

# SELF-DUAL UNIFORM MATROIDS ON INFINITE SETS

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ABSTRACT. We extend the notion of a uniform matroid to the infinitary case and construct, using weak fragments of Martin's Axiom, self-dual uniform matroids on infinite sets. In 1969, Higgs showed that, assuming the Generalised Continuum Hypothesis (GCH), any two bases of a fixed matroid have the same size. We show that this cannot be proved from the usual axioms of set theory, ZFC, alone: in fact, we show that it is consistent with ZFC that there is a uniform self-dual matroid with two bases of different size.

Self-dual uniform matroids on infinite sets also provide examples of infinitely connected matroids, answering a question of Bruhn and Wollan under additional set-theoretic assumptions. While we do not know whether the existence of a self-dual uniform matroid on an infinite set can be proved in ZFC alone, we show that ZF, Zermelo-Fraenkel Set Theory without the Axiom of Choice, is not enough. Finally, we observe that there is a model of set theory in which GCH fails while any two bases of a matroid have the same size. This answers a question of Higgs.

## 1. INTRODUCTION

Results of Oxley ([12]) and Bruhn et al. ([5]) show that Higgs' *B-matroids* (defined in [6]) provide a suitable extension of finite matroids to the infinite. We use the notation of Bruhn et al. and call B-matroids simply *matroids*. Matroids can be described many different ways, for example in terms of their *independent sets* or in terms of their *bases*. If the bases of a matroid are known, then a set is independent iff it is included in a basis. If the independent sets are known, then the bases are the maximal independent sets. The definition of matroids in terms of bases given in [5] is the following:

**Definition 1.** Let  $E$  be a set and  $\mathcal{B} \subseteq \mathcal{P}(E)$ . Then  $\mathcal{B}$  is the set of *bases* of a *matroid*  $\mathcal{M}$  on  $E$  if the following hold:

- (B1)  $\mathcal{B} \neq \emptyset$
- (B2) Whenever  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \setminus B_2$ , there is  $y \in B_2 \setminus B_1$  such that  $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ .
- (BM) Whenever  $I \subseteq B$  for some  $B \in \mathcal{B}$  and  $I \subseteq X \subseteq E$ , then the collection  $\{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\}$  has a maximal element.

A simple example of a matroid on a set  $E$  is the matroid  $U_n(E)$  whose bases are all subsets of  $E$  of a fixed finite size  $n$  with  $0 \leq n \leq |E|$ . Such matroids are called *uniform*. These uniform matroids are all *finitary*, that is, a set is independent iff all of its finite subsets are independent.

The following is a natural generalization of uniformity to the infinitary case.

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**Definition 2.** Let  $\mathcal{B}$  be the set of bases of a matroid  $\mathcal{M}$  on a set  $E$ . Then  $\mathcal{M}$  is *uniform* if the following strengthening of (B2) holds:

(U) Whenever  $B \in \mathcal{B}$ ,  $x \in B$ , and  $y \in E \setminus B$ , then  $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ .

Every matroid  $\mathcal{M}$  has a dual  $\mathcal{M}^*$ , whose set of bases consists of the complements of bases of  $\mathcal{M}$ . A matroid is *self-dual* if it is dual to itself. (Note that our notion of self-duality is more restrictive than what is often found in the literature: frequently, matroids are considered self-dual if they are just isomorphic to their duals.) A matroid is *cofinitary* if its dual is finitary. In the next section we will observe that a uniform matroid is finitary iff it has a finite basis. If a matroid has a finite basis, then by (B2) all bases are finite and of the same size. It follows that self-dual matroids on infinite sets are neither finitary nor cofinitary.

In [4] Bruhn and Wollan studied connectedness of infinite matroids and asked whether there is an infinitely connected matroid on an infinite set. It turns out that a uniform matroid with an infinite basis and a basis with an infinite complement is infinitely connected. We show that the Continuum Hypothesis (CH, i.e.,  $2^{\aleph_0} = \aleph_1$ ) implies the existence of a self-dual uniform matroid on a countably infinite set, solving the question of Bruhn and Wollan under some set-theoretic assumptions.

In [7], Higgs showed that, assuming the Generalised Continuum Hypothesis (GCH, i.e., for every infinite cardinal  $\kappa$ ,  $2^\kappa = \kappa^+$ ), any two bases of a fixed matroid have the same size. Using a fragment of Martin's Axiom together with the negation of CH we obtain a self-dual uniform matroid on an uncountable set that has two bases of different size. This shows that Higgs' result cannot be proved without any additional assumption beyond the usual axioms of set theory, ZFC.

It remains an open question whether uniform self-dual matroids on infinite sets can be constructed in ZFC. However, we show that the existence of a self-dual uniform matroid on a countably infinite set implies the existence of a set of reals without the Baire property. By a result of Shelah [13], the existence of a set of reals without the Baire property, and hence of a uniform self-dual matroid on a countably infinite set, cannot be proved in ZF alone, i.e., without the Axiom of Choice.

Finally, we answer two questions of Higgs ([6]). First we show that the statement "all bases of a fixed matroid have the same size" does not imply GCH. Then we show that a certain property shared by collections of bases of matroids characterizes collections of bases of matroids on finite sets, but not on infinite sets.

## 2. UNIFORM MATROIDS

Let us collect some basic facts on uniform matroids.

If  $\mathcal{M}$  is a matroid on a set  $E$  and  $X \subseteq E$ , then the *restriction* of  $\mathcal{M}$  to  $X$  is the matroid  $\mathcal{M}|X$  on  $X$  whose independent sets are the independent subsets of  $X$  (in the sense of  $\mathcal{M}$ ).

Now let  $B$  be a basis of  $\mathcal{M}|X$ . The *contraction* of  $\mathcal{M}$  by  $X$  is the matroid  $\mathcal{M}/X$  on  $E \setminus X$  whose bases are the sets  $A \subseteq E \setminus X$  such that  $B \cup A$  is a basis of  $\mathcal{M}$ .

**Lemma 3.** *Contractions, restrictions, and duals of uniform matroids are again uniform.*

The proof of this lemma is straightforward. The following lemma collects the main combinatorial properties of uniform matroids.

**Lemma 4.** *Let  $\mathcal{M}$  be a uniform matroid on  $E$ .*

- a) *For every set  $X \subseteq E$  there is a basis  $B$  of  $\mathcal{M}$  such that  $X \subseteq B$  or  $B \subseteq X$ .*
- b) *A set  $X \subseteq E$  is dependent iff it properly includes a basis.*
- c) *If  $I \subseteq X \subseteq E$ , there is a basis  $B$  of  $\mathcal{M}$  such that one of the following holds:*
  - (i)  $B \subseteq I$
  - (ii)  $I \subseteq B \subseteq X$
  - (iii)  $X \subseteq B$

*Proof.* For a) let  $\mathcal{A}$  be a maximal element of  $\mathcal{A} = \{X \cap B : B \text{ is a basis of } \mathcal{M}\}$ . Let  $B$  be a basis of  $\mathcal{M}$  such that  $A = X \cap B$ . If  $A = X$  then  $X \subseteq B$  and we are done. Now assume  $A \neq B$ . Then either  $B \subseteq X$  or there is  $y \in B \setminus X$ . Assume there is  $y \in B \setminus X$ . Let  $x \in X \setminus B$ . By (U),  $(B \setminus \{y\}) \cup \{x\}$  is a basis of  $\mathcal{M}$ . This shows that  $A \cup \{x\} \in \mathcal{A}$ , contradicting the maximality of  $A$ . Hence  $B \subseteq X$ .

b) follows immediately from a). For c) assume that there is no basis of  $\mathcal{M}$  that is properly included in  $I$ . Then by b),  $I$  is independent. Now  $I$  itself is a basis of  $\mathcal{M}/I$ . Since  $\mathcal{M}/I$  is uniform, by a) there is a basis  $A$  of  $\mathcal{M}/I$  such that  $A \subseteq X \setminus I$  or  $X \setminus I \subseteq A$ . Now  $B = I \cup A$  is a basis of  $\mathcal{M}$  such that  $I \subseteq B \subseteq X$  or  $X \subseteq B$ .  $\square$

Since the existence of a finite basis of a matroid implies that all bases are finite and of the same size, every matroid with a finite basis is finitary. The converse is true for uniform matroids that have more than one basis.

**Corollary 5.** *Let  $\mathcal{M}$  be a finitary uniform matroid on a set  $E$  such that  $E$  is not a basis of  $\mathcal{M}$ . Then  $\mathcal{M}$  has a finite basis.*

*Proof.* Since  $E$  is not a basis of  $\mathcal{M}$ , it is dependent. Since  $\mathcal{M}$  is finitary, every dependent set contains a finite dependent set. Let  $D$  be a finite dependent subset of  $E$ . By part b) of Lemma 4,  $D$  includes a basis of  $\mathcal{M}$ . It follows that  $\mathcal{M}$  has a finite basis.  $\square$

Part c) of Lemma 4 can be used to characterize uniform matroids. First observe that a matroid on  $E$  is uniform provided its set  $\mathcal{B}$  of bases is closed under the equivalence relation  $\sim$  where for  $A, B \subseteq E$  we let  $A \sim B$  iff  $A \setminus B$  and  $B \setminus A$  are both finite and of the same size.

**Theorem 6.** *Let  $\mathcal{B}$  be a nonempty collection of subsets of  $E$ . Then  $\mathcal{B}$  is the set of bases of a uniform matroid on  $E$  iff the following hold:*

- (1) *No element of  $\mathcal{B}$  is properly included in another.*
- (2)  *$\mathcal{B}$  is closed under  $\sim$ .*
- (3) *For all  $I, X \subseteq E$  with  $I \subseteq X$  and  $X \setminus I$  infinite there is  $B \in \mathcal{B}$  such that one of the following holds:*
  - (i)  $B \subseteq I$
  - (ii)  $I \subseteq B \subseteq X$
  - (iii)  $X \subseteq B$

*Proof.* Suppose  $\mathcal{B}$  is the collection of bases of a matroid  $\mathcal{M}$  on  $E$ . Then no element of  $\mathcal{B}$  is properly included in another. If  $\mathcal{M}$  is uniform, then  $\mathcal{B}$  is closed under  $\sim$  and  $\mathcal{B}$  satisfies (3) by Lemma 4 c).

Now suppose  $\mathcal{B}$  satisfies (1)–(3). Since  $\mathcal{B}$  is closed under  $\sim$ , it satisfies (U). It remains to show (BM).

Let  $I \subseteq E$  be such that for some  $B \in \mathcal{B}$ ,  $I \subseteq B$  and let  $X \subseteq E$  be such that  $I \subseteq X$ . Let

$$\mathcal{A} = \{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\}.$$

Note that  $\mathcal{A}$  is nonempty since  $I \subseteq B$  for some  $B \in \mathcal{B}$ . We have to show that  $\mathcal{A}$  has a maximal element. But this is obvious if  $X \setminus I$  is finite. Therefore we may assume that  $X \setminus I$  is infinite.

Since  $I$  is included in a member of  $\mathcal{B}$ , (1) implies that there is no  $B \in \mathcal{B}$  with  $B \subsetneq I$ . By (3), there is  $B \in \mathcal{B}$  such that  $I \subseteq B \subseteq X$  or  $X \subseteq B$ . In the first case,  $B$  is a maximal element of  $\mathcal{A}$ . In the second case,  $X$  is a maximal element of  $\mathcal{A}$ .  $\square$

In [4], Bruhn and Wollan defined *k-connectedness* for infinite matroids and asked whether there is a matroid on an infinite set that is *k-connected* for all finite  $k$ .

**Definition 7.** Let  $\mathcal{M}$  be a matroid on a set  $E$ . For independent sets  $I, J \subseteq E$  let

$$\text{del}_{\mathcal{M}}(I, J) = \min\{|F| : F \subseteq I \cup J \wedge (I \cup J) \setminus F \text{ is independent}\},$$

where  $\text{del}_{\mathcal{M}}(I, J)$  is defined to be  $\infty$  if there is no finite set  $F$  such that  $(I \cup J) \setminus F$  is independent.

Now for  $X \subseteq E$  let  $\kappa_{\mathcal{M}}(X)$  be  $\text{del}_{\mathcal{M}}(B, B')$  where  $B$  is a basis of  $\mathcal{M}|_X$  and  $B'$  is a basis of  $\mathcal{M}|_{(E \setminus X)}$ . (Bruhn and Wollan have shown that  $\kappa_{\mathcal{M}}(X)$  does not depend on the choice of  $B$  and  $B'$ . Hence  $\kappa_{\mathcal{M}}(X)$  is well defined.)

The matroid  $\mathcal{M}$  is *k-connected* for a natural number  $k$ , unless for some  $\ell < k$  there is an  $\ell$ -separation, i.e., a set  $X \subseteq E$  such that  $\kappa_{\mathcal{M}}(X) < \ell$  and  $|X|, |E \setminus X| \geq \ell$ .  $\mathcal{M}$  is *infinitely connected* if it is *k-connected* for every  $k \in \mathbb{N}$ .

**Theorem 8.** *Let  $\mathcal{M}$  be a uniform matroid that is neither finitary nor cofinitary. Then  $\mathcal{M}$  is infinitely connected.*

*Proof.* By Corollary 5, every basis of  $\mathcal{M}$  is infinite and co-infinite. It follows from (U) together with the existence of an infinite basis that every finite subset of  $E$  is independent. We have to show that  $\mathcal{M}$  has no  $\ell$ -separation for any  $\ell \in \mathbb{N}$ .

Let  $X \subseteq E$  and assume  $|X|, |E \setminus X| \geq \ell$  for some  $\ell \in \mathbb{N}$ . We compute  $\kappa_{\mathcal{M}}(X)$ . By Lemma 4 b), either  $X$  properly includes a basis of  $\mathcal{M}$  or  $X$  is independent. Hence we can choose a basis  $B$  of  $\mathcal{M}|_X$  such that  $B = X$  or  $B \subseteq X$  is a basis of  $\mathcal{M}$ . By the same argument, there is a basis  $B'$  of  $\mathcal{M}|_{(E \setminus X)}$  such that  $B' = E \setminus X$  or  $B'$  is a basis of  $\mathcal{M}$ .

**Case 1.**  $B$  or  $B'$  is a basis of  $\mathcal{M}$ .

Without loss of generality we may assume that  $B$  is a basis of  $\mathcal{M}$ . By (U), for every finite set  $F \subseteq B \cup B'$ ,  $(B \cup B') \setminus F$  is independent iff  $|F| \geq |B'|$ . This shows that  $\text{del}_{\mathcal{M}}(B, B')$ , and hence  $\kappa_{\mathcal{M}}(X)$ , is at least  $\ell$ .

**Case 2.**  $B = X$  and  $B' = E \setminus X$ .

In this case  $B \cup B' = E$ . Since every basis of  $\mathcal{M}$  has an infinite complement, every independent set has an infinite complement. It follows that  $\text{del}_{\mathcal{M}}(B, B')$ , and therefore  $\kappa_{\mathcal{M}}(X)$ , is infinite.

In both cases it follows that  $X$  is not an  $\ell$ -separation of  $\mathcal{M}$ .  $\square$

Our final observation in this section is that uniform matroids always satisfy the following conjecture of Nash-Williams (see [1]).

**Conjecture 9** (The infinite matroid intersection conjecture). Any two matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$  on a common ground set  $E$  have a common independent set  $I$  admitting a partition  $I = J_1 \cup J_2$  such that  $\text{cl}_{\mathcal{M}_1}(J_1) \cup \text{cl}_{\mathcal{M}_2}(J_2) = E$ .

The closure operator  $\text{cl}_{\mathcal{M}}$  associated with a matroid  $\mathcal{M}$  on a set  $E$  has a very simple definition on independent sets  $I \subseteq E$ :

$$\text{cl}_{\mathcal{M}}(I) = \{e \in E : I \cup \{e\} \text{ is dependent}\}$$

**Theorem 10.** *Let  $\mathcal{M}_1$  be a uniform matroid on a set  $E$ . If  $\mathcal{M}_2$  is any matroid on  $E$ , then there is set  $I \subseteq E$  that is independent with respect to both  $\mathcal{M}_1$  and  $\mathcal{M}_2$  such that  $\text{cl}_{\mathcal{M}_1}(I) = E$  or  $\text{cl}_{\mathcal{M}_2}(I) = E$ . In particular,  $\mathcal{M}_1$  and  $\mathcal{M}_2$  satisfy the infinite matroid intersection conjecture.*

*Proof.* Let  $B_2 \subseteq E$  be a basis of  $\mathcal{M}_2$ . By Lemma 4 a), there is basis  $B_1$  of  $\mathcal{M}_1$  such that  $B_1 \subseteq B_2$  or  $B_2 \subseteq B_1$ . Now  $I = B_1 \cap B_2$  is an independent subset of both  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and a basis of at least one of them. Hence  $\text{cl}_{\mathcal{M}_1}(I) = E$  or  $\text{cl}_{\mathcal{M}_2}(I) = E$ . If  $I = B_1$ , let  $J_1 = B_1$  and  $J_2 = \emptyset$ . If  $I = B_2$  and  $B_1 \neq B_2$ , let  $J_1 = \emptyset$  and  $J_2 = B_2$ . In either case,  $\text{cl}_{\mathcal{M}_1}(J_1) \cup \text{cl}_{\mathcal{M}_2}(J_2) = E$ .  $\square$

### 3. MARTIN'S AXIOM

We introduce the fragment of Martin's Axiom that we will use in the construction of self-dual uniform matroids. The set-theoretic background and in particular the proof of the consistency of full Martin's Axiom with  $\neg\text{CH}$  can be found in either [8] or [10].

Let  $(\mathbb{P}, \leq)$  be a partial order. For  $p, q \in \mathbb{P}$  we say that  $p$  *extends*  $q$  if  $p \leq q$ .  $F \subseteq \mathbb{P}$  is a *filter* if any two elements of  $F$  have a common extension in  $F$  and for all  $p \in F$  and  $q \in \mathbb{P}$  with  $p \leq q$ ,  $q \in F$ . A set  $D \subseteq \mathbb{P}$  is *dense* in  $\mathbb{P}$  if every  $p \in \mathbb{P}$  has an extension in  $D$ . A filter  $F \subseteq \mathbb{P}$  is *generic* for a collection  $\mathcal{D}$  of dense subsets of  $\mathbb{P}$  if  $F$  has a nonempty intersection with every  $D \in \mathcal{D}$ .

Given a partial order  $\mathbb{P}$ ,  $\text{MA}(\mathbb{P})$  is the statement that for every collection  $\mathcal{D}$  of size  $< 2^{\aleph_0}$  of dense subsets of  $\mathbb{P}$  there is a  $\mathcal{D}$ -generic filter  $F \subseteq \mathbb{P}$ . For every partial order  $\mathbb{P}$  and every countable collection  $\mathcal{D}$  of dense subsets of  $\mathbb{P}$  there is a  $\mathcal{D}$ -generic filter  $F \subseteq \mathbb{P}$ . This is the Rasiowa-Sikorski Theorem. Hence for every partial order  $\mathbb{P}$ ,  $\text{MA}(\mathbb{P})$  follows from the Continuum Hypothesis ( $\text{CH}$ ,  $2^{\aleph_0} = \aleph_1$ ).

We will be interested in partial orders of the following form:

For a cardinal  $\kappa$  let  $\text{Fn}(\kappa, 2)$  denote the set

$$\{p : \text{there is a finite set } A \subseteq \kappa \text{ such that } p : A \rightarrow \{0, 1\}\}$$

ordered by reverse inclusion. For all infinite cardinals  $\kappa$  and  $\lambda$  with  $\kappa \leq \lambda$ ,  $\text{MA}(\text{Fn}(\kappa, 2))$  follows from  $\text{MA}(\text{Fn}(\lambda, 2))$ . The statement  $\text{MA}(\text{Fn}(\aleph_0, 2))$  is usually denoted by  $\text{MA}(\text{countable})$ . *Martin's Axiom* is the statement that  $\text{MA}(\mathbb{P})$  holds for all partial orders  $\mathbb{P}$  that satisfy the so called *countable chain condition* (c.c.c.). For all infinite cardinals  $\kappa$ ,  $\text{Fn}(\kappa, 2)$  satisfies the c.c.c.

Gödel showed that if the usual system of axioms for set theory, ZFC, is consistent, then so is ZFC together with CH. Of course, we have no reason to doubt the consistency of ZFC and, following usual practice, assume it throughout the whole article. Solovay and Tennenbaum constructed a model of ZFC that satisfies both MA and  $2^{\aleph_0} = \aleph_2$ . Also, it is known that MA implies  $2^{\aleph_0} = 2^{\aleph_1}$ . It follows that the statement

$$\text{MA}(\text{Fn}(\aleph_1, 2)) \wedge 2^{\aleph_1} = 2^{\aleph_0} = \aleph_2$$

is consistent with ZFC.

## 4. THE CONSTRUCTION

Fix an infinite set  $E$ . Since we want to construct a self-dual matroid, we want to talk about subsets of  $E$  and their complements at the same time. In other words, we consider partitions of  $E$  into two classes  $A^0$  and  $A^1$ . The following lemma isolates the combinatorial effect of  $\text{MA}(\text{Fn}(\kappa, 2))$  that we will use in our construction of a uniform matroid.

**Lemma 11.** *Let  $\kappa$  be an infinite cardinal  $< 2^{\aleph_0}$  and let  $\mathcal{S} \subseteq \mathcal{P}(\kappa)$  be a collection of infinite sets such that  $|\mathcal{S}| < 2^{\aleph_0}$ . Then  $\text{MA}(\text{Fn}(\kappa, 2))$  implies that there is a partition  $\{A^0, A^1\}$  of  $\kappa$  such that for all  $S \in \mathcal{S}$  the sets  $S \cap A^0$  and  $S \cap A^1$  are infinite.*

*Proof.* For all infinite sets  $S \subseteq \kappa$ , all finite sets  $F \subseteq \kappa$ , and all  $i \in \{0, 1\}$  let

$$D_F^i(S) = \{p \in \text{Fn}(\kappa, 1) : \exists m \in S \setminus F (p(m) = i)\}.$$

It is easily checked that the sets  $D_F^i(S)$  are dense subsets of  $\text{Fn}(\kappa)$ .

We may assume that  $\kappa \in \mathcal{S}$ . Let

$$\mathcal{D} = \{D_F^0(S) : S \in \mathcal{S} \wedge F \in [\kappa]^{<\aleph_0}\} \cup \{D_F^1(S) : S \in \mathcal{S} \wedge F \in [\kappa]^{\aleph_0}\}.$$

Since  $\kappa < 2^{\aleph_0}$ ,  $|\mathcal{D}| < 2^{\aleph_0}$ . Hence, by  $\text{MA}(\text{Fn}(\kappa, 2))$  there is a  $\mathcal{D}$ -generic filter  $G \subseteq \text{Fn}(\kappa, 2)$ . Let  $x = \bigcup G$ . Since  $G$  is a filter,  $x$  is a function. For  $i \in \{0, 1\}$  let  $A = x^{-1}(i)$ . By the choice of  $G$  and  $\mathcal{D}$ ,  $A^0$  and  $A^1$  have an infinite intersection with all  $S \in \mathcal{S}$ .  $\square$

We call two partitions  $\{A^0, A^1\}$  and  $\{B^0, B^1\}$  of  $E$  *independent* if the sets  $A^i \cap B^j$ ,  $i, j \in \{0, 1\}$ , are all nonempty. We define the equivalence relation  $\sim$  on partitions of  $E$  into two classes in the natural way:

$$\{A^0, A^1\} \sim \{B^0, B^1\} \text{ iff for some } i \in \{0, 1\}, A^0 \sim B^i.$$

Note that two partitions  $P$  and  $P'$  of  $E$  into infinite parts with  $P \sim P'$  are independent unless they are equal.

**Lemma 12.** *Suppose  $2^{|E|} = 2^{\aleph_0}$ . Then  $\text{MA}(\text{Fn}(|E|, 2))$  implies that there is a set  $\mathcal{P}$  of partitions of  $E$  into two infinite classes with the following properties:*

- (1)  $\mathcal{P}$  is closed under the equivalence relation  $\sim$ .
- (2) The elements of  $\mathcal{P}$  are pairwise independent.
- (3) Whenever  $I^0, I^1 \subseteq E$  are disjoint with  $E \setminus (I^0 \cup I^1)$  infinite, then there is a partition  $\{B^0, B^1\} \in \mathcal{P}$  such that one of the following holds:
  - (i)  $B^0 \subseteq I^0$
  - (ii)  $I^0 \subseteq B^0$  and  $I^1 \subseteq B^1$ .
  - (iii)  $B^1 \subseteq I^1$

*Proof.* Let  $((I_\alpha^0, I_\alpha^1))_{\alpha < 2^{\aleph_0}}$  be an enumeration of all pairs  $(I^0, I^1)$  of subsets of  $E$  with  $I^0 \cap I^1 = \emptyset$  and  $E \setminus (I^0 \cup I^1)$  infinite. We recursively choose partitions  $P_\alpha = \{B_\alpha^0, B_\alpha^1\}$ ,  $\alpha < 2^{\aleph_0}$ , of  $E$  into infinite sets.

Suppose that for some  $\alpha < 2^{\aleph_0}$  for all  $\beta < \alpha$ ,  $P_\beta$  has been chosen. Let  $\mathcal{P}_\alpha$  denote the closure of the family  $\{P_\beta : \beta < \alpha\}$  under  $\sim$  and let

$$\mathcal{B}_\alpha = \{B \subseteq E : \exists P \in \mathcal{P}_\alpha (B \in P)\}.$$

We distinguish two cases:

**Case 1.** There is a partition  $\{B^0, B^1\} \in \mathcal{P}_\alpha$  such that one of the following holds:

- (i)  $B^0 \subseteq I_\alpha^0$
- (ii)  $I_\alpha^0 \subseteq B^0$  and  $I_\alpha^1 \subseteq B^1$ .
- (iii)  $B^1 \subseteq I_\alpha^1$

In this case let  $B_\alpha^i = B^i$  for  $i = 0, 1$ .

**Case 2.** There is no partition  $\{B^0, B^1\} \in \mathcal{P}_\alpha$  as in Case 1.

We construct a partition  $P_\alpha = \{B_\alpha^0, B_\alpha^1\}$  of  $E$  such that  $I_\alpha^0 \subseteq B_\alpha^0$ ,  $I_\alpha^1 \subseteq B_\alpha^1$ , and  $P_\alpha$  is independent of all  $P \in \mathcal{P}_\alpha$ .

Let  $\{A^0, A^1\}$  be a partition of  $E$  such that  $I_\alpha^0 \subseteq A^0$  and  $I_\alpha^1 \subseteq A^1$  and let  $\{B^0, B^1\} \in \mathcal{B}_\alpha$ . If for some  $i, j \in \{0, 1\}$ ,  $B^i$  intersects  $I_\alpha^j$ , then  $B^i \cap A^j \neq \emptyset$ . It follows that for  $\{B_\alpha^0, B_\alpha^1\}$  to be independent of all  $P \in \mathcal{P}_\alpha$ , we have to make sure that for all  $i \in \{0, 1\}$  and all  $B \in \mathcal{B}_\alpha$ , if  $B \cap I_\alpha^i = \emptyset$ , then  $B \cap B_\alpha^i \neq \emptyset$ .

**Claim 13.** Suppose for some  $i \in \{0, 1\}$ ,  $B \in \mathcal{B}_\alpha$  is disjoint from  $I_\alpha^i$ . Then  $B \setminus I_\alpha^{1-i}$  is infinite.

For the proof of the claim assume that  $B \setminus I_\alpha^{1-i}$  is finite. If  $|I_\alpha^{1-i} \setminus B| \leq |B \setminus I_\alpha^{1-i}|$ , there is  $B' \sim B$  such that  $I_\alpha^{1-i} \subseteq B'$  and  $B' \cap I_\alpha^i = \emptyset$ . Since  $\mathcal{B}_\alpha$  is closed under  $\sim$ ,  $B' \in \mathcal{B}_\alpha$ . Now the partition  $\{B', E \setminus B'\} \in \mathcal{P}_\alpha$  contradicts the fact that we are in Case 2.

If  $|I_\alpha^{1-i} \setminus B| > |B \setminus I_\alpha^{1-i}|$ , then there is  $B' \sim B$  such that  $B' \subseteq I_\alpha^{1-i}$ . As before,  $B' \in \mathcal{B}_\alpha$ . Again the partition  $\{B', E \setminus B'\}$  contradicts the fact that we are in Case 2. This finishes the proof of the claim.

Let

$$\mathcal{S} = \{B \setminus I_\alpha^{1-i} : B \in \mathcal{B}_\alpha \wedge i \in \{0, 1\} \wedge B \cap I_\alpha^i = \emptyset\} \cup \{E \setminus (I_\alpha^0 \cup I_\alpha^1)\}.$$

Then by the claim, all elements of  $\mathcal{S}$  are infinite subsets of  $E \setminus (I_\alpha^0 \cup I_\alpha^1)$ . Also,  $0 < |\mathcal{S}| < 2^{\aleph_0}$ .

By Lemma 11, there is a partition  $\{A^0, A^1\}$  of  $E \setminus (I_\alpha^0 \cup I_\alpha^1)$  such that  $A^0$  and  $A^1$  have an infinite intersection with all elements of  $\mathcal{S}$ . In particular,  $A^0$  and  $A^1$  are both infinite. For  $i \in \{0, 1\}$  let  $B_\alpha^i = I_\alpha^i \cup A_i$ . By the previous discussion and by the choice of  $\mathcal{S}$ , the partition  $\{B_\alpha^0, B_\alpha^1\}$  of  $E$  is independent of all the partitions in  $\mathcal{P}_\alpha$ .

This finishes the recursive construction of the partitions  $\{B_\alpha^0, B_\alpha^1\}$ . Observe that since the  $\mathcal{P}_\alpha$  are closed under  $\sim$  and  $\{B_\alpha^0, B_\alpha^1\}$  is independent of all partitions in  $\mathcal{P}_\alpha$ , also every partition  $\{B^0, B^1\} \sim \{B_\alpha^0, B_\alpha^1\}$  is independent of all partitions in  $\mathcal{P}_\alpha$ . It follows that with our choice of  $\{B_\alpha^0, B_\alpha^1\}$ ,  $\mathcal{P}_{\alpha+1}$  consists of pairwise independent partitions if  $\mathcal{P}_\alpha$  does.

Finally let  $\mathcal{P} = \bigcup_{\alpha < 2^{\aleph_0}} \mathcal{P}_\alpha$ . It is clear that  $\mathcal{P}$  is closed under  $\sim$ . By the previous discussion, the elements of  $\mathcal{P}$  are pairwise independent. If  $I^0, I^1 \subseteq E$  are disjoint and such that  $E \setminus (I^0 \cup I^1)$  is infinite, then there is  $\alpha < 2^{\aleph_0}$  such that  $I^0 = I_\alpha^0$  and  $I^1 = I_\alpha^1$ . Now  $\{B_\alpha^0, B_\alpha^1\}$  witnesses (3) for  $I_\alpha^0$  and  $I_\alpha^1$ . It follows that  $\mathcal{P}$  satisfies the conditions (1)–(3).  $\square$

**Lemma 14.** *If  $\mathcal{P}$  is a set of partitions of  $E$  into two infinite classes such that (1)–(3) of Lemma 12 are satisfied, then  $\mathcal{B} = \{B \subseteq E : \{B, E \setminus B\} \in \mathcal{P}\}$  is the set of bases of a self-dual uniform matroid on  $E$ .*

*Proof.* (3) together with the fact that  $E$  is infinite implies that  $\mathcal{P}$ , and therefore  $\mathcal{B}$ , is nonempty. By (1),  $\mathcal{P}$  is closed under  $\sim$ . Hence  $\mathcal{B}$  is closed under  $\sim$ . By (2), the partitions in  $\mathcal{P}$  are pairwise independent. It follows that no element of  $\mathcal{B}$  is properly included in another.

Now let  $I \subseteq X \subseteq E$  and assume that  $X \setminus I$  is infinite. Let  $I^0 = I$  and  $I^1 = E \setminus X$ . Let  $\{B^0, B^1\} \in \mathcal{P}$  be a witness of (3) for  $I^0$  and  $I^1$  and let  $B = B^0$ . Now  $B \in \mathcal{B}$  and one of the following holds:

- (i)  $B \subseteq I$
- (ii)  $I \subseteq B \subseteq X$
- (iii)  $X \subseteq B$

By Theorem 6,  $\mathcal{B}$  is the collection of bases of a uniform matroid. Since  $\mathcal{B}$  is closed under complementation, this matroid is self-dual.  $\square$

**Theorem 15.** *a) CH implies the existence of a self-dual uniform matroid on a countably infinite set.*

*b) The existence of a self-dual uniform matroid on a countably infinite set is consistent with an arbitrarily large value of  $2^{\aleph_0}$ .*

*c) It is consistent that there is a self-dual uniform matroid on an uncountable set that has one basis of size  $\aleph_0$  and another basis of size  $\aleph_1$ .*

*Proof.* By Lemma 12 together with Lemma 14,  $\text{MA}(\text{countable})$  implies the existence of a self-dual uniform matroid on a countably infinite set. But  $\text{MA}(\text{countable})$  follows from CH. This shows a). Also,  $\text{MA}(\text{countable})$  is consistent with arbitrarily large values of  $2^{\aleph_0}$ . This implies b).

For c) let  $E$  be a set of size  $\aleph_1$ . We modify the construction in Lemma 12 a little bit. We may assume that the enumeration  $((I_\alpha^0, I_\alpha^1))_{\alpha < 2^{\aleph_0}}$  is chosen so that  $(I_0^0, I_0^1) = (\emptyset, \emptyset)$ . Now choose a partition of  $E$  into a countably infinite set  $B_0^0$  and a set  $B_0^1$  of size  $\aleph_1$ . We continue the construction as in the proof of Lemma 12 and obtain a set  $\mathcal{P}$  of partitions of  $E$  into two infinite classes satisfying (1)–(3). Now the set  $\mathcal{B} = \{B \subseteq E : \exists P \in \mathcal{P} (B \in P)\}$  is the set of bases of a self-dual uniform matroid on  $E$  and one basis,  $B_0^0$ , is countable, while another basis,  $B_0^1$ , is of size  $\aleph_1$ .  $\square$

**Remark 16.** The *reaping number*  $\mathfrak{r}$  is the least size of a family  $\mathcal{S}$  of infinite subsets of  $\mathbb{N}$  such that there is no  $A \subseteq \mathbb{N}$  such that for all  $S \in \mathcal{S}$ , both  $S \setminus A$  and  $S \cap A$  are infinite (see [3]). By Lemma 11,  $\text{MA}(\text{Fn}(\aleph_0, 2))$  implies  $\mathfrak{r} = 2^{\aleph_0}$ . This is well known. On the other hand,  $\mathfrak{r} = 2^{\aleph_0}$  is all we need for the construction of a self-dual uniform on a countably infinite set.

Together with Higgs result about the equicardinality of a bases of matroids under GCH, Theorem 15 c) yields the following corollary:

**Corollary 17.** *Whether or not any two bases of a matroid have the same size cannot be decided in ZFC alone.*

## 5. THE COMPLEXITY OF SELF-DUAL UNIFORM MATROIDS

In this section we work in ZF. The background in descriptive set theory used in this section can be found in either [8] or [9].

Flutters were introduced and studied by Delhommé, Mathias, and Morillon [11]. They can be constructed in ZFC, but their existence does not follow from ZF alone. We consider a notion that is formally slightly weaker than that of a 2-flutter.

**Definition 18.** A  $(\sim, 2)$ -flutter is a set  $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$  that is closed under  $\sim$  and has the property that for each  $A \subseteq \mathbb{N}$ , exactly one of the sets  $A$  and  $\mathbb{N} \setminus A$  is a member of  $\mathcal{A}$ .



We translate the notion of a  $(\sim, 2)$ -flutter into a more topological setting. Instead of  $\mathcal{P}(\mathbb{N})$  we consider the Cantor space  $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ . Each set  $A \in \mathcal{P}(\mathbb{N})$  corresponds to its characteristic function in  $\mathcal{C}$ . The relation  $\sim$  translates to an equivalence relation on  $\mathcal{C}$ , also denoted by  $\sim$ , where for all  $x, y \in \mathcal{C}$  we have  $x \sim y$  iff the sets  $\{n \in \mathbb{N} : x(n) \neq y(n) \wedge x(n) = 0\}$  and  $\{n \in \mathbb{N} : x(n) \neq y(n) \wedge x(n) = 1\}$  are finite and of the same size.

We also consider the equivalence relation  $\text{Comp}$  on  $\mathcal{C}$  that identifies every function  $x \in \mathcal{C}$  with the function  $\bar{x} : \mathbb{N} \rightarrow \{0, 1\}; n \mapsto 1 - x(n)$ . In this setting, a  $(\sim, 2)$ -flutter is a subset of  $\mathcal{C}$  that intersects each  $\text{Comp}$ -class in exactly one element and is closed under  $\sim$ .

**Definition 19.** Recall that the topology on  $\mathcal{C}$  is generated by the sets

$$[s] = \{x \in \mathcal{C} : s \subseteq x\},$$

where  $s : S \rightarrow \{0, 1\}$  for some finite set  $S \subseteq \mathbb{N}$ . This topology is compatible with a complete metric.

If  $X$  is any complete metric space, a subset  $N$  of  $X$  is *nowhere dense* if its closure has empty interior. A subset  $M$  of  $X$  is *meager* if it is a countable union of nowhere dense sets. Finally, a subset  $A$  of  $X$  has the *Baire property* if there is an open set  $O \subseteq X$  such that the symmetric difference  $A \Delta O$  is meager.

The Baire category theorem implies that no nonempty open subset of  $\mathcal{C}$  is meager. In particular, no nonempty open subset of  $\mathcal{C}$  is the union of two meager sets. In other words, if  $O \subseteq \mathcal{C}$  is open and nonempty and  $A_0, A_1 \subseteq O$  are *comeager* in  $O$ , i.e., have a meager complement relative to  $O$ , then  $A_0 \cap A_1 \neq \emptyset$ . Also, the collection of sets with the Baire property is closed under complementation.

**Theorem 20.** A  $(\sim, 2)$ -flutter on the Cantor space  $\mathcal{C}$  does not have the Baire property.

*Proof.* Let  $X \subseteq \mathcal{C}$  be a  $(\sim, 2)$ -flutter and suppose that  $X$  has the Baire property. Now  $\mathcal{C} \setminus X$  is a  $(\sim, 2)$ -flutter as well and has the Baire property. At most one of  $X$  and  $\mathcal{C} \setminus X$  is meager. Hence we may assume that  $X$  is not meager.

Since  $X$  has the Baire property, there is an open set  $O \subseteq \mathcal{C}$  such that  $X \Delta O$  is meager. Since  $X$  is not meager,  $O$  is nonempty. Hence there is a finite set  $S \subseteq \mathbb{N}$  and a function  $s : S \rightarrow \{0, 1\}$  such that  $[s] \subseteq O$ .

Choose an extension  $t$  of  $s$  to some finite subset  $T$  of  $\mathbb{N}$  such that  $t^{-1}(0)$  and  $t^{-1}(1)$  have the same size. Let  $n$  be the minimal element of  $\mathbb{N} \setminus \text{dom}(t)$ . Let  $t_0 = t \cup \{(n, 0)\}$  and  $t_1 = t \cup \{(n, 1)\}$ . For each  $x \in [t_0]$  let  $h(x) \in [t_1]$  be defined by letting  $h(x) \upharpoonright \text{dom}(t) = x \upharpoonright t$  and  $h(x) \upharpoonright (\mathbb{N} \setminus \text{dom}(t)) = \bar{x} \upharpoonright (\mathbb{N} \setminus \text{dom}(t))$ . The map  $h : [t_0] \rightarrow [t_1]$  is a homeomorphism. Since the set  $[t_0] \cap X$  is comeager in  $[t_0]$ ,  $h[[t_0] \cap X]$  is comeager in  $[t_1]$ . Also,  $[t_1] \cap X$  is comeager in  $[t_1]$ . Hence there is  $x \in [t_0] \cap X$  such that  $h(x) \in X$ . Since  $t^{-1}(0)$  and  $t^{-1}(1)$  are of the same size,  $h(x) \sim \bar{x}$ .

Since  $X$  is closed under  $\sim$ ,  $\bar{x} \in X$ . Hence  $X$  contains both  $x$  and  $\bar{x}$ . Therefore  $X$  is not a  $(\sim, 2)$ -flutter, a contradiction.  $\square$

We will show that every self-dual uniform matroid on a countable set gives rise to a  $(\sim, 2)$ -flutter. First we observe the following:

**Lemma 21.** Let  $\mathcal{M}$  be a uniform matroid on a set  $E$ . Then every dependent subset of  $E$  includes a basis.

*Proof.* Let  $X \subseteq E$  be dependent and let  $\mathcal{B}$  be the set of bases of  $\mathcal{M}$ . By (BM), the collection  $\{X \cap B : B \in \mathcal{B}\}$  has a maximal element  $A$ . If  $X = A$ , then  $X$  is independent, contradicting our assumption on  $X$ . Hence  $X \neq A$ . Let  $B \in \mathcal{B}$  be such that  $A = B \cap X$ .

We have to show that  $B \subseteq X$ . Suppose not. Then there are  $x \in B \setminus X$  and  $y \in X \setminus B$ . By (U),  $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ . But now  $B \cap X \subsetneq ((B \setminus \{x\}) \cup \{y\}) \cap X$ , contradicting the fact that  $A = B \cap X$  is maximal in  $\{X \cap B : B \in \mathcal{B}\}$ .

This shows that  $X$  includes a basis and finishes the proof of the lemma.  $\square$

**Theorem 22.** *If there is a self-dual uniform matroid on a countable set, then there is a  $(\sim, 2)$ -flutter.*

*Proof.* Let  $\mathcal{M}$  be a self-dual uniform matroid on  $E = \mathbb{N} \cup \{\infty\}$ . Let

$$\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ includes a basis of } \mathcal{M}\}.$$

Since the set of bases of  $\mathcal{M}$  is closed under  $\sim$ , so is  $\mathcal{A}$ .

Now let  $\{A_0, A_1\}$  be a partition of  $\mathbb{N}$ . If  $A_0$  is dependent, then there is a basis  $B \subseteq A_0$ . Now  $E \setminus B$  is a basis as well. Hence there is no basis included in  $A_1$  as  $A_1$  is a proper subset of  $E \setminus B$ .

If  $A_0$  is independent, then there is a basis  $B$  with  $A_0 \subseteq B$ . Now  $E \setminus B$  is a basis. If  $A_0 = B$ , then  $A_1$  is properly included in the basis  $E \setminus B$  and thus  $A_1$  does not include a basis. If  $A_0 \neq B$ , then  $E \setminus B$  is a proper subset of  $A_1 \cup \{\infty\}$ . There is  $B' \sim E \setminus B$  such that  $\infty \notin B'$  and  $B' \subseteq A_1 \cup \{\infty\}$ . Now  $B'$  is a basis that is included in  $A_1$ .

It follows that exactly one of  $A_0$  and  $A_1$  includes a basis. Hence  $\mathcal{A}$  is a  $(\sim, 2)$ -flutter.  $\square$

**Corollary 23.** *The existence of a self-dual uniform matroid on a countable set is not provable in ZF+DC.*

*Proof.* If there is a self-dual uniform matroid on a countable set then there is a subset of  $\mathcal{C}$  that does not have the Baire property. However, in [13] Shelah proved that if ZF is consistent, then there is a model of ZF+DC where every subset of  $\mathcal{C}$  has the Baire property. In this model there is no self-dual uniform matroid on a countable set.  $\square$

A subset  $A$  of a Hausdorff space  $X$  is *analytic* if it is the continuous image of a Borel subset of a complete metric space. Analytic subsets of complete metric spaces have the Baire property. For any countably infinite set  $E$  we can identify  $\mathcal{P}(E)$  with  $\mathcal{C}$  and then we know when a set  $\mathcal{A} \subseteq \mathcal{P}(E)$  is analytic.

**Corollary 24.** *The set of bases of a self-dual uniform matroid on a countably infinite set  $E$  is not analytic.*

*Proof.* Let  $\mathcal{B}$  be the set of bases of a self-dual uniform matroid on  $E$ . As in Theorem 22 we can assume  $E = \mathbb{N} \cup \{\infty\}$ . In the proof of Theorem 22 we defined

$$\mathcal{A} = \{A \subseteq \mathbb{N} : \exists B \in \mathcal{B} (B \subseteq A)\}.$$

Now assume that  $\mathcal{B}$  is analytic in  $\mathcal{P}(E)$ . Then the set

$$\{(A, B) : A \subseteq \mathbb{N} \wedge B \in \mathcal{B} \wedge B \subseteq A\}$$

is an analytic subset of  $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(E)$ . The set  $\mathcal{A}$  is the projection of this set to the first coordinate and hence analytic. But by Theorem 22 and Theorem 20,  $\mathcal{A}$  does

not have the Baire property and hence is not analytic, a contradiction. It follows that  $\mathcal{B}$  is not analytic.  $\square$

## 6. TWO QUESTIONS OF HIGGS

We continue our discussion of Higgs' result about the equicardinality of bases under GCH. Higgs actually proved the following stronger statement:

**Theorem 25** ([7]). *Assume GCH. Let  $E$  be a set and  $\mathcal{B}$  be a collection of subsets of  $E$  such that*

- (i) *no one member of  $\mathcal{B}$  is properly included in another, and*
- (ii) *if  $B_1, B_2 \in \mathcal{B}$  and  $I, X \subseteq E$  are such that  $I \subseteq X$ ,  $I \subseteq B_1$ , and  $B_2 \subseteq X$ , then there is  $B \in \mathcal{B}$  such that  $I \subseteq B \subseteq X$ .*

*Then the members of  $\mathcal{B}$  all have the same cardinality.*

The collection of bases of a matroid satisfy (i) and (ii). This is obvious in the case of (i). For (ii) suppose that  $I \subseteq X \subseteq E$  and there are bases  $B_1$  and  $B_2$  with  $I \subseteq B_1$  and  $B_2 \subseteq X$ . Let  $B$  be a maximal element of

$$\mathcal{A} = \{A \subseteq X : I \subseteq A \text{ and } A \text{ is independent}\}.$$

If  $B$  is not a basis, then there are a basis  $B'$  and  $y \in B'$  such that  $B' \cap X = B$  and  $y \in B' \setminus X$ . By (B2), there is  $x \in B_2 \setminus B'$  such that  $(B' \setminus \{y\}) \cup \{x\}$  is a basis. Now  $B \cup \{x\} = X \cap ((B' \setminus \{y\}) \cup \{x\}) \in \mathcal{A}$  witnesses that  $B$  is not maximal in  $\mathcal{A}$ , a contradiction.

Hence the equicardinality of bases of matroids under GCH follows from Theorem 25. Higgs asked whether the conclusion of Theorem 25 implies GCH.

The proof of Theorem 25 in [7] uses two different consequences of GCH:

- (1) The continuum function  $\kappa \mapsto 2^\kappa$  is 1-1 on infinite cardinals.
- (2) For every infinite cardinal  $\kappa$ , the partial order  $(\mathcal{P}(\kappa), \subseteq)$  has a chain of size  $2^\kappa$ .

**Theorem 26.** *If ZFC is consistent then so is ZFC together with the statements (1) and (2) above and the negation of CH.*

*Proof.* We use Easton forcing (see [8, Theorem 15.18]) over a model of GCH to obtain a model of ZFC in which for each  $n \in \mathbb{N}$ ,  $2^{\aleph_n} = \aleph_{n+2}$ . This can be done by a forcing of size  $2^{\aleph_\omega} = \aleph_\omega^+$ . This forcing does not change the size of  $2^\kappa$  for any  $\kappa \geq \aleph_\omega$ . It follows that the continuum function is 1-1 in the resulting model.

We now work inside this forcing extension. Baumgartner and Mitchell independently showed that  $\mathcal{P}(\kappa)$  includes a chain of size  $2^\kappa$  iff there is a linear order of size  $2^\kappa$  that has a dense subset of size  $\kappa$  ([2, Theorem 2.1]). From [2, Theorem 3.5] together with [2, Theorem 2.2] it follows that if  $2^{\aleph_n} = \aleph_{n+2}$  for all  $n \in \mathbb{N}$ , then for all  $n \in \omega$  there is a linear order of size  $2^{\aleph_n}$  with a dense subset of size  $\aleph_n$ . In our model GCH holds from  $\aleph_\omega$  on. Moreover,  $\aleph_\omega$  is the least cardinal  $\mu$  such that  $\aleph_\omega < 2^\mu$ . Now [2, Corollary 2.4] shows that for all  $\kappa \geq \aleph_\omega$  there is a linear order of size  $2^\kappa$  and density  $\kappa$ . It follows that for all infinite  $\kappa$ ,  $\mathcal{P}(\kappa)$  includes a chain of size  $2^\kappa$ .  $\square$

**Corollary 27.** *If ZFC is consistent, then conclusion of Theorem 25 does not imply GCH.*

In [7], Higgs also asked whether every nonempty collection  $\mathcal{B} \subseteq \mathcal{P}(E)$  that satisfies (i) and (ii) in Theorem 25 is the set of bases of a matroid on  $E$ . We show that this is not the case in general, but it is true if  $E$  is finite.

**Theorem 28.** *a) There is a nonempty collection  $\mathcal{B}$  of subsets of a countably infinite set  $E$  satisfying (i) and (ii) in Theorem 25 such that  $\mathcal{B}$  is not the set of bases of a matroid.*

*b) If  $E$  is finite and  $\mathcal{B} \subseteq \mathcal{P}(E)$  is nonempty and satisfies (i) and (ii) in Theorem 25, then  $\mathcal{B}$  is the set of bases of a matroid on  $E$ .*

*Proof.* a) Let  $E$  be a countably infinite set. Let  $B \subseteq E$  be an infinite, co-infinite set and let  $\mathcal{B}$  be the  $\sim$ -class of  $B$ . Then  $\mathcal{B}$  is not the set of bases of a matroid.

Namely, let  $I = \emptyset$  and let  $X \subseteq E$  be such that both  $X \cap B$  and  $B \setminus X$  are infinite. Then for all  $B' \in \mathcal{B}$ ,  $(B' \setminus B) \cap X$  is finite and for all finite sets  $F \subseteq X \setminus B$  there is  $B' \in \mathcal{B}$  with  $B' \cap X = (X \cap B) \cup F$ . In particular, the set  $\{X \cap B' : B' \in \mathcal{B}\}$  does not have a maximal element and hence  $\mathcal{B}$  does not satisfy (BM).

On the other hand,  $\mathcal{B}$  satisfies (i) and (ii) in Theorem 25. This can be seen as follows:

Let  $I \subseteq X \subseteq E$  be such that for some  $B_1, B_2 \in \mathcal{B}$  we have  $I \subseteq B_1$  and  $B_2 \subseteq X$ . Since  $B_1 \sim B_2$ , the sets  $B_1 \setminus B_2$  and  $B_2 \setminus B_1$  are finite and of the same size. We have  $B_1 \setminus X \subseteq B_1 \setminus B_2$  and  $B_2 \setminus B_1 \subseteq X \setminus I$ . Let  $C = B_1 \setminus X$  and let  $D \subseteq B_2 \setminus B_1$  be a set of size  $|B_1 \setminus X|$ . Now let  $B = (B_1 \setminus C) \cup D$ . We have  $B \sim B_1$  and therefore  $B \in \mathcal{B}$ . Also,  $I \subseteq B \subseteq X$ .

b) We have to show that  $\mathcal{B}$  satisfies (B2). Let  $B_1, B_2 \in \mathcal{B}$  and suppose  $x \in B_1 \setminus B_2$ . We first show that there is a set  $Y \subseteq B_2 \setminus B_1$  such that  $(B_1 \setminus \{x\}) \cup Y \in \mathcal{B}$ . Let  $I = B_1 \setminus \{x\}$  and  $X = I \cup B_2$ . By our assumptions on  $\mathcal{B}$  there is  $B \in \mathcal{B}$  such that  $I \subseteq B \subseteq X$ . Let  $Y = B \setminus B_1$ . Now  $B = (B_1 \setminus \{x\}) \cup Y$  and  $Y \subseteq B_2$ .

We show that  $Y$  is a singleton. Since no two elements of  $\mathcal{B}$  are properly included in another,  $Y$  is nonempty. Let  $y \in Y$ . By the same argument as before, there is a set  $X \subseteq B_1 \setminus B$  such that  $(B \setminus \{y\}) \cup X \in \mathcal{B}$ . Since  $B_1 \setminus B = \{x\}$ , we must have  $X = \{x\}$ . Now  $(B \setminus \{y\}) \cup \{x\}$  and  $B_1 = (B \setminus Y) \cup \{x\}$  are both in  $\mathcal{B}$ . Since no two elements of  $\mathcal{B}$  are properly included in another,  $Y = \{y\}$ . This finishes the proof of the theorem.  $\square$

We finish with an observation that follows easily from Higgs' proof of Theorem 25 without any instances of GCH.

**Theorem 29.** *Let  $\mathcal{B}$  be a nonempty collection of subsets of a countable set  $E$  satisfying (i) and (ii) in Theorem 25. Then  $\mathcal{B}$  is either countable or of size  $2^{\aleph_0}$ . In particular, this dichotomy applies to the collection of bases of a matroid on a countable set.*

*Proof.* Suppose  $\mathcal{B}$  is uncountable. Then there are  $B, B' \in \mathcal{B}$  such that the symmetric difference

$$B \triangle B' = (B \setminus B') \cup (B' \setminus B)$$

is infinite. Without loss of generality we may assume that  $B \setminus B'$  is infinite. Let  $\mathcal{S} = \mathcal{P}(B \setminus B')$ . For each  $S \in \mathcal{S}$  we have  $I_S = (B \cap B') \cup S \subseteq B$  and  $B' \subseteq X_S = S \cup B'$ . Hence, by (ii), for each  $S \in \mathcal{S}$  there is  $B_S \in \mathcal{B}$  such that  $I_S \subseteq B_S \subseteq X_S$ . Since  $S = B_S \setminus B'$  for every  $S \in \mathcal{S}$ , the  $B_S$  are pairwise distinct. Hence  $|\mathcal{B}| \geq |\mathcal{S}| = 2^{\aleph_0}$ . On the other hand,  $\mathcal{B}$  consists of subsets of the countable set  $E$  and therefore  $|\mathcal{B}| \leq 2^{\aleph_0}$ .  $\square$

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