compared to the order of $\{B \cup C, A \cap D\}$ by exactly one. So $\{B \cup C \cup e, A \cap D\}$ has order $< k$. We now show that none of its orientations can be contained in the tangle $\tilde{\tau}$, which then yields the desired contradiction.

On the one hand, the maximality of $(B, A)$ in $\tilde{\tau} \cap \vec{S}_k$ implies that $(B \cup C \cup e, A \cap D) \notin \tilde{\tau}$. On the other hand, since $\{C, D\}$ is not a separation of $G$, but $(C, D)$ is maximal in $\tau'$, the orientation $(C, D)$ of $\{C, D\}$ cannot be forced by $\tau$. Hence by construction of $\tau'$, we put $(C, D) \in \tau'$ because of $(C \cup e, D) \in \tilde{\tau}$. But $\{(B, A), (C \cup e, D), (A \cap D, B \cup C \cup e)\}$ is forbidden triple in $G$, so $(A \cap D, B \cup C \cup e) \notin \tilde{\tau}$. ■

Using theorem 5.4.6, we can now find a forbidden triple $\{(C_1, D_1), (C_2', D_2'), (C_3', D_3')\}$ in $\tau$ as follows: As shown above, $(C_1, D_1)$ is in $\tau$ and satisfies $(A, B) \leq (C_1, D_1)$. If $(C_i, D_i)$ with $i \in \{2, 3\}$ is a separation of $G$, then it also is in $\tau$, as $\tau$ extends to $\tau'$, and we set $(C_i', D_i') := (C_i, D_i)$. If it is not a separation of $G$, then $\{C_i, D_i\}$ must cross $\{A, B\}$, as shown above theorem 5.4.6, and theorem 5.4.6 yields $(C_i', D_i') := (B \cap C_i, A \cup D_i) \in \tau$. To see that $\{(C_1, D_1), (C_2', D_2'), (C_3', D_3')\} \subseteq \tau$ is indeed a forbidden triple in $G$, note that

$$G[C_1] \cup G[C_2'] \cup G[C_3'] \supseteq G[C_1] \cup G[C_2 \cap B] \cup G[C_3 \cap B] \supseteq G[C_1] \cup G[C_2 \cap D_1] \cup G[C_3 \cap D_1] = G,$$

where the last equation holds because $e \in G[A] \subseteq G[C_1]$ and $\{(C_i, D_i) : i \in \{1, 2, 3\}\}$ is a forbidden triple in $G' = G - e$. This concludes the proof. □

*Proof of theorem 5.4.1.* This follows immediately from theorem 5.4.3 and theorem 5.4.4. □

Our proof of theorem 5.4.1 heavily relies on the fact that the order of the additional tangle $\tilde{\tau}$ is greater than the one of $\tau$. However, we do not know whether similar proof techniques could also work if the order of $\tilde{\tau}$ does not exceed the one of $\tau$:

**Problem 5.4.7.** *Let $\tau$ be a tangle in $G$ of order $k \geq 3$. Suppose further that there exists another $k$-tangle $\tau^*$ in $G$ with $\tau^* \nsubseteq \tau$ and $G$ has minimum degree at least 3. Is there an edge $e \in G$ such that $\tau$ extends to a $k$-tangle $\tau'$ in $G - e$?*

## 5.5 Rainbow-Cloud-Decompositions in the absence of high-order tangles

Recall that theorem 5.4.1 in section 5.4 immediately yields Theorem 5 if the graph has a $(k+1)$-tangle. So from now on we work towards a proof of Theorem 5 for graphs without tangles of high order. In this section, we show that, in the absence of $(k+1)$-tangles, a large graph admits a certain type of decomposition, which we will call 'rainbow-cloud decomposition'; this decomposition is inspired by [52]. We will later use that this decomposition exhibits a substructure of the graph, the 'rainbow', which is a long linear structure that is fairly independent of the rest of the graph and internally made up of similar enough parts such that deleting an edge in one of the parts does not change the overall structure of the graph. In particular, it will allow us to understand how the separations of the graph change after deleting such an edge and hence how to find a tangle of this smaller graph to which our given tangle extends.

We begin by building up to the definition of a 'rainbow-cloud decomposition'. Let $G$ be a graph. First, a *linear decomposition*[3] of $G$ of *length* $M \in \mathbb{N}$ of $G$ is a family $\mathcal{W} = (W_0, W_1, \dots, W_M)$ of sets $W_i$ of vertices of $G$ such that

(L1) $\bigcup_{i=0}^{M} G[W_i] = G$,

(L2) if $0 \le i \le j \le k \le M$, then $W_i \cap W_k \subseteq W_j$, and

(L3) there is an integer $\ell$ such that $|W_{i-1} \cap W_i| = \ell$ for every $i \in [M]$, and

(L4) $W_{i-1} \neq W_{i-1} \cap W_i \neq W_i$ for all $i \in [M]$.

We call the sets $W_i$ the *bags* and and the induced subgraphs $G[W_i]$ the *parts* of the linear decomposition $\mathcal{W}$. Note that the bags of a linear decomposition of length at least 1 are non-empty by 4. The *adhesion sets* of a linear decomposition $\mathcal{W}$ are the sets $U_i := W_{i-1} \cap W_i$ for $i \in [M]$. The size of the adhesion sets $U_i$ is the *adhesion* of $\mathcal{W}$. We emphasise that adhesion 0 is allowed. Whenever we introduce a linear decomposition as $\mathcal{W}$ without specifying its bags, then we will tacitly assume the bags to be denoted by $W_0, \dots, W_M$ and the adhesion sets by $U_1, \dots, U_M$.

Next we turn to the definition of 'rainbow-decompositions' which are special linear decompositions whose adhesion sets are minimal $U_1$–$U_M$ separators of $G$ as witnessed by respective families of disjoint paths. To make this formal, a *linkage* in a graph $G$ is a set $\mathcal{P}$ of disjoint paths in $G$. If $A$ and $B$ are sets of vertices of $G$ such that $\mathcal{P}$ consists of $A$–$B$ paths, i.e. such paths that meet $A$ precisely in one endvertex and $B$ precisely in the other endvertex, then $\mathcal{P}$ forms an *$A$–$B$ linkage*. A linear decomposition $\mathcal{W}$ of adhesion $\ell$ and length $M$ is called a *rainbow-decomposition* of *adhesion* $\ell$ and *length* $M$ if it has the following three properties:

(R1) There exists a $U_i$–$U_{i+1}$ linkage of cardinality $\ell$ in $G[W_i]$ for every $i \in [M-1]$.

(R2) Every part $G[W_i]$ of $\mathcal{W}$ is connected.

(R3) Every two consecutive adhesion sets $U_i, U_{i+1}$ are disjoint.

[3]Decompositions satisfying 1 and 2 are often known as *path-decompositions* (cf. [50]*§12.6). In [52] these are refereed to as *linear decompositions*. In these paper, linear decompositions will always not only satisfy 1 and 2 but also 3 and 4.

We may combine the linkages of cardinality $\ell$ from 1 to obtain a $U_1$–$U_M$ linkage $\mathcal{P}$ of cardinality $\ell$ in $G$. We call such a linkage $\mathcal{P}$ a *foundational linkage* of the rainbow-decomposition.

Finally, we define 'rainbow-cloud-decompositions', which consist of a rainbow-decomposition of a subgraph of $G$ that interacts with the remainder of $G$, the 'cloud', in a very controlled manner (see fig. 5.2 for an illustration). Formally, a *rainbow-cloud-decomposition* (or *RC-decomposition* for short) of $G$ of *adhesion* $\ell$ and *length* $M$ is a quadruple $(R, \mathcal{W}, Z, C)$ consisting of two induced subgraphs $R$ and $C$ of a graph $G$, a vertex set $Z \subseteq V(C)$ disjoint from $V(R)$ such that $G[V(R) \cup Z] \cup C = G$ and a rainbow-decomposition $\mathcal{W} = (W_0, \dots, W_M)$ of $R$ of adhesion $\ell$ and length $M$ with adhesion sets $U_1, \dots, U_M$ and two additional adhesion sets $U_0 := V(C) \cap W_0$ and $U_{M+1} := V(C) \cap W_M$ such that

(RC1) $V(R) \cap V(C) = U_0 \cup U_{M+1}$,

(RC2) $|U_0| = \ell = |U_{M+1}|$ and $U_0 \cap U_1 = \emptyset = U_M \cap U_{M+1}$,

(RC3) there exists a $U_0$–$U_1$ linkage in $G[W_0]$ and a $U_M$–$U_{M+1}$ linkage in $G[W_M]$, both of cardinality $\ell$, and

(RC4) $Z \subseteq N_G(W_i)$ for every $i \in \{0, \dots, M\}$.

We refer to $R$ as the *rainbow*, to $C$ as the *cloud* of the RC-decomposition and to $Z$ as the *sun* of the RC-decomposition. Whenever we introduce an RC-decomposition $(R, \mathcal{W}, Z, C)$, we tacitly assume that $\mathcal{W} = (W_0, \dots, W_M)$ and $U_0, \dots, U_{M+1}$ are defined as above.

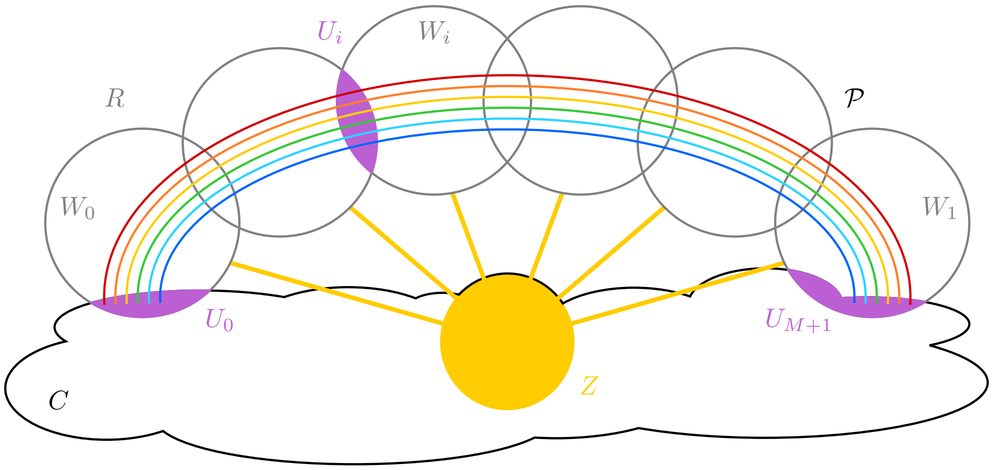


**Figure 5.2:** A rainbow-cloud decomposition $(R, \mathcal{W}, Z, C)$: the rainbow $R$ together with its decomposition $\mathcal{W}$ is indicated in grey, the foundational linkage $\mathcal{P}$ of $\mathcal{W}$ is depicted in rainbow colours, and the cloud $C$ is depicted in black. Further, indicated in yellow, is the sun $Z \subseteq V(C)$ together with the $Z$–$R$ edges required for 4. The adhesion sets $U_0, U_i, U_{M+1}$ are depicted in brown.

With the definition of rainbow-cloud-decompositions at hand, we can state the main result of this section:

**Theorem 5.5.1.** *For every two integers $k, M \geq 1$, there exists some integer $N = N(k, M) \geq 1$ such that every connected graph $G$ with at least $N$ vertices and no $(k+1)$-tangle admits an RC-decomposition $(R, \mathcal{W}, Z, C)$ of length at least $M$ and adhesion $\ell$ such that $|Z| + \ell \geq 1$.*

The remainder of this section is devoted to the proof of theorem 5.5.1, which roughly proceeds as follows: We will start by using the tangle-tree duality theorem, one of the two central tangle theorems, to get a long nested sequence of separations which will allow us to construct a still long linear decomposition. This linear decomposition can subsequently be refined into an RC-decomposition, by putting the 'unnecessary bits' of its bags into the cloud.

As a first step, theorem 5.5.3 asserts that every sufficiently long sequence of separations contains a long subsequence which induces a linear decomposition of some adhesion $\ell$ satisfying 1. In other words, we find a subsequence such that all its elements have the same order and such that for every two successive separations, there exists a linkage between its separators. By Menger's theorem (e.g. [50]*Theorem 3.3.1), this second property is equivalent to the absence of any separation of smaller order between two successive separations.

We start with a lemma about sequences of positive integers, which we will later apply to the order of the separations in the sequence. In what follows all sequences will be finite.

**Lemma 5.5.2.** *Let $n, m, p \geq 1$ be integers with $p \geq n^m$. Then every sequence $(a_1, \dots, a_p)$ of integers of length $p$ with $a_i \in \{0, \dots, m-1\}$ for every $i \in [p]$ has a subsequence $(a_{i_1}, \dots, a_{i_n})$ of length $n$ such that*

1. $\ell := a_{i_1} = \dots = a_{i_n}$, *and*
2. $a_j \geq \ell$ *for all* $i_1 \leq j \leq i_n$.

*Proof.* We proceed by induction on $m$. For $m = 1$, we have $a_i = 0$ for all $1 \leq i \leq n \leq p$ which immediately yields the statement. So consider $m \geq 2$. If at least $n$ of the $a_i$ equal 0, then any $n$ of them form the desired sequence. So suppose that at most $n' < n$ of the $a_i$ equal 0 and let $i_1, \dots, i_{n'}$ be the respective indices. Now consider the $n' + 1$ subsequences

$$(a_1, \dots, a_{i_1 - 1}), (a_{i_1+1}, \dots, a_{i_2-1}), \dots, (a_{i_{n'}+1}, \dots, a_p) \tag{5.1}$$

of consecutive $a_1, \dots, a_p$; note that some of these subsequences may be empty. If each of these subsequences has length less than $n^{m-1}$, then we obtain a contradiction via

$$n^m \leq p \leq (n'+1)(n^{m-1} - 1) + n' \leq n(n^{m-1} - 1) + (n-1) = n^m - 1.$$

So one of the subsequences in (5.1), let us call it $(b_1, \dots, b_{p'})$, must have length at least $p' \geq n^{m-1}$, and we thus can apply the induction hypothesis to $(b_1 - 1, \dots, b_{p'} - 1)$ with integers $n, m-1, p'$ to obtain a subsequence of $(b_1, \dots, b_{p'})$ which is as desired. □

Given a graph $G$, a sequence $((A_i, B_i))_{i \in [p]})$ of (oriented) separations is *strictly increasing* if $(A_i, B_i) < (A_j, B_j)$ for every two elements $i < j$ of $[p]$.

**Lemma 5.5.3.** *Let $G$ be a graph and $n, m, p \geq 1$ be integers with $p \geq n^m$. If there is a strictly increasing sequence of length $p$ in $\vec{S}_m(G)$, then there is also a strictly increasing sequence $((A_i, B_i))_{i \in [n]}$ of length $n$ in $\vec{S}_m(G)$ such that*

1. $\ell := |A_1, B_1| = \dots = |A_n, B_n|$, *and*
2. *for every separation $(A, B) \in \vec{S}_\ell(G)$ there is no $i \in [n-1]$ with $(A_i, B_i) < (A, B) < (A_{i+1}, B_{i+1})$.*

*Proof.* With each strictly increasing sequence $T$ in $\vec{S}_m(G)$, we associate a sequence $(n_i(T))_{i\in[m-1]}$ of integers where $n_i(T)$ denotes the number of separations in the strictly increasing sequence that have exactly order $i$. Whenever we will compare sequences of integers in this proof, we do so with respect to the lexicographic order. Let $\mathcal{T}$ denote the set of all strictly increasing sequence in $\vec{S}_m(G)$ of length at least $p$, and let $T = ((C_i, D_i))_{i\in[q]}$ be an element in $\mathcal{T}$ whose associated sequence of integers is maximal among all such sequences associated with elements in $\mathcal{T}$. By assumption there exists some such strictly increasing sequence of length at least $p$ and, as the graph $G$ is finite, there is a maximal sequence of integers among such associated to elements in $\mathcal{T}$.

Applying theorem 5.5.2 to $(|C_i, D_i|)_{i\in[q]}$, we obtain a subsequence $((C_{i_j}, D_{i_j}))_{j\in[n]}$ which we will show to be as desired. A subsequence of a strictly increasing sequence is again strictly increasing. It also satisfies 1, as all the separations in the subsequence have the same order $\ell$ by theorem 5.5.2. It remains to show that 2 holds as well. To do so, we show that if 2 does not hold for $((C_{i_j}, D_{i_j}))_{j\in[n]}$, then we find an element in $\mathcal{T}$ whose associated sequence of integers is larger than the one associated to $((C_i, D_i))_{i\in[q]}$, which contradicts our choice.

So suppose that there exists a separation $(A, B) \in \vec{S}_m(G)$ of order less than $\ell$ and an integer $j \in [n-1]$ such that $(C_{i_j}, D_{i_j}) < (A, B) < (C_{i_{j+1}}, D_{i_{j+1}})$. Consider the subsequence

$$((C_{i_j}, D_{i_j})), (C_{i_j+1}, D_{i_j+1}), \dots, (C_{i_{j+1}}, D_{i_{j+1}}))$$

of $((C_i, D_i))_{i\in[p']}$; for notational simplicity, we also denote this subsequence by $R = ((A_i, B_i))_{i\in[r]}$. Then we can obtain a new strictly increasing sequence $R' = ((A'_i, B'_i))_{i\in[r']}$ by removing duplicates, which will only appear consecutively, from the sequence

$$(A_1 \cap A, B_1 \cup B), \dots, (A_q \cap A, B_q \cup B), (A, B), (A_1 \cup A, B_1 \cap B), \dots, (A_r \cup A, B_r \cap B).$$

We now consider the strictly increasing sequence $T'$ which is obtained from $T$ by replacing its subsequence $R$ with $R'$. As every element of the distributive lattice $\vec{U}(G)$ is uniquely determined by its infimum and supremum with any given other element in $\vec{U}(G)$, $T'$ has length at least $q+1 \geq p$; thus, $T' \in \mathcal{T}$. Moreover, the sequence of integers associated to $T'$ is larger than the one associated with $T$, which contradicts the choice of $T$. Indeed by the choice of $R$, all separations $(C_g, D_g)$ with $i_j \leq g \leq i_{j+1}$ have order at least $\ell$. Hence, $n_i(T') \geq n_i(T)$ for every $i < \ell$ and moreover $n_{|A,B|}(T') \geq n_{|A,B|}(T) + 1$, since the new sequence $T'$ additionally contains $(A, B)$. □

To proceed to the proof of theorem 5.5.1, we recall the tangle-tree duality theorem and all necessary definition: For a tree $T$ and two nodes or edges $x, y$ of $T$, we denote by $xTy$ the (unique) $\subseteq$-minimal path in $T$ which contains $x$ and $y$. A *tree-decomposition* of a graph $G$ is a pair $(T, \mathcal{V})$ of a tree $T$ and a family $\mathcal{V}$ of subsets $V_t$ of $V(G)$ indexed by the nodes of $T$ such that

(T1) $\bigcup_{t\in V(T)} G[V_t] = G$, and

(T2) for every three nodes $r, s, t \in T$ with $s \in rTt$ we have $V_r \cap V_t \subseteq V_s$.

The maximum of the sizes of the $V_t$ minus 1 is the *width* of $(T, \mathcal{V})$. The *adhesion set* $V_e$ corresponding to an edge $e = t_1t_2$ of $T$ is $V_{t_1} \cap V_{t_2}$. The maximum of the sizes of the $V_e$ is the *adhesion* of $(T, \mathcal{V})$. Every orientation $(t_1, t_2)$ of an edge $e$ of $T$ *induces* a separation $(\bigcup_{t\in T_1} V_t, \bigcup_{t\in T_2} V_t)$ of $G$ with separator $V_e$ where $T_1 \ni t_1$ and $T_2 \ni t_2$

are the two components of $T - t_1 t_2$ (cf. [50]*Lemma 12.3.1). It is immediate from the definition that the set of separations induced by a tree-decomposition is nested. Conversely, it is well-known (e.g. proof of [53]*Lemma 2.7 or [21, 25]) that every nested set $N$ of separations of a finite graph *induces* a tree-decomposition whose induced separations are in bijection with $N$. We remark that tree-decompositions $(T, \mathcal{V})$ where $T$ is a path of length $M$ correspond precisely to the families $\mathcal{W} = (W_0, \dots, W_M)$ which satisfy 1 and 2. We now recall the tangle-tree duality theorem, rephrased here in the version which we need later:

**Theorem 5.5.4** (e.g. [50]*Theorem 12.5.1). *Every graph $G$ with no $(k+1)$-tangle admits a tree-decomposition $(T, \mathcal{V})$ of adhesion at most $k$ such that every two induced separations are distinct and for every node $t \in T$ the set of (oriented) separations induced by the oriented edges $(s, t)$ of $T$ is in $\mathcal{T}$. In particular, every node of $T$ has degree at most 3 and the width of $(T, \mathcal{V})$ is less than $3k$.*

The next lemma assets that in the absence of high-order tangles, the graph admits a certain type of linear decomposition. To prove it, we first apply the tangle-tree duality to obtain a tree-decomposition of small width and then sort out a suitable linear decomposition by theorem 5.5.3.

**Lemma 5.5.5.** *For every two integers $k, M \geq 1$, there exists an integer $N_1 = N_1(k, M) \geq 1$ such that if a graph $G$ with more than $N_1$ vertices has no $(k+1)$-tangle, then there exists a linear decomposition $\mathcal{W}$ of $G$ of length at least $M$ and adhesion at most $k$ such that $\mathcal{W}$ satisfies 1.*

We remark that the proof will show that $N_1(k, M) = 3k \cdot 3^{(M+2)^{k+1}}$ suffices.

*Proof of theorem 5.5.5.* We set $M_1 := M + 2$, $M_2 := {M_1}^{k+1}$, $M_3 := 3^{M_2}$ and $N_1 := N_1(k, M) := 3kM_3$. Let $G$ be a graph with more than $N_1$ vertices and no $(k+1)$-tangle. Since $G$ has no $(k+1)$-tangle, it admits a tree-decomposition $(T, \mathcal{V})$ of width at most $3k$ and adhesion at most $k$ such that $T$ has maximum degree $\leq 3$. Moreover, we may choose $(T, \mathcal{V})$ such that all its induced separations are distinct. Then $T$ contains at least $|G|/3k \geq N_1/3k \geq M_3$ vertices. Hence, $T$ contains a path of length at least $M_2$, as all the nodes of $T$ have degree at most 3.

Fix a path $P = p_0 \dots p_{M_2}$ in $T$, and let $(A'_i, B'_i)$ be the separation of $G$ induced by the oriented edge $(p_{i-1}, p_i)$ of $T$ for all $i \in [M_2]$. As $(T, \mathcal{V})$ has adhesion at most $k$, all these separations have order at most $k$. It is also immediate from the definition of inducing a separation that $(A'_i, B'_i) < (A'_{i+1}, B'_{i+1})$ for all $i \in [M_2 - 1]$. Thus, $((A'_i, B'_i))_{i \in [M_2]}$ is a strictly increasing sequence of length $M_2$ in $\vec{S}_k(G)$.

Hence, by theorem 5.5.3, we obtain a new strictly increasing sequence $((A_i, B_i))_{i \in [M_1]}$ of length $M_1$ in $\vec{S}_k(G)$ whose elements all have the same order $\ell \leq k$ and such that there is no separation $(A, B) \in \vec{S}_\ell(G)$ with $(A_i, B_i) < (A, B) < (A_{i+1}, B_{i+1})$ for all $i \in [M_1 - 1]$. From this sequence, we now construct a linear decomposition $\mathcal{W} = (W_0, \dots, W_{M_1})$ via

$$W_0 := A_1,\ W_i := B_i \cap A_{i+1} \text{ for } i \in [M_1 - 1], \text{ and } W_{M_1} := B_{M_1}.$$

As we have noted above for tree-decompositions, $\mathcal{W}$ indeed satisfies 1 and 2. Note that the adhesion set $U_i$ equals $W_{i-1} \cap W_i = A_i \cap B_i$; thus, 3 holds as well. Moreover, $\mathcal{W}$ has length $M_1$ and adhesion $\ell \leq k$.

Before we prove 4, let us check that $\mathcal{W}$ satisfies 1. By Menger's theorem (e.g. [50]*Theorem 3.3.1), it suffices to show that for $i \in [M_1 - 1]$, the part $G[W_i]$ contains no separation $(C', D')$ of order less than $\ell$ with $U_i \subseteq C'$ and $U_{i+1} \subseteq D'$. Suppose for a contradiction that such a separation exists for some $i \in [M_1 - 1]$. Since $\mathcal{W}$ satisfies 1 and 2, the ordered pair

$$(C, D) := (C' \cup \bigcup_{j<i} W_j, D' \cup \bigcup_{j>i} W_j)$$

is a separation of $G$ and has the same order as $(C', D')$; in particular, its order is less than $\ell$. The construction of $\mathcal{W}$ ensures that for $i \in [M_1]$ we have

$$(A_i, B_i) = (\bigcup_{j<i} W_j, \bigcup_{j\geq i} W_j).$$

Thus, $(A_i, B_i) < (C, D) < (A_{i+1}, B_{i+1})$, which contradicts our choice of the $(A_j, B_j)$ via theorem 5.5.3.

We finally turn to 4. If $W_{i-1} \subseteq W_i$ for some $i \in \{2, \dots, M_1\}$, then $W_{i-1} \subseteq A_i \cap B_i$, as we have noted above that $W_i \subseteq A_i \cap B_i$. Since $\ell = |A_{i-1}, B_{i-1}| = |A_i, B_i|$, we thus have $A_{i-1} \cap B_{i-1} = W_{i-1} = A_i \cap B_i$. Together with $W_{i-1} = B_{i-1} \cap A_i$, this implies $(A_{i-1}, B_{i-1}) = (A_i, B_i)$, which contradicts that the $(A_i, B_i)$ are distinct by choice. A symmetrical argument shows that $W_{i-1} \not\supseteq W_i$ for all $i \in \{1, \dots, M_1 - 1\}$. Thus, we have $W_i \neq W_i \cap W_{i+1} \neq W_{i+1}$ for all $i \in \{1, \dots, M_1 - 2\}$. However, we still might have $W_0 \subseteq W_1$ and $W_{M_1-1} \supseteq W_{M_1}$. In any of these cases, we remove $W_0$ or $W_{M_1}$, respectively, from $\mathcal{W}$. This operation does not affect any of the properties of $\mathcal{W}$, except that its length might decrease by at most 2; however, we have $M_1 = M + 2$, so the length of the obtained linear decomposition of $G$ is still at least $M_1$, as desired. This completes the proof. □

We will now transform a linear decomposition such as the one in theorem 5.5.5 into the desired rainbow-cloud-decomposition. As an intermediate step, let us find a linear decomposition $\mathcal{W} = (W_0, W_1, \dots, W_M)$ with a foundational linkage $\mathcal{P}$ which satisfies two additional properties[4]:

(FL1) For every $P \in \mathcal{P}$, if there exists $i \in [M-1]$ such that $P[W_i] = P \cap G[W_i]$ is a trivial path, then $P[W_i]$ is a trivial path for all $i \in [M - 1]$.

(FL2) For every two distinct $P, P' \in \mathcal{P}$, if there exists $i \in [M - 1]$ such that there is path in $G[W_i]$ with one endvertex in $P$ and the other in $P'$ and whose internal vertices avoid every path in $\mathcal{P}$, then this holds for every $i \in [M - 1]$.

The next lemma yields that the existence of a linear decomposition of some suitable length whose foundational linkage satisfies 1 and 2 is ensured by a long enough linear decomposition as in theorem 5.5.5.

**Lemma 5.5.6** ([52]*Lemma 3.5[5]). *For every two integers $M \geq 1$ and $\ell \geq 0$, there exists an integer $M_1 = M_1(\ell, M) \geq 1$ such that if a linear-decomposition $\mathcal{W} = (W_0, \dots, W_{M_1})$ of a graph $G$ has length $M_1$, adhesion $\ell$ and pairwise distinct $W_i$, and $\mathcal{W}$ satisfies 1, then $G$ has a linear decomposition $\mathcal{W}'$ of length at least $M$ which additionally has a foundational linkage satisfying 1 and 2.*

[4] 1 and 2 are (L7) and (L8) in [52], respectively.

[5] More precisely, it follows from the proof of [52]*Lemma 3.5, as (L6) is only assumed to guarantee that the resulting decomposition also satisfies (L6).

We remark that the proof of theorem 5.5.6 in [52] shows that $M_1(\ell, M) = (M^{\binom{\ell}{2}+1} \cdot \binom{\ell}{2}!)^{\ell+1} \cdot l!$ suffices.

We now turn to the proof of theorem 5.5.1. For this, we use the previous lemmas to find a linear decomposition with strong structural properties, which we then refine into the desired rainbow-cloud-decomposition.

*Proof of theorem 5.5.1.* Suppose that $G$ has no $(k+1)$-tangle. We set $N(k, M) := N_1(k, M_1(k, M+2))$, where $N_1$ and $M_1$ are as in theorem 5.5.5 and theorem 5.5.6, respectively. Then theorem 5.5.5 yields a linear-decomposition of $G$ of length at least $M_1(k, M+2)$ and adhesion $\ell \leq k$ which satisfies 1; by merging the first $i$ bags for some suitable integer $i$, we may assume that this linear decomposition has length exactly $M_1(\ell, M+2) \leq M_1(k, M+2)$. By theorem 5.5.6 ,there is a linear-decomposition $\mathcal{W}''$ of $G$ of length $M' \geq M+2$ which additionally has a foundational linkage $\mathcal{P}''$ satisfying 1 and 2. We note that $\ell > 0$ since $G$ is a connected graph and $M' \geq 1$.

Consider the adhesion sets $U''_i$ of $\mathcal{W}''$ and let $Z' := \bigcap_{i \in \{1,\dots,M'\}} U''_i$. By 1, the set $Z'$ consists precisely of all trivial paths in $\mathcal{P}''$. Now $\mathcal{W}''$ induces the linear decomposition $\mathcal{W}'$ of $G - Z$ by setting $W'_i := W''_i \setminus Z'$ together with its foundational linkage $\mathcal{P}' \subseteq \mathcal{P}''$. Now the adhesion $\ell'$ of $\mathcal{W}'$ satisfies $\ell = \ell' + |Z'| > 0$; in particular, we might have $\ell' = 0$. Clearly, $\mathcal{P}''$ again satisfies 1 and 2. Moreover, $\mathcal{W}'$ still satisfies 1, and it now also satisfies 3 by 1. Finally, 2 implies that if $z' \in Z'$ is adjacent to some $W_i$ in $G$, then it is adjacent to all $W_i$.

We now define an auxiliary graph $H_{\mathcal{P}'}$ with vertex set $\mathcal{P}'$ and an edge joining two distinct paths $P, P' \in \mathcal{P}'$ if there exists a path in some, and by 2 hence every, $G[W'_i]$ with one endvertex in $P$ and the other in $P'$ and whose internal vertices avoid every path in $\mathcal{P}'$. Let $C_H$ be an arbitrary component of $H_{\mathcal{P}'}$. For $i \in \{1, \dots, M'-1\} = [M'-1]$ we then let $W_i$ be the vertex set of the component of $G[W'_i]$ which contains all $V(P) \cap W'_i$ for $P \in C_H$; note that this is well-defined by the construction of $H_{\mathcal{P}'}$. If $C_H$, and thus $H_{\mathcal{P}'}$, is empty, we let $Z := Z'$; otherwise, we let $Z$ consist of all those $z \in Z'$ which are adjacent to some $W_i$, and hence all $W_i$ by 2. We further set

$$R := G[\bigcup_{i \in [M'-1]} W_i]$$

and

$$C := G[Z' \cup \bigcup_{i \in [M'-1]} (W'_i \setminus W_i) \cup W'_0 \cup W'_{M'}].$$

This ensures that $Z \subseteq V(C)$ as well as $G = G[V(R) \cup Z] \cup C$, as we chose the $W_i$ as components of the $G[W'_i]$. We now claim that $(R, \mathcal{W}, Z, C)$ where $\mathcal{W} := (W_1, \dots, W_{M'-1})$ is the desired rainbow-cloud-decomposition of $G$.

Clearly, $\mathcal{W}$ is a linear decomposition of $R$ of adhesion $|C_H|$ and length $M' - 2 \geq M$; note that $|C_H| > 0$ if $\ell' > 0$ and hence $|Z| + |C_H| > 0$ by the choice of $Z$. We now verify that $\mathcal{W}$ is even a rainbow-decomposition of $R$. The foundational linkage $\mathcal{P}'$ of $\mathcal{W}'$ induces a foundational linkage of $\mathcal{W}$, as we have chosen a component $C_H$ of $H_{\mathcal{P}'}$; thus, 1 holds. For 2, note that the $W_i$ are components of the $G[W'_i]$ and hence connected by construction. Finally, 3 transfers from $\mathcal{W}'$.

It remains to check 1 to 4. So let $U_1 := V(C) \cap W_1$ and $U_{M'} := V(C) \cap W_{M'-1}$; note that our construction implies $U_1 = W'_0 \cap W_1$ and $U_{M'} = W'_{M'} \cap W_{M'-1}$. By definition, we have $V(R) \cap V(C) = (W_1 \cup W_{M'-1}) \cap V(C) = U_1 \cup U_{M'}$, so 1 holds, and 3 for $\mathcal{W}'$ implies 2 as well as 1 implies 3. For 4, we recall that this holds by the definition of $Z$. This completes the proof. □

## 5.6 RC-decompositions and separations

In this section, we investigate how a fixed RC-decomposition of a graph $G$ interacts with the separations of $G$. On the one hand, we will analyse in what ways a separation of $G$ may meet the rainbow of an RC-decomposition. On the other hand, we look at the separations of $G$ induced by the structure of the RC-decomposition. With this we build a set of tools that we will later apply in the proof of our structural main result, Theorem 5. These tools will allow us to control the new low-order separations that arise when we delete an edge deep inside the rainbow.

### 5.6.1 Separations and bags

In this subsection, we prove two general lemmas that describe in which ways a separation of a graph meets the bags of a linear decomposition or an RC-decomposition. The first lemma asserts that if a linear decomposition satisfies 2 and 3, then the strict sides of a separation contain most of its bags.

**Lemma 5.6.1.** *Let $\{A, B\}$ be a separation of a graph $G$ of order $k$ and let $\mathcal{W}$ be a linear decomposition of a subgraph $R$ of $G$ that satisfies 3. Then the separator $A \cap B$ meets at most $2k$ bags of $\mathcal{W}$. Moreover, if $\mathcal{W}$ additionally satisfies 2, then at most $2k$ bags of $\mathcal{W}$ are not contained in either $A \setminus B$ or $B \setminus A$.*

*Proof.* Each vertex in $R$ is contained in at most one adhesion set of $\mathcal{W}$ by 3 and thus in at most two bags of $\mathcal{W}$. So since $\{A, B\}$ has order $k$, the separator $A \cap B$ meets at most $2k$ bags of $\mathcal{W}$. As each part $R[W_i]$ of $\mathcal{W}$ is connected by 2, every bag of $\mathcal{W}$ that is disjoint from $A \cap B$ is included in precisely one of $A \setminus B$ and $B \setminus A$. Hence, at most $2k$ bags of $\mathcal{W}$ are not contained in either $A \setminus B$ or $B \setminus A$. □

The second lemma states that if a strict side of a separation has a component which meets both the cloud and a bag $W_i$ of a fixed RC-decomposition, then it also contains most bags of the rainbow along one of the two connections from $W_i$ to the cloud.

**Lemma 5.6.2.** *Let $(R, \mathcal{W}, Z, C)$ be an RC-decomposition of a graph $G$ of length $M$, and let $\{A, B\}$ be a separation of $G$ of order $k$. Suppose that $D \subseteq G[A \setminus B]$ is a component of $G - (A \cap B)$ which meets both $C$ and some bag $W_i$. Then $D$ contains all but at most $2k$ of $W_0, \dots, W_i$ or $W_i, \dots, W_M$. Moreover, $D$ meets either $Z$ or every bag of $W_0, \dots, W_i$ or of $W_i, \dots, W_M$, respectively.*

*Proof.* By theorem 5.6.1 applied to the restriction $\{A \cap V(R), B \cap V(R)\}$ of $\{A, B\}$ to the rainbow $R$ at most $2k$ of the bags $W_i$ meet $A \cap B$. If $D$ contains a vertex $z$ of $Z$, then every bag which does not meet $A \cap B$ is contained in $D$ by 4 and 2 because $D$ is a component of $G - (A \cap B)$, as desired. So let us now assume that $V(D)$ is disjoint from $Z$. Since $D$ is connected and meets both $C$ and $W_i$, the component $D$ contains a path $P$ from $W_i$ to $C$. This path $P$ has to first enter $C$ either through $U_0$ or $U_{M+1}$, as $V(D) \cap Z = \emptyset$ and also $G[V(R) \cup Z] \cup C = G$ and $V(R) \cap V(C) = U_0 \cup U_{M+1}$ by the definition of an RC-decomposition. Thus, we may assume that the only vertex of $P$ in $C$ is its endvertex which is in $U_0$, as the other case is symmetrical. Hence, it meets all bags $W_j$ with $0 \le j \le i$, because $\mathcal{W}$ is a linear decomposition of $R$. Thus, every bag $W_j$ with $0 \le j \le i$ which avoids $A \cap B$ is contained in the component $D$ of $G - (A \cap B)$ by 2, as desired. □

### 5.6.2 Separating along the rainbow

We can easily turn an RC-decomposition $(R, \mathcal{W}, Z, C)$ of a graph $G$ of length $M$ into a new one of shorter length and of the same adhesion by restricting the linear decomposition $\mathcal{W}$ of $R$ to some interval in $\{0, \dots, M\}$ and adding the remaining bags to $C$ as follows. For $0 \leq i \leq j \leq M$, we set

$$\begin{aligned} R_{i,j} &:= G\left[\bigcup_{h=i}^{j} W_h\right] = \bigcup_{h=i}^{j} G[W_h], \\ \mathcal{W}_{i,j} &:= (W_i, \dots, W_j), \\ C_{i,j} &:= G\left[V(C) \cup \left(\bigcup_{h=0}^{i-1} W_h\right) \cup \left(\bigcup_{h=j+1}^{M} W_h\right)\right], \text{ and} \\ (R, \mathcal{W}, Z, C)_{i,j} &:= (R_{i,j}, \mathcal{W}_{i,j}, Z, C_{i,j}). \end{aligned}$$

We remark that we follow the convention that an empty union, such as $\bigcup_{h=0}^{-1} W_h$ or $\bigcup_{h=M+1}^{M} W_h$, is the empty set.

It is straight-forward from the definition that $(R, \mathcal{W}, Z, C)_{i,j}$ is again an $RC$-decomposition of $G$:

**Lemma 5.6.3.** *Let $G$ be a graph with an RC-decomposition $(R, \mathcal{W}, Z, C)$ of length $M$ and of adhesion $\ell$. Then $(R, \mathcal{W}, Z, C)_{i,j}$ is an RC-decomposition of $G$ of length $j - i$ and of adhesion $\ell$ for all $0 \leq i \leq j \leq M$.* □

It follows immediately that we obtain separations of $G$ along the rainbow of a fixed RC-decomposition:

**Lemma 5.6.4.** *Let $G$ be a graph with an RC-decomposition $(R, \mathcal{W}, Z, C)$ of length $M$ and adhesion $\ell$, and let $0 \leq i \leq j \leq M$. Then $\{V(R_{i,j}) \cup Z, V(C_{i,j})\}$ is a separation of $G$ with separator $U_i \cup U_{j+1} \cup Z$. In particular, its order is $2\ell + |Z|$.* □

### 5.6.3 Rainbow-crossing separations

We now investigate separations that 'cross' the rainbow of a given RC-decomposition in that both strict sides of the separation contain bags of the linear decomposition of the rainbow: Let $(R, \mathcal{W}, Z, C)$ be an RC-decomposition of a graph $G$ of length $M$. An oriented separation $(A, B)$ of $G$ of order $k$ *crosses the rainbow $R$ clockwise* if there exist integers $i \in \{0, \dots, 2k\}$ and $j \in \{M - 2k, \dots, M\}$ such that $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$. We denote the minimal such $i$ by $i_{A,B}$ and the maximal such $j$ by $j_{A,B}$. If an oriented separation $(A, B)$ crosses the rainbow clockwise, we say that its other orientation $(B, A)$ *crosses the rainbow $R$ counter-clockwise* and the underlying unoriented separation $\{A, B\}$ *crosses the rainbow $R$*. We remark that a separation $\{A, B\}$ which crosses the rainbow contains $Z$ in its separator $A \cap B$ by 4, $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$. Note that in the above definition, we may have $j \leq i$, since we do not assume any lower bound on $M$. In most applications, however, we will have $M \geq 4k$ and thus $i < j$.

Given a separation which crosses the rainbow, we may construct several separations that separate the cloud in the same way, but split the rainbow along the adhesion sets of its linear decomposition (see fig. 5.3): More precisely, let $(R, \mathcal{W}, Z, C)$ be an RC-decomposition of a graph $G$. For a separation $(A, B)$ of $G$ of order $k$ that crosses

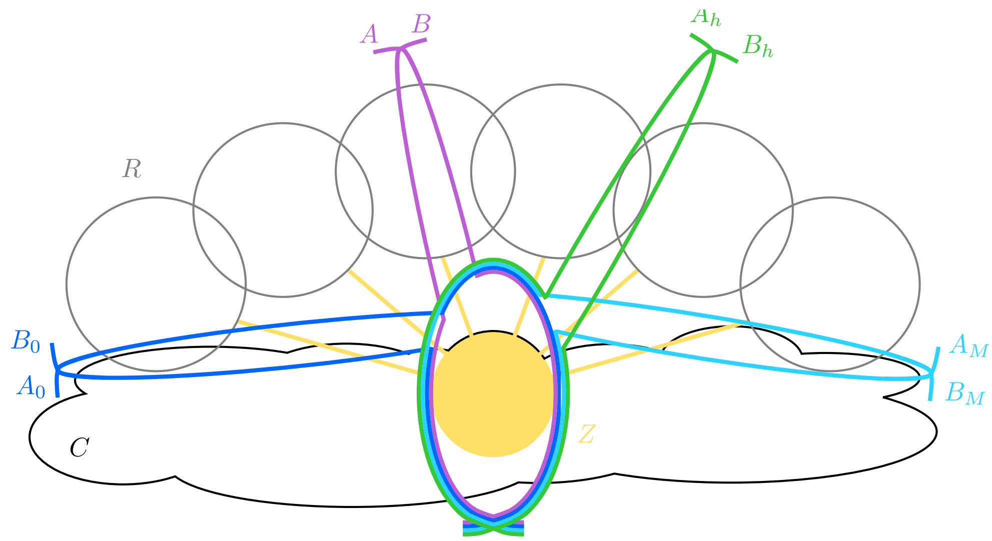


**Figure 5.3**: A rainbow-crossing separation $(A, B)$ and the arising separations $(A^0, B^0)$, $(A^h, B^h)$ and $(A^M, B^M)$.

the rainbow clockwise, we let $i := i_{A,B} \in \{0, \dots, 2k\}$ be the smallest and $j := j_{A,B} \in \{M-2k, \dots, M\}$ the largest number such that $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$, which are well-defined by the definition of crossing the rainbow clockwise. For $h \in \{i+1, \dots, j\}$ we set

$$(A^h, B^h) := ((A \cap V(C_{i,j})) \cup V(R_{i,h-1}),\ V(R_{h,j}) \cup (B \cap V(C_{i,j}))).$$

If $(A, B)$ crosses the rainbow counter-clockwise, we use the fact that $(B, A)$ crosses the rainbow clockwise to define $(A^h, B^h)$ accordingly.

It is immediate from the definition that $(A^{i+1}, B^{i+1}) \leq (A, B) \leq (A^j, B^j)$ and that the $(A^h, B^h)$ form a strictly increasing sequence. Indeed, the $(A^h, B^h)$ also are separations of $G$:

**Lemma 5.6.5.** *Let $(R, \mathcal{W}, Z, C)$ be an RC-decomposition of a graph $G$ of adhesion $\ell$, and let $(A, B)$ be a separation of $G$ of order $k$ which crosses the rainbow $R$ clockwise. Then $(A^h, B^h)$ is a separation of $G$ of order $k$ which crosses the rainbow $R$ clockwise for every $h \in \{i_{A,B} + 1, \dots, j_{A,B}\}$.*

*Proof.* We abbreviate $i := i_{A,B}$ and $j := j_{A,B}$. As $C_{i,j}$ is a subgraph of $G$, the separation $\{A, B\}$ of $G$ induces the separation $\{A \cap V(C_{i,j}), B \cap V(C_{i,j})\}$ of $C_{i,j}$. Note that $Z \subseteq A \cap B$, since $Z$ is contained in the neighbourhood of both $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$ by 4. Moreover, by theorem 5.6.4, both $U_i \cup Z \cup U_h$ and $U_h \cup Z \cup U_{j+1}$ separate $R_{i,h-1}$ and $R_{h,j}$, respectively, from the rest of $G$. As $U_i \cup Z \cup U_h \subseteq A^h$ and $U_h \cup Z \cup U_{j+1} \subseteq B^h$, it follows that $\{A^h, B^h\}$ is a separation of $G$.

It remains to show that $|A^h \cap B^h| \leq k$. The foundational linkage of the RC-decomposition induces a $U_{i+1}$–$U_j$ linkage of cardinality $\ell$ in $R_{i+1,j-1}$. Since $\{A, B\}$ is a separation of $G$ and $U_{i+1} \subseteq W_i \subseteq A \setminus B$ and $U_j \subseteq W_j \subseteq B \setminus A$, the separator $A \cap B$ has to contain one vertex of every $U_{i+1}$–$U_j$ paths. But since $V(C_{i,j}) \cap V(R_{i+1,j-1}) = \emptyset$, these $\ell$ vertices are contained in $(A \cap B) \setminus V(C_{i,j})$. Hence, we obtain $|(A \cap B) \cap V(C_{i,j})| \leq |A \cap B| - \ell = k - \ell$. With this, we may now bound the order of $\{A^h, B^h\}$:

$$\begin{aligned}
|A^h \cap B^h| &= |((A \cap V(C_{i,j})) \cup V(R_{i,h-1})) \cap (V(R_{h,j}) \cup (B \cap V(C_{i,j})))| \\
&\leq |(A \cap B) \cap V(C_{i,j})| + |A \cap (V(C_{i,j}) \cap V(R_{h,j}))| + |B \cap (V(C_{i,j}) \cap V(R_{i,h-1}))| + |V(R_{i,h-1}) \cap V(R_{h,j})| \\
&\leq (k - \ell) + |A \cap W_j| + |B \cap W_i| + |U_h| = (k - \ell) + 0 + 0 + \ell = k,
\end{aligned}$$

where the penultimate equation holds because $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$. This completes the proof. □

### 5.6.4 Rainbow-slicing separations

In this section, we study separations that 'slice' the rainbow in that they cut bags in the middle of the rainbow away:

Let $(R, \mathcal{W}, Z, C)$ be an RC-decomposition of a graph $G$ of length $M$. A separation $\{A, B\}$ of $G$ of order $k$ *slices the rainbow R* if there are integers $i < h < j$ with $i \in \{0, \dots, 2k\}$ and $j \in \{M-2k, \dots, M\}$ such that $W_i, W_j \subseteq A \setminus B$ and $W_h \subseteq B \setminus A$ or such that $W_i, W_j \subseteq B \setminus A$ and $W_h \subseteq A \setminus B$.

We now prove a lower bound on the order of separations that slice the rainbow. Towards this, we first show such a lower bound for a slightly more general class of separations for later use, which in particular yields the desired bound for rainbow-slicing separations.

**Lemma 5.6.6.** *Let G be a graph with an RC-decomposition $(R, \mathcal{W}, Z, C)$ of adhesion $\ell$, and let $\{A, B\}$ be a separation of G. Suppose that there are integers $0 \leq i < h < j \leq M+1$ such that $B \setminus A$ contains $W_h$ and A contains the adhesion sets $U_i$ and $U_j$. Then $|A \cap B \cap V(R)| \geq 2\ell$. Moreover, if additionally $Z \subseteq A$, then $|A \cap B| \geq 2\ell + |Z|$.*

*Proof.* The fundamental linkage induces a $U_i$–$U_h$ linkage $\mathcal{P}_i$ and a $U_h$–$U_j$ linkage $\mathcal{P}_j$; both have cardinality $\ell$ and are in $R$. We remark that the paths in $\mathcal{P}_i$ meet the paths in $\mathcal{P}_j$ only in $U_h$. Hence, $W_h \subseteq B \setminus A$ and $U_i, U_j \subseteq A$ yields that the separator $A \cap B$ contains at least one vertex of each of these $2\ell$ paths. As every two paths in $\mathcal{P}_i$ and also every two paths in $\mathcal{P}_j$ are pairwise disjoint, we thus have $|A \cap B \cap V(R)| \geq 2\ell$.

If additionally $Z \subseteq A$, then $Z \subseteq A \cap B$ since $Z \subseteq N_G(W_h)$ by 4 and $W_h \subseteq B \setminus A$. As all the above $2\ell$ paths lie in $R$ and hence avoid $Z$, the claim follows. □

Since every rainbow-slicing separation contains $Z$ in its separator by 4, we obtain the following lower bound on their order:

**Corollary 5.6.7.** *Let G be a graph with an RC-decomposition $(R, \mathcal{W}, Z, C)$ of adhesion $\ell$. If a separation $\{A, B\}$ of G slices the rainbow R, then $|A \cap B \cap V(R)| \geq 2\ell$ and $|A \cap B| \geq 2\ell + |Z|$.* □

We conclude this section of preparatory lemmas by showing that every separation which separates two bags of an RC-decomposition either crosses or slices its rainbow.

**Lemma 5.6.8.** *Let G be a graph with an RC-decomposition $(R, \mathcal{W}, Z, C)$ of length M, and let $\{A, B\}$ be a separation of G. If there are bags $W_i$ and $W_j$ of $\mathcal{W}$ such that $W_i \subseteq A \setminus B$ and $W_j \subseteq B \setminus A$, then $\{A, B\}$ either crosses or slices the rainbow.*

*Proof.* Let $k := |A, B|$. If $M \leq 2k$, then $W_i$ and $W_j$ witness that $\{A, B\}$ crosses the rainbow. So we may assume $M \geq 2k$. By theorem 5.6.1, all but at most $2k$ bags of $\mathcal{W}$ are contained in either $A \setminus B$ or $B \setminus A$. In particular, there exist bags $W_h$ with $h \in \{0, \dots, 2k\}$ and $W_s$ with $s \in \{M-2k, \dots, M\}$ that are each contained in either $A \setminus B$ or $B \setminus A$.

If one of $W_h$ and $W_s$ is contained in $A \setminus B$ and the other one in $B \setminus A$, then $\{A, B\}$ crosses the rainbow. Otherwise, both $W_h$ and $W_s$ are contained in the same side of $\{A, B\}$. By symmetry, we may assume $W_h, W_s \subseteq A \setminus B$. Now if $j \in \{0, \dots, 2k\}$, then $W_j$ and $W_s$ witness that $(B, A)$ crosses the rainbow clockwise, and if $j \in \{M-2k, \dots, M\}$, then $W_h$ and $W_j$ witness that $(A, B)$ crosses the rainbow clockwise. Otherwise, $j \in \{2k+1, \dots, M-2k-1\}$; in particular $h < j < s$. Therefore $\{A, B\}$ slices the rainbow. □

## 5.7 Proof of Theorem 5 if $G$ has no higher-order tangle

In this section, we complete the proof of Theorem 5. Based on the results in sections 5.3 to 5.5, it suffices to consider graphs that admit an RC-decomposition with certain properties. More precisely, the technical main result of this section, which will allow us to finish the proof of Theorem 5, reads as follows:

**Theorem 5.7.1.** *Let $k \geq 1$ be an integer, let $G$ be a graph of minimum degree at least 3, and let $\tau$ be a $k$-tangle in $G$. Suppose that $G$ admits an RC-decomposition with sun $Z$ which has length $\geq 18k$ and adhesion $\ell$ such that $\ell + |Z| \geq 1$. Then there exists an edge $e \in E(G)$ such that $\tau$ extends to a $k$-tangle $\tau'$ in $G - e$.*

Most of this section is devoted to the proof of theorem 5.7.1. In the very end we combine the previous results to prove Theorem 5.

### 5.7.1 Living in the rainbow

In the proof of theorem 5.7.1, we will delete an edge $e$ deep inside the rainbow $R$ of the given long RC-decomposition. With this, we aim to make use of the regular structure of $R$ to orient the newly arising separations in $G' := G - e$ in such a way that we find a $k$-tangle $\tau'$ in $G'$ to which $\tau$ extends. As we want to orient these new separations of $G'$ in line with $\tau$, the tangle $\tau$ should ideally be 'regular' or 'monotonic' along $R$. Then we could use this monotonicity of $\tau$ along $R$ to orient the newly arising separations in $G'$ by consistency.

In this section, we extract from a given RC-decomposition another RC-decomposition such that $\tau$ has the desired monotonic behaviour along the rainbow. More precisely, the tangle $\tau$ will 'point away' from the rainbow, which will turn out to be equivalent to the desired monotonicity, as we see below. Formally, we enclose this in a definition of when a tangle $\tau$ does not 'live in the rainbow $R$' of an RC-decomposition. The intuition behind this definition is as follows.

Clearly, a tangle $\tau$ should live in the rainbow $R$, if there is a separation $(A, B) \in \tau$ whose strict big side $B \setminus A$ is contained in the rainbow $R$ and does not meet the cloud $C$. For our proof of theorem 5.7.1, this case is not the only relevant one: we also need to regard $\tau$ as living in $R$ if it orients two separations in such a way that they point towards each other and to a piece of $R$. It turns out that for this second case, we can even restrict our attention to rainbow-crossing separations, as follows.

Let $\tau$ be a $k$-tangle in a graph $G$. Consider a given RC-decomposition $(R, \mathcal{W}, Z, C)$ of $G$ and a separation $(A, B)$ of $G$ of order $k' < k$ that crosses the rainbow $R$ clockwise or counter-clockwise. Then $\tau$ *orients $\{A, B\}$ monotonically over the rainbow* if $\tau$ orients all the $\{A^h, B^h\}$ with $h \in \{i_{A,B} + 1, \dots, j_{A,B}\}$ such that they are pairwise comparable, i.e. either all as $(A^h, B^h)$ or all as $(B^h, A^h)$. We remark that if $\tau$ does not orient a rainbow-crossing separation $\{A, B\}$ of order less than $k$ monotonically over the rainbow $R$, then the consistency of $\tau$ ensures that there is a (unique) index $h^* \in \{0, \dots, M-1\}$ such that $(A^{h^*}, B^{h^*}) \in \tau$ and $(B^{h^*+1}, A^{h^*+1}) \in \tau$, as the $(A^h, B^h)$ form an increasing or decreasing sequence as $(A, B)$ crosses the rainbow clockwise or counter-clockwise, respectively. We then call $h^*$ its *turning point*; note that $h^* \in \{0, \dots, M-1\}$. Moreover, we say that $\tau$ *lives in the rainbow $R$* if at least one of the following holds:

(LR1) there is a separation $(A, B) \in \tau$ such that $B \setminus A \subseteq V(R) \setminus V(C)$, or

(LR2) $\tau$ orients at least one rainbow-crossing separation of order less than $k$ not monotonically over the rainbow.

The next theorem asserts that we can shorten an RC-decomposition of sufficient length so that a given tangle does not live in the shortened rainbow:

**Theorem 5.7.2.** *Let $k \geq 1$, $M \geq 6k$ and $\ell \geq 0$ be integers, and let $\tau$ be a $k$-tangle in a graph $G$. If $G$ admits an RC-decomposition $(R, \mathcal{W}, Z, C)$ of $G$ of length at least $M$ and of adhesion $\ell$, then there exist $0 \leq i < j \leq M$ with $j - i \geq M/2 - k$ such that $\tau$ does not live in the rainbow of the RC-decomposition $(R, \mathcal{W}, Z, C)_{i,j}$.*

*Proof.* If $\tau$ does not live in $R$, then we are done by setting $i := 0$ and $j := M$; so suppose that $\tau$ lives in $R$. By definition, there are two possible reasons for that, 1 and 2, which we will treat separately.

First, suppose that $\tau$ lives in $R$ because of 1, i.e. there is a separation $(A, B) \in \tau$ with $B \setminus A \subseteq V(R \setminus C)$. Then there also exists such a separation in $\tau$ with an even stronger property:

**Sublemma 5.7.3.** *There is a separation $(X, Y) \in \tau$ such that $Y \setminus X \subseteq \bigcup_{t=r}^{s} W_t$ for integers $r \leq s$ with $s - r < 2k - 2$.*

*Proof.* First, suppose that $2\ell + |Z| < k$. Then $\{V(C), V(R) \cup Z\}$ is a separation of $G$ of order $2\ell + |Z| < k$ and hence the $k$-tangle $\tau$ has to contain one of its orientations. By the consistency of the tangle $\tau$, $(V(C), V(R) \cup Z) \leq (A, B) \in \tau$ yields $(V(C), V(R) \cup Z) \in \tau$. Every 'slice' $\{V(R_{h,h}) \cup Z, V(C_{h,h})\}$ of the rainbow $R$ with $h \in \{0, \dots, M\}$ is also a separation of $G$ of order $2\ell + |Z| < k$ by theorem 5.6.4.

Suppose for a contradiction that they are all oriented as $(V(R_{h,h}) \cup Z, V(C_{h,h}))$ by $\tau$. By iteratively using the fact that the tangle $\tau$ avoids the triples in $\mathcal{T}$ and for every $0 \leq h \leq i \leq j \leq M$ the separations $(V(R_{h,i} \cup Z, V(C_{h,i})), (V(R_{i,j}) \cup Z, V(C_{i,j}))$ and $(V(C_{h,j}), V(R_{h,j} \cup Z)\}$ of order $2\ell + |Z|$ form a triple in $\mathcal{T}$, we obtain that $(V(R_{0,M} \cup Z, V(C_{0,M})) = (V(R) \cup Z, V(C))$ is in $\tau$, which contradicts that its other orientation is also in $\tau$. Therefore, there exists $h \in \{0, \dots, M\}$ with $(V(C) \cup (V(R) \setminus W_h) \cup U_h \cup U_{h+1}, W_h \cup Z) \in \tau$, which is the desired separation $(X, Y)$ with $r := h =: s$.

Secondly, suppose that $2\ell + |Z| \geq k$, and let $(A, B) \in \tau$. We may assume that the separation $(A, B) \in \tau$ with $B \setminus A \subseteq V(R) \setminus V(C)$ is $\leq$-maximal in $\tau$ with that property; in particular, $G[B \setminus A]$ is connected. Then $B \setminus A$ cannot contain a bag of $\mathcal{W}$ by theorem 5.6.6, since $B \setminus A \subseteq V(R) \setminus V(C)$ yields that $U_0, U_{M+1}, Z \subseteq V(C) \subseteq A$, and $\{A, B\}$ has order $< k \leq 2\ell + |Z|$. Thus, every bag which meets $B \setminus A$ also meets the separator $A \cap B$. Thus, theorem 5.6.1 ensures that at most $2k - 2$ bags of $\mathcal{W}$ meet $B \setminus A$. Further, since $G[B \setminus A] \subseteq R$ is connected and each bag $W_i$ separates $R$, the bags which are met by $B \setminus A$ are consecutive bags of $\mathcal{W}$. Thus, $(X, Y) := (A, B)$ is the desired separation, as witnessed by the respective indices $r$ and $s$ of the first and last bag of $\mathcal{W}$ which are met by $B \setminus A$. □

Let $(X, Y) \in \tau$ and $r \leq s$ with $s - r < 2k - 2$ be given by theorem 5.7.3. Now if $r > \frac{M}{2} - k$, then we set $i := 0$ and $j := r - 1$, and otherwise, if $r \leq \frac{M}{2} - k$, then we set $i := s + 1$ and $j := M$. We remark that in the latter case $s < \frac{M}{2} + k$. By theorem 5.6.3, $(R, \mathcal{W}, Z, C)_{i,j}$ is an RC-decomposition of $G$ of adhesion $\ell$ and length $j - i \geq \frac{M}{2} - k$. We claim that $\tau$ does not live in $R_{i,j}$, as desired.

Since $Y \setminus X \subseteq \bigcup_{t=r}^{s} W_t$ and therefore $Y \setminus X$ is disjoint from $R_{i,j}$ by definition of $i, j$, we have $V(R_{i,j}) \subseteq X$. Thus, every two separations $(X_1, Y_1), (X_2, Y_2)$ of $G$ with $(Y_1 \cap Y_2) \setminus (X_1 \cup X_2) \subseteq V(R_{i,j})$ form a forbidden triple together with $(X, Y)$. But every separation as in 1 (taken as both $(X_i, Y_i)$) as well as every pair of separations witnessing the non-monotonicity in 2 are such two separations. Therefore, $\tau$ cannot live in $R_{i,j}$, as desired.

Secondly, suppose that $\tau$ does live in the rainbow $R$ because of 2, i.e. there is a separation of $G$ of order $< k$ that crosses the rainbow $R$ and $\tau$ does not orient it monotonically over the rainbow $R$. It turns out that all such separations $\{A, B\}$ have the same turning point:

**Sublemma 5.7.4.** *If a rainbow-crossing separation $\{A, B\}$ which $\tau$ does not orient monotonically over the rainbow $R$ has turning point $h^*$, then the turning point of every such separation is $h^*$.*

*Proof.* Suppose for a contradiction that there is another rainbow-crossing separation $\{X, Y\}$ which $\tau$ does not orient monotonically over the rainbow $R$ and has turning point $h' \neq h^*$. Let $(A, B)$ and $(X, Y)$ be the orientations which cross the rainbow clockwise. By possibly interchanging $\{A, B\}$ and $\{X, Y\}$, we may assume $h^* < h'$. Now the definition of the $(X^h, Y^h)$ immediately yields that $W_{h^*} \subseteq X^{h'}$. But it also guarantees that $G = G[A^{h^*}] \cup G[B^{h^*+1}] \cup G[W_{h^*}]$. Thus $(X^{h'}, Y^{h'})$ together with $(A^{h^*}, B^{h^*})$ and $(B^{h^*+1}, A^{h^*+1})$ forms a forbidden triple in $\tau$, which is a contradiction. □

If $h^* \geq M/2$, then we set $i = 0$ and $j = h^* - 1$, and if $h^* < M/2$, then we set $i = h^* + 1$ and $j = M$. By theorem 5.6.3, $(R, \mathcal{W}, Z, C)_{i,j}$ then is an RC-decomposition of $G$ of length $j - i \geq M/2 - 1$ and adhesion $\ell$. We claim that $\tau$ does not live in $R_{i,j}$. Since $\tau$ does not live in $R$ because of 1 by assumption, $\tau$ does neither live in $R_{i,j} \subseteq R$ because of 1, either. Moreover, $\tau$ can also not live in $R_{i,j}$ because of 2: Indeed, every separation that crosses $R_{i,j}$ also crosses $R$, and if $\tau$ does not orient it monotonically over $R_{i,j}$, it does not do so over $R$, as well. But since all such separations have turning point $h^*$ with respect to $R$ by theorem 5.7.4, the choice of $i$ and $j$ ensure that $\tau$ orients them all monotonically over $R_{i,j}$ by the choice of $i$ and $j$, as desired. □

### 5.7.2 Deleting an edge and orienting the new separations

This subsection is dedicated to the proof of theorem 5.7.1. We remark that this section is the only part of the proof of Theorem 5 for graphs $G$ which do not have a $(k + 1)$-tangle in which we make use of the assumption that $G$ has no vertex of degree $\leq 2$. In the remainder of this section, we will always assume that the given graph satisfies the premise of theorem 5.7.1:

**Setting 5.7.5.** Let $G$ be a graph of minimum degree at least 3 that admits an RC-decomposition with sun $Z$ which has length $M_0$ and adhesion $\ell$ such that $\ell + |Z| \geq 1$. Let $\tau$ be a $k$-tangle in $G$ with $k \geq 1$.

First we find our desired edge deep inside some rainbow such that the RC-decomposition is unaffected by its deletion:

**Lemma 5.7.6.** *If we assume theorem 5.7.5 with $M_0 \geq 6k$, then there is an edge $e$ of $G$ and an RC-decomposition $(R, \mathcal{W}, Z, C)$ of $G$ which has even length $M \geq M_0/2 - k - 2$*

*and adhesion $\ell$ such that $\tau$ does not live in the rainbow and $e$ has one endvertex in $W_{M/2}$ and one in $W_{M/2}$ or $Z$. Moreover, $(R, \mathcal{W}, Z, C)$ is also an RC-decomposition of $G - e$ after deleting $e$ from $R$, if $e$ has both its endvertices in $W_{M/2}$.*

*Proof.* theorem 5.7.2 ensures that $G$ admits an RC-decomposition $(R', \mathcal{W}', Z, C')$ of length $M' \geq M_0/2 - k$ and adhesion $\ell$ such that $\tau$ does not live in the rainbow $R'$; by theorem 5.6.3, we may assume without loss of generality that $M'$ is even. To find the desired edge $e$, we merge the middle bags $W'_{M'/2-1}$, $W'_{M'/2}$, $W'_{M'/2+1}$ of $\mathcal{W}'$ into one bag $W_{M/2}$ in $\mathcal{W}$ to make it robust against the deletion of an edge, and keep all other bags of $\mathcal{W}'$ in $\mathcal{W}^*$. It is immediate that this yields an RC-decomposition $(R, \mathcal{W}, Z, C)$ in whose rainbow $\tau$ does not live, where $R := R', C := C'$; in particular, it has length $M = M' - 2$ and adhesion $\ell$.

First, assume that $|Z| \geq 1$. We fix an edge $E$ of $G$ between $Z$ and $W_{M/2}$, which exists by 4. Note that we still have $Z \subseteq N_{G'}(W_{M/2})$: Since $(R', \mathcal{W}', Z, C')$ is an RC-decomposition, each of the $Z \subseteq N_G(W'_{M'/2+i})$ for every $i \in \{-1, 0, +1\}$ by 4 and $W'_{M'/2-1} \cap W'_{M'/2+1} = \emptyset$ by 3. Thus, each vertex of $Z$, in particular the endvertex of $e$ in $Z$, is joined to $W_{M/2}$ by two distinct edges of $G$ of which one persists in $G - e$. Thus, it is now easy to see that $(R, \mathcal{W}, Z, C)$ is also an RC-decomposition of $G - e$.

Secondly, assume that $Z = \emptyset$. Then $|Z| + \ell \geq 1$ ensures that $\ell \geq 1$. By 1, there exists a $U_{M/2}$–$U_{M/2+1}$ linkage $\mathcal{P}$ of cardinality $\ell$ in $G[W_{M/2}]$; note that $U'_{M'/2+i} \subseteq V(\bigcup \mathcal{P})$ for every $i \in \{-1, 0, +1, +2\}$ by the construction of $W_{M/2}$ and since $(R', \mathcal{W}', Z', C')$ also has adhesion $\ell$. We aim to fix an edge $e$ of $G[W_{M/2}] - E(\bigcup \mathcal{P})$ such that $G[W_{M/2}] - e$ is still connected. If this is possible, it is easy to check that $(R - e, \mathcal{W}, Z, C)$ is an RC-decomposition of $G - e$, i.e. $e$ is as desired. We claim that such an edge $e$ exists:

Consider any vertex $v \in U'_{M'/2}$, which exists as $\ell \geq 1$. All its neighbours and itself are contained in $W'_{M'/2-1} \cup W'_{M'/2} \subseteq W_{M/2}$, as $Z = \emptyset$ and by the definition of RC-decompositions, in particular by 3. Since $v$ has degree at least 3 by assumption on $G$ and every vertex has at most 2 incident edges in the linkage $\mathcal{P}$, there is a neighbour $w$ of $v$ with $vw \notin E(\mathcal{P})$. If $G[W^*_{M^*/2}] - vw$ is connected, $e := vw$ is as desired. So assume that $G[W_{M/2}] - vw$ is disconnected. We note that the linkage $\mathcal{P}$ is contained in one component of $G[W_{M/2}] - vw$, since $G[W'_{M'/2+1}] \subseteq G[W_{M/2}]$ is connected by 2, meets every path in $\mathcal{P}$ as it contains $U'_{M'/2+1}$ and also avoids $vw$ by the choice of $vw$. Thus, the component $C$ of $G[W_{M/2}] - vw$ containing $w$ does not meet $V(\mathcal{P})$, as $v \in U'_{M'/2} \subseteq V(\mathcal{P})$. Now $U_{M/2} \cup U_{M/2-1} = U'_{M'/2+2} \cup U'_{M'/2-1} \subseteq V(\mathcal{P})$ is the separator of a separation of $G$ with precisely $W_{M/2}$ on one of its sides by theorem 5.6.4. Thus, $C$ is not only a component of $G[W_{M/2} - vw]$, but also a component of $G - vw$, since $C$ avoids $\mathcal{P}$. In particular, all vertices in $C$ have degree at least 2 in $C$ as $G$ has minimum degree 3. Hence, $C$ contains a cycle, and we fix $e$ as an edge on this cycle. Then $C - e$ is still connected which by the previous description implies that also $G[W_{M/2}] - e$ is connected, as desired. □

We emphasise that in theorem 5.7.6 the possibly removed edge $e$ from $R$ is the only difference between the two RC-decompositions of $G$ and $G' := G - e$; in particular, all corresponding vertex sets such as $V(R)$ and the $V(R_{i,j})$ are the same, and a separation crosses or slices the rainbow in $G'$ if and only if it does so in $G$.

For the proof of theorem 5.7.1, we have to construct an orientation $\tau'$ of $S_k(G')$ that is a $k$-tangle in $G'$ and extends $\tau$. Clearly, any such extension $\tau'$ has to contain not only $\tau$, but also all orientations of separations of $S_k(G')$ that are forced by the request that $\tau'$ shall again be a tangle and hence especially consistent. More formally, recall that $\tau$ *forces* an orientation of a separation $\{A, B\}$ of $G'$ if there exists a separation $(E, F) \in \tau$ such that $(A, B) \leq (E, F)$ or $(B, A) \leq (E, F)$ (cf. the proof of theorem 5.4.4). Let us also recall that $\tau$ forces an orientation of every separation in $S_k(G')$ that is also a separation of $G$.

Towards a suitable construction of $\tau'$, we now prove that rainbow-crossing and rainbow-slicing separations are forced by $\tau$ (theorem 5.7.8 and theorem 5.7.9, respectively). This then allows us to show that an even broader class of separations are forced by $\tau$ (theorem 5.7.10). For all these subsequent three lemmas, we assume the following setting:

**Setting 5.7.7.** Assume theorem 5.7.5 with $M_0 \geq 18k$. Fix some edge $e$ of $G$ and some RC-decomposition $(R, \mathcal{W}, Z, C)$ of $G$ of length $M$ which theorem 5.7.6 yields. In particular, $M \geq 8k$.

**Lemma 5.7.8.** *Assume theorem 5.7.7. If $\{A, B\} \in S_k(G')$ crosses the rainbow, then there is $(X, Y) \in \tau$ with $(A, B) \leq (X, Y)$ or $(B, A) \leq (X, Y)$ such that $V(R_{2k-1,M-2k+1}) \subseteq X$. In particular, $\tau$ forces an orientation of $\{A, B\}$.*

*Proof.* By possibly reversing the rainbow, we may assume that $(A, B)$ crosses the rainbow clockwise. Let $k' < k$ be the order of $\{A, B\}$, and set $i := i_{A,B} \in \{0, \dots, 2k'\}$ and $j := j_{A,B} \in \{M - 2k', \dots, M\}$. By theorem 5.6.5, $(A^{i+1}, B^{i+1}), (A^j, B^j)$ are again separations of $G'$ of order at most $k'$ which cross the rainbow clockwise. They are also separations of $G$ by the choice of $e$: The bag $W_{M/2}$ is in $B^{i+1} \cap A^j$, since $i \leq 2k' \leq M/2 \leq M - 2k' \leq j$ because $M \geq 4k$ and $k' \leq k - 1$. As both $(A^{i+1}, B^{i+1}), (A^j, B^j)$ cross the rainbow, $Z$ is contained in each of their separators. Thus, both endvertices of $e$ are in any of the cases contained in $B^{i+1} \cap A^j$, as claimed.

Since $\tau$ does not live in the rainbow by the choice of $(R, \mathcal{W}, Z, C)$, the separation $\{A, B\}$ of $G$ is oriented by $\tau$ monotonically over the rainbow. In particular, we have either $(A^{i+1}, B^{i+1}), (A^j, B^j) \in \tau$ or $(B^{i+1}, A^{i+1}), (B^j, A^j) \in \tau$. By definition of the $(A^h, B^h)$ and the choice of $i, j$, we have $(A^{i+1}, B^{i+1}) \leq (A, B) \leq (A^j, B^j)$ and $V(R_{2k'+1,M-2k'-1}) \subseteq B^{i+1}, A^j$. So either $(A^j, B^j)$ or $(B^{i+1}, A^{i+1})$ can be chosen as the desired separation $(X, Y) \in \tau$, as $k' < k$. □

**Lemma 5.7.9.** *Assume theorem 5.7.7. If $\{A, B\} \in S_k(G')$ slices the rainbow, then there is $(X, Y) \in \tau$ with $(A, B) \leq (X, Y)$ or $(B, A) \leq (X, Y)$ such that $V(R) \subseteq X$. In particular, $\tau$ forces an orientation of $\{A, B\}$.*

*Proof.* As the separation $\{A, B\}$ of order $k' < k$ slices the rainbow, we may, by possibly renaming the sides of $\{A, B\}$, assume that there exists integers $i < h < j$ with $i \in \{0, \dots, 2k'\}$ and $j \in \{M - 2k', \dots, M\}$ with $W_i, W_j \subseteq A \setminus B$ and $W_h \subseteq B \setminus A$. If $\{A, B\}$ is only a separation of $G'$, but not a separation of $G$, we let $D$ be the component of $G'[B \setminus A]$ that contains an endvertex of $e$; otherwise, we let $D$ be the empty graph. Since $\{A, B\}$ slices the rainbow, we have $Z \subseteq A \cap B$ and thus $Z \cap V(D) = \emptyset$. So by the choice of the edge $e$ that we deleted from $G$ to get $G'$, we observe that if $V(D) \neq \emptyset$, then it meets $W_{M/2}$. By theorem 5.6.2 applied in $G'$, we thus obtain that either $V(D) \subseteq V(R) \setminus V(C)$ (which is in particular the case if $V(D) = \emptyset$) or $D$ meets every bag of either $W_0, \dots, W_{M/2}$ or $W_{M/2}, \dots, W_M$. But since $W_i, W_j \subseteq A \setminus B$ and $i \leq 2k' \leq M/2 \leq M - 2k' \leq j$, the second case cannot occur; so we have $V(D) \subseteq V(R) \setminus V(C)$.

Further, theorem 5.6.7 yields $2\ell + |Z| \leq |A, B| < k$ and therefore $|V(R) \cup Z, V(C)| = 2\ell + |Z| < k$. Thus, $\tau$ orients $\{V(R) \cup Z, V(C)\}$, and we have $(V(R) \cup Z, V(C)) \in \tau$ since $\tau$ does not live in the rainbow by construction of the RC-decomposition $(R, \mathcal{W}, Z, C)$.

By the choice of $D$, the pair $(A', B') := (A \cup V(D), B \setminus V(D))$ is a separation of $G$, which again has the same order $< k$ as $\{A, B\}$. Thus, $\tau$ contains an orientation of $\{A', B'\}$. If $(A', B') \in \tau$, we define

$$\begin{aligned}(E, F) &:= (A', B') \vee (V(R) \cup Z, V(C))\\ &= ((A \cup V(D)) \cup V(R), (B \setminus V(D)) \cap V(C))\\ &= (A \cup V(R), B \cap V(C)).\end{aligned}$$

If $(B', A') \in \tau$, we define

$$\begin{aligned}(E, F) &:= (B', A') \vee (V(R) \cup Z, V(C))\\ &= ((B \setminus V(D)) \cup V(R), (A \cup V(D)) \cap V(C))\\ &= (B \cup V(R), A \cap V(C)).\end{aligned}$$

We remark that in both definition the second equality holds due to $Z \subseteq A \cap B \setminus V(D)$ and the third holds due to $V(D) \subseteq V(R) \setminus V(C)$. We remark that the proof in the case of $(A', B') \in \tau$ is analogous to the one below in the case of $(B', A') \in \tau$ by swapping $A$ and $B$; thus, one obtains $(A, B) \leq (A \cup V(R), B \cap V(C) = (E, F) \in \tau$, if $(A', B') \in \tau$.

So let us assume that $(B', A') \in \tau$. The pair $(E, F)$ is again a separation of $G$, since it is the supremum of two separations of $G$. We claim that $\{E, F\}$ has order $< k$. Indeed, we have

$$\begin{aligned}|E, F| &= |(B \cup V(R)) \cap (A \cap V(C))| = |(B \cap A \cap V(C)) \cup (V(R) \cap A \cap V(C))|\\ &= |B \cap A \cap (V(C) \setminus V(R))| + |V(R) \cap V(C) \cap A| \leq |B \cap A \cap (V(C) \setminus V(R))| + |V(R) \cap V(C)|\\ &= |B \cap A \cap (V(C) \setminus V(R))| + 2\ell \leq |B \cap A \cap (V(C) \setminus V(R))| + |B \cap A \cap V(R)|\\ &= |B \cap A| = |B, A| < k,\end{aligned}$$

where $2\ell \leq |B \cap A \cap V(R)|$ holds by theorem 5.6.7. Hence, $\tau$ has to orient $\{E, F\}$ and it does so as $(E, F)$ by the profile property of the tangle $\tau$, since $(E, F)$ is the supremum of the two separations $(B', A')$ and $(V(R) \cup Z, V(C))$, which are both contained in $\tau$. Thus, $(B, A) \leq (B \cup V(R), A \cap V(C)) = (E, F) \in \tau$ and $V(R) \subseteq E$, so $(X, Y) := (E, F)$ is as desired. □

**Lemma 5.7.10.** *Assume theorem 5.7.7. Let $\{A, B\} \in S_k(G') \setminus S_k(G)$, and let $C_A \subseteq G'[A \setminus B]$ and $C_B \subseteq G'[B \setminus A]$ be the two components of $G' - (A \cap B)$ that contain an endvertex of $e$. If either both or none of $C_A$ and $C_B$ meet $C$, then $\tau$ forces an orientation of $\{A, B\}$.*

*Proof.* Let $k' < k$ be the order of $\{A, B\}$. We first argue that we are in the case in which both endvertices of $e$ are in $W_{M/2}$. Suppose for a contradiciton that $e$ otherwise has one endvertex in $W_{M/2}$ and the other one in $Z$. Then one of $C_A$ and $C_B$, say $C_A$, contains the endvertex $z$ of $e$ in $Z$ and thus meets $V(C) \supseteq Z$. Then $C_B$ meets $C$ as well by assumption. Since $C_B$ also contains the other endvertex of $e$, i.e. the one in $W_{M/2}$, and $M \geq 4k'$, theorem 5.6.2 yields that $C_B$ contains a bag $W_i$. By 4, $G'$ contains an edge joining $z \in V(C_A) \subseteq A \setminus B$ and $W_i \subseteq V(C_B) \subseteq B \setminus A$, which is a contradiction. So we may assume by the choice of $e$ that both endvertices of $e$ lie in $W_{M/2}$; in particular, both $C_A$ and $C_B$ meet $W_{M/2}$.

The assumptions on $C_A$ and $C_B$ ensure that $\{A \cup V(C_B), B \setminus V(C_B)\}$ and $\{A \setminus V(C_A), B \cup V(C_A)\}$ are both separations of $G$. Also their order $|A \cap B|$ is $< k$; thus, $\tau$ orients them. If $\tau$

orients one of them as $(A \cup V(C_B), B \setminus V(C_B)) \geq (A, B)$ or $(B \cup V(C_A), A \setminus V(C_A)) \geq (B, A)$, then $\tau$ forces an orientation of $\{A, B\}$, as desired. So assume that $\tau$ orients them as $(B \setminus V(C_B), A \cup V(C_B))$ and$(A \setminus V(C_A), B \cup V(C_A))$. Then their supremum $(V(G) \setminus (V(C_A) \cup V(C_B)), (A \cap B) \cup V(C_A) \cup V(C_B))$ is also contained in $\tau$, as its order is the same as $|A, B| = k' < k$ and the tangle $\tau$ has the profile property. Since $\tau$ does not live in $R$ by construction pf $(R, \mathcal{W}, Z, C)$, it follows that $(V(C_A) \cup V(C_B)) \cap V(C) \neq \emptyset$. As either both or none of $C_A$ and $C_B$ meets $C$ by assumption, both $C_A$ and $C_B$ meet $C$.

All in all, both $C_A$ and $C_B$ meet $C$ and $W_{M/2}$. Hence, theorem 5.6.2 yields that each of $C_A$ and $C_B$ contains all but at most $2k'$ bags of $W_0, \dots, W_{M/2}$ or of $W_{M/2}, \dots, W_M$; in particular, they contain some $W_i$ and some $W_j$ respectively, as $M \geq 4k'$. Now theorem 5.6.8 ensures that $\{A, B\}$ either crosses or slices the rainbow in $G'$. Thus, the previous theorems 5.7.8 and 5.7.9 guarantee taht $\tau$ forces an orientation of $\{A, B\}$. □

With these tools at hand, we are now ready to prove theorem 5.7.1:

*Proof of theorem 5.7.1.* We may assume theorem 5.7.7, as theorem 5.7.5 with $M_0 \geq 18k$ is precisely the premise of theorem 5.7.1. We claim that $\tau$ extends to a $k$-tangle in $G'$. For this, we begin by defining an orientation $\tau'$ of $S_k(G')$. So let $\{A, B\}$ be an arbitrary separation in $S_k(G')$. If $\{A, B\}$ is not a separation of $G$, then we let $C_A \subseteq G'[A \setminus B]$ and $C_B \subseteq G'[B \setminus A]$ be the components of $G - (A \cap B)$ which contain the respective endvertex of $e$.

(1) If $\tau$ forces an orientation of $\{A, B\}$, then we let $(A, B) \in \tau'$ if and only if $(A, B)$ is forced by $\tau$.

(2) If $\tau$ does not force an orientation of $\{A, B\}$, then $\{A, B\} \notin S_k(G)$. If $C_B$ meets $C$, then we let $(A, B) \in \tau'$, and if $C_A$ meets $C$, we let $(B, A) \in \tau'$.

Note that $\tau'$ contains at least one orientation of every separation in $S_k(G')$. Moreover, $\tau'$ is an orientation of $S_k(G')$: for a separation in case 1 $\tau$ forces at most one of its orientations due to the consistency of $\tau$, and theorem 5.7.10 ensures that precisely one of $C_A$ and $C_B$ meet $C$ for a separation $\{A, B\}$ in case 2. It is immediate form 1that $\tau'$ contains $\tau$; so $\tau$ extends to $\tau'$. It remains to show that $\tau'$ is indeed a $k$-tangle in $G'$.

Suppose for a contradiction that $\tau'$ is not a tangle in $G'$, i.e. the orientation $\tau'$ of $S_k(G')$ contains a forbidden triple $\{(A_i', B_i') : i \in [3]\} \in \mathcal{T}(G')$. By the definition of $\mathcal{T}(G')$, we may assume without loss of generality that all the $(A_i', B_i')$ are $\leq$-maximal in $\tau'$. In the remainder of the proof, we will construct from the forbidden triple $\{(A_i', B_i') : i \in [3]\} \subseteq \tau'$ a forbidden triple $\{(A_i, B_i) : i \in [3]\} \subseteq \tau$, which then contradicts that $\tau$ is a tangle in $G$.

To this end, we will make use of the RC-decomposition $(R, \mathcal{W}, Z, C)$. We first show that one of the $(A_i', B_i')$ contains $V(R_{2k-1, M-2k+1}) \cup Z$ in its respective small side $A_i'$; in particular, the edge $e$ is contained in $G[A_i']$ and thus it is a separation of $G$. We then show for the other two separations $(A_i', B_i')$ in the forbidden triple that either they are separations of $G$, too, or the component $C_{A_i'}$ of $G[A_i' \setminus B_i']$ containing an endvertex of $e$ in fact contains also $V(R_{2k-1, M-2k+1})$. Moving the $C_{A_i'}$ to the respective big side $B_i'$ if necessary, these three separations of $G$ will then yield a forbidden triple in $\tau$.

So we first show the following:

**Sublemma 5.7.11.** *If $\{(A_i', B_i') : i \in [3]\} \in \mathcal{T}(G')$ is contained in $\tau'$ where each $(A_i', B_i')$ is $\leq$-maximal in $\tau'$, then some $(A_j', B_j')$ is also a separation of $G$ with $V(R_{2k-1, M-2k+1}) \cup Z \subseteq A_j'$. In particular, $(A_j', B_j') \in \tau$.*

*Proof.* For every $(A'_i, B'_i)$, at most $2k-2$ bags of $\mathcal{W}$ are contained neither in $A'_i \setminus B'_i$ nor in $B'_i \setminus A'_i$ by theorem 5.6.1. Thus, all but at most $6k-6$ bags are contained in one of the strict sides of every $(A'_i, B'_i)$. As $\{(A'_i, B'_i) : i \in [3]\}$ is a forbidden triple in $G'$, we in particular have that $R[A'_1]$, $R[A'_2]$ and $R[A'_3]$ together cover the rainbow $R$ in $G'$. Thus, no such bag may be contained in every big side $B'_i$. Since $M \geq 6k-6$, there thus must be some bag $W_h$ of $\mathcal{W}$ that is contained in the strict small side $A'_j \setminus B'_j$ of some $(A'_j, B'_j)$; by renaming the $(A'_i, B'_i)$, we may assume that $j = 1$. Note that as $W_h \subseteq A'_1 \setminus B'_1$ and $Z \subseteq N_{G'}(W_h)$ by 4, we have $Z \subseteq A'_1$. We claim that $(A'_1, B'_1)$ is as desired, i.e. it is also a separation of $G$ and $V(R_{2k-1,M-2k+1}) \subseteq A'_1$.

Assume that $B'_1 \setminus A'_1$ also contains some bag $W_s$. By theorem 5.6.8, $(A'_1, B'_1)$ then either crosses or slices the rainbow in $G'$, and thus, the orientation $(A'_1, B'_1)$ of $\{A'_1, B'_1\}$ was forced by $\tau$ by theorems 5.7.8 and 5.7.9. But if a separation is both $\leq$-maximal in $\tau'$ and forced by $\tau$, then it is also contained in $\tau$, and thus a separation of $G$. Hence, $(A'_1, B'_1)$ is a separation of $G$ and also $\leq$-maximal in $\tau$. Moreover, then theorems 5.7.8 and 5.7.9 yield $V(R_{2k-1,M-2k+1}) \subseteq A'_1$, as desired.

So now assume that no bag of $\mathcal{W}$ is contained in $B'_1 \setminus A'_1$. By theorem 5.6.1, the strict side $A'_1 \setminus B'_1$ then contains all but at most $2k-2$ many bags of $\mathcal{W}$. Suppose for a contradiction that $\{A'_1, B'_1\}$ was not already a separation of $G$. Since $(A'_1, B'_1)$ is $\leq$-maximal in $\tau'$, the tangle $\tau$ did thus not force an orientation of $\{A'_1, B'_1\}$. Hence we have $(A'_1, B'_1) \in \tau'$ due to 2. Hence, $C_{B'_1}$ meets $C$. Since we have seen above that $Z \subseteq N_{G'}(W_m) \subseteq A'_1$, the choice of $e$ yields that the endvertex of $e$ contained in $C_{B'_1} \subseteq B'_1 \setminus A'_1$ lies in $W_{M/2}$. Thus, theorem 5.6.2 yields that $C_{B'_1} \subseteq B'_1 \setminus A'_1$ contains some bag of $\mathcal{W}$, which contradicts our assumption on $B'_1 \setminus A'_1$. Thus, $\{A'_1, B'_1\}$ is also a separation of $G$. It remains to show $V(R_{2k-1,M-2k+1}) \subseteq A'_1$. Since $(A'_1, B'_1)$ is a separation of $G$ and $\leq$-maximal in $\tau'$, it is also $\leq$-maximal in the tangle $\tau \subseteq \tau'$. Thus, $G[B'_1 \setminus A'_1]$ is connected, and thus equal to $C_{B'_1}$. As $C_{B'_1}$ meets $C$ but $G[B'_1 \setminus A'_1] = C_{B'_1}$ does not contain a bag of $\mathcal{W}$, theorem 5.6.2 yields that $C_{B'_1} = G[B'_1 \setminus A'_1]$ does meet at most the first and last $2k-2$ bags of $\mathcal{W}$. Thus, $V(R_{2k-1,M-2k+1}) \subseteq A'_1$, as desired. □

So by theorem 5.7.11, there is some $j \in [3]$ such that $(A'_j, B'_j) \in \tau$ and $V(R_{2k-1,M-2k+1}) \cup Z \subseteq A'_j$; by symmetry we may assume $j = 1$. Next, we aim to obtain from the other two separations $(A'_2, B'_2)$ and $(A'_3, B'_3)$ of the forbidden triple two separations $(A_2, B_2)$ and $(A_3, B_3)$ of $G$ that are contained in $\tau$ and differ from $(A'_2, B'_2)$ or $(A'_3, B'_3)$, respectively, only in a subset of $A'_1$. This will then be the desired forbidden triple in $\tau$.

**Sublemma 5.7.12.** *Every $\leq$-maximal separation $(A, B)$ in $\tau'$ is either also contained in $\tau$ or we have $(A \setminus V(C_A), B \cup V(C_A)) \in \tau$ and $V(C_A) \subseteq V(R_{2k-1,M-2k+1})$.*

*Proof.* If the tangle $\tau$ forces the orientation $(A, B)$ of $\{A, B\}$, then we already have $(A, B) \in \tau$ by the $\leq$-maximality of $(A, B)$ in $\tau'$. Thus, we may assume that $(A, B) \in \tau'$ due to 2, i.e. $C_A$ avoids $C$ and $C_B$ meets $C$. As each of $C_A$ and $C_B$ contains one endvertex of $e$, $\{A \setminus V(C_A), B \cup V(C_A)\}$ is a separation of $G$. It also has separator $A \cap B$, and thus is oriented by the $k$-tangle $\tau$ in $G$. Since $\tau$ does not force an orientation of $\{A, B\}$, it follows from the consistency of the tangle $\tau$ that $(B \cup V(C_A), A \setminus V(C_A)) \notin \tau$. Hence, $(A \setminus V(C_A), B \cup V(C_A)) \in \tau$. It remains to show $V(C_A) \subseteq V(R_{2k-1,M-2k+1})$.

First we claim that $C_B$ contains a bag of $\mathcal{W}$: Since $C_B$ contains an endvertex $x$ of the edge $e$ by definition, the choice of $e$ ensures that either $x \in W_{M/2}$ or $x \in Z$. If $x \in W_{M/2}$, then theorem 5.6.2 yields that $C_B$ contains a bag of $\mathcal{W}$, as $C_B$ meets $C$ and $M \geq 4k-4$. So assume that $x \in Z$. By 4, $x$ has neighbours in every bag of $\mathcal{W}$. In particular, every bag of $\mathcal{W}$ is adjacent to $V(C_B)$. At most $2k-2$ many bags of $\mathcal{W}$ meet $A \cap B$ by theorem 5.6.1. Thus, the component $C_B$ of $G' - (A \cap B)$ contains some bag of $\mathcal{W}$, since $M \geq 2k-2$ all the $G'[W_i]$ are connected by 2.

Secondly, we claim that $C_A$ does not contain a bag of $\mathcal{W}$. Suppose that $C_A$ contains a bag of $\mathcal{W}$. Then $\{A, B\}$ would either cross or slice the rainbow by theorem 5.6.8. Thus, $\tau$ forces an orientation of $\{A, B\}$ by theorems 5.7.8 and 5.7.9, which contradicts our assumption on $\{A, B\}$.

Hence, as all the $G'[W_i]$ is connected by 2, every bag which meets $C_A$ also meets $A \cap B$. theorem 5.6.1 shows that there are at most $2k-2$ such bags of $\mathcal{W}$ . Since $C_A$ contains an endvertex of $e$ but avoids $V(C) \supseteq Z$, it contains the endvertex of $e$ in $W_{M/2}$. Hence, $C_A$ can only meet $W_i$ with $|M/2 - i| \leq 2k-3$, as $C_A$ is connected and $\mathcal{W}$ is a linear decomposition of $R$. Thus, $M \geq 8k$ yields $V(C_A) \subseteq V(R_{2k-1,M-2k+1})$, as desired □

If a separation $(A, B)$ of $G'$ is also a separation of $G$, let $C_A$ be the empty graph. Recall that we have obtained earlier from theorem 5.7.11 that $(A'_1, B'_1) \in \tau$ and $V(R_{2k-1,M-2k+1}) \cup Z \subseteq A'_1$. By theorem 5.7.12, we now also have $(A_i, B_i) := (A'_i \setminus V(C_{A'_i}), B'_i \cup V(C_{A'_i})) \in \tau$ for $i = 2, 3$.

We claim that the triple $\{(A'_1, B'_1), (A_2, B_2), (A_3, B_3)\} \subseteq \tau$ is a forbidden triple in $G$. Since $\{(A'_i, B'_i) : i \in \{1, 2, 3\}\}$ is a forbidden triple in $G'$, we have

$$G[A'_1] \cup G[A_2] \cup G[A_3] = G[A'_1] \cup G[A'_2 \setminus V(C_{A'_2})] \cup G[A'_3 \setminus V(C_{A'_3})] \supseteq G - e - C_{A'_2} - C_{A'_3}.$$

Since the endvertices of $e$ are contained in $W_{M/2} \cup Z$, and $W_{M/2} \cup Z \subseteq V(R_{2k-1,M-2k+1}) \cup Z \subseteq A'_1$ as $M \geq 4k$, we have that

$$G[A'_1] \cup G[A_2] \cup G[A_3] \supseteq G - C_{A'_2} - C_{A'_3}.$$

By theorem 5.7.12, each of $V(C_{A'_2})$ and $V(C_{A'_3})$ is either empty or contained in $V(R_{2k-1,M-2k+1})$. So since $V(R_{2k-1,M-2k+1}) \subseteq A'_1$, it follows that both $C_{A'_2}$ and $C_{A'_3}$ are subgraphs of $G[A'_1]$ and hence $G[A'_1] \cup G[A_2] \cup G[A_3] = G$. Thus, $\{(A'_1, B'_1), (A_2, B_2), (A_3, B_3)\}$ is a forbidden triple in $\tau$, which contradicts that $\tau$ is a $k$-tangle in $G$. □

*Proof of Theorem 5.* Choose $M(k)$ to be $N(k, 18k)$ as in theorem 5.5.1. Let $k \geq 1$ be an integer, let $G$ be a connected graph with at least $M(k)$ edges, and let $\tau$ be a $k$-tangle in $G$. We may assume $k \geq 3$; otherwise, we are done by theorems 5.3.1 and 5.3.2. If there exists a $(k+1)$-tangle in $G$, then we are done by theorem 5.4.1. Therefore, we may assume that there is no tangle in $G$ of order $> k$. Then, the connected graph $G$ admits an RC-decomposition with sun $Z$ which has length $\geq 18k$ and adhesion $\ell$ such that $|Z| + \ell \geq 1$ by theorem 5.5.1. If $G$ has a vertex of degree at most 2, then we are done by theorems 5.3.4 and 5.3.5. Thus, theorem 5.7.1 concludes the proof. □

We remark that one may calculate that $M(k) \in O(3^{k^{k^5}})$.

## 5.8 The inductive proof method and its applications

This section consists of three parts: we first collect all above auxiliary results to conclude the formal proof of our inductive proof method, Theorem 4. Next, we deduce from Theorem 4 our reduction of theorem 5.1.1 to small graphs, Theorem 1, and then derive Theorems 2 and 3 from it. Finally, we present a further application of Theorem 4 in theorem 5.8.4, which bounds the size of a subgraph 'witnessing' a $k$-tangle.

Let us first prove our inductive proof method, which we restate here for the reader's convenience:

**Theorem 4.** *For every integer $k \geq 1$ there is some $M(k) \in O(3^{k^{k^5}})$ such that the following holds: Let $\tau$ be a $k$-tangle in a graph $G$. Then there exists a sequence $G_0, \dots, G_m$ of graphs and $k$-tangles $\tau_i$ in $G_i$ for every $i \in \{0, \dots, m\}$ such that*

- *$G_0 = G$, $\tau_0 = \tau$;*
- *$G_i$ is obtained from $G_{i-1}$ by deleting an edge, suppressing a vertex, or taking a proper component;*
- *the $k$-tangle $\tau_{i-1}$ in $G_{i-1}$ survives as the $k$-tangle $\tau_i$ in $G_i$ for every $i \in [m]$;*
- *$G_m$ is connected and has less than $M(k)$ edges.*

*Proof.* Let $M(k)$ be given by Theorem 5. Suppose that $G_0, \dots, G_{i-1}$ and $\tau_0, \dots, \tau_{i-1}$ are already defined. If $G_{i-1}$ is disconnected, then we apply theorem 5.3.3 to obtain $G_i$ and $\tau_i$. If $k \leq 2$, then we apply theorem 5.3.1 or theorem 5.3.2. If $k \geq 3$, but $G_{i-1}$ has a vertex of degree $\leq 2$, then we apply theorem 5.3.4 or theorem 5.3.5. Thus, we may assume that $G_{i-1}$ is a connected graph with minimum degree $\geq 3$ and $k \geq 3$. In this case, we apply Theorem 5, if $G_{i-1}$ has at least $M(k)$ edges. Otherwise, we set $m := i - 1$, completing the proof. □

### 5.8.1 Application I: sets and functions inducing tangles

In this section, we address theorem 5.1.1.

**Problem 1.1.** *Is every tangle in a graph $G$ induced by some set $X \subseteq V(G)$?*

We will use Theorem Theorem 4 to prove Theorem 1, our reduction of Theorem Problem 1.1 for $k$-tangles to graphs of size bounded in $k$. Let us briefly recall the relevant definitions, for which we mostly follow [46].

Given a tangle $\tau$ in a graph $G$, a set $X \subseteq V(G)$ *induces* $\tau$ if for every separation $(A, B) \in \tau$, we have $|X \cap A| < |X \cap B|$; in this case, we also say that $X$ *induces* the orientation $(A, B)$ of the separation $\{A, B\}$. As a natural relaxation, a *weight function* on $V(G)$ is a map $w \colon V(G) \to \mathbb{N}$, and we say that it *induces* $\tau$ if $w(A) < w(B)$ for all $(A, B) \in \tau$; in this case, we also say that $w$ *induces* the orientation $(A, B)$ of the separation $\{A, B\}$. We remark that a set $X \subseteq V(G)$ induces a tangle $\tau$ if and only if its indicator function $\mathbb{1}_X$ induces $\tau$. This allows us to focus on weight functions in what follows.

Recall that Theorem 1 reads as follows:

**Theorem 1.** *For every integer $k \geq 1$, there exists $M = M(k) \in O(3^{k^{k^5}})$ such that for every $k$-tangle $\tau$ in a graph $G$, there exists a $k$-tangle $\tau'$ in a connected topological minor $G'$ of $G$ with less than $M$ edges such that if a weight function $w'$ on $V(G')$ induces the tangle $\tau'$, then the weight function $w$ on $V(G)$ which extends $w'$ by zero induces the tangle $\tau$. In particular, a set of vertices which induces $\tau'$ also induces $\tau$.*

For the proof of Theorem Theorem 1 via Theorem Theorem 4, we need to consider the following setting: Let $G'$ be a graph which arises from a graph $G$ by deleting an edge, suppressing a vertex or by passing to a component such that a tangle $\tau$ in $G$ survives as a tangle $\tau'$ in $G'$. We now aim to transfer a weight function inducing the tangle $\tau'$ of $G'$ to a weight function inducing the tangle $\tau$ of $G$. The subsequent three lemmas show that the extension by zero always works.

**Lemma 5.8.1.** *If a $k$-tangle $\tau$ in a graph $G$ extends to a $k$-tangle $\tau'$ in $G - e$ for an edge $e \in G$, then every weight function $w$ on $V(G) = V(G')$ which induces $\tau'$ also induces $\tau$. In particular, a set of vertices which induces $\tau'$ also induces $\tau$.*

*Proof.* As $\tau$ extends to the tangle $\tau'$ in $G - e$, we have $\tau \subseteq \tau'$. Thus, $w$ induces $\tau$ as well. □

**Lemma 5.8.2.** *Let $\tau$ be a $k$-tangle in a graph $G$ with $k \geq 3$, and let $\tau'$ be the induced $k$-tangle in a graph $G' = G - v + xy$ obtained by suppressing a vertex $v$ with its two neighbours $x, y$ in $G$. If a weight function $w'$ on $V(G')$ induces $\tau'$, then the weight function $w$ on $V(G)$ which extends $w'$ by zero induces $\tau$. In particular, a set of vertices which induces $\tau'$ also induces $\tau$.*

*Proof.* Throughout this proof, we will use that, by definition of $\tau'$, a separation $(A', B')$ of $G'$ is in $\tau'$ if and only if at least one of $(A' \cup \{v\}, B')$ or $(A', B' \cup \{v\})$ is in $\tau$. Let $(A, B) \in \tau$. Our aim is to find a separation $(A', B') \in \tau'$ such that $A \subseteq A' \cup \{v\}$ and $B' \subseteq B \cup \{v\}$, which then implies that $w(A) \leq w'(A') + w(v) = w'(A') < w'(B') = w'(B') + w(v) \leq w(B)$; so $w$ induces the orientation $(A, B)$ of $\{A, B\}$ as desired.

If $x, y \in A$ and $v \notin B$, then $(A', B') := (A \setminus \{v\}, B)$ is a separation of $G'$, and $(A, B) \in \tau$ witnesses $(A', B') \in \tau'$, as desired. Similarly, if $x, y \in B$ and $v \notin A$, we have $(A', B') := (A, B \setminus \{v\}) \in \tau'$, as desired.

Let us now assume that $x, y \in A$ and $v \in B$. Then $(A, B \setminus \{v\})$ is a separation of $G$. Since the only neighbours $x, y$ of $v$ are in $A$, the separations $(A, B)$ and $(B \setminus \{v\}, A)$ form a forbidden tuple in $G$. Hence, $(A, B \setminus \{v\})$ is in the tangle $\tau$. Now the above described case yields $(A', B') := (A \setminus \{v\}, B \setminus \{v\}) \in \tau'$, as desired. Similarly, if $u, w \in B$ and $v \in A$, then $(A', B') := (A \setminus \{v\}, B \setminus \{v\}) \in \tau'$.

So to conclude the proof, we may assume that $x \in A \setminus B$ and $y \in B \setminus A$ by possibly renaming $x, y$; in particular $v \in A \cap B$, as $xv, vy \in E(G)$. Hence, $(C, D) := (A \cup \{y\}, B \setminus \{v\})$ is a separation of $G$. It suffices to show that $(C, D) \in \tau$ because the above described case for $x, y \in C$ and $v \notin D$ yields that $(A', B') := ((A \setminus \{v\}) \cup \{y\}, B \setminus \{v\}) = (C \setminus \{v\}, D) \in \tau'$, as desired. We now show that $(C, D) \in \tau$: The regularity of the tangle $\tau$ of order $k \geq 3$ yields that the separation $(\{v, y\}, V(G))$ is in $\tau$. Since the separation $(D, C)$ of $G$ together with $(A, B)$ and $(\{v, y\}, V(G))$ forms a forbidden triple in $G$, we have $(C, D) \in \tau$. □

**Lemma 5.8.3.** *Let $\tau$ be a $k$-tangle $\tau$ in a graph $G$, and let $\tau'$ be its induced $k$-tangle $\tau'$ in some component $G'$ of $G$. If a weight function $w'$ on $V(G')$ induces $\tau'$, then the weight function $w$ on $V(G)$ which extends $w'$ by zero induces $\tau$. In particular, a set of vertices which induces $\tau'$ also induces $\tau$.*

*Proof.* Consider an arbitrary separation $(A, B) \in \tau$. Then $(A', B') := (A \cap V(G'), B \cap V(G')) \in \tau'$, as $\tau$ induces the tangle $\tau'$ in the component $G'$ of $G$. Since $w'$ induces $\tau'$ and due to the definition of $w$, we have $w(A) = w'(A') < w'(B') = w(B)$, as desired. □

*Proof of Theorem Theorem 1.* Let $\tau_m$ be the $k$-tangle in the graph $G_m$ with less than $M(k)$ edges as described in Theorem Theorem 4. Let $w'$ be a weight function on $V(G')$ which induces the tangle $\tau_m$. Iteratively applying theorems 5.8.1 to 5.8.3 yields that the extension of the weight function $w'$ by zero induces the tangle $\tau_0 = \tau$ in the graph $G_0 = G$. □

As direct consequences of Theorem Theorem 1, we now deduce Theorem Corollary 2 and Theorem Corollary 3:

**Corollary 2.** *For $k \geq 1$, there exists $M = M(k) \in O(3^{k^{k^5}})$ such that Theorem Problem 1.1 holds for $k$ if it holds for all $k$-tangles in connected graphs $G$ with fewer than $M$ edges.*

*Proof.* This follows immediately from Theorem Theorem 1. □

**Corollary 3.** *For every integer $k \geq 1$, there exists $K = K(k)$ such that for every $k$-tangle $\tau$ in a graph $G$ there exists a weight function $V(G) \to \mathbb{N}$ which induces $\tau$ and whose total weight $w(V(G))$ is bounded by $K$. In particular, the support of $w$ has size $\leq K$.*

*Moreover, if Theorem Problem 1.1 holds for $k$, then every $k$-tangle in a graph is induced by a set of at most $M(k)$ vertices, where $M(k)$ is given by Theorem 1.*

*Proof.* We enumerate all the finitely many non-isomorphic connected graphs $G_1, \dots, G_m$ with fewer than $M$ edges. As every finite graph has at most finitely many tangles, there are only finitely many $k$-tangles $\tau$ in any such $G_i$. Elbracht, Kneip and Teegen showed with theorem 5.1.2 that every tangle in a graph is induced by a weight function, so we may fix for every such $k$-tangle $\tau$ a weight function $w_\tau$ which induces $\tau$. We then set $K(k)$ to be the maximum over all the total weights of these weight functions $w_\tau$. Theorem Theorem 1 yields that every $k$-tangle in a graph $G$ is induced by some weight function which extends one of the weight functions $w_\tau$ by zero, and thus has total weight $\leq K(k)$.

The moreover-part follows immediately from Theorem Theorem 1 by choosing the weight functions as indicator functions of the inducing sets given by the assumed positive answer to Theorem Problem 1.1. □

We remark that the proof of Theorem Corollary 3 in fact shows that the support of the weight function inducing a $k$-tangle may actually be bounded by $M(k)$ as given in Theorem Theorem 1.

### 5.8.2 Application II: subgraphs witnessing a tangle

In this section, we demonstrate another application of our inductive proof method Theorem Theorem 4 by bounding the size of a subgraph 'witnessing' a tangle. We say that a subgraph $H$ of a graph $G$ *witnesses* that an orientation $\tau$ of $S_k(G)$ is a tangle if $H \not\subseteq \bigcup_{i=1}^{3} G[A_i]$ for every three (not necessarily distinct) $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \tau$. Indeed, $\tau$ is a tangle if and only if such a witnessing subgraph $H$ exists, since every tangle in $G$ is witnessed by $G$ itself.

Grohe and Schweitzer [54]*Lemma 3.1[6] proved that every $k$-tangle is witnessed by a set of edges whose size can be bounded in $k$. However, their bound is defined recursively

[6]We remark that triple covers in their paper are precisely the witnessing sets here. In fact, they proved a more general result about tangles on bipartitions in a more general setting. However, every $k$-tangle $\tau$ in $G$ induces a tangle $\tau'$ on the set of bipartitions of the edge set $E(G)$ of order $< k$: let $(C, D) \in \tau'$ if and only if there exists a separation $(A, B) \in \tau$ with $C \subseteq E(G[A])$ and $D \subseteq E(G[B])$. Then their result yields the described conclusion.

and yields a power tower of height $k-1$. By Theorem Theorem 4, we obtain a new bound which is significantly better for sufficiently large $k$:

**Corollary 5.8.4.** *For every integer $k \geq 1$, there is an integer $M' = M'(k) \in O(3^{k^{k^5}})$ such that every $k$-tangle in a graph $G$ is witnessed by some subgraph $H$ of $G$ of size at most $M'$.*

*Proof of theorem 5.8.4.* Let $M(k)$ be given by Theorem Theorem 4, and set $M'(k) := 2M(k)$. Let $\tau$ be a $k$-tangle in some graph $G$. Theorem 4 yields a $k$-tangle $\tau' := \tau_m$ in a topological minor $G' := G_m$ of $G$ which is connected and has fewer than $M(k)$ edges; in particular, $G'$ has at most $M(k)$ vertices. Now the lift of $\tau'$ to $G$ is indeed $\tau$, as one checks by following the lifts along the inductive structure given by Theorem Theorem 4.

Let $H'$ be the subdivision of $G'$ in $G$, i.e. the subgraph of $G$ from which we obtain $G'$ by vertex suppressions. Consider the subgraph $H$ of $H'$ consisting of the branch vertices $V(G')$ together with precisely one edge of $H'$ per $V(G')$-path in $H'$. We may choose these edges such that they are incident with at least one branch vertex. Since $G'$ witnesses $\tau'$, one again checks along the inductive structure given by Theorem Theorem 4 that $H$ witnesses $\tau$. Note that $H$ has at most $|E(G')|$ edges and, as each edge of $H$ is incident to a branch vertex of $H'$, $H$ has at most $|V(G')| + |E(G')| \leq M'(k)$ vertices. □

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# Appendices

# A
# Summary

In chapter 2 we introduce a new perspective on the structure of order-function-based abstract $\mathcal{F}$-tangles, inspired by our work on tangle applications [4]. The goal of this chapter is to build a theoretical framework around the core concept of the 'tangle structure tree'. Our core results are theorems 2.4.6 and 2.6.1, which together show that tangle structure trees exist precisely when the forbidden sets $\mathcal{F}$ satisfy a natural richness condition. This opens new possibilities for proving results about tangles.

In chapter 3 we build on this theoretical foundation to derive the two pillars of tangle theory: the tree-of-tangles theorem in theorems 3.5.1 and 3.5.8, and the tangle-tree duality theorem in theorem 3.3.6. We also show that this framework applies to large classes of theoretical and practical tangle applications, in particular to graphs and features.

In chapter 4 we present a concrete application of tangles in the social sciences by using them to cluster the questions of the Big Five questionnaire. In particular, we study how the additional structure provided by tangles, as opposed to traditional clustering methods, can be used to gain deeper insights into the questionnaire.

In the final chapter 5, we reduce the problem of whether every $k$-tangle in a graph is induced by a set of vertices by majority vote to graphs of bounded size.

# B
# Deutschsprachige Zusammenfassung

In Kapitel 2 führen wir eine neue Perspektive auf die Struktur von auf Ordnungsfunktionen basierenden abstrakten $\mathcal{F}$-Knäueln ein, inspiriert durch unsere Arbeit zu Tangle-Anwendungen in [4]. Das Ziel dieses Teils der Arbeit ist es, einen theoretischen Rahmen um das Kernkonzept des ‘Knäuel-Strukturbaums’ zu entwickeln. Unsere Kernresultate sind theorems 2.4.6 and 2.6.1; diese zeigen, dass Knäuel-Strukturbäume genau dann existieren, wenn die verbotenen Mengen $\mathcal{F}$ eine natürliche Reichhaltigkeitsbedingung erfüllen. Dies eröffnet neue Möglichkeiten, Resultate über Knäuel zu beweisen.

In Kapitel 3 bauen wir auf dieser theoretischen Grundlage auf und leiten die beiden Säulen der Knäueltheorie her: das Knäuel-Baum-Theorem in theorems 3.5.1 and 3.5.8 und das Knäuel-Baum-Dualitätstheorem in theorem 3.3.6. Wir zeigen außerdem, dass dieser Ansatz auf große Klassen theoretischer und praktischer Anwendungen von Knäueln angewendet werden kann.

In Kapitel 4 präsentieren wir eine konkrete Anwendung von Knäueln in den Sozialwissenschaften, indem wir sie zur Clusterung der Fragen des "Big-Five"-Fragebogens verwenden. Insbesondere untersuchen wir, wie die durch Knäuel bereitgestellte zusätzliche Struktur im Gegensatz zu traditionellen Clusterverfahren genutzt werden kann, um tiefere Einblicke in den Fragebogen zu gewinnen.

Im abschließenden Kapitel 5 reduzieren wir das Problem, ob jedes $k$-Knäuel in einem Graphen durch Mehrheitsabstimmung von einer Knotenmenge induziert wird, auf Graphen beschränkter Größe.

# C
# Publications related to this thesis

The following publications are related to this thesis:

Chapter 2 is based on [23].

Chapter 3, excluding section 3.7, is based on [15]. Section 3.7 is based on [24].

Chapter 4 is based on [22].

Chapter 5 is based on [55].

# D

# Declaration of contributions

Chapter 2 is based on [23], which was developed in collaboration with Reinhard Diestel. The main idea of the paper was developed by me, inspired by a suggestion from Reinhard Diestel to consider whether we could have an algorithm implementing the Tangle-tree duality theorem for forbidden subsets that need not consist of stars. I developed an initial version of the tangle structure tree framework, as well as the structure of the paper and its proofs and theorems, which were then significantly refined by Diestel.

Chapter 3 is mostly based on [15], which was developed in collaboration with Reinhard Diestel. The main idea of the paper was again developed by me, based on my work on the previous chapter. In this paper, Diestel did a great deal of work in thinking through my rough ideas and turning them into a coherent mathematical direction. The final section 3.7 is based on [24], which was written entirely by myself.

Chapter 4 is based on [22], which was written in collaboration with Reinhard Diestel. I carried out the implementation of the experiments and wrote a first draft of the paper, while Diestel contributed much of the paper's structure and significantly refined it.

Chapter 5 is based on [55], which was written in collaboration with Sandra Albrechtsen, Raphael W. Jacobs, Paul Knappe and Paul Wollan. We developed a rough draft of the paper together during a visit by Paul Wollan in October 2021. Sandra Albrechtsen and I then expanded the proof. In particular I expanded the proof in section 5.5 and restructured and simplified section 5.6 and section 5.7. Afterwards Sandra Albrechtsen, Raphael W. Jacobs and Paul Knappe worked out the full details of the proofs and developed the final presentation.

# E
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# F
# Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde und das dieses Exemplar und die beim für das Promotionsverfahren zuständigen Fach-Promotionsausschuss eingereichten elektronischen und ausgedruckten Exemplare identisch sind.

Ich versichere, dass ich generative Künstliche Intelligenz (KI)-Tools nur in dem Umfang verwendet habe, der mit den Prüfenden vereinbart wurde.

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Ich verantworte die Übernahme jeglicher von mir verwendeter maschinell generierter Textpassagen vollumfänglich selbst und dokumentiere im Folgenden listenartig, für welche Aufgaben ich KI-Tools genutzt habe.

In der hier vorliegenden Arbeit habe ich gKI-Systeme, konkret Codex (GPT-5.4), wie folgt genutzt:

- zum Korrekturlesen hinsichtlich Rechtschreibung, Grammatik und mathematischer Sprache in chapter 2, chapter 3 sowie in den Formalia dieser Arbeit,

- zum Korrekturlesen auf technische Unstimmigkeiten in chapter 2 und chapter 3. Hierbei wurden die Vorschläge der KI nicht unkritisch übernommen, sondern nur als Inspiration benutzt. Die Bewertung, ob es sich tatsächlich um Fehler handelte, wurde von mir vorgenommen.

Ich versichere, alle Nutzungen vollständig angegeben zu haben. Mir ist bekannt, dass fehlende oder fehlerhafte Angaben als Täuschungsversuch gewertet werden können. Zudem bin ich bereit, auf Nachfrage meine Vorgehensweise (z.B. durch Arbeitsnotizen, Prompts oder mündliche Erläuterungen u.Ä.) offen zu legen.

Sofern für die Anfertigung der Abschlussarbeit die Nutzung von KI-Schreibwerkzeugen nicht ausdrücklich erlaubt wurde, sind diese kennzeichnungspflichtig.

Mir ist bekannt, dass ein Verstoß gegen die oben genannten Punkte prüfungsrechtliche Konsequenzen haben und insbesondere dazu führen kann, dass die Prüfungsleistung mit "nicht ausreichend" bewertet wird und bei mehrfachem oder schwerwiegendem Täuschungsversuch eine Exmatrikulation erfolgen kann.

***Hamburg, 27.03.2026***

Date

Signature
(author)