PLANAR TRANSITIVE GRAPHS

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Abstract. We prove that the first homology group of every planar transitive locally finite graph G is a finitely generated $\mathrm{Aut}(G)$-module. Corollaries of our main theorem include Droms's theorem that planar groups are finitely presented and Dunwoody's theorem that planar transitive locally finite graphs are accessible.

1. Introduction

A finitely generated group is *planar* if it has some locally finite planar Cayley graph. Droms [2] proved that finitely generated planar groups are finitely presented. This is a hint that a similar result is true for transitive planar graphs. Indeed, we shall show the following:

Theorem 1.1. *Let G be a locally finite transitive planar graph. Then the first homology group of G is a finitely generated* $\mathrm{Aut}(G)$*-module.*

Theorem 1.1 directly implies Droms's theorem. Whereas Droms's proof uses an accessibility result of Maskit [9] for planar groups, our self-contained proof does not. Instead, our proof will be based on Theorem 1.2. But before we state that theorem, we have to make some definitions first.

We call a graph *finitely separable* if no two distinct vertices are joined by infinitely many edge disjoint paths, or equivalently, any two vertices are separated by finitely many edges.

Let G be a planar graph with planar embedding $\varphi\colon G \to \mathbb{R}^2$. Two cycles C_1, C_2 in G are *nested* if no C_i has vertices or edges in distinct faces of $\varphi(C_{3-i})$. A set of cycles is *nested* if every two of its elements are nested.

Theorem 1.2. *Every 3-connected finitely separable planar graph has a canonical nested set of cycles generating the first homology group.*

Here, *canonical* means mostly that the set of cycles is invariant under the automorphisms of the graph. But in addition, our proof is constructive and this construction commutes with graph isomorphisms, i. e. whenever we run this construction for two isomorphic graphs G and H, then this isomorphism maps the set of cycles in G we obtain to that of H.

Note that Theorem 1.2 is easy to prove if the graph has no accumulation points in the plane, i. e. if it is *VAP-free*, as you may then take the finite face boundaries as generating set, see e. g. [5, Lemma 3.2].

Theorem 1.2 has various analogues in the literature: in [7] the author proved the corresponding result for the cycle space[1] of 3-connected finitely separable planar

[1]The *cycle space* of a graph is the set of finite sums of cycles over $\mathbb{F}_2$.

graphs and, previously, Dicks and Dunwoody [1] proved the analogous result for the cut space[2] of arbitrary graphs.

The mentioned theorem of Dicks and Dunwoody is one of the central theorems for the investigation of transitive graphs with more than one end and hence of accessible graphs and of accessible groups. (We refer to Section 6 for definitions.) Even though, accessibility has a priori more in common with the cut space than with the cycle space or the first homology group, the main result of [6] exhibited a connection between accessibility and the cycle space:

Theorem 1.3. [6] *Every transitive graph G whose cycle space is a finitely generated* $\mathrm{Aut}(G)$*-module is accessible.*

As an application of our results and Theorem 1.3 we shall obtain Dunwoody's [4] theorem that locally finite transitive planar graphs are accessible.

2. Indecomposable cycles

We call a cycle *indecomposable* if it is not generated in $H_1(G)$ by cycles of strictly smaller length. Note that no indecomposable cycle C has a *shortcut*, i.e. a path between any two of its vertices that has smaller length than their distance on C. Indeed, let P be a shortest shortcut of C and Q_1, Q_2 be the two subpaths of C whose end vertices are those of P. Then $Q_1 \cup P$ and $Q_2 \cup P$ sum to C if we run through P in different directions. (This is necessary as we are taking sums over $\mathbb{Z}$ for the first homology group.) For a path P and x, y on P, we denote by xPy the subpath of P from x to y.

Lemma 2.1. *Let G be a planar graph and let $C_1, C_2 \subseteq G$ be two indecomposable cycles of lengths n_1, n_2, respectively, that are not nested. Let $P_1 \subseteq C_1$ be a non-trivial path of shortest length that meets C_2 in precisely its end vertices. Let $P_2 \subseteq C_2$ be a shortest path with the same end vertices as P_1. Then one of the following is true.*

(i) $|P_1| = |P_2|$ *and P_2 meets C_1 only in its end vertices;*
(ii) $|P_1| \geq |P_2|$ *and* $P_1 \cup P_2 = C_1$*;*
(iii) $|P_1| \geq |P_2|$ *and* $(C_1 - P_1) \cup P_2 = C_2$*.*

Proof. Let v, w be the end vertices of P_1. Note that C_2 is the sum of vP_1wQ_1v and vQ_2wP_1v where Q_1 and Q_2 are the two subpaths of C_2 with end vertices v and w. First, assume $|P_1| < |P_2|$. By the choice of P_2, we have $|P_2| \leq |Q_1|$ and $|P_2| \leq |Q_2|$, so P_1 is a shortcut of C_2, which is impossible. Hence, we have $|P_1| \geq |P_2|$.

If $P_2 \subseteq C_1$, then we directly have $P_2 \cup P_1 = C_1$ and (ii) holds. So we may assume that P_2 contains an edge not on C_1.

Let us suppose that P_2 has an inner vertex on C_1. So any subpath xP_2y that intersects C_1 in precisely its end vertices has shorter length than P_1. Note that such a subpath exists as P_2 has an edge outside C_1. But xP_2y cannot be a shortcut of C_1. So the distance between x and y on C_1 is at most $|xP_2y|$. The subpath Q of C_1 realising the distance of x and y on C_1 together with xP_2y does not contain V and W, so it cannot be C_2. Thus, some edge of Q does not lie on C_2 and hence Q contains some subpath that contradicts the choice of P_1.

So P_2 meets C_1 only in its end vertices. Then C_1 is the sum of $C_1 - P_1 + P_2$ and $P_1 + P_2$. As C_1 is indecomposable, P_2 is not a shortcut of C_1 and thus we have

[2]The *cut space* of a graph is the set of finite sums over $\mathbb{F}_2$ of minimal separating edge sets.

either $|P_2| = |P_1|$ or $|P_2| = |C_1 - P_1|$. The first case implies (i) while, if the first case does not hold, we have $|P_1| > |P_2| = |C_1 - P_1|$. Thus, the minimality of $|P_1|$ implies that $C_1 - P_1$ lies on C_2. So we have $(C_1 - P_1) \cup P_2 = C_2$ as P_2 meets C_1 only in its end vertices. This shows (iii) in this situation. □

If C is a cycle in a planar graph G, we denote by f_C^0 the bounded face of C and by f_C^1 the unbounded face.

For two cycles $C, D \subseteq G$, we call a non-trivial maximal subpath P of C that has precisely its end vertices in D a *D-path in C*. By $n(C, D)$ we denote the number of C-paths in D.

Lemma 2.2. *Let G be a planar graph and let $C, D \subsetneq G$ be two indecomposable cycles. Then there are nested indecomposable cycles $\widetilde{C}$ and $\widetilde{D}$ with $|C| = |\widetilde{C}|$ and $|D| = |\widetilde{D}|$ that are either the boundaries of $f_C^0 \cap f_D^0$ and of $f_C^1 \cap f_D^1$ or the boundaries of $f_C^0 \cap f_D^1$ and $f_C^1 \cap f_D^0$.*

In addition, we may choose $\widetilde{C}$ and $\widetilde{D}$ so that, if $\mathcal{E}$ is a set of cycles generating all cycles of length smaller than $|C|$, then $\mathcal{E}$ generates C or D as soon as it generates $\widetilde{C}$ or $\widetilde{D}$.

Proof. If C and D are nested, then the assertion holds trivially. This covers the situation that $n(D, C)$ is either 0 or 1, as it implies that C and D are nested. Thus, C contains some smallest D-path P_1. Note that the cases (ii) and (iii) of Lemma 2.1 imply $n(D, C) = 1$. Hence, Lemma 2.1 implies that D contains a C-path Q_1 with the same end vertices as P_1 and with $|P_1| = |Q_1|$. By definition, neither P_1 nor Q_1 has an inner vertex of D or C, respectively. Let $D' := D$ and $C' := (C - P_1) \cup Q_1$. Inductively, we obtain two sequences $(P_i)_{i \leq n}$ and $(Q_i)_{i \leq n}$ of D-paths in C and C-paths in D, respectively, which are ordered by the length of the paths P_i. Note that – just as above – Lemma 2.1 ensures $|P_i| = |Q_i|$ for all but at most one $i \leq n$. (The case with $|P_i| \neq |Q_i|$ occurs if C and D are nested and either (ii) or (iii) of Lemma 2.1 holds.)

Consider a cyclic ordering of C and let $i_1, \dots, i_n \in \{1, \dots, n\}$ be pairwise distinct such that $P_{i_1}, \dots, P_{i_n}$ appear on C in this order. Then, using planarity, it immediately follows by their definitions as C-path or D-path, respectively, that $Q_{i_1}, \dots, Q_{i_n}$ appear in this order on D. Note that one face of $P_i \cup Q_i$ contains no vertices or edges of $C \cup D$. The assertion follows except for the fact that the obtained cycles are indecomposable and the additional statement.

Let $\mathcal{E}$ be a set of cycles generating in $H_1(G)$ all cycles of length smaller than $|C|$. Assume that the boundaries C' and D' of $f_C^0 \cap f_D^0$ and $f_C^1 \cap f_D^1$, respectively, have the desired property up to being indecomposable. Let us assume that C' is generated by $\mathcal{E}$. (Note that this covers also the case that C' is not indecomposable.)

If all cycles $P_i \cup Q_i$ have length less than $|C|$ and $|D|$, then we add every cycle $P_i \cup Q_i$ to C' for which Q_i lies on the boundary of $f_C^0 \cap f_D^0$ and we obtain C. So C is generated by $\mathcal{E}$ as C' and all of the added cycles are generated by $\mathcal{E}$.

If all but exactly one of the cycles $P_i \cup Q_i$ have length less than $|C|$ and $|D|$, then $P_n \cup Q_n$ has largest length of all those cycles. If P_n lies on the boundary of $f_C^0 \cap f_D^0$, then we add every cycle $P_i \cup Q_i$ to C' for which Q_i lies on the boundary of $f_C^0 \cap f_D^0$. As before, we obtain that C is generated by $\mathcal{E}$. If P_n lies on the boundary of $f_C^1 \cap f_D^1$, then we add every cycle $P_i \cup Q_i$ to C' for which P_i lies on the boundary of $f_C^0 \cap f_D^0$ and obtain D. So D is generated by $\mathcal{E}$.

If at least two cycles $P_i \cup Q_i$ have length at least $\min\{|C|, |D|\}$, then $n = 2$ follows immediately. Hence, also the boundaries C'' and D'' of $f_C^0 \cap f_D^1$ and $f_C^1 \cap f_D^0$, respectively, are cycles. So we may have chosen them instead of C' and D'. If one of them, C'' say, is generated by $\mathcal{E}$, too, then $C' + C''$ is generated by $\mathcal{E}$. As this sum is either C or D, the assertion follows. □

Note that it follows from the proof of Lemma 2.2 that there is a canonical bijection between the C-paths in D and the D-paths in C. In particular, we have $n(C, D) = n(D, C)$.

3. Counting crossing cycles

Our restiction to finitely separable graphs implies that each cycle in such a graph is nested with all but finitely many cycles of bounded length. In general, this is not true if we omit that assumption.

Proposition 3.1. *Let $i \in \mathbb{N}$. Every cycle in a finitely separable planar graph is nested with all but finitely many cycles of length at most i.*

Proof. Let us assume that some cycle C is not nested with infinitely many cycles of length at most i. Then there are two vertices x_1, x_2 of C that lie on infinitely many of these cycles and thus we obtain infinitely many distinct x_1–x_2 paths of length at most $i - 1$. Either there are already infinitely many edge disjoint x_1–x_2 paths or infinitely many share another vertex x_3. In the latter situation, there are either infinitely many distinct x_1–x_3 or x_2–x_3 paths of length at most $i - 2$. Continuing this process, we end up at some point with two distinct vertices and infinitely many edge disjoint paths between them, since we reduce the length of the involved paths in each step by at least 1. So we obtain a contradiction to finite separability. □

Let $\mathcal{E}$ be a set of cycles of length at most i in a finitely separable graph G and $C \subseteq G$ be a cycle. We define $\mu_{\mathcal{E}}(C)$ to be the number of cycles in $\mathcal{E}$ that are not nested with C. Note that Proposition 3.1 says that $\mu_{\mathcal{E}}(C)$ is finite. If $\mathcal{F}$ is another set of cycles of length at most i, we set $\mu_{\mathcal{E}}(\mathcal{F})$ as minimum over all $\mu_{\mathcal{E}}(C)$ with $C \in \mathcal{F}$.

Proposition 3.2. *Let G be a finitely separable planar graph. Let $\mathcal{E}$ be a set of cycles in G of length at most $i \in \mathbb{N}$ and let C, D be two indecomposable cycles in G that are not nested. Then we have*

$$\mu_{\mathcal{E}}(C) + \mu_{\mathcal{E}}(D) \geq \mu_{\mathcal{E}}(\widetilde{C}) + \mu_{\mathcal{E}}(\widetilde{D}),$$

where $\widetilde{C}$ and $\widetilde{D}$ are the cycles obtained by Lemma 2.2. Furthermore, if $D \in \mathcal{E}$, then the inequality is a strict inequality.

Proof. Using homeomorphisms of the sphere, we may assume that $\widetilde{C}$ is the boundary of $f_C^0 \cap f_D^0$ and $\widetilde{D}$ is the boundary of $f_C^1 \cap f_D^1$. Let $F \in \mathcal{E}$ be nested with C and D. We may assume that F avoids f_C^0. Thus, it is nested with $\widetilde{C}$. If F avoids f_D^0, too, then it lies in $f_C^1 \cap f_D^1$ with its boundary and is nested with $\widetilde{D}$. So let us assume that it avoids f_D^1. Thus, F does not contain any points of $f_C^1 \cap f_D^1$ and thus is nested with $\widetilde{D}$.

Now consider the case that $F \in \mathcal{E}$ is nested with C but not with D. We may assume that F avoids f_C^0. Hence, it avoids $f_C^0 \cap f_D^0$, too, and is nested with $\widetilde{C}$.

This shows that every $F \in \mathcal{E}$ that is not counted on the left side of the inequality is not counted on the right side either and that every $F \in \mathcal{E}$ that is counted on the left side precisely once is counted on the right side at most once. This shows the first part of the assertion.

To see the additional statement, just note that D is counted on the left for $\mu_{\mathcal{E}}(C)$ but not for $\mu_{\mathcal{E}}(D)$ and that both cycles $\widetilde{C}$ and $\widetilde{D}$ are nested with D. □

4. Finding a nested generating set

The main theorem of [7] says that the cycle space of any 3-connected finitely separable planar graph G is generated by some canonical nested set of cycles as $\mathbb{F}_2$-vector space. We will prove the analogous result for the first (simplicial) homology group $H_1(G)$ as module over $\mathbb{Z}$.

Throughout this section, let G be a 3-connected planar finitely separable graph.

Let $\mathcal{H}_i := \mathcal{H}_i(G)$ be the submodule of $H_1(G)$ generated by all cycles of length at most i. So $H_1(g) = \bigcup_{i\in\mathbb{N}} \mathcal{H}_i$. We shall recursively define canonical nested subsets $\mathcal{C}_i$ of $\mathcal{H}_i$ that generate $\mathcal{H}_i$ and consist only of indecomposable cycles of length at most i. So $\bigcup_{i\in\mathbb{N}} \mathcal{C}_i$ will generate $H_1(G)$. For the start, let $\mathcal{C}_i = \emptyset$ for $i \leq 2$. Let us assume that we already defined $\mathcal{C}_{i-1}$. Then we shall define $\mathcal{C}_i$ recursively.

In order to define $\mathcal{C}_i$, we construct another sequence of nested $\mathrm{Aut}(G)$-invariant sets $\mathcal{C}_i^\kappa$ of indecomposable cycles. Set $\mathcal{C}_i^0 := \mathcal{C}_{i-1}$. Let κ be some ordinal such that $\mathcal{C}_i^\lambda$ is defined for all $\lambda < \kappa$. If κ is a limit ordinal, then set $\mathcal{C}_i^\kappa = \bigcup_{\lambda<\kappa} \mathcal{C}_i^\lambda$. So let κ be a successor ordinal, say $\kappa = \nu + 1$. Any cycle of length i that is not generated by $\mathcal{C}_i^\nu$ must be indecomposable by definition of $\mathcal{C}_{i-1}$. If there is not such a cycle, set $\mathcal{C}_i := \mathcal{C}_i^\nu$. So in the following, we assume that there is at least one indecomposable cycle of length i that is not generated by $\mathcal{C}_i^\nu$. Hence, the set $\mathcal{D}_i^\kappa$ of all indecomposable cycles that are not generated by $\mathcal{C}_i^\nu$ and that have length i is not empty.

Lemma 4.1. *The set $\mathcal{D}_i^\kappa \neq \emptyset$ contains a cycle that is nested with $\mathcal{C}_i^\nu$.*

Proof. Let $C \in \mathcal{D}_i^\kappa$ with minimum $\mu_{\mathcal{C}_i^\nu}(C)$. We shall show $\mu_{\mathcal{C}_i^\nu}(C) = 0$. So let us suppose that C is not nested with some $D \in \mathcal{C}_i^\nu$. Since C and D are indecomposable cycles, we obtain by Lemma 2.2 two indecomposable cycles $\widetilde{C}$ and $\widetilde{D}$ with $|C| = |\widetilde{C}|$ and $|D| = |\widetilde{D}|$ such that Proposition 3.2 implies

$$\mu_{\mathcal{C}_i^\nu}(C) = \mu_{\mathcal{C}_i^\nu}(C) + \mu_{\mathcal{C}_i^\nu}(D) > \mu_{\mathcal{C}_i^\nu}(\widetilde{C}) + \mu_{\mathcal{C}_i^\nu}(\widetilde{D}).$$

Note that, if $\widetilde{C}$ and $\widetilde{D}$ are generated by $\mathcal{C}_i^\nu$, then C being the sum of D, $\widetilde{C}$, and $\widetilde{D}$ is generated by $\mathcal{C}_i^\nu$, too. But then it lies outside $\mathcal{D}_i^\kappa$. As this is not the case, either $\widetilde{C}$ or $\widetilde{D}$ is not generated by $\mathcal{C}_i^\nu$. In particular, this cycle must lie in $\mathcal{D}_i^\kappa$, a contradiction to the choice of C. □

Let $\mathcal{E}_i^\kappa$ be the set of all cycles in $\mathcal{D}_i^\kappa$ that are nested with $\mathcal{C}_i^\nu$. By Lemma 4.1, this set is not empty.

For a set $\mathcal{E}$ of cycles of length at most i, we call $C \in \mathcal{E}$ *optimally nested* in $\mathcal{E}$ if $\mu_{\mathcal{E}}(C) = \mu_{\mathcal{E}}(\mathcal{E})$. Note that $\mu_{\mathcal{E}}(\mathcal{E})$ is finite by Proposition 3.1 and, furthermore, as 3-connected planar graphs have (up to homeomorphisms) unique embeddings into the sphere due to Whitney [12] for finite graphs and Imrich [8] for infinite graphs, $\mu_{\mathcal{E}}(C) = \mu_{\mathcal{E}}(C\alpha)$ for all $\alpha \in \mathrm{Aut}(G)$.

Lemma 4.2. *The set $\mathcal{F}_i^\kappa$ of optimally nested cycles in $\mathcal{E}_i^\kappa$ is non-empty and nested.*

Proof. Since $\mathcal{E}_i^\kappa$ is non-empty, the same is true for $\mathcal{F}_i^\kappa$. Let us suppose that $\mathcal{F}_i^\kappa$ contains two cycles C, D that are not nested. Let $\widetilde{C}$ and $\widetilde{D}$ be the indecomposable cycles obtained by Lemma 2.2 with $|C| = |\widetilde{C}|$ and $|D| = |\widetilde{D}|$ each of which is not generated by $\mathcal{C}_i^\nu$ and such that Proposition 3.2 yields

$$\mu_{\mathcal{C}_i^\nu}(C) + \mu_{\mathcal{C}_i^\nu}(D) \geq \mu_{\mathcal{C}_i^\nu}(\widetilde{C}) + \mu_{\mathcal{C}_i^\nu}(\widetilde{D}).$$

As $\mathcal{E}_i^\kappa$ is nested with $\mathcal{C}_i^\nu$ by definition, we have $\mu_{\mathcal{C}_i^\nu}(C) + \mu_{\mathcal{C}_i^\nu}(D) = 0$. Note that $\widetilde{C}$ and $\widetilde{D}$ lie in $\mathcal{D}_i^\kappa$ by definition. As both are nested with $\mathcal{C}_i^\nu$, they lie in $\mathcal{E}_i^\kappa$. We apply Proposition 3.2 once more and obtain

$$\mu_{\mathcal{E}_i^\kappa}(C) + \mu_{\mathcal{E}_i^\kappa}(D) > \mu_{\mathcal{E}_i^\kappa}(\widetilde{C}) + \mu_{\mathcal{E}_i^\kappa}(\widetilde{D}).$$

Thus either $\widetilde{C}$ or $\widetilde{D}$ is not nested with less cycles in $\mathcal{E}_i^\kappa$ than C. This contradiction to the choice of C shows that $\mathcal{E}_i^\kappa$ is nested. □

So we set $\mathcal{C}_i^\kappa := \mathcal{C}_i^\nu \cup \mathcal{F}_i^\kappa$. Then $\mathcal{C}_i^\kappa$ is nested as $\mathcal{C}_i^\nu$ is nested and by the choice of $\mathcal{E}_i^\kappa$ all elements of $\mathcal{C}_i^\kappa$ are indecomposable.

This process will terminate at some point as we strictly enlarge the sets $\mathcal{C}_\kappa$ in each step but we cannot put in more cycles than there are in G. Let $\mathcal{C}_i$ be the union of all $\mathcal{C}_i^\kappa$. Note that we made no choices at any point, i. e. all sets $\mathcal{C}_i$ are $\mathrm{Aut}(G)$-invariant and canonical. Thus, we proved Theorem 1.2. More precisely, we have proved the following theorem.

Theorem 4.3. *For every finitely separable 3-connected planar graph G there is a sequence $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of sets of cycles in G such that*

(i) $\mathcal{C}_{i-1} \subseteq \mathcal{C}_i$;
(ii) $\mathcal{C}_i \smallsetminus \mathcal{C}_{i-1}$ *consists of indecomposable cycles of length i;*
(iii) $\mathcal{C}_i$ *generates $\mathcal{H}_i(G)$;*
(iv) $\mathcal{C}_i$ *is canonical and nested.*

Note that the only situation where we used 3-connectivity was when we concluded that we have $\mu_\mathcal{E}(C) = \mu_\mathcal{E}(C\alpha)$ for any cycle C, set $\mathcal{E}$ of cycles of bounded length and automorphism α. That is, the above proof also give us the existence of a nested generating set for lower connectivity, but we lose canonicity. Note that, in general, the statement of Theorem 1.2 is false if we do not require the graph to be 3-connected: let G be the graph obtained by two vertices joined by four internally disjoint paths of length 2. Then all cycles have length 4 and lie in the same $\mathrm{Aut}(G)$-orbit, but it is not hard to find two of them which are not nested. So you cannot find a canonical nested generating set of $H_1(G)$.

5. Finding a finite generating set

Let us introduce the notion of a degree sequence of orbits because the general idea to prove Theorem 1.1 will mainly be done by induction on this notion.

For a connected locally finite quasi-transitive graph G with $|V(G)| > 1$ we call a tupel $(d_1, \dots, d_m)$ of positive integers with $d_i \geq d_{i+1}$ for all $i < m$ the *degree sequence of the orbits of G* if for some set $\{v_1, \dots, v_m\}$ of vertices that contains precisely one vertex from each $\mathrm{Aut}(G)$-orbit the degree of v_i is d_i. We consider the lexicographic order on the finite tupels of positive integers (and thus on the degree sequences of orbits), that is, we set

$$(d_1, \dots, d_m) \leq (c_1, \dots, c_n)$$

if either $m \leq n$ and $d_i = c_i$ for all $i \leq m$ or $d_i < c_i$ for the smallest $i \leq m$ with $d_i \neq c_i$. Note that any two finite tupels of positive integers are $\leq$-comparable.

A direct consequence of this definition is the following lemma.

Lemma 5.1. *Any strictly decreasing sequence in the set of finite tupels of positive integers is finite.* □

We call a graph *quasi-transitive* if its automorphism group has only finitely many orbits on the vertex set.

Lemma 5.1 for quasi-transitive graphs and their degree sequences of orbits reads as follows and enables us to use induction on the degree sequence of the orbits of graphs:

Lemma 5.2. *Every sequence of locally finite quasi-transitive graphs whose corresponding sequence of degree sequences of the orbits is strictly decreasing is finite.* □

Lemma 5.3. *Let G be a locally finite quasi-transitive graph and let $S \subseteq V(G)$ and $H \subseteq G$ be such that the following conditions hold:*

(i) *$G - S$ is disconnected;*
(ii) *each $S\alpha$ with $\alpha \in \mathrm{Aut}(G)$ meets at most one component of $G - S$;*
(iii) *such that no vertex of S has all its neighbours in S;*
(iv) *H is a maximal subgraph of G such that no $S\alpha$ with $\alpha \in \mathrm{Aut}(G)$ disconnects H.*

Then the degree sequence of the orbits of H is smaller than the one of G.

Proof. First we show that all vertices in H that lie in a common $\mathrm{Aut}(G)$-orbit of G and whose degrees in G and in H are the same also lie in a common $\mathrm{Aut}(H)$-orbit. Let x, y be two such vertices and $\alpha \in \mathrm{Aut}(G)$ with $x\alpha = y$. Suppose that $H\alpha \neq H$. Then there is some $S\beta$ that separates some vertex of H from some vertex of $H\alpha$ by the maximality of H. But as y and all its neighbours lie in H and in $H\alpha$, they lie in $S\beta$, which is a contradiction to (iii). Thus, α fixes H and induces an automorphism of H that maps x to y.

We consider vertices x such that $\{x\} \cup N(x)$ lies in no $H\alpha$ with $\alpha \in \mathrm{Aut}(G)$ and such that x has maximum degree with this property. Let $\{x_1, \ldots, x_m\}$ be a maximal set that contains precisely one vertex from each orbit of those vertices. If x_i lies outside every $H\alpha$, then no vertex of its orbit is considered for the degree sequence of the orbits of H. If x_i lies in H, then its degree in some $H\alpha$ is smaller than its degree in G. By replacing x_i by $x_i\alpha^{-1}$, if necessary, we may assume $d_H(x_i) < d_G(x_i)$. So its value in the degree sequence of orbits of H is smaller than its value in the degree sequence of orbits of G; but it may be counted multiple times now as the $\mathrm{Aut}(G)$-orbit containing x_i may be splitted into multiple $\mathrm{Aut}(H)$-orbits. Nevertheless, the degree sequence of orbits of H is smaller than that of G. □

Remember that a *block* of a graph is a maximal 2-connected subgraph. As any cycle lies completely in some block and as any locally finite quasi-transitive graph has only finitely many orbits of blocks, we directly have:

Proposition 5.4. *For a locally finite quasi-transitive graph G, we have that $H_1(G)$ is a finitely generated $\mathrm{Aut}(G)$-module over $\mathbb{Z}$ if and only if the same is true for every block.* □

Remark 5.5. In the situation of Proposition 5.4 we can take the orbits of the cutvertices one-by-one and apply Lemma 5.3 for each such orbit. It follows recursively that each block has a smaller degree sequence of its orbits than the original graph.

For the reduction to the 3-connected case for graphs of connectivity 2, we apply Tutte's decomposition of 2-connected graphs into '3-connected parts' and cycles. Tutte [11] proved it for finite graphs. Later, it was extended by Droms et al. [3] to locally finite graphs.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t\in T}$ of vertex sets $V_t \subseteq V(G)$, one for each vertex of T, such that

(T1) $V = \bigcup_{t\in T} V_t$;
(T2) for every edge $e \in G$ there exists a $t \in V(T)$ such that both ends of e lie in V_t;
(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1–t_3 path in T.

The sets V_t are the *parts* of $(T, \mathcal{V})$ and the intersections $V_{t_1} \cap V_{t_2}$ for edges t_1t_2 of T are its *adhesion sets*; the maximum size of such a set is the *adhesion* of $(T, \mathcal{V})$. Given a part V_t, its *torso* is the graph with vertex set V_t and whose edge set is

$$\{xy \in E(G) \mid x, y \in V_t\} \;\cup\; \{xy \mid \{x, y\} \subseteq V_t \text{ lies in an adhesion set}\}.$$

The automorphisms of G act canonically on vertex sets of G. If every part of the tree-decomposition is mapped to another of its parts and this map induces an automorphism of T then we call the tree-decomposition $\mathrm{Aut}(G)$*-invariant*.

Theorem 5.6. [3, Theorem 1] *Every locally finite* 2*-connected graph* G *has an* $\mathrm{Aut}(G)$*-invariant tree-decomposition of adhesion* 2 *each of whose torsos is either* 3*-connected or a cycle or a complete graph on two vertices.* □

Remark 5.7. In addition to the conclusion of Theorem 5.6, we may assume that the tree-decomposition is such that the torsos of tree vertices of degree 2 are either 3-connected or cycles and that no two torsos of adjacent tree vertices t_1, t_2 are cycles if $V_{t_1} \cap V_{t_2}$ is no edge of G. (Remember that edges are two-element vertex sets.) We call a tree-decomposition as Theorem 5.6 with this additional property a *Tutte decomposition*.

Now we reduce the problem for finding a finite set of cycles generating $H_1(G)$ as $\mathrm{Aut}(G)$-module over $\mathbb{Z}$ from 2-connected graphs to 3-connected ones.

Proposition 5.8. *The cycles space of a locally finite quasi-transitive* 2*-connected graph* G *is a finitely generated* $\mathrm{Aut}(G)$*-module if and only if the same is true for each of its torsos in every Tutte decomposition.*

Proof. Let $(T, \mathcal{V})$ be a Tutte decomposition of G. Note that every vertex lies in only finitely many 2-separators (cf. [10, Proposition 4.2]). Thus, the graph H given by G together with all edges xy, where $\{x, y\}$ forms an adhesion set, is also locally finite and quasi-transitive. There are only finitely many orbits of (the action induced by) $\mathrm{Aut}(G)$ on T, since any 2-separator of G uniquely determines the parts V_t of $(T, \mathcal{V})$ it is contained in and since there are only finitely many $\mathrm{Aut}(G)$-orbits of 2-separators. Obviously, the restriction of H to any $V_t \in \mathcal{V}$ is the torso of V_t.

Let us assume that $H_1(G)$ is finitely generated as $\mathrm{Aut}(G)$-module and let $\mathcal{C}$ be a finite set of cycles generating it over $\mathbb{Z}$. Every $C \in \mathcal{C}$ can be written as the sum of (finitely many) induced cycles $C_1, \ldots, C_n$ in H. So the set $\mathcal{D}$ of all those C_i for all

$C \in \mathcal{C}$ generates $H_1(H)$ as $\mathrm{Aut}(H)$-module. Each of the cycles C_i lies in a unique part V_t of $(T, \mathcal{V})$ as they are induced and as every adhesion set in H is complete. Note that cycles which lie in the same $\mathrm{Aut}(G)$-orbit and in some V_t also lie in the same orbit with respect to the automorphisms of the torso G_t of V_t. Let $\mathcal{D}_t$ be the set of all cycles in $\mathcal{D}$ that lie in G_t. Let C be a cycle in G_t. Then it is the sum of $C_1, \ldots, C_n \in \mathcal{D}$. Since all $C_i \nsubseteq G_t$ sum to 0, those $C_i \subseteq G_t$ sum to C. Thus, $H_1(G_t)$ is a finitely generated $\mathrm{Aut}(G_t)$-orbit.

For the converse, let $H_1(G_t)$ for every torso G_t of $(T, \mathcal{V})$ be a finitely generated $\mathrm{Aut}(G_t)$-orbit and let $\mathcal{C}_t$ be a finite set of cycles in G_t that generates $H_1(G_t)$ as $\mathrm{Aut}(G_t)$-module. We may choose the sets $\mathcal{C}_t$ so that $\mathcal{C}_t = \mathcal{C}_{t'}\alpha$ if $\alpha \in \mathrm{Aut}(G)$ maps V_t to $V_{t'}$. Let $\mathcal{A}$ be a set of ordered adhesion sets (x, y) of $(T, \mathcal{V})$ consisting of one element for each $\mathrm{Aut}(G)$-orbit. For every $(x, y) \in \mathcal{A}$ with $xy \notin E(G)$ we fix an x–y path P_{xy} in G. Then $P_{xy} + xy$ is a cycle C_{xy} in H. If $xy \in E(G)$, let $P_{xy} = xy$ and, for later conveniences, $C_{xy} = \emptyset$. Note that for an adhesion set $\{x, y\}$ we may have fixed two distinct paths P_{xy} and P_{yx}. We canonically extend the definition of the paths P_{xy} and cycles C_{xy} to all ordered adhesion sets (x, y), i.e. if $(x, y) = (x', y')\alpha$ with $(x', y') \in \mathcal{A}$, set $P_{xy} := P_{(x'y')}\alpha$ and $C_{xy} := C_{(x'y')}\alpha$.

We notice that every orbit of cycles in G_t contains elements of only finitely many $\mathrm{Aut}(H)$-orbits since each of its cycles has the same length ℓ and by local finiteness and quasi-transitivity there are only finitely many orbits of cycles in H whose cycles have length ℓ. Note that there are only finitely many $\mathrm{Aut}(G)$-orbits and hence only finitely many $\mathrm{Aut}(H)$-orbits of parts of $(T, \mathcal{V})$. So the union $\mathcal{C}$ of all $\mathcal{C}_t$ is a set of cycles in H meeting only finitely many $\mathrm{Aut}(H)$-orbits and generating $H_1(H)$, as it has a generating set of induced cycles, each of those lies in some G_t and thus is generated by $\mathcal{C}$. For every $C \in \mathcal{C}$ let $\mathcal{C}_C$ be the set of all elements of $H_1(G)$ that are obtained from C by adding for the edges xy that form an adhesion set $\{x, y\}$ of $(T, \mathcal{V})$ the cycle C_{xy} in any possible way such that xy does not lie in the sum.[3] We can write each element of $\mathcal{C}_C$ as sum of cycles of G – possibly in more than just one way. Let $\mathcal{C}'_C$ be obtained from $\mathcal{C}_C$ by replacing each element by any cycle in G that occurs in any of the just mentioned sums. Let $\mathcal{C}' := \bigcup_{C \in \mathcal{C}} \mathcal{C}'_C$. Then each element in any $\mathcal{C}_C$ has a bounded number of edges and thus the cycles in $\mathcal{C}'$ have bounded length and lie in finitely many $\mathrm{Aut}(G)$-orbits. Obviously, every element of $\mathcal{C}_C$ can be generated by $\mathcal{C}'$.

To see that $\mathcal{C}'$ generates $H_1(G)$, let C be any cycle of G. Thus it is also a cycle of H and can be written as sum of cycles $C_1, \ldots, C_m \in \mathcal{C}$. Now let us fix for each edge xy on any of these C_i that form an adhesion set of $(T, \mathcal{V})$ an orientation (x, y) and thus a cycle C_{xy}. Let C'_i be the sum of C_i with all our fixed cycles C_{xy}. Then C'_i is an element of $H_1(G)$ since it contains no edge of $H \setminus G$. Note that $\sum_{i=1}^m C_i = \sum_{i=1}^m C'_i$ and that each C'_i lies in $\mathcal{C}_{C_i}$. So C can be generated by $\bigcup_{i=1}^m \mathcal{C}_{C_i}$ and thus by $\mathcal{C}'$. □

Remark 5.9. Unfortunately, we are not able to apply Lemma 5.3 directly for Proposition 5.8 to see that the torsos in a Tutte decomposition have a smaller degree sequence of orbits, as the orbits are not subgraphs of G. But as not both vertices of any adhesion set have degree 2, it is possible to follow the argument

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[3] Note that if C contains only one such edge xy, we may add either C_{xy} or (the negative of) C_{yx} and thus can obtain up to two possibilities. So if C has n edges that form adhesion sets of $(T, \mathcal{V})$, we may obtain up to 2^n possibilities.

of the proof of Lemma 5.3 for each of the finitely many orbits of the 2-separators one-by-one to see that each torso has a smaller degree sequence of orbits than G.

Now we are able to attack the general VAP-free case.

Proposition 5.10. *Let G be a locally finite quasi-transitive VAP-free planar graph. Then $H_1(G)$ is a finitely generated* $\mathrm{Aut}(G)$*-module.*

Proof. Due to Propositions 5.4 and 5.8, it suffices to show the assertion if G is 3-connected. As 3-connected planar graphs have (up to homeomorphisms) unique embeddings into the sphere, every automorphism of G induces a homeomorphism of the plane. So faces are mapped to faces and cycles that are face boundaries are mapped to such cycles. As G is quasi-transitive and locally finite, there are only finitely many $\mathrm{Aut}(G)$-orbits of finite face boundaries.

Every cycle in G determines an inner face and an outer face in the plane. The inner face contains only finitely many vertices as G is VAP-free. Hence, every cycle is the sum of all face boundaries of the faces that lie in its inner part in the plane and the assertion follows. □

Now we are able to prove a strengthened version of our main theorem, Theorem 1.1.

Theorem 5.11. *Let G be a locally finite quasi-transitive planar graph. Then $H_1(G)$ is a finitely generated* $\mathrm{Aut}(G)$*-module.*

Proof. Due to Propositions 5.4 and 5.8, we may assume that G is 3-connected and due to Proposition 5.10 we may assume that G is not VAP-free. Let $\varphi\colon G \to \mathbb{R}^2$ be a planar embedding of G. Let $\mathcal{C}$ be a non-empty $\mathrm{Aut}(G)$-invariant nested set of cycles that generates $H_1(G)$, which exists by Theorem 1.2. Since G is not VAP-free, there is some cycle C of G such that both faces of $\mathbb{R}^2 \setminus \varphi(C)$ contain infinitely many vertices of G. As $\mathcal{C}$ generates $H_1(G)$ as $\mathrm{Aut}(G)$-module, one of the cycles in $\mathcal{C}$ has the same property as C. Hence, we may assume $C \in \mathcal{C}$. In particular, $\{C\alpha \mid \alpha \in \mathrm{Aut}(G)\}$ is nested.

We consider maximal subgraphs H of G such that no $C\alpha$ with $\alpha \in \mathrm{Aut}(G)$ disconnects H. In particular, H is connected and for every $C\alpha$ with $\alpha \in \mathrm{Aut}(G)$ one of the faces of $\mathbb{R}^2 \setminus \varphi(C\alpha)$ is disjoint from H. Note that there are only finitely many $\mathrm{Aut}(G)$-orbits of such subgraphs H as we find in each orbit some element that contains vertices of C by maximality of H. Due to Lemma 5.3, the graph H has a strictly smaller degree sequence of its orbits than G as C disconnects G. As H is again a locally finite quasi-transitive planar graph, we conclude by induction on the degree sequence of the orbits of such graphs (cf. Lemma 5.2) with base case if G is VAP-free that $H_1(H)$ is a finitely generated $\mathrm{Aut}(H)$-module. Let $\mathcal{E}_H$ be an $\mathrm{Aut}(H)$-invariant set of cycles with only finitely many $\mathrm{Aut}(H)$-orbits generating $H_1(H)$.

There are only finitely many pairwise non-$\mathrm{Aut}(G)$-equivalent such subgraphs H. So let $\mathcal{H}$ be a finite set of such subgraphs consisting of one per $\mathrm{Aut}(G)$-orbit. Let

$$\mathcal{E} := \bigcup_{H \in \mathcal{H}} \bigcup_{\alpha \in \mathrm{Aut}(G)} \mathcal{E}_H \alpha.$$

Then $\mathcal{E}$ is $\mathrm{Aut}(G)$-invariant and has only finitely many orbits. We shall show that $\mathcal{E}$ generates $H_1(G)$.

Let D be a cycle of G. If D lies entirely inside some of the subgraphs $H \in \mathcal{H}$ or its $\mathrm{Aut}(G)$-images, then, obviously, it is generated by $\mathcal{E}$. So let us assume that there is some $\alpha \in \mathrm{Aut}(G)$ such that both faces of $C\alpha$ contain vertices or edges of D. By considering $D\alpha^{-1}$ instead of D, we may assume $\alpha = id$. We add all edges of C to D that lie in the bounded face of D to obtain a subgraph F of G. Then D is the sum over all boundaries $C_1, \ldots, C_k$ of bounded faces of F.

Assume that $C\beta$ with $\beta \in \mathrm{Aut}(G)$ is not nested with C_i and suppose that it is nested with D. Remember that C and $C\beta$ are nested. Since $C\beta$ contains points in both faces of C_i, there is some (possibly trivial) common path P of C_i and $C\beta$ such that the edges on $C\beta$ incident with the end vertices of P lie in different faces of C_i and also the edges of C_i incident with the end vertices of P lie in different faces of $C\beta$. As $C\beta$ is nested with C and with D, one of these edges belongs to C and the other to D. Thus, C and D must lie in distinct faces of $C\beta$ and hence must be nested. This contradiction shows that every $C\beta$ that is not nested with C_i is not nested with D either.

As C is not nested with D but with every C_i, every C_i is not nested with less cycles $C\beta$ than D and this is a finite number by Proposition 3.1 as all cycles $C\beta$ have the same length. Induction on the number of cycles $C\beta$ the current cycle is not nested with implies that each C_i is generated by $\mathcal{E}$ and so is D. □

6. Applications

Droms [2] proved that planar groups are finitely presented. His proof uses an accessibility result of Maskit [9]. We prove his result without any accessibility result. But before, we recall the following well-known lemma.

Lemma 6.1. *Let Γ be a Cayley graph of the group G with presentation $\langle S \mid R\rangle$. Then the set of walks in Γ induced by relators in R generates $H_1(\Gamma)$.*

Conversely, if R' is a set of relations of G over S such that the set of closed walks of Γ induced by R' generates $H_1(\Gamma)$, then $\langle S \mid R'\rangle$ is a presentation of G. □

As an application of Theorem 1.1 we obtain a self-contained proof of Droms's result.

Theorem 6.2. [2] *Every finitely generated planar group is finitely presented.* □

A *ray* is a one-way infinite path and two rays are *equivalent* if they lie in the same component whenever we remove a finite vertex set. This is an equivalence relation whose classes are the *ends* of the graph. We call a quasi-transitive graph *accessible* if there is some $n \in \mathbb{N}$ such that any two ends can be separated by removing at most n vertices.

In [6] the author proved the following accessibility result for quasi-transitive graphs.

Theorem 6.3. [6, Theorem 3.2] *Every quasi-transitive graph G whose cycle space is a finitely generated $\mathrm{Aut}(G)$-module is accessible.*

As a further corollary of Theorem 5.11 together with Theorem 6.3, we obtain Dunwoody's theorem of the accessibility of locally finite quasi-transitive planar graphs, a strengthened version of Theorem 1.3. (Note that any generating set of the first homology group of a graph is also a generating set of its cycle space.)

Theorem 6.4. [4] *Every locally finite quasi-transitive planar graph is accessible.* □

Note that, in order to prove Theorem 6.4, we do not need the full strength of a nested canonical generating set for the first homology group. Indeed, instead of applying Theorem 1.2, we could just do the same arguments as in Section 5 using a nested canonical generating set for the cycle space obtained from [7, Theorem 1] to obtain a finite set of cycles generating the cycle space as module.

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References

1. W. Dicks and M.J. Dunwoody, *Groups Acting on Graphs*, Cambridge Stud. Adv. Math., vol. 17, Cambridge Univ. Press, 1989.
2. C. Droms, *Infinite-ended groups with planar Cayley graphs*, J. Group Theory **9** (2006), 487–496.
3. C. Droms, B. Servatius, and H. Servatius, *The structure of locally finite two-connected graphs*, Electron. J. of Comb. **2** (1995), research paper 17.
4. M.J. Dunwoody, *Planar graphs and covers*, preprint, 2007.
5. A. Georgakopoulos, *Characterising planar Cayley graphs and Cayley complexes in terms of group presentations*, European J. Combin. **36** (2014), 282–293.
6. M. Hamann, *Accessibility in transitive graphs*, submitted, arXiv:1404.7677.
7. ———, *Generating the cycle space of planar graphs*, Electron. J. of Comb. **22** (2015), #2.34.
8. W. Imrich, *On Whitney's theorem on the unique imbeddability of* 3*-connected planar graphs*, Recent Advances in Graph Theory (M. Fiedler, ed.), Academia Praha, Prague, 1975, pp. 303–306.
9. B. Maskit, *A theorem on planar covering surfaces with applications to* 3*-manifolds*, Ann. of Math. **81** (1965), 341–355.
10. C. Thomassen and W. Woess, *Vertex-transitive graphs and accessibility*, J. Combin. Theory (Series B) **58** (1993), no. 2, 248–268.
11. W.T. Tutte, *Graph Theory*, Cambridge University Press, Cambridge, 1984.
12. H. Whitney, *Congruent graphs and the connectivity of graphs*, American J. of Mathematics **54** (1932), no. 1, 150–168.

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