PLANAR TRANSITIVE GRAPHS

MATTHIAS HAMANN

Abstract. We prove that the first homology group of every planar transitive locally finite graph $G$ is a finitely generated $\text{Aut}(G)$-module. Corollaries of our main theorem include Droms’s theorem that planar groups are finitely presented and Dunwoody’s theorem that planar transitive locally finite graphs are accessible.

1. Introduction

A finitely generated group is planar if it has some locally finite planar Cayley graph. Droms [2] proved that finitely generated planar groups are finitely presented. This is a hint that a similar result is true for transitive planar graphs. Indeed, we shall show the following:

Theorem 1.1. Let $G$ be a locally finite transitive planar graph. Then the first homology group of $G$ is a finitely generated $\text{Aut}(G)$-module.

Theorem 1.1 directly implies Droms’s theorem. Whereas Droms’s proof uses an accessibility result of Maskit [9] for planar groups, our self-contained proof does not. Instead, our proof will be based on Theorem 1.2. But before we state that theorem, we have to make some definitions first.

We call a graph finitely separable if no two distinct vertices are joined by infinitely many edge disjoint paths, or equivalently, any two vertices are separated by finitely many edges.

Let $G$ be a planar graph with planar embedding $\varphi: G \to \mathbb{R}^2$. Two cycles $C_1, C_2$ in $G$ are nested if no $C_i$ has vertices or edges in distinct faces of $\varphi(C_{3-i})$. A set of cycles is nested if every two of its elements are nested.

Theorem 1.2. Every 3-connected finitely separable planar graph has a canonical nested set of cycles generating the first homology group.

Here, canonical means mostly that the set of cycles is invariant under the automorphisms of the graph. But in addition, our proof is constructive and this construction commutes with graph isomorphisms, i.e. whenever we run this construction for two isomorphic graphs $G$ and $H$, then this isomorphism maps the set of cycles in $G$ we obtain to that of $H$.

Note that Theorem 1.2 is easy to prove if the graph has no accumulation points in the plane, i.e. if it is VAP-free, as you may then take the finite face boundaries as generating set, see e.g. [5, Lemma 3.2].

Theorem 1.2 has various analogues in the literature: in [7] the author proved the corresponding result for the cycle space\(^1\) of 3-connected finitely separable planar graphs.

\(^1\)The cycle space of a graph is the set of finite sums of cycles over $\mathbb{F}_2$. 

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graphs and, previously, Dicks and Dunwoody [1] proved the analogous result for the cut space\(^2\) of arbitrary graphs.

The mentioned theorem of Dicks and Dunwoody is one of the central theorems for the investigation of transitive graphs with more than one end and hence of accessible graphs and of accessible groups. (We refer to Section 6 for definitions.) Even though, accessibility has a priori more in common with the cut space than with the cycle space or the first homology group, the main result of [6] exhibited a connection between accessibility and the cycle space:

**Theorem 1.3.** [6] *Every transitive graph* \(G\) *whose cycle space is a finitely generated* \(\text{Aut}(G)\)-*module is accessible.*

As an application of our results and Theorem 1.3 we shall obtain Dunwoody’s [4] theorem that locally finite transitive planar graphs are accessible.

2. Indecomposable cycles

We call a cycle *indecomposable* if it is not generated in \(H_1(G)\) by cycles of strictly smaller length. Note that no indecomposable cycle \(C\) has a *shortcut*, i.e. a path between any two of its vertices that has smaller length than their distance on \(C\). Indeed, let \(P\) be a shortest shortcut of \(C\) and \(Q_1, Q_2\) be the two subpaths of \(C\) whose end vertices are those of \(P\). Then \(Q_1 \cup P\) and \(Q_2 \cup P\) sum to \(C\) if we run through \(P\) in different directions. (This is necessary as we are taking sums over \(\mathbb{Z}\) for the first homology group.) For a path \(P\) and \(x, y\) on \(P\), we denote by \(xPy\) the subpath of \(P\) from \(x\) to \(y\).

**Lemma 2.1.** Let \(G\) be a planar graph and let \(C_1, C_2 \subseteq G\) be two indecomposable cycles of lengths \(n_1, n_2\), respectively, that are not nested. Let \(P_1 \subseteq C_1\) be a non-trivial path of shortest length that meets \(C_2\) in precisely its end vertices. Let \(P_2 \subseteq C_2\) be a shortest path with the same end vertices as \(P_1\). Then one of the following is true.

(i) \(|P_1| = |P_2|\) and \(P_2\) meets \(C_1\) only in its end vertices;
(ii) \(|P_1| \geq |P_2|\) and \(P_1 \cup P_2 = C_1\);
(iii) \(|P_1| \geq |P_2|\) and \((C_1 \setminus P_1) \cup P_2 = C_2\).

**Proof.** Let \(v, w\) be the end vertices of \(P_1\). Note that \(C_2\) is the sum of \(vP_1wQ_1v\) and \(vQ_2wP_1v\) where \(Q_1\) and \(Q_2\) are the two subpaths of \(C_2\) with end vertices \(v\) and \(w\). First, assume \(|P_1| < |P_2|\). By the choice of \(P_2\), we have \(|P_2| \leq |Q_1|\) and \(|P_2| \leq |Q_2|\), so \(P_1\) is a shortcut of \(C_2\), which is impossible. Hence, we have \(|P_1| \geq |P_2|\).

If \(P_2 \subseteq C_1\), then we directly have \(P_2 \cup P_1 = C_1\) and (ii) holds. So we may assume that \(P_2\) contains an edge not on \(C_1\).

Let us suppose that \(P_2\) has an inner vertex on \(C_1\). So any subpath \(xP_2y\) that intersects \(C_1\) in precisely its end vertices has shorter length than \(P_1\). Note that such a subpath exists as \(P_2\) has an edge outside \(C_1\). But \(xP_2y\) cannot be a shortcut of \(C_1\). So the distance between \(x\) and \(y\) on \(C_1\) is at most \(|xP_2y|\). The subpath \(Q\) of \(C_1\) realising the distance of \(x\) and \(y\) on \(C_1\) together with \(xP_2y\) does not contain \(V\) and \(W\), so it cannot be \(C_2\). Thus, some edge of \(Q\) does not lie on \(C_2\) and hence \(Q\) contains some subpath that contradicts the choice of \(P_1\).

So \(P_2\) meets \(C_1\) only in its end vertices. Then \(C_1\) is the sum of \(C_1 = P_1 + P_2\) and \(P_1 + P_2\). As \(C_1\) is indecomposable, \(P_2\) is not a shortcut of \(C_1\) and thus we have

\(^2\)The cut space of a graph is the set of finite sums over \(\mathbb{F}_2\) of minimal separating edge sets.
either \(|P_2| = |P_1|\) or \(|P_2| = |C_1 - P_1|\). The first case implies (i) while, if the first case does not hold, we have \(|P_1| > |P_2| = |C_1 - P_1|\). Thus, the minimality of \(|P_1|\) implies that \(C_1 - P_1\) lies on \(C_2\). So we have \((C_1 - P_1) \cup P_2 = C_2\) as \(P_2\) meets \(C_1\) only in its end vertices. This shows (iii) in this situation.

If \(C\) is a cycle in a planar graph \(G\), we denote by \(f_{C}^0\) the bounded face of \(C\) and by \(f_{C}^1\), the unbounded face.

For two cycles \(C, D \subseteq G\), we call a non-trivial maximal subpath \(P\) of \(C\) that has precisely its end vertices in \(D\) a \(D\)-path in \(C\). By \(n(C, D)\) we denote the number of \(C\)-paths in \(D\).

**Lemma 2.2.** Let \(G\) be a planar graph and let \(C, D \subseteq G\) be two indecomposable cycles. Then there are nested indecomposable cycles \(\bar{C}\) and \(\bar{D}\) with \(|C| = |\bar{C}|\) and \(|D| = |\bar{D}|\) that are either the boundaries of \(f_{C}^0 \cap f_{D}^0\) and of \(f_{C}^1 \cap f_{D}^1\) or the boundaries of \(f_{C}^0 \cap f_{D}^1\) and \(f_{C}^1 \cap f_{D}^0\).

In addition, we may choose \(\bar{C}\) and \(\bar{D}\) so that, if \(\mathcal{E}\) is a set of cycles generating all cycles of length smaller than \(|C|\), then \(\mathcal{E}\) generates \(C\) or \(D\) as soon as it generates \(\bar{C}\) or \(\bar{D}\).

**Proof.** If \(C\) and \(D\) are nested, then the assertion holds trivially. This covers the situation that \(n(D, C)\) is either 0 or 1, as it implies that \(C\) and \(D\) are nested. Thus, \(C\) contains some smallest \(D\)-path \(P_1\). Note that the cases (ii) and (iii) of Lemma 2.1 imply \(n(D, C) = 1\). Hence, Lemma 2.1 implies that \(D\) contains a \(C\)-path \(Q_1\) with the same end vertices as \(P_1\) and with \(|P_1| = |Q_1|\). By definition, neither \(P_1\) nor \(Q_1\) has an inner vertex of \(D\) or \(C\), respectively. Let \(D' := D \cup C\) and \(C' := (C - P_1) \cup Q_1\).

Inductively, we obtain two sequences \((P_i)_{i \leq n}\) and \((Q_i)_{i \leq n}\) of \(D\)-paths in \(C\) and \(C\)-paths in \(D\), respectively, which are ordered by the length of the paths \(P_i\). Note that, just as above, Lemma 2.1 ensures \(|P_i| = |Q_i|\) for all but at most one \(i \leq n\). (The case with \(|P_i| \neq |Q_i|\) occurs if \(C\) and \(D\) are nested and either (ii) or (iii) of Lemma 2.1 holds.)

Consider a cyclic ordering of \(C\) and let \(i_1, \ldots, i_n \in \{1, \ldots, n\}\) be pairwise distinct such that \(P_{i_1}, \ldots, P_{i_n}\) appear on \(C\) in this order. Then, using planarity, it immediately follows by their definitions as \(C\)-path or \(D\)-path, respectively, that \(Q_{i_1}, \ldots, Q_{i_n}\) appear in this order on \(D\). Note that one face of \(P_i \cup Q_i\) contains no vertices or edges of \(C \cup D\). The assertion follows except for the fact that the obtained cycles are indecomposable and the additional statement.

Let \(\mathcal{E}\) be a set of cycles generating in \(H_1(G)\) all cycles of length smaller than \(|C|\). Assume that the boundaries \(C'\) and \(D'\) of \(f_{C}^0 \cap f_{D}^0\) and \(f_{C}^1 \cap f_{D}^1\), respectively, have the desired property up to being indecomposable. Let us assume that \(C'\) is generated by \(\mathcal{E}\). (Note that this covers also the case that \(C'\) is not indecomposable.)

If all cycles \(P_i \cup Q_i\) have length less than \(|C|\) and \(|D|\), then we add every cycle \(P_i \cup Q_i\) to \(C'\) for which \(Q_i\) lies on the boundary of \(f_{C}^0 \cap f_{D}^0\) and we obtain \(C\). So \(C\) is generated by \(\mathcal{E}\) as \(C'\) and all of the added cycles are generated by \(\mathcal{E}\).

If all but exactly one of the cycles \(P_i \cup Q_i\) have length less than \(|C|\) and \(|D|\), then \(P_i \cup Q_i\) has largest length of all those cycles. If \(P_i\) lies on the boundary of \(f_{C}^0 \cap f_{D}^0\), then we add every cycle \(P_i \cup Q_i\) to \(C'\) for which \(Q_i\) lies on the boundary of \(f_{C}^0 \cap f_{D}^0\). As before, we obtain that \(C\) is generated by \(\mathcal{E}\). If \(P_i\) lies on the boundary of \(f_{C}^1 \cap f_{D}^1\), then we add every cycle \(P_i \cup Q_i\) to \(C'\) for which \(P_i\) lies on the boundary of \(f_{C}^1 \cap f_{D}^1\) and obtain \(D\). So \(D\) is generated by \(\mathcal{E}\).
If at least two cycles \( P_i \cup Q_i \) have length at least \( \min\{|C|, |D|\} \), then \( n = 2 \) follows immediately. Hence, also the boundaries \( C'' \) and \( D'' \) of \( f_C^3 \cap f_D^3 \) and \( f_C^1 \cap f_D^1 \), respectively, are cycles. So we may have chosen them instead of \( C' \) and \( D' \). If one of them, \( C'' \) say, is generated by \( E \), too, then \( C' + C'' \) is generated by \( E \). As this sum is either \( C \) or \( D \), the assertion follows. \( \square \)

Note that it follows from the proof of Lemma 2.2 that there is a canonical bijection between the \( C \)-paths in \( D \) and the \( D \)-paths in \( C \). In particular, we have \( n(C, D) = n(D, C) \).

3. Counting Crossing Cycles

Our restriction to finitely separable graphs implies that each cycle in such a graph is nested with all but finitely many cycles of bounded length. In general, this is not true if we omit that assumption.

Proposition 3.1. Let \( i \in \mathbb{N} \). Every cycle in a finitely separable planar graph is nested with all but finitely many cycles of length at most \( i \).

Proof. Let us assume that some cycle \( C \) is not nested with infinitely many cycles of length at most \( i \). Then there are two vertices \( x_1, x_2 \) of \( C \) that lie on infinitely many of these cycles and thus we obtain infinitely many distinct \( x_1-x_2 \) paths of length at most \( i - 1 \). Either there are already infinitely many edge disjoint \( x_1-x_2 \) paths or infinitely many share another vertex \( x_3 \). In the latter situation, there are either infinitely many distinct \( x_1-x_3 \) or \( x_2-x_3 \) paths of length at most \( i - 2 \). Continuing this process, we end up at some point with two distinct vertices and infinitely many edge disjoint paths between them, since we reduce the length of the involved paths in each step by at least 1. So we obtain a contradiction to finite separability. \( \square \)

Let \( \mathcal{E} \) be a set of cycles of length at most \( i \) in a finitely separable graph \( G \) and \( C \subseteq G \) be a cycle. We define \( \mu_{\mathcal{E}}(C) \) to be the number of cycles in \( \mathcal{E} \) that are not nested with \( C \). Note that Proposition 3.1 says that \( \mu_{\mathcal{E}}(C) \) is finite. If \( \mathcal{F} \) is another set of cycles of length at most \( i \), we set \( \mu_{\mathcal{E}}(\mathcal{F}) \) as minimum over all \( \mu_{\mathcal{E}}(C) \) with \( C \in \mathcal{F} \).

Proposition 3.2. Let \( G \) be a finitely separable planar graph. Let \( \mathcal{E} \) be a set of cycles in \( G \) of length at most \( i \in \mathbb{N} \) and let \( C, D \) be two indecomposable cycles in \( G \) that are not nested. Then we have

\[
\mu_{\mathcal{E}}(C) + \mu_{\mathcal{E}}(D) \geq \mu_{\mathcal{E}}(\tilde{C}) + \mu_{\mathcal{E}}(\tilde{D}),
\]

where \( \tilde{C} \) and \( \tilde{D} \) are the cycles obtained by Lemma 2.2. Furthermore, if \( D \in \mathcal{E} \), then the inequality is a strict inequality.

Proof. Using homeomorphisms of the sphere, we may assume that \( \tilde{C} \) is the boundary of \( f_C^0 \cap f_C^0 \) and \( \tilde{D} \) is the boundary of \( f_D^1 \cap f_D^1 \). Let \( F \in \mathcal{E} \) be nested with \( C \) and \( D \). We may assume that \( F \) avoids \( f_C^0 \). Thus, it is nested with \( \tilde{C} \). If \( F \) avoids \( f_D^1 \), too, then it lies in \( f_C^1 \cap f_D^1 \) with its boundary and is nested with \( \tilde{D} \). So let us assume that it avoids \( f_D^1 \). Thus, \( F \) does not contain any points of \( f_C^1 \cap f_D^1 \) and thus is nested with \( \tilde{D} \).

Now consider the case that \( F \in \mathcal{E} \) is nested with \( C \) but not with \( D \). We may assume that \( F \) avoids \( f_C^0 \). Hence, it avoids \( f_C^0 \cap f_D^0 \), too, and is nested with \( \tilde{C} \).
This shows that every $F \in \mathcal{E}$ that is not counted on the left side of the inequality is not counted on the right side either and that every $F \in \mathcal{E}$ that is counted on the left side precisely once is counted on the right side at most once. This shows the first part of the assertion.

To see the additional statement, just note that $D$ is counted on the left for $\mu_{\mathcal{E}}(C)$ but not for $\mu_{\mathcal{E}}(D)$ and that both cycles $\tilde{C}$ and $\tilde{D}$ are nested with $D$.

4. Finding a Nested Generating Set

The main theorem of [7] says that the cycle space of any 3-connected finitely separable planar graph $G$ is generated by some canonical nested set of cycles as $\mathbb{F}_2$-vector space. We will prove the analogous result for the first (simplicial) homology group $H_1(G)$ as module over $\mathbb{Z}$.

Throughout this section, let $G$ be a 3-connected planar finitely separable graph.

Let $\mathcal{H}_i := \mathcal{H}_i(G)$ be the submodule of $H_1(G)$ generated by all cycles of length at most $i$. So $H_1(y) = \bigcup_{i \geq n} \mathcal{H}_i$. We shall recursively define canonical nested subsets $\mathcal{C}_i$ of $\mathcal{H}_i$ that generate $\mathcal{H}_i$ and consist only of indecomposable cycles of length at most $i$. So $\bigcup_{i \geq n} \mathcal{C}_i$ will generate $H_1(G)$. For the start, let $\mathcal{C}_1 = \emptyset$ for $i \leq 2$. Let us assume that we already defined $\mathcal{C}_{i-1}$. Then we shall define $\mathcal{C}_i$ recursively.

In order to define $\mathcal{C}_i$, we construct another sequence of nested $\text{Aut}(G)$-invariant sets $\mathcal{C}_i^{\kappa}$ of indecomposable cycles. Set $\mathcal{C}_1^0 := \mathcal{C}_{i-1}$. Let $\kappa$ be some ordinal such that $\mathcal{C}_i^\lambda$ is defined for all $\lambda < \kappa$. If $\kappa$ is a limit ordinal, then set $\mathcal{C}_i^\kappa = \bigcup_{\lambda < \kappa} \mathcal{C}_i^\lambda$.

So let $\kappa$ be a successor ordinal, say $\kappa = \nu + 1$. Any cycle of length $i$ that is not generated by $\mathcal{C}_i^\nu$ must be indecomposable by definition of $\mathcal{C}_{i-1}$. If there is not such a cycle, set $\mathcal{C}_i := \mathcal{C}_i^\nu$. So in the following, we assume that there is at least one indecomposable cycle of length $i$ that is not generated by $\mathcal{C}_i^\nu$. Hence, the set $\mathcal{D}_i^\nu$ of all indecomposable cycles that are not generated by $\mathcal{C}_i^\nu$ and that have length $i$ is not empty.

**Lemma 4.1.** The set $\mathcal{D}_i^\nu \neq \emptyset$ contains a cycle that is nested with $\mathcal{C}_i^\nu$.

**Proof.** Let $C \in \mathcal{D}_i^\nu$ with minimum $\mu_{\mathcal{C}_i^\nu}(C)$. We shall show $\mu_{\mathcal{C}_i^\nu}(C) = 0$. So let us suppose that $C$ is not nested with some $D \in \mathcal{C}_i^\nu$. Since $C$ and $D$ are indecomposable cycles, we obtain by Lemma 2.2 two indecomposable cycles $\tilde{C}$ and $\tilde{D}$ with $|C| = |\tilde{C}|$ and $|D| = |\tilde{D}|$ such that Proposition 3.2 implies

$$
\mu_{\mathcal{C}_i^\nu}(C) = \mu_{\mathcal{C}_i^\nu}(\tilde{C}) + \mu_{\mathcal{C}_i^\nu}(\tilde{D}) > \mu_{\mathcal{C}_i^\nu}(\tilde{C}) + \mu_{\mathcal{C}_i^\nu}(\tilde{D}).
$$

Note that, if $\tilde{C}$ and $\tilde{D}$ are generated by $\mathcal{C}_i^\nu$, then $C$ being the sum of $D$, $\tilde{C}$, and $\tilde{D}$ is generated by $\mathcal{C}_i^\nu$, too. But then it lies outside $\mathcal{D}_i^\nu$. As this is not the case, either $\tilde{C}$ or $\tilde{D}$ is not generated by $\mathcal{C}_i^\nu$. In particular, this cycle must lie in $\mathcal{D}_i^\nu$, a contradiction to the choice of $C$. \hfill \Box

Let $\mathcal{E}_i^\kappa$ be the set of all cycles in $\mathcal{D}_i^\kappa$ that are nested with $\mathcal{C}_i^\nu$. By Lemma 4.1, this set is not empty.

For a set $\mathcal{E}$ of cycles of length at most $i$, we call $C \in \mathcal{E}$ optimally nested in $\mathcal{E}$ if $\mu_{\mathcal{E}}(C) = \mu_{\mathcal{E}}(\mathcal{E})$. Note that $\mu_{\mathcal{E}}(\mathcal{E})$ is finite by Proposition 3.1 and, furthermore, as 3-connected planar graphs have (up to homeomorphisms) unique embeddings into the sphere due to Whitney [12] for finite graphs and Imrich [8] for infinite graphs, $\mu_{\mathcal{E}}(C) = \mu_{\mathcal{E}}(\mathcal{E})$ for all $\alpha \in \text{Aut}(G)$.

**Lemma 4.2.** The set $\mathcal{F}_i^\kappa$ of optimally nested cycles in $\mathcal{E}_i^\kappa$ is non-empty and nested.
Proof. Since $\mathcal{E}_i^\kappa$ is non-empty, the same is true for $\mathcal{F}_i^\kappa$. Let us suppose that $\mathcal{F}_i^\kappa$ contains two cycles $C, D$ that are not nested. Let $\tilde{C}$ and $\tilde{D}$ be the indecomposable cycles obtained by Lemma 2.2 with $|C| = |\tilde{C}|$ and $|D| = |\tilde{D}|$ each of which is not generated by $\mathcal{C}_i^\kappa$ and such that Proposition 3.2 yields

$$\mu_C(C) + \mu_D(D) \geq \mu_{\tilde{C}}(\tilde{C}) + \mu_{\tilde{D}}(\tilde{D}).$$

As $\mathcal{E}_i^\kappa$ is nested with $\mathcal{C}_i^\kappa$ by definition, we have $\mu_C(C) + \mu_D(D) = 0$. Note that $\tilde{C}$ and $\tilde{D}$ lie in $\mathcal{D}_i^\kappa$ by definition. As both are nested with $\mathcal{C}_i^\kappa$, they lie in $\mathcal{E}_i^\kappa$. We apply Proposition 3.2 once more and obtain

$$\mu_{\tilde{C}}(\tilde{C}) + \mu_{\tilde{D}}(\tilde{D}) > \mu_{C}(C) + \mu_{D}(D).$$

Thus either $\tilde{C}$ or $\tilde{D}$ is not nested with less cycles in $\mathcal{E}_i^\kappa$ than $C$. This contradiction to the choice of $C$ shows that $\mathcal{E}_i^\kappa$ is nested.

So we set $\mathcal{C}_i^\kappa := \mathcal{C}_i^\nu \cup \mathcal{F}_i^\kappa$. Then $\mathcal{C}_i^\kappa$ is nested as $\mathcal{C}_i^\nu$ is nested and by the choice of $\mathcal{E}_i^\kappa$ all elements of $\mathcal{C}_i^\kappa$ are indecomposable.

This process will terminate at some point as we strictly enlarge the sets $\mathcal{C}_i$ in each step but we cannot put in more cycles than there are in $G$. Let $\mathcal{C}_i$ be the union of all $\mathcal{C}_i^\kappa$. Note that we made no choices at any point, i.e. all sets $\mathcal{C}_i$ are $\text{Aut}(G)$-invariant and canonical. Thus, we proved Theorem 1.2. More precisely, we have proved the following theorem.

**Theorem 4.3.** For every finitely separable 3-connected planar graph $G$ there is a sequence $(\mathcal{C}_i)_{i \in \mathbb{N}}$ of sets of cycles in $G$ such that

1. $\mathcal{C}_{i-1} \subseteq \mathcal{C}_i$;
2. $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ consists of indecomposable cycles of length $i$;
3. $\mathcal{C}_i$ generates $\mathcal{H}_i(G)$;
4. $\mathcal{C}_i$ is canonical and nested.

Note that the only situation where we used 3-connectivity was when we concluded that we have $\mu(C) = \mu(C \alpha)$ for any cycle $C$, set $E$ of cycles of bounded length and automorphism $\alpha$. That is, the above proof also give us the existence of a nested generating set for lower connectivity, but we lose canonicity. Note that, in general, the statement of Theorem 1.2 is false if we do not require the graph to be 3-connected: let $G$ be the graph obtained by two vertices joined by four internally disjoint paths of length 2. Then all cycles have length 4 and lie in the same $\text{Aut}(G)$-orbit, but it is not hard to find two of them which are not nested. So you cannot find a canonical nested generating set of $\mathcal{H}_1(G)$.

5. Finding a finite generating set

Let us introduce the notion of a degree sequence of orbits because the general idea to prove Theorem 1.1 will mainly be done by induction on this notion. For a connected locally finite quasi-transitive graph $G$ with $|V(G)| > 1$ we call a tuple $(d_1, \ldots, d_m)$ of positive integers with $d_i \geq d_{i+1}$ for all $i < m$ the degree sequence of the orbits of $G$ if for some set $\{v_1, \ldots, v_m\}$ of vertices that contains precisely one vertex from each $\text{Aut}(G)$-orbit the degree of $v_i$ is $d_i$. We consider the lexicographic order on the finite tuples of positive integers (and thus on the degree sequences of orbits), that is, we set 

$$(d_1, \ldots, d_m) \leq (e_1, \ldots, e_n).$$
if either \( m \leq n \) and \( d_i = c_i \) for all \( i \leq m \) or \( d_i < c_i \) for the smallest \( i \leq m \) with \( d_i \neq c_i \). Note that any two finite tuples of positive integers are \( \leq \)-comparable.

A direct consequence of this definition is the following lemma.

**Lemma 5.1.** Any strictly decreasing sequence in the set of finite tuples of positive integers is finite. \( \square \)

We call a graph quasi-transitive if its automorphism group has only finitely many orbits on the vertex set.

Lemma 5.1 for quasi-transitive graphs and their degree sequences of orbits reads as follows and enables us to use induction on the degree sequence of the orbits of graphs:

**Lemma 5.2.** Every sequence of locally finite quasi-transitive graphs whose corresponding sequence of degree sequences of the orbits is strictly decreasing is finite. \( \square \)

**Lemma 5.3.** Let \( G \) be a locally finite quasi-transitive graph and let \( S \subseteq V(G) \) and \( H \subseteq G \) be such that the following conditions hold:

(i) \( G - S \) is disconnected;
(ii) each \( S \alpha \) with \( \alpha \in \text{Aut}(G) \) meets at most one component of \( G - S \);
(iii) such that no vertex of \( S \) has all its neighbours in \( S \);
(iv) \( H \) is a maximal subgraph of \( G \) such that no \( S \alpha \) with \( \alpha \in \text{Aut}(G) \) disconnects \( H \).

Then the degree sequence of the orbits of \( H \) is smaller than the one of \( G \).

**Proof.** First we show that all vertices in \( H \) that lie in a common \( \text{Aut}(G) \)-orbit of \( G \) and whose degrees in \( G \) and in \( H \) are the same also lie in a common \( \text{Aut}(H) \)-orbit. Let \( x, y \) be two such vertices and \( \alpha \in \text{Aut}(G) \) with \( x\alpha = y \). Suppose that \( H\alpha \neq H \).

Then there is some \( S\beta \) that separates some vertex of \( H \) from some vertex of \( H\alpha \) by the maximality of \( H \). But as \( y \) and all its neighbours lie in \( H \) and in \( H\alpha \), they lie in \( S\beta \), which is a contradiction to (iii). Thus, \( \alpha \) fixes \( H \) and induces an automorphism of \( H \) that maps \( x \) to \( y \).

We consider vertices \( x \) such that \( \{x\} \cup N(x) \) lies in no \( H\alpha \) with \( \alpha \in \text{Aut}(G) \) and such that \( x \) has maximum degree with this property. Let \( \{x_1, \ldots, x_m\} \) be a maximal set that contains precisely one vertex from each orbit of those vertices. If \( x_i \) lies outside every \( H\alpha \), then no vertex of its orbit is considered for the degree sequence of the orbits of \( H \). If \( x_i \) lies in \( H \), then its degree in some \( H\alpha \) is smaller than its degree in \( G \). By replacing \( x_i \) by \( x_i\alpha^{-1} \), if necessary, we may assume \( d_H(x_i) < d_G(x_i) \). So its value in the degree sequence of orbits of \( H \) is smaller than its value in the degree sequence of orbits of \( G \); but it may be counted multiple times now as the \( \text{Aut}(G) \)-orbit containing \( x_i \) may be split into multiple \( \text{Aut}(H) \)-orbits. Nevertheless, the degree sequence of orbits of \( H \) is smaller than that of \( G \). \( \square \)

Remember that a block of a graph is a maximal 2-connected subgraph. As any cycle lies completely in some block and as any locally finite quasi-transitive graph has only finitely many orbits of blocks, we directly have:

**Proposition 5.4.** For a locally finite quasi-transitive graph \( G \), we have that \( H_1(G) \) is a finitely generated \( \text{Aut}(G) \)-module over \( \mathbb{Z} \) if and only if the same is true for every block. \( \square \)
Remark 5.5. In the situation of Proposition 5.4 we can take the orbits of the cutvertices one-by-one and apply Lemma 5.3 for each such orbit. It follows recursively that each block has a smaller degree sequence of its orbits than the original graph.

For the reduction to the 3-connected case for graphs of connectivity 2, we apply Tutte’s decomposition of 2-connected graphs into ‘3-connected parts’ and cycles. Tutte [11] proved it for finite graphs. Later, it was extended by Droms et al. [3] to locally finite graphs.

A tree-decomposition of a graph $G$ is a pair $(T, \mathcal{V})$ of a tree $T$ and a family $\mathcal{V} = \{V_t\}_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for each vertex of $T$, such that

- $(T1)$ $V = \bigcup_{t \in T} V_t$;
- $(T2)$ for every edge $e \in G$ there exists a $t \in V(T)$ such that both ends of $e$ lie in $V_t$;
- $(T3)$ $V_{t_1} \cap V_{t_2} \subseteq V_{t_3}$ whenever $t_2$ lies on the $t_1$-$t_3$ path in $T$.

The sets $V_t$ are the parts of $(T, \mathcal{V})$ and the intersections $V_{t_1} \cap V_{t_2}$ for edges $t_1t_2$ of $T$ are its adhesion sets; the maximum size of such a set is the adhesion of $(T, \mathcal{V})$. Given a part $V_t$, its torso is the graph with vertex set $V_t$ and whose edge set is

$$\{xy \in E(G) \mid x, y \in V_t\} \cup \{xy \mid \{x, y\} \subseteq V_t \text{ lies in an adhesion set}\}.$$ 

The automorphisms of $G$ act canonically on vertex sets of $G$. If every part of the tree-decomposition is mapped to another of its parts and this map induces an automorphism of $T$ then we call the tree-decomposition Aut($G$)-invariant.

Theorem 5.6. [3, Theorem 1] Every locally finite 2-connected graph $G$ has an Aut($G$)-invariant tree-decomposition of adhesion 2 each of whose torsos is either 3-connected or a cycle or a complete graph on two vertices.

Remark 5.7. In addition to the conclusion of Theorem 5.6, we may assume that the tree-decomposition is such that the torsos of tree vertices of degree 2 are either 3-connected or cycles and that no two torsos of adjacent tree vertices $t_1, t_2$ are cycles if $V_{t_1} \cap V_{t_2}$ is no edge of $G$. (Remember that edges are two-element vertex sets.) We call a tree-decomposition as Theorem 5.6 with this additional property a Tutte decomposition.

Now we reduce the problem for finding a finite set of cycles generating $H_1(G)$ as Aut($G$)-module over $\mathbb{Z}$ from 2-connected graphs to 3-connected ones.

Proposition 5.8. The cycles space of a locally finite quasi-transitive 2-connected graph $G$ is a finitely generated Aut($G$)-module if and only if the same is true for each of its torsos in every Tutte decomposition.

Proof. Let $(T, \mathcal{V})$ be a Tutte decomposition of $G$. Note that every vertex lies in only finitely many 2-separators (cf. [10, Proposition 4.2]). Thus, the graph $H$ given by $G$ together with all edges $xy$, where $\{x, y\}$ forms an adhesion set, is also locally finite and quasi-transitive. There are only finitely many orbits of (the action induced by) Aut($G$) on $T$, since any 2-separator of $G$ uniquely determines the parts $V_t$ of $(T, \mathcal{V})$ it is contained in and since there are only finitely many Aut($G$)-orbits of 2-separators. Obviously, the restriction of $H$ to any $V_t \in \mathcal{V}$ is the torso of $V_t$.

Let us assume that $H_1(G)$ is finitely generated as Aut($G$)-module and let $\mathcal{C}$ be a finite set of cycles generating it over $\mathbb{Z}$. Every $C \in \mathcal{C}$ can be written as the sum of (finitely many) induced cycles $C_1, \ldots, C_n$ in $H$. So the set $\mathcal{D}$ of all those $C_i$ for all
$C \in \mathcal{C}$ generates $H_1(H)$ as Aut$(H)$-module. Each of the cycles $C_i$ lies in a unique part $V_i$ of $(T, V)$ as they are induced and as every adhesion set in $H$ is complete. Note that cycles which lie in the same Aut$(G)$-orbit and in some $V_i$ also lie in the same orbit with respect to the automorphisms of the torso $G_i$ of $V_i$. Let $D_i$ be the set of all cycles in $D$ that lie in $G_i$. Let $C$ be a cycle in $G_i$. Then it is the sum of $C_1, \ldots, C_n \in D$. Since all $C_i \not\subseteq G_i$ sum to 0, those $C_i \subseteq G_i$ sum to $C$. Thus, $H_1(G_i)$ is a finitely generated Aut$(G_i)$-orbit.

For the converse, let $H_1(G_i)$ for every torso $G_i$ of $(T, V)$ be a finitely generated Aut$(G_i)$-orbit and let $C_i$ be a finite set of cycles in $G_i$ that generates $H_1(G_i)$ as Aut$(G_i)$-module. We may choose the sets $C_i$ so that $C_i = C_{i', \alpha}$ if $\alpha \in$ Aut$(G)$ maps $V_i$ to $V_{i'}$. Let $\mathcal{A}$ be a set of ordered adhesion sets $(x, y)$ of $(T, V)$ consisting of one element for each Aut$(G)$-orbit. For every $(x, y) \in \mathcal{A}$ with $xy \not\in$ E(G) we fix an $x$-$y$ path $P_{xy}$ in $G$. Then $P_{xy} + xy$ is a cycle $C_{xy}$ in $H$. If $xy \in$ E(G), let $P_{xy} = xy$ and, for later conveniences, $C_{xy} = 0$. Note that for an adhesion set $(x, y)$ we may have fixed two distinct paths $P_{xy}$ and $P_{yx}$. We can extend the definition of the paths $P_{xy}$ and cycles $C_{xy}$ to all ordered adhesion sets $(x, y)$, i.e. if $(x, y) = (x', y')/\alpha$ with $(x', y') \in \mathcal{A}$, set $P_{xy} = P_{x'y'} \alpha$ and $C_{xy} = C_{x'y'} \alpha$.

We notice that every orbit of cycles in $G_i$ contains elements of only finitely many Aut$(H)$-orbits since each of its cycles has the same length $t$ and by local finiteness and quasi-transitivity there are only finitely many orbits of cycles in $H$ whose cycles have length $t$. Note that there are only finitely many Aut$(G)$-orbits and hence only finitely many Aut$(H)$-orbits of parts of $(T, V)$. So the union $\mathcal{C}$ of all $C_i$ is a set of cycles in $H$ meeting only finitely many Aut$(H)$-orbits and generating $H_1(H)$, as it has a generating set of induced cycles, each of those lies in some $G_i$ and thus is generated by $\mathcal{C}$. For every $C \in \mathcal{C}$ let $\mathcal{C}_C$ be the set of all elements of $H_1(G)$ that are obtained from $C$ by adding for the edges $xy$ that form an adhesion set $(x, y)$ of $(T, V)$ the cycle $C_{xy}$ in any possible way such that $xy$ does not lie in the sum.\(^3\)

We can write each element of $\mathcal{C}_C$ as sum of cycles of $G$ – possibly in more than just one way. Let $\mathcal{C}'_C$ be obtained from $\mathcal{C}_C$ by replacing each element by any cycle in $G$ that occurs in any of the just mentioned sums. Let $\mathcal{C}' := \bigcup_{C \in \mathcal{C}} C'_C$. Then each element in any $\mathcal{C}_C$ has a bounded number of edges and thus the cycles in $\mathcal{C}'$ have bounded length and lie in finitely many Aut$(G)$-orbits. Obviously, every element of $\mathcal{C}_C$ can be generated by $\mathcal{C}'$.

To see that $\mathcal{C}'$ generates $H_1(G)$, let $C$ be any cycle of $G$. Thus it is also a cycle of $H$ and can be written as sum of cycles $C_1, \ldots, C_m \in \mathcal{C}$. Now let us fix for each edge $xy$ on any of these $C_i$ that form an adhesion set of $(T, V)$ an orientation $(x, y)$ and thus a cycle $C_{xy}$. Let $C'_i$ be the sum of $C_i$ with all our fixed cycles $C_{xy}$. Then $C'_i$ is an element of $H_1(G)$ since it contains no edge of $H \setminus G$. Note that $\sum_{i=1}^m C_i = \sum_{i=1}^m C'_i$ and that each $C'_i$ lies in $\mathcal{C}_C$. So $C$ can be generated by $\bigcup_{i=1}^m \mathcal{C}_C$, and thus by $\mathcal{C}'$.

**Remark 5.9.** Unfortunately, we are not able to apply Lemma 5.3 directly for Proposition 5.8 to see that the torsos in a Tutte decomposition have a smaller degree sequence of orbits, as the orbits are not subgraphs of $G$. But as not both vertices of any adhesion set have degree 2, it is possible to follow the argument.

\(^3\)Note that if $C$ contains only one such edge $xy$, we may add either $C_{xy}$ or (the negative of) $C_{yx}$ and thus can obtain up to two possibilities. So if $C$ has $n$ edges that form adhesion sets of $(T, V)$, we may obtain up to $2^n$ possibilities.
of the proof of Lemma 5.3 for each of the finitely many orbits of the 2-separators one-by-one to see that each torso has a smaller degree sequence of orbits than \( G \).

Now we are able to attack the general VAP-free case.

**Proposition 5.10.** Let \( G \) be a locally finite quasi-transitive VAP-free planar graph. Then \( H_1(G) \) is a finitely generated \( \text{Aut}(G) \)-module.

**Proof.** Due to Propositions 5.4 and 5.8, it suffices to show the assertion if \( G \) is 3-connected. As 3-connected planar graphs have (up to homeomorphisms) unique embeddings into the sphere, every automorphism of \( G \) induces a homeomorphism of the plane. So faces are mapped to faces and cycles that are face boundaries are mapped to such cycles. As \( G \) is quasi-transitive and locally finite, there are only finitely many \( \text{Aut}(G) \)-orbits of finite face boundaries.

Every cycle in \( G \) determines an inner face and an outer face in the plane. The inner face contains only finitely many vertices as \( G \) is VAP-free. Hence, every cycle is the sum of all face boundaries of the faces that lie in its inner part in the plane and the assertion follows.

Now we are able to prove a strengthened version of our main theorem, Theorem 1.1.

**Theorem 5.11.** Let \( G \) be a locally finite quasi-transitive planar graph. Then \( H_1(G) \) is a finitely generated \( \text{Aut}(G) \)-module.

**Proof.** Due to Propositions 5.4 and 5.8, we may assume that \( G \) is 3-connected and due to Proposition 5.10 we may assume that \( G \) is not VAP-free. Let \( \varphi : G \to \mathbb{R}^2 \) be a planar embedding of \( G \). Let \( \mathcal{C} \) be a non-empty \( \text{Aut}(G) \)-invariant nested set of cycles that generates \( H_1(G) \), which exists by Theorem 1.2. Since \( G \) is not VAP-free, there is some cycle \( C \) of \( G \) such that both faces of \( \mathbb{R}^2 \setminus \varphi(C) \) contain infinitely many vertices of \( G \). As \( \mathcal{C} \) generates \( H_1(G) \) as \( \text{Aut}(G) \)-module, one of the cycles in \( \mathcal{C} \) has the same property as \( C \). Hence, we may assume \( C \in \mathcal{C} \). In particular, \( \{ C\alpha \mid \alpha \in \text{Aut}(G) \} \) is nested.

We consider maximal subgraphs \( H \) of \( G \) such that no \( C\alpha \) with \( \alpha \in \text{Aut}(G) \) disconnects \( H \). In particular, \( H \) is connected and for every \( C\alpha \) with \( \alpha \in \text{Aut}(G) \) one of the faces of \( \mathbb{R}^2 \setminus \varphi(C\alpha) \) is disjoint from \( H \). Note that there are only finitely many \( \text{Aut}(G) \)-orbits of such subgraphs \( H \) as we find in each orbit some element that contains vertices of \( C \) by maximality of \( H \). Due to Lemma 5.3, the graph \( H \) has a strictly smaller degree sequence of its orbits than \( G \) as \( C \) disconnects \( G \). As \( H \) is again a locally finite quasi-transitive planar graph, we conclude by induction on the degree sequence of the orbits of such graphs (cf. Lemma 5.2) with base case if \( G \) is VAP-free that \( H_1(H) \) is a finitely generated \( \text{Aut}(H) \)-module. Let \( \mathcal{E}_H \) be an \( \text{Aut}(H) \)-invariant set of cycles with only finitely many \( \text{Aut}(H) \)-orbits generating \( H_1(H) \).

There are only finitely many pairwise non-\( \text{Aut}(G) \)-equivalent such subgraphs \( H \). So let \( \mathcal{H} \) be a finite set of such subgraphs consisting of one per \( \text{Aut}(G) \)-orbit. Let

\[
\mathcal{E} := \bigcup_{H \in \mathcal{H}} \bigcup_{\alpha \in \text{Aut}(G)} \mathcal{E}_{H\alpha}.
\]

Then \( \mathcal{E} \) is \( \text{Aut}(G) \)-invariant and has only finitely many orbits. We shall show that \( \mathcal{E} \) generates \( H_1(G) \).
Let $D$ be a cycle of $G$. If $D$ lies entirely inside some of the subgraphs $H \in \mathcal{H}$ or its $\text{Aut}(G)$-images, then, obviously, it is generated by $E$. So let us assume that there is some $\alpha \in \text{Aut}(G)$ such that both faces of $C\alpha$ contain vertices or edges of $D$. By considering $D\alpha^{-1}$ instead of $D$, we may assume $\alpha = \text{id}$. We add all edges of $C$ to $D$ that lie in the bounded face of $D$ to obtain a subgraph $F$ of $G$. Then $D$ is the sum over all boundaries $C_1, \ldots, C_k$ of bounded faces of $F$.

Assume that $C\beta$ with $\beta \in \text{Aut}(G)$ is not nested with $C_i$ and suppose that it is nested with $D$. Remember that $C$ and $C\beta$ are nested. Since $C\beta$ contains points in both faces of $C_i$, there is some (possibly trivial) common path $P$ of $C_i$ and $C\beta$ such that the edges on $C\beta$ incident with the end vertices of $P$ lie in different faces of $C_i$ and also the edges of $C_i$ incident with the end vertices of $P$ lie in different faces of $C\beta$. As $C\beta$ is nested with $C$ and with $D$, one of these edges belongs to $C$ and the other to $D$. Thus, $C$ and $D$ must lie in distinct faces of $C\beta$ and hence must be nested. This contradiction shows that every $C\beta$ that is not nested with $C_i$ is not nested with $D$ either.

As $C$ is not nested with $D$ but with every $C_i$, every $C_i$ is not nested with less cycles $C\beta$ than $D$ and this is a finite number by Proposition 3.1 as all cycles $C\beta$ have the same length. Induction on the number of cycles $C\beta$ the current cycle is not nested with implies that each $C_i$ is generated by $E$ and so is $D$. 

\section{Applications}

Droms [2] proved that planar groups are finitely presented. His proof uses an accessibility result of Maskit [9]. We prove his result without any accessibility result. But before, we recall the following well-known lemma.

Lemma 6.1. Let $\Gamma$ be a Cayley graph of the group $G$ with presentation $\langle S \mid R \rangle$. Then the set of walks in $\Gamma$ induced by relations in $R$ generates $H_1(\Gamma)$.

Conversely, if $R'$ is a set of relations of $G$ over $S$ such that the set of closed walks of $\Gamma$ induced by $R'$ generates $H_1(\Gamma)$, then $\langle S \mid R' \rangle$ is a presentation of $G$. 

As an application of Theorem 1.1 we obtain a self-contained proof of Droms’s result.

Theorem 6.2. [2] Every finitely generated planar group is finitely presented. 

A \textit{ray} is a one-way infinite path and two rays are \textit{equivalent} if they lie in the same component whenever we remove a finite vertex set. This is an equivalence relation whose classes are the \textit{ends} of the graph. We call a quasi-transitive graph \textit{accessible} if there is some $n \in \mathbb{N}$ such that any two ends can be separated by removing at most $n$ vertices.

In [6] the author proved the following accessibility result for quasi-transitive graphs.

Theorem 6.3. [6, Theorem 3.2] Every quasi-transitive graph $G$ whose cycle space is a finitely generated $\text{Aut}(G)$-module is accessible.

As a further corollary of Theorem 5.11 together with Theorem 6.3, we obtain Dunwoody’s theorem of the accessibility of locally finite quasi-transitive planar graphs, a strengthened version of Theorem 1.3. (Note that any generating set of the first homology group of a graph is also a generating set of its cycle space.)

Theorem 6.4. [4] Every locally finite quasi-transitive planar graph is accessible.
Note that, in order to prove Theorem 6.4, we do not need the full strength of a nested canonical generating set for the first homology group. Indeed, instead of applying Theorem 1.2, we could just do the same arguments as in Section 5 using a nested canonical generating set for the cycle space obtained from [7, Theorem 1] to obtain a finite set of cycles generating the cycle space as module.

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Matthias Hamann, Department of Mathematics, University of Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany