

MATROIDS WITH AN INFINITE CIRCUIT-COCIRCUIT INTERSECTION

Nathan Bowler Johannes Carmesin

February 16, 2012

Abstract

We construct some matroids that have a circuit and a cocircuit with infinite intersection.

This answers a question of Bruhn, Diestel, Kriesell, Pendavingh and Wollan. It further shows that the axiom system for matroids proposed by Dress does not axiomatize all infinite matroids. We show that one of the matroids we define is a thin sums matroid whose dual isn't a thin sums matroid.

1 Introduction

In [5], Bruhn, Diestel, Kriesell, Pendavingh and Wollan introduced axioms for infinite matroids in terms of independent sets, bases, circuits, closure and relative rank. These axioms allow for duality of infinite matroids as known from finite matroid theory, which settles an old problem of Rado, and generalizes dualities that can be observed in infinite graphs. They ask the following.

Question 1.1 ([5]). *Is the intersection of a circuit with a cocircuit in an infinite matroid always finite?*

We call a matroid *tame* if the intersection of any circuit with any cocircuit is finite, and otherwise *wild*. While many infinite matroids are either finitary (have no infinite circuits) or co-finitary (have no infinite cocircuits), matroids containing both infinite circuits and infinite cocircuits do exist; the first non-trivial examples were constructed in [5]. However, in those examples every intersection of a circuit with a cocircuit is finite. Indeed, Dress [7] thought that the aim to have infinite matroids with duality would make it necessary that circuit-cocircuit intersection were finite. He proposed a

theory of matroids which had the finiteness of circuit-cocircuit intersections built into the definition of a matroid, in order to facilitate duality.

The axioms proposed by Dress describe a superclass of the class of all infinite matroids if and only if Question 1.1 has a positive answer [10]. We prove that the assertion of Question 1.1 is false and consequently that the axiom system for matroids proposed by Dress does not capture all matroids.

Theorem 1.2. *There exists a matroid M that has a circuit C and a cocircuit D such that $|C \cap D| = \infty$.*

To construct such matroids M , we use some recent result from an investigation of matroid union [2].

We hope that the matroids we construct here will be sufficiently badly behaved to serve as counterexamples and so allow the resolution of other open problems. To illustrate this potential, we shall show that we do obtain a counterexample to a natural open question about thin sums matroids, a generalisation of representable matroids.

If we have a family of vectors in a vector space, we get a matroid structure on that family whose independent sets are given by the linearly independent subsets of the family. Matroids arising in this way are called *representable* matroids. Although many interesting finite matroids (eg. all graphic matroids) are representable, it is clear that any representable matroid is finitary and so many interesting examples of infinite matroids are not of this type. However, since the construction of many of these examples, including the algebraic cycle matroids of infinite graphs, is suggestively similar to that of representable matroids, the notion of *thin sums matroids* was introduced in [4]: it is a generalisation of representability which captures these infinite examples.

Since thin sums matroids need not be finitary, and the duals of many thin sums matroids are again thin sums matroids, it is natural to ask whether the class of thin sums matroids itself is closed under duality. It is shown in [3] that the class of tame thin sums matroids is closed under duality, so that any counterexample must be wild. We show below that one of the wild matroids we have constructed does give a counterexample.

Theorem 1.3. *There exists a thin sums matroid whose dual is not a thin sums matroid.*

The paper is organised as follows. In Section 2, we recall some basic matroid theory. After this, in Section 3, we give the first example of a wild matroid. In Section 4, we give a second example, which is obtained by taking

the union of a matroid with itself. In Section 5, we show that the class of thin sums matroids is not closed under duality by construct a suitable wild thin sums matroid whose dual is not a thin sums matroid.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [6], for matroids that of [9, 5]. A set system \mathcal{I} is the set of independent sets of a matroid if it satisfies the following *independence axioms* [5].

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) \mathcal{I} is closed under taking subsets.
- (I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.
- (IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

M always denotes a matroid and $E(M)$, $\mathcal{I}(M)$, $\mathcal{B}(M)$, $\mathcal{C}(M)$ and $\mathcal{S}(M)$ denote its ground set and its sets of independent sets, bases, circuits and spanning sets, respectively. A matroid is called *finitary* if every circuit is finite.

In our constructions, we will make use of algebraic cycle matroid $M_A(G)$ of a graph G . The circuits of $M_A(G)$ are the edge sets of finite cycles of G and the edge sets of double rays¹. If G is locally finite, then $M_A(G)$ is *cofinitary*, that is, its dual is finitary [2]. If G is not locally finite, then this is no longer true [5]. Higgs [8] characterized those graphs G that have an algebraic cycle matroid, that is, whose finite circuits and double rays from the circuits of a matroid: G has an algebraic cycle matroid if and only if G does not contain a subdivision of the Bean-graph, see Figure 1.

3 First construction: the matroid M^+

In this example, we will need the following construction from [2]:

Definition 3.1. Let M be a matroid, in which \emptyset isn't a base. Then the matroid M^- , on the same groundset, is that whose bases are those obtained by removing a point from a base of M . That is, $\mathcal{B}(M^-) = \{B - e \mid B \in$

¹A *double ray* is a two sided infinite path

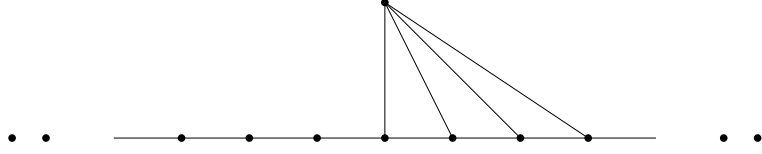


Figure 1: The Bean-graph

$\mathcal{B}(M), e \in B\}$. Dually, if M is a matroid whose ground set E isn't a base, we define M^+ by $\mathcal{B}(M^+) = \{B + e \mid B \in \mathcal{B}(M), e \in E \setminus B\}$.

Thus $(M^+)^* = (M^*)^-$.

We shall show that the matroids constructed in this way are very often wild.

Since M^- is obtained from M by making the bases of M into dependent sets, we may expect that $\mathcal{C}(M^-) = \mathcal{C}(M) \cup \mathcal{B}(M)$: that is, the set of circuits of M^- contains exactly the circuits and the bases of M . This is essentially true, but there is one complication: an M -circuit might include an M -base, which would prevent it being an M^- -circuit. Let O be a circuit of M^- . If O is M -independent, it is clear that O must be an M -base. Conversely, any M -base is a circuit of M^- . If O is M -dependent, then since all proper subsets of O are M^- -independent and so M -independent, O must be an M -circuit. Conversely, an M -circuit not including an M -base is an M^- -circuit.

On the other hand, none of the circuits of M is a circuit of M^+ : for any circuit O of M , pick any $e \in O$ and extend $O - e$ to a base B of M . Then $O \subseteq B + e$, so $O \in \mathcal{I}(M^+)$. In fact, a circuit of M^+ is a set minimal with the property that at least two elements must be removed before it becomes M -independent. To see this note that the independent sets of M^+ are those sets from which an M -independent set can be obtained by removing at most one element.

Now we are in a position to construct a wild matroid: let M be the algebraic cycle matroid of the graph in Figure 2. Then the dashed edges form a circuit in M^+ , and the bold edges form a circuit in $(M^+)^* = (M^*)^-$ (they form a base in M^* since their complement forms a base in M). The intersection, consisting of the dotted bold edges, is evidently infinite.

For the remainder of this section, we will generalize this example to construct a large class of wild matroids. To do so, we first have a closer look at the circuits of M^+ . It is clear that if M is the finite cycle matroid of a graph G , then we get as circuits of M^+ any subgraphs which are subdivisions of those in Figure 3.



Figure 2: A circuit and a cocircuit with infinite intersection

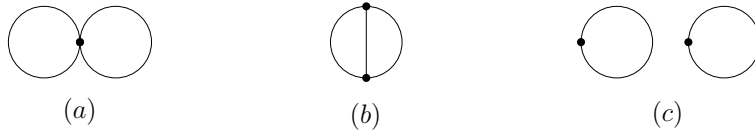


Figure 3: Shapes of circuits in M^+ , with M a finite cycle matroid

More generally, we can make precise a sense in which every circuit of M^+ is obtained by sticking together two circuits.

Lemma 3.2. *Let O be a circuit of M , and $I \subseteq E(M) \setminus O$. Then $O \cup I$ is M^+ -independent iff I is M/O -independent.*

Proof. If: Extend I to a base B of M/O . Pick any $e \in O$. Then $B' = B \cup O - e$ is a basis of M and $O \cup I \subseteq B' + e$.

Only if: Pick B a base of M and $e \in E \setminus B$ such that $O \cup I \subseteq B \cup e$. Since O is dependent, we must have $e \in O$, and so $I \subseteq B \setminus O$. Finally, $B \setminus O$ is a base of M/O , since $B \cap O = O - e$ is a base of O . \square

Lemma 3.3. *Let O_1 be a circuit of M , and O_2 a circuit of M/O_1 . Then $O_1 \cup O_2$ is a circuit of M^+ . Every circuit of M^+ arises in this way.*

Proof. $O_1 \cup O_2$ is M^+ -dependent by Lemma 3.2. Next, we shall show that any set $O_1 \cup O_2 - e$ obtained by removing a single element from $O_1 \cup O_2$ is M^+ -independent, and so that $O_1 \cup O_2$ is a *minimal* dependent set (a circuit) in M^+ . The case $e \in O_2$ is immediate by Lemma 3.2. If $e \in O_1$, then we pick any $e' \in O_2$. Now extend $O_2 - e'$ to a base B of M/O_1 . Then $B' = B \cup O_1 - e$ is a base of M and $O_1 \cup O_2 - e \subseteq B' + e'$.

Finally, we need to show that any circuit O of M^+ arises in this way. O must be M -dependent, and so we can find a circuit $O_1 \subseteq O$ of M . Let $O_2 = O \setminus O_1$: O_2 is a circuit of M/O_1 by Lemma 3.2. \square

Corollary 3.4. *Any union of two distinct circuits of M is dependent in M^+ .*

It follows from Lemma 3.3 that the subgraphs of the types illustrated in Figure 3 give all of the circuits of M^+ for M a finite cycle matroid. Similarly, subdivisions of the graphs in Figure 3 and Figure 4 give circuits in the algebraic cycle matroid of a graph.

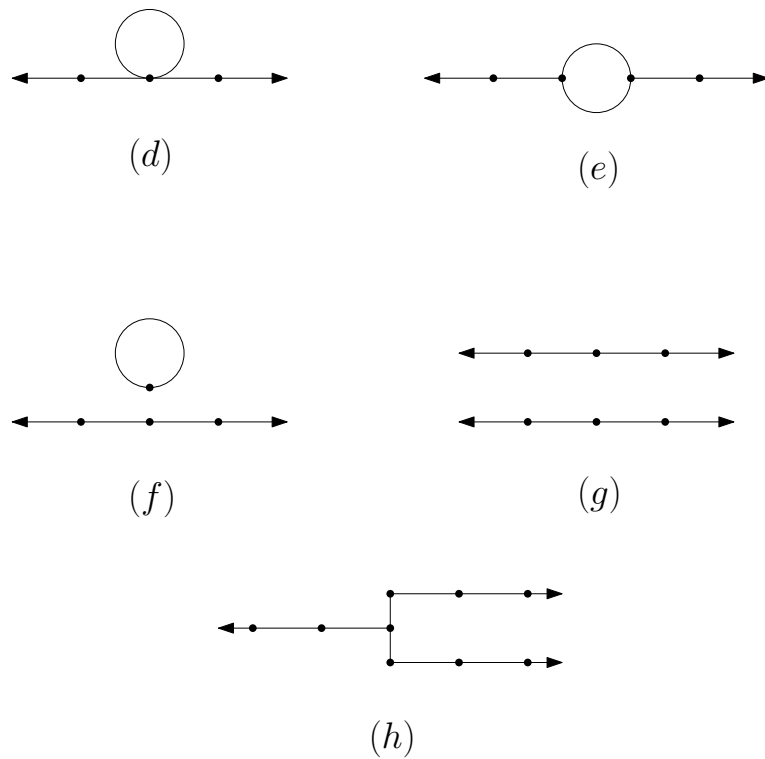


Figure 4: Shapes of circuits in M^+ , with M an algebraic cycle matroid

Now that we have a good understanding of the circuits of matroids constructed this way, we can find many matroids M such that M^+ is wild.

Theorem 3.5. *Let M be a matroid such that*

1. M contains at least two circuits;
2. M has a base B and a circuit O such that $O \setminus B$ is infinite.

Then M^+ is wild.

Proof. Let O' be any circuit other than O . As O' is dependent in M/O , there is an M/O -circuit O'' contained in O' . By Lemma 3.3, $O \cup O''$ is an M^+ -circuit.

Since $E \setminus B$ is an M^* -base, it is a circuit of $(M^*)^- = (M^+)^*$. Now $(O \cup O'') \cap (E \setminus B)$ includes $O \setminus B$ and so it is infinite. □

4 Second construction: matroid union

The union of two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ is the pair $(E_1 \cup E_2, \mathcal{I}_1 \vee \mathcal{I}_2)$, where

$$\mathcal{I}_1 \vee \mathcal{I}_2 := \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$$

The finitarization M^{fin} of a matroid M is the matroid whose circuits are precisely the finite circuits of M . In [2] it is shown that M^{fin} is always a matroid. Note that every base of M^{fin} contains some base of M and conversely every base of M is contained in some base of M^{fin} . A matroid M is called *nearly finitary* if for every base of M , it suffices to add finitely many elements to that base to obtain some base of M^{fin} . It is easy to show that M is nearly finitary if and only if for every base of M^{fin} it suffices to delete finitely many elements from that base to obtain some base of M .

The main tool for this example is the following theorem.

Theorem 4.1 ([2]). *The union of two nearly finitary matroids is a matroid, and in fact nearly finitary.*

Note that there are two matroids whose union is not a matroid [1].

One can also define M^+ using matroid union: $M^+ = M \vee U_{1, E(M)}$. Here $U_{1, E(M)}$ is the matroid with groundset $E(M)$, whose bases are the 1-element subsets of $E(M)$. In this section, we will obtain a wild matroid as union of some non-wild matroid M with itself.

Let us start constructing M . We obtain the graph H from the infinite one-sided ladder L by doubling every edge, see Figure 5.

As in the figure, we fix the following notation for the edges of H : In L , call the edges on the upper side of the ladder u_1, u_2, \dots , the edges on the

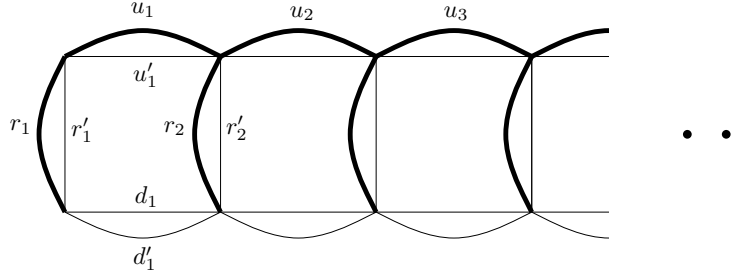


Figure 5: The graph H

lower side d_1, d_2, \dots and the rungs r_1, r_2, \dots . For every edge e of L , call its clone e' .

Let $M_A(H)$ be the algebraic cycle matroid of H . Note that $M_A(H)$ is a matroid by the results mentioned in the Preliminaries. Now we define M as the union of $M_A(H)$ with itself. To show that M is a matroid, by Theorem 4.1 it suffices to show the following.

Lemma 4.2. $M_A(H)$ is nearly finitary.

Proof. First note that the finitarization of $M_A(H)$ is the finite cycle matroid $M_F(H)$, whose circuits are the finite cycles of H . To see that $M_A(H)$ is nearly finitary, it suffices to show that each base B of $M_F(H)$ contains at most one double ray. It is easy to see that a double ray R of H contains precisely one rung r_i or r'_i . From this rung onwards, R contains precisely one of u_j or u'_j for $j \geq i$ and one of d_j or d'_j for $j \geq i$. Let R and S be two distinct double rays with unique rung edges e_R and e_S . Wlog assume that the index of e_R is less or equal than the index of e_S . Then already $R + e_S$ contains a finite circuit, which consists of e_R , e_S and all edges of R with smaller index than that of e_S . So each base B of $M_F(H)$ contains at most one double ray, proving the assumption. \square

Having proved that $M \vee M$ is a matroid, we next prove that it is wild.

Theorem 4.3. The matroid $M \vee M$ is wild.

To prove this, we will construct a circuit C and a cocircuit D with infinite intersection. Let us start with C , which we define as the set of all horizontal edges in Figure 5 together with the rung r_1 .

Lemma 4.4. $C := \{u_i, u'_i, d_i, d'_i | i = 1, 2, \dots\} + r_1$ is a circuit of M .

Proof. First, we show that C is dependent. To this end, it suffices to show that $C - r_1 = \{u_i, u'_i, d_i, d'_i | i = 1, 2, \dots\}$ is a basis of M . As $I_1 = \{u_i, d_i | i = 1, 2, \dots\}$ and $I_2 = \{u'_i, d'_i | i = 1, 2, \dots\}$ are both independent in $M_A(H)$, their union $C - r_1$ is independent in M . All other representations $C - r_1 = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{I}(M_A(H))$ are the upper one up to exchanging parallel edges since from u_i and u'_i precisely one is in I_1 and the other is in I_2 . Similarly, the same is true for d_i and d'_i . So $C - r_1$ is a base and C is dependent, as desired.

It remains to show that $C - e$ is independent for every $e \in C$. The case $e = r_1$, was already consider above. By symmetry, we may else assume that $e = u_i$. Then $C - u_i = I_1 \cup I_2$ where $I_1 = \{u_i, d_i | i = 1, 2, \dots\} - u_i + r_1$ and $I_2 = \{u'_i, d'_i | i = 1, 2, \dots\}$ and I_1 and I_2 are both independent in $M_A(H)$, proving the assumption. \square

Next we turn to D , drawn bold in Figure 5.

Lemma 4.5. $D := \{u_i, r_i | i = 1, 2, \dots\}$ is a cocircuit of M .

Proof. To this end, we show that $E \setminus D$ is a hyperplane, that is, $E \setminus D$ is non-spanning and $E \setminus D$ together with any edge is spanning in M . To see that $E \setminus D$ is non-spanning, we properly cover it by the following two bases B_1 and B_2 of $M_A(H)$, see Figure 6. Formally, $B_1 := \{d_i | i = 1, 2, \dots\} \cup \{r'_i | i \text{ odd}\} \cup \{u'_i | i \text{ odd}\}$, $B_2 := \{d'_i | i = 1, 2, \dots\} \cup \{r'_i | i \text{ even}\} \cup \{u'_i | i \text{ even}\} + r_1$.

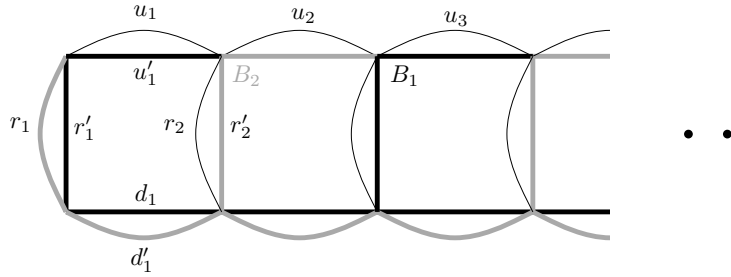


Figure 6: The two bases B_1 and B_2 properly cover $E \setminus D$

To see that $E \setminus D$ together with any edge is spanning in M , we even show that $E \setminus D$ together with any edge is a base of M . This is done in two steps: first we show that $E \setminus D$ together with any edge e is independent in

M and then that $E \setminus D$ together with any two edges is dependent in M . Concerning the first assertion, we distinguish between the cases $e = u_n$ for some n and $e = r_n$ for some n . In both cases we assume that n is odd. If n is even, then the argument is similar. In both cases we will cover $E \setminus D + e$ with two bases of $M_A(H)$, which arise from a slight modification of B_1 and B_2 , see Figures 7 and 8.

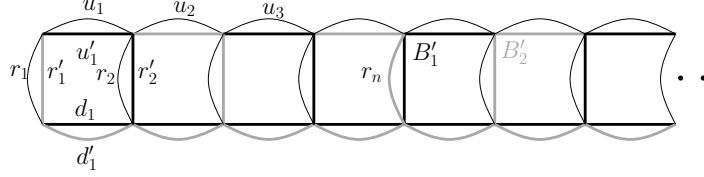


Figure 7: The two bases B'_1 and B'_2 cover $E \setminus D + r_n$

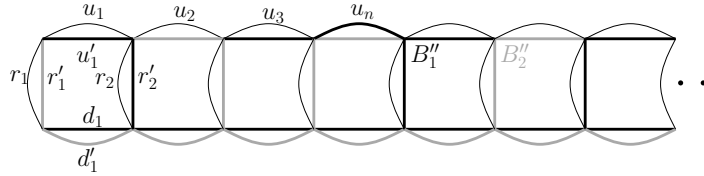


Figure 8: The two bases B''_1 and B''_2 cover $E \setminus D + u_n$

In the first case the bases are

$$B'_1 := B_1 \setminus \{r'_i | i < n \text{ and odd}\} \cup \{r'_i | i < n \text{ and even}\},$$

$$B'_2 := B_2 \cup \{r'_i | i < n \text{ and odd}\} \setminus \{r'_i | i < n \text{ and even}\} - r_1 + r_n$$

In the second case, the bases arise from B'_1 and B'_2 as follows:

$$B''_1 := B'_1 + u_n - r'_{n-1}, \quad B''_2 := B'_2 + r'_{n-1} - r_n$$

Having shown that $E \setminus D + e$ is independent for every $e \in D$, it remains to show for any two $e_1, e_2 \in D$ that $E \setminus D + e_1 + e_2$ cannot be covered by two bases of $M_A(H)$. In fact we prove the slightly stronger fact that $E \setminus D + e_1 + e_2$ cannot be covered by two bases of $M_F(H)$, that is by two spanning trees T_1 and T_2 of H .

Let H_n be the subgraph of H consisting of those $2n$ vertices that have the least distance to r_1 . Choose n large enough so that $e_1, e_2 \in H_n$. An

induction argument shows that $E \setminus D$ has $4n - 3$ edges in H_n since $E \setminus D$ has 1 edge in H_1 and $E \setminus D$ has 4 edges in $H_n \setminus H_{n-1}$. On the other hand, $T_1 \cup T_2$ can have at most $2(2n - 1)$ edges in H_n since H_n has $2n$ vertices. This shows that T_1 and T_2 cannot cover $E \setminus D + e_1 + e_2$ because they cannot cover $H_n \setminus D + e_1 + e_2$. So for any $e \in D$ the set $E \setminus D + e$ is a base of M , proving the assumption. \square

As $|C \cap D| = \infty$, this completes the proof of Theorem 4.3.

In the previous section, we were able to generalise our example and give a necessary condition under which M^+ is wild. Here, we do not see a way to do this, because the description of C and D made heavy use of the structure of M . It would be nice to have a large class of matroids M , as in the previous section, such that $M \vee M$ is wild.

Open Question 4.6. *For which matroids M is $M \vee M$ wild?*

5 A thin sums matroid whose dual isn't a thin sums matroid

The constructions introduced so far give us examples of matroids which are wild, and so badly behaved. We therefore believe they will be a fruitful source of counterexamples in matroid theory. In this section, we shall illustrate this by giving a counterexample for a very natural question.

First we recall the notion of a thin sums matroid.

Definition 5.1. Let A be a set, and k a field. Let $f = (f_e | e \in E)$ be a family of functions from A to k , and let $\lambda = (\lambda_e | e \in E)$ be a family of elements of k . We say that λ is a *thin dependence* of f iff for each $a \in A$ we have

$$\sum_{e \in E} \lambda_e f_e(a) = 0,$$

where the equation is taken implicitly to include the claim that the sum on the left is well defined, that is, that there are only finitely many $e \in E$ with $\lambda_e f_e(a) \neq 0$.

We say that a subset I of E is *thinly independent* for f iff the only thin dependence of f which is 0 everywhere outside I is $(0 | e \in E)$. The *thin sums system* M_f of f is the set of such thinly independent sets. This isn't always the set of independent sets of a matroid [5], but when it is we call it the *thin sums matroid* of f .

This definition is deceptively similar to the definition of the representable matroid corresponding to f considered as a family of vectors in the k -vector space k^A . The difference is in the more liberal definition of dependence: it is possible for λ to be a thin dependence even if there are infinitely many $e \in E$ with $\lambda_e \neq 0$, provided that for each $a \in A$ there are only finitely many $e \in E$ such that *both* $\lambda_e \neq 0$ and $f_e(a) \neq 0$.

Indeed, the notion of thin sums matroid was introduced as a generalisation of the notion of representable matroid: every representable matroid is finitary, but this restriction does not apply to thin sums matroids. Thus, although it is clear that the class of representable matroids isn't closed under duality, the question of whether the class of thin sums matroids is closed under duality remained open. It is shown in [3] that the class of tame thin sums matroids is closed under duality, so that any counterexample must be wild. We show below that one of the wild matroids we have constructed does give a counterexample.

There are many natural examples of thin sums matroids: for example, the algebraic cycle matroid of any graph not including a subdivision of the Bean graph is a thin sums matroid, as follows:

Definition 5.2. Let G be a graph with vertex set V and edge set E , and k a field. We can pick a direction for each edge e , calling one of its ends its *source* $s(e)$ and the other its *target* $t(e)$. Then the family $f^G = (f_e^G | e \in E)$ of functions from V to k is given by $f_e = \chi_{t(e)} - \chi_{s(e)}$, where for any vertex v the function χ_v takes the value 1 at v and 0 elsewhere.

Theorem 5.3. *Let G be a graph not including any subdivision of the Bean graph. Then M_{f^G} is the algebraic cycle matroid of G .*

This theorem, which motivated the definition of M_f , is proved in [3].

For the rest of this section, M will denote the algebraic cycle matroid for the graph G in Figure 9, in which we have assigned directions to all the edges and labelled them for future reference. We showed in the Section 3 that M^+ is wild. We shall devote the rest of this Section 3 to showing that in fact it gives an example of a thin sums matroid whose dual isn't a thin sums matroid.

As usual, we denote the vertex set of G by V and the edge set by E . We call the unique vertex lying on the loop at the left $*$.

Theorem 5.4. *M^+ is a thin sums matroid over the field \mathbb{Q} .*

Proof. We begin by specifying the family $(f_e | e \in E)$ of functions from V to \mathbb{Q} for which $M^+ = M_f$. We take f_e to be f_e^G as in Definition 5.2 if e is one of the p_i or q_i , to be χ_* if $e = l$, and to be $f_e^G + i \cdot \chi_*$ if $e = r_i$.

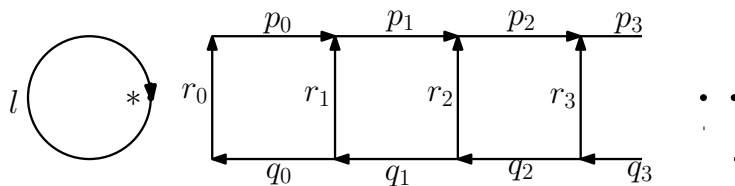


Figure 9: The graph G

First, we have to show that every circuit of M^+ is dependent in M_f . There are a variety of possible circuit types: in fact, types (b), (c), (e) and (f) from Figures 3 and 4 can arise. We shall only consider type (f): the proofs for the other types are very similar. Figure 10 shows the two ways a circuit of type (f) can arise.

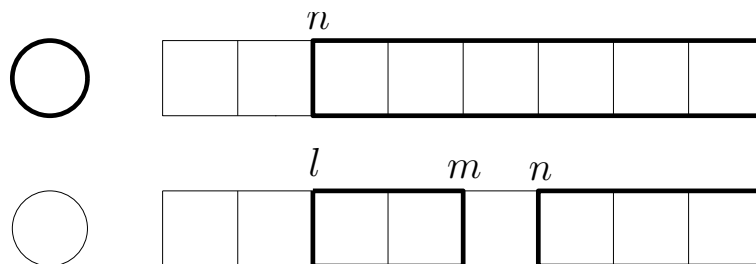


Figure 10: The two ways of obtaining a circuit of type (f)

The first includes the edge l , together with r_n for some n and all those p_i and q_i with $i \geq n$. We seek a thin dependence λ such that λ is nonzero on precisely these edges.

We shall take $\lambda_{r_n} = 1$. We can satisfy the equations $\sum_{e \in E} \lambda_e f_e(v)$ with $v \neq *$ by taking $\lambda_{p_i} = \lambda_{q_i} = 1$ for all $i \geq n$. The equation $\sum_{e \in E} \lambda_e f_e(*) = 0$ reduces to $\lambda_* + n\lambda_{r_n} = 0$, which we can satisfy by taking $\lambda_* = -n$. It is immediate that this gives a thin dependence of f .

The second way a circuit of type (f) can arise includes the edges r_l , r_m and r_n , together with those p_i and q_i with either $l \leq i < m$ or $n \leq i$. We seek a thin dependence λ such that λ is nonzero on precisely these edges.

The equations $\sum_{e \in E} \lambda_e f_e(v)$ with $v \neq *$ may be satisfied by taking $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_l} = -\lambda_{r_m}$ for $l \leq i \leq m$ and $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_n}$ for $i \geq n$.

The equation $\sum_{e \in E} \lambda_e f_e(*) = 0$ reduces to $l\lambda_{r_l} + m\lambda_{r_m} + n\lambda_{r_n} = 0$, which since $\lambda_{r_m} = -\lambda_{r_l}$ reduces further to $(m-l)\lambda_{r_m} = n\lambda_{r_n}$. We can satisfy this equation by taking $\lambda_{r_m} = n$ and $\lambda_{r_n} = m-l$. Taking the remaining λ_e to be given as above then gives a thin dependence of f . Note that $\lambda \neq 0$ since $m \neq l$ and thus $\lambda_{r_n} \neq 0$.

Next, we need to show that every dependent set of M_f is also dependent in M^+ , completing the proof. Let D be such a dependent set, as witnessed by a nonzero thin dependence λ of f which is 0 outside D . Let $D' = \{e \mid \lambda_e \neq 0\}$, the *support* of D . Using the equations $\sum_{e \in E} \lambda_e f_e(v)$ with $v \neq *$, we may deduce that the degree of D' at each vertex (except possibly $*$) is either 0 or at least 2. Therefore any edge (except possibly l) contained in D' is contained in some circuit of M included in D' . Since $\{l\}$ is already a circuit of M , we can even drop the qualification 'except possibly l '.

Since D' is nonempty, it must include some circuit O of M . Suppose first of all for a contradiction that $D' = O$. The intersection of D' with the set $\{l\} \cup \{r_i \mid i \in \mathbb{N}_0\}$ is nonempty, so by the equation $\sum_{e \in E} \lambda_e f_e(*) = 0$ this intersection must have at least 2 elements. The only way this can happen with D' a circuit is if there are $m < n$ such that D' consists of r_m, r_n , and the p_i and q_i with $m \leq i < n$. We now deduce, since λ is a thin dependence, that $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_m} = -\lambda_{r_n}$ for $m \leq i < n$. In particular, the equation $\sum_{e \in E} \lambda_e f_e(*) = 0$ reduces to $(m-n)\lambda_{r_m} = 0$, which is the desired contradiction as by assumption $\lambda_{r_m} \neq 0$ and $m < n$. Thus $D' \neq O$, and we can pick some $e \in D' \setminus O$. As above, D' includes some M -circuit O' containing e . Then the union $O \cup O' \subseteq D'$ is M^+ -dependent by Corollary 3.4. \square

Theorem 5.5. $(M^+)^*$ is not a thin sums matroid over any field.

Proof. Suppose for a contradiction that it is a thin sums matroid M_f , with f a family of functions $A \rightarrow k$. For each circuit O of $(M^+)^*$, we can find a nonzero thin dependence λ of f which is nonzero only on O - it must be nonzero on the whole of O by minimality of O .

The circuits of $(M^+)^* = (M^*)^-$ are precisely the circuits and the bases of M^* , the dual of the algebraic cycle matroid of G , since no circuit in M^* includes a base. This dual M^* , called the *skew cuts* matroid of G , is known to have as its circuits those cuts of G which are minimal subject to the condition that one side contains no rays.

Thus since $\{r_0, p_0\}$ is a skew cut, we can find a thin dependence λ^0 which is nonzero precisely at r_0 and q_0 . Similarly, for each $i > 0$ we can find a thin dependence λ^i which is nonzero precisely at q_{i-1}, r_i and q_i . Since the set of

bold edges in Figure 2 is also a circuit of $(M^+)^*$, there is a thin dependence λ which is nonzero on precisely those edges.

To obtain a contradiction, we will show that $\{r_i | i \in \mathbb{N}\}$ is dependent in M_f . The idea behind the following calculations is to consider $\{r_i | i \in \mathbb{N}\}$ as the limit of the M_f -circuits $\{r_i | 0 \leq i \leq n\} \cup \{p_n\}$ and then to use the properties of thin sum representations to show that the “limit” $\{r_i | i \in \mathbb{N}\}$ inherits the dependence.

Now define the sequences $(\mu_i | i \in \mathbb{N})$ and $(\nu_i | i \in \mathbb{N})$ inductively by $\nu_0 = 1$, $\nu_i = -(\lambda_{p_i}^i / \lambda_{p_{i-1}}^i) \nu_{i-1}$ for $i > 0$ and $\mu_i = -(\lambda_{r_i}^i / \lambda_{p_i}^i) \nu_i$. Pick any $a \in A$. Then we have $0 = \sum_{e \in E} \lambda_e^0 f_e(a) = \lambda_{r_0}^0 f_{r_0}(a) + \lambda_{p_0}^0 f_{p_0}(a)$, and rearranging gives

$$\nu_0 f_{p_0}(a) = \mu_0 f_{r_0}(a).$$

Similarly, $0 = \sum_{e \in E} \lambda_e^i f_e(a) = \lambda_{p_{i-1}}^i f_{p_{i-1}}(a) + \lambda_{r_i}^i f_{r_i}(a) + \lambda_{p_i}^i f_{p_i}(a)$, and rearranging gives

$$\nu_i f_{p_i}(a) = \nu_{i-1} f_{p_{i-1}}(a) + \mu_i f_{r_i}(a).$$

So by induction on i we get the formula

$$\nu_i f_{p_i}(a) = \sum_{j=0}^i \mu_j f_{r_j}(a).$$

The formula $\sum_{e \in E} \lambda_e f_e(a) = 0$ implicitly includes the statement that the sum is well defined, so only finitely many summands can be nonzero. In particular, there can only be finitely many i for which $f_{p_i}(a) \neq 0$. It then follows by the formula above that there are only finitely many i such that $f_{r_i}(a)$ is nonzero, since if $f_{r_i} \neq 0$, then as $\mu_i \neq 0$ we have $\nu_i f_{p_i}(a) \neq \nu_{i-1} f_{p_{i-1}}(a)$. So as $\nu_i \neq 0$ and $\nu_{i-1} \neq 0$, one of $f_{p_i}(a)$ or $f_{p_{i-1}}(a)$ is not equal to zero. Therefore all but finitely many $f_{r_i}(a)$ are zero since all but finitely many $f_{p_i}(a)$ are zero. So the following sum is well defined and evaluates to zero.

$$\sum_{i=0}^{\infty} \mu_i f_{r_i}(a) = 0.$$

Therefore, if we define a family $(\lambda'_e | e \in E)$ by $\lambda'_{r_i} = \mu_i$ and $\lambda'_e = 0$ for other values of e , then we have

$$\sum_{e \in E} \lambda'_e f_e(a) = 0.$$

Since $a \in A$ was arbitrary, this implies that λ' is a thin dependence of f . Note that $\lambda' \neq 0$ since $\lambda'_{r_0} \neq 0$. Thus the set $\{r_i | i \in \mathbb{N}\}$ is dependent in $M_f = (M^*)^-$. But it is also an $(M^*)^-$ -basis, since adding l gives a basis of M^* . This is the desired contradiction. \square

References

- [1] E. Aigner-Horev, J. Carmesin, and J. Fröhlich. Infinite matroid union I. Preprint (2011).
- [2] E. Aigner-Horev, J. Carmesin, and J. Fröhlich. Infinite matroid union II. Preprint (2011).
- [3] N. Bowler and H. Afzali. Thin sums matroids and duality. In preparation.
- [4] H. Bruhn and R. Diestel. Infinite matroids in graphs. arXiv:1011.4749 [math.CO], 2010.
- [5] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, and P. Wollan. Axioms for infinite matroids. arXiv:1003.3919 [math.CO], 2010.
- [6] R. Diestel. *Graph Theory* (4th edition). Springer-Verlag, 2010. Electronic edition available at: <http://diestel-graph-theory.com/index.html>.
- [7] A. Dress. Duality theory for finite and infinite matroids with coefficients. *Advances in Mathematics*, 59:97–123, 1986.
- [8] D.A. Higgs. Infinite graphs and matroids. Proceedings Third Waterloo Conference on Combinatorics, Academic Press, 1969, pp. 245 - 53.
- [9] J. Oxley. *Matroid Theory*. Oxford University Press, 1992.
- [10] M. Wagowski. Strong duality property for matroids with coefficients. *Europ. J. Comb.*, 15:293–302, 1994.