Peripheral circuits in infinite binary matroids

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Abstract

A *peripheral circuit* in a (possibly infinite) matroid M is a circuit C of M such that M/C is connected. In the case of a 3-connected graph G, this is equivalent to C being chordless and $G - V(C)$ being connected. Two classical theorems of Tutte assert that, for a 3-connected graph G: (i) every edge e of G is in two peripheral cycles that intersect just on e and its incident vertices; and (ii) the peripheral cycles generate the cycle space of G [12].

Bixby and Cunningham generalized these to binary matroids, with (i) requiring a small adaptation. Bruhn generalized (i) and (ii) to the Freudenthal compactification of a locally finite graph. We unify these two generalizations to "cofinitary, binary B-matroids". (Higgs introduced the B-matroid as an infinite matroid; recent works show this should now be accepted as the right notion of infinite matroid. Cofinitary means every cocircuit is finite.)
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1 Introduction

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Bixby and Cunningham [1] generalized these to binary matroids, with (i) requiring a small adaptation. Bruhn [4] generalized (i) and (ii) to the Freudenthal compactification of a locally finite graph. We unify these two generalizations to "cofinitary, binary B-matroids". (Higgs [8] introduced the notion of B-matroid as an infinite matroid; in view of [5], this should now be accepted as the right notion of infinite matroid. Cofinitary means every cocircuit is finite; binary means no $U_{2,4}$-minor.)

All these previous results are proved by considering the "C-bridges" and proving that, from any circuit C that is not peripheral, there is a circuit C' so that the C'-bridges are

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"better" than the C-bridges. For example, to prove that every edge e in a 3-connected graph is in a peripheral cycle, we start with any circuit C containing e, and fix a C-bridge B. If there is a second C-bridge B', then a new circuit C' can be found so that, for some C'-bridge $\widehat{B}$, $B \subsetneq \widehat{B}$. Our proof takes the same tack, with many aspects being modifications of Bruhn's argument.

The next section introduces infinite matroids and Section 3 presents several basic facts about them that we need. Section 4 introduces C-bridges of a subset C of the elements of a matroid; these are the foundation for all proofs about peripheral cycles. Section 5 gives the proof of Theorem 2.3, while Section 6 discusses the specialization of our result to graph-like continua, giving a positive answer to an open question from Bruhn [4]: his theorem generalizes perfectly to a very natural setting.

This article was originally prepared by Christian and Richter. Bowler was the referee, and, over the course of several iterations of reviewing the article, made many substantial suggestions for improving the accuracy, scope, and style of the article. This final version is very different, and much better, than the original. Therefore, after the acceptance of the article for publication, the original authors invited the referee to be a co-author.

2 Matroids

sc:matroids

In this section, we introduce the basic notions of a matroid and precisely state our main result. The definition here is Oxley's characterization of B-matroid [9].

Definition 2.1 *A* matroid *is an ordered pair $(S, \mathcal{I})$ consisting of a set S and a set $\mathcal{I}$ of subsets of S satisfying:*

it:notEmpty *(M1)* $\varnothing \in \mathcal{I}$;

it:subsets *(M2) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;*

it:maximal *(M3) if $I \subseteq X \subseteq S$ and $I \in \mathcal{I}$, then there is a maximal subset J of X so that $I \subseteq J$ and $J \in \mathcal{I}$ (J is a* base *of X);*

it:exchange *(M4) if $X \subseteq S$ and $B_1, B_2 \subseteq X$ are two bases of X, then, for any $x \in B_1 \setminus B_2$, there is a $y \in B_2 \setminus B_1$ so that $(B_1 \setminus \{x\}) \cup \{y\}$ is a base of X.*

We remark that, when X is finite, (M3) is a triviality. For infinite X, this condition is required for the existence of bases. Also, this is a slightly different version of the independence axioms of [5]. Axiom (I3) from [5] applies to all non-maximal independent sets, whereas (M4) applies only to those of the form $B \setminus x$, for B a base.

As usual, let M denote the matroid $(S, \mathcal{I})$. The *dual of M* is the matroid (Higgs proved this) $(S, \mathcal{J})$, where the maximal elements of $\mathcal{J}$ are precisely the complements of the maximal elements of $\mathcal{I}$. The dual of M is denoted M^* and, evidently, $(M^*)^* = M$. A subset X of S is *independent* if $X \in \mathcal{I}$ and *dependent* if $X \notin \mathcal{I}$. A *circuit* of M is minimally dependent in $\mathcal{I}$ and a *cocircuit* if X is a circuit of M^*. The matroid M is *finitary* if each of its circuits is finite and *cofinitary* if each of its cocircuits is finite.

The matroid $M = (S, \mathcal{I})$ is *connected* if, for every partition (X, Y) of S, there is a circuit C of M so that $C \cap X$ and $C \cap Y$ are both non-empty. More generally, let

(X,Y) be a partition of S and let B_X and B_Y be bases of X and Y, respectively. Let $d_{(X,Y)}(B_X, B_Y) = \min\{|F| : F \subseteq X \text{ and } (B_X \cup B_Y) \setminus F \in \mathcal{I}\}$. The following facts are proved by Bruhn and Wollan [6, Lemmas 3 and 14].

Lemma 2.2 ([6]) *Let B and B' be bases of a matroid $M = (S, \mathcal{I})$, let (X,Y) be a partition of S, and let B_X and B_Y be bases of X and Y, respectively.*

1. *If $|B \setminus B'| < \infty$, then $|B \setminus B'| = |B' \setminus B|$.*
2. *If $F \subseteq B_X$ and $(B_X \cup B_Y) \setminus F$ is a base for M, then $d_{(X,Y)}(B_X, B_Y) = |F|$.*
3. *$d_{(X,Y)}(B_X, B_Y) = d_{(Y,X)}(B_Y, B_X)$.*
4. *If B'_X and B'_Y are bases of X and Y, respectively, then*
$$d_{(X,Y)}(B_X, B_Y) = d_{(X,Y)}(B'_X, B'_Y).$$ ■

This lemma allows us to unambiguously introduce the connectivity function $\lambda_M(X) = d_{(X,S\setminus X)}(B_X, B_{S\setminus X})$, where B_X and $B_{S\setminus X}$ are bases of X and $S \setminus X$, respectively. For a positive integer k, the matroid is *k-connected* if, for every subset X of S with both $|X| \geq k$ and $|S \setminus X| \geq k$, then $\lambda_M(X) \geq k$.

A matroid M is *binary* if it does not have $U_{2,4}$ as a minor. If M is binary, the *cycle space* $\mathcal{Z}(M)$ of M is the set of all subsets of M that are edge-disjoint unions of circuits of M. We shall see in the next section that $\mathcal{Z}(M)$ is closed under symmetric differences, and so is a vector space. We will need something slightly stronger than this.

Let T be a set of elements in a matroid M. A *T-bridge* in M is a component (that is, a maximal connected submatroid) of M/T. The set T is *peripheral* if there is only one T-bridge.

main **Theorem 2.3** *Let M be a 3-connected, binary, cofinitary matroid with ground set S.*

it:exist 1. *For distinct $e, f \in S$, there is a peripheral circuit in M containing e and not containing f.*

it:span 2. *If M is countable, then the peripheral circuits generate the cycle space of M.*

Some condition on M, in addition to being 3-connected and binary, is required for the conclusions in Theorem 2.3. Bruhn [4] gives some discussion of limitations (see the end of P. 239 and Figures 3 and 4). It would be interesting to have the right hypothesis here. It would also be interesting to know if Theorem 2.3 (2) holds in case the matroid is also uncountable.

We note that Bruhn asks if his version of Theorem 2.3 for the Freudenthal compactification of a locally finite graph generalizes. Theorem 2.3 does so to matroids; we discuss briefly in Section 6 that his theorem generalizes to graph-like continua. This generalization is an easy consequence of our matroidal result.

3 Elementary facts

sc:matroidFacts

In this section, we provide some elementary properties of circuits, cocircuits, T-bridges, and binary matroids. Our first elementary result is the following useful fact about T-bridges.

lm:Tbridge

Lemma 3.1 *Let T and T' be sets of elements of a matroid M and let B be a T-bridge in M. If $B \cap T' = \varnothing$, then there is a T'-bridge B' in M such that $B \subseteq B'$.*

Proof. Since B is a T-bridge, $(B \cup T)/T$ is connected. Thus, any two elements of B lie in a cocircuit of $(B \cup T)/T$, and so in a cocircuit of M. Since $B \cap T' = \varnothing$, any two elements of B lie in a cocircuit of $(B \cup T')/T'$, so B is contained in a T'-bridge. ■

Our next lemma gives a necessary condition for a matroid of finite rank to be binary. Recall that, for a matroid M, the *simplification* $\mathrm{si}(M)$ of M is obtained from M by removing all loops and all but one element from each parallel class in M.

lm:simpleU24

Lemma 3.2 *Suppose M is a matroid with finite rank. Then either* $\mathrm{si}(M)$ *is finite or M has a $U_{2,4}$-minor.*

Proof. Let B be a base of $\mathrm{si}(M)$ (and so of M). For each element e of $\mathrm{si}(M)$ not in B, let C_e denote the unique circuit in $B \cup \{e\}$. We first show that, if there are distinct elements e, f of $\mathrm{si}(M)$ for which $C_e \setminus \{e\} = C_f \setminus \{f\}$, then M has a $U_{2,4}$-minor. Set $B_{e,f} = C_e \setminus \{e\} = C_f \setminus \{f\}$.

For each $x \in B_{e,f}$, there is a circuit $C_x \subseteq (C_e \cup C_f) \setminus \{x\}$. Note that $e \in C_x$, as otherwise, $C_x \subseteq (B \cup \{f\}) \setminus \{x\}$ and $x \in C_f \setminus C_x$, contradicting the uniqueness of the circuit in $B \cup \{f\}$. Likewise $f \in C_x$.

Since $\{e, f\}$ is independent, there is a $y \in C_x \setminus \{e, f\}$. Then $y \in (C_e \cup C_f) \setminus \{e, f\} = B_{e,f}$, so $y \in C_e \cap C_f$. Therefore, there is a circuit $C_y \subseteq (C_e \cup C_f) \setminus \{y\}$.

Note that $y \in C_x \setminus C_y$, so there is some z in $C_y \setminus C_x$. Evidently, e, f, y, z are all distinct. Let N be the minor of M obtained by deleting all the elements of M not in $C_e \cup C_f$ and then contracting all the elements of $(C_e \cup C_f) \setminus \{e, f, x, z\}$. Thus, the ground set of N is $\{e, f, x, z\}$.

Notice that $C_e \cap \{e, f, x, z\} = \{e, x, z\}$, $C_f \cap \{e, f, x, z\} = \{f, x, z\}$, $C_x \cap \{e, f, x, z\} = \{e, f, z\}$, and $C_y \cap \{e, f, x, z\} = \{e, f, z\}$. It follows that all four 3-subsets of $\{e, f, x, z\}$ are dependent in N.

On the other hand, because each is properly contained in either C_e or C_f, the sets $\{e, y\}$, $\{e, z\}$, $\{f, y\}$, $\{f, z\}$, and $\{y, z\}$ are all independent in N. The proof that N is $U_{2,4}$ is completed by showing that $\{e, f\}$ is independent in N. Otherwise, it is a circuit in N, so there is a circuit C contained in $(C_e \cup C_f) \setminus \{y, z\}$; as above, C necessarily contains e and f. Therefore, there is a circuit contained in $(C_x \cup C) \setminus \{e\}$. But this is contained in $C_f \setminus \{z\}$, which is an independent set.

Thus, if there are elements e, f of $\mathrm{si}(M)$ for which $C_e \setminus \{e\} = C_f \setminus \{f\}$, then M contains $U_{2,4}$. Otherwise, each element of $\mathrm{si}(M)$ distinctly determines one of the finitely many subsets of B, so $\mathrm{si}(M)$ has the finitely many elements of B plus finitely many elements not in B. ■

To move a little deeper into the theory, we have the following general paradigm. For a set $\mathcal{X}$ of matroids, $\mathrm{ex}(\mathcal{X})$ denotes the set of all matroids not having an element of $\mathcal{X}$ as a minor. This and the subsequent lemma were developed in joint work with Paul Wollan.

lm:finiteMinor **Lemma 3.3** *Let $\mathcal{X}$ be a set of finite matroids. Let $\mathcal{P}$ be a minor-closed set of matroids. If: (i) the finite matroids in $\mathrm{ex}(\mathcal{X})$ are the same as the finite matroids in $\mathcal{P}$; and (ii) for every matroid $M \notin \mathcal{P}$, there is a finite minor N of M so that $N \notin \mathcal{P}$, then $\mathcal{P} = \mathrm{ex}(\mathcal{X})$.*

Proof. Since $\mathcal{P}$ is closed under minors, if $M \in \mathcal{P}$, then any finite minor N of M is also in $\mathcal{P}$. Hypothesis (i) implies N is also in $\mathrm{ex}(\mathcal{X})$. Since each element of $\mathcal{X}$ is finite, we conclude that $M \in \mathrm{ex}(\mathcal{X})$.

Conversely, if $M \in \mathrm{ex}(\mathcal{X})$, then no finite minor of M is in $\mathcal{X}$. Therefore, each finite minor of M is in $\mathcal{P}$. Hypothesis (ii) implies that M is in $\mathcal{P}$. ■

The immediate application of Lemma 3.3 is the following characterization of binary matroids. We remark that these are only some of the known characterizations for finite binary matroids. A more comprehensive discussion can be found in [3, Sec. 3].

haracterization **Lemma 3.4** *The following are equivalent for a matroid M:*

it:u24 1. *M does not have $U_{2,4}$ as a minor;*

it:even 2. *for every circuit C and cocircuit K of M, $|C \cap K|$ is either infinite or even; and*

it:not3 3. *for every circuit C and cocircuit K of M, $|C \cap K| \neq 3$.*

Proof. These statements are known to be equivalent for finite matroids [10]. We will first show:

:yesDownToMinor **Claim 1** *If M is a matroid satisfying either (1), (2), or (3), then every minor of M satisfies the same statement.*

Proof. We note that the claim is obvious for Statement 1. Let N be the minor of M obtained by deleting the elements in D and contracting the independent set I: $N = M/I \setminus D$. Suppose N does not satisfy Statement 2 (respectively, Statement 3). Then there is a circuit C and a cocircuit K of N such that $|C \cap K|$ is odd (respectively, equal to 3). There is a subset I_C of I so that $C \cup I_C$ is a circuit of M and there is a subset D_K of D such that $K \cup D_K$ is a cocircuit of M. Evidently, $|(C \cup I_C) \cap (K \cup D_K)| = |C \cap K|$ is odd (respectively, equal to 3), showing that M does not satisfy Statement 2 (respectively, 3). □

The proof for the equivalence of (2) and (3) is essentially completed by proving the next claim.

wnToFiniteMinor **Claim 2** *If M is a matroid that does not satisfy either (2) or (3), then either M has $U_{2,4}$ as a minor or some finite minor of M does not satisfy the same statement.*

Proof. Suppose that M does not satisfy Statement 2 (respectively, 3). Then there is a circuit C and a cocircuit K of M such that $|C \cap K|$ is odd (respectively, equal to 3). In particular, $|C \cap K|$ is finite and non-empty.

Letting x be any element of $C \cap K$, we see that $C \setminus \{x\}$ is independent and so extends to a base B of M that is disjoint from the coindependent set $K \setminus C$. Set $N = M/(B \setminus (C \cap K))$.

Since $(C \cap K) \setminus \{x\}$ is a base for N, N has finite rank. Also, K is disjoint from the set $B \setminus (C \cap K)$ of contracted elements, so K is a cocircuit of N. Since C is the only circuit in $B \cup \{x\}$, $C \cap K$ is the contraction of C and so is dependent in N. On the other hand, every proper subset of $C \cap K$ is independent in N, so $C \cap K$ is a circuit in N. Evidently, $|(C \cap K) \cap K| = |C \cap K|$. Since N is a finite rank minor of M, Lemma 3.2 shows that $\mathrm{si}(N)$ either has a $U_{2,4}$-minor or is finite. If $\mathrm{si}(N)$ is finite, then some finite minor of N has a circuit and cocircuit with intersection odd (respectively, equal to 3) and so has a $U_{2,4}$-minor. □

To complete the proof of the lemma, we first consider the equivalence of Statements 1 and 2; the equivalence with (3) is similar. If $M \in \mathrm{ex}(\{U_{2,4}\})$, then no finite minor of M has a circuit C and a cocircuit K such that $|C \cap K|$ is odd. Claim 2 shows that M has no such circuit and cocircuit, as required.

Conversely, suppose M satisfies Statement 2. Then Claim 1 shows every minor of M satisfies Statement 2. From the equivalence for finite matroids, no finite minor of M has a $U_{2,4}$-minor, and, therefore, M has no $U_{2,4}$ minor. ■

For two sets A, B, the set $A \triangle B$ is the symmetric difference $(A \cup B) \setminus (A \cap B)$ of A and B. It is not clear to us that the cofinitary assumption is required for the first assertion of the following corollary. This form is sufficient for our purposes.

co:symmDiff

Corollary 3.5 *Let M be a matroid. If M is cofinitary and binary, then, for every pair of distinct circuits C_1, C_2 of M, $C_1 \triangle C_2$ is dependent.*

If M is not binary, then M has two circuits C_1 and C_2 such that $C_1 \triangle C_2$ is independent.

Proof. For the first assertion, every cocircuit K of M is finite. Since M is binary, Lemma 3.4 (2) implies that, for any two circuits C_1 and C_2 of M, $|K \cap C_1|$ and $|K \cap C_2|$ are both even. Thus, for every cocircuit K of M, $|K \cap (C_1 \triangle C_2)|$ is even, whence $C_1 \triangle C_2$ is dependent.

For the second assertion, let I be independent in M and let D be disjoint from I so that $M/I \setminus D$ is a $U_{2,4}$ minor of M. Let x_1, x_2, x_3, x_4 be its four elements and let C_1 and C_2 be the circuits x_1, x_3, x_4 and x_2, x_3, x_4, respectively. Since $\{x_1, x_2\}$ is independent in $U_{2,4}$, $\{x_1, x_2\} \cup I$ is independent in M.

On the other hand, for $i = 1, 2$, there is a subset I_i of I such that $C_i \cup I_i$ is a circuit in M. But now $(C_1 \cup I_1) \triangle (C_2 \cup I_2) \subseteq \{x_1, x_2\} \cup I$ is independent. ■

The proof of our next lemma is based on Vella and Richter [14, Thm. 14].

lm:vellaRichter

Lemma 3.6 *Let $M = (S, \mathcal{I})$ be a cofinitary, binary matroid, and $X \subseteq S$. If, for every cocircuit K of M, $|X \cap K|$ is even, then X is a disjoint union of circuits of M.*

Proof. Let $\mathcal{X}$ denote the set of all sets of disjoint circuits contained in X, ordered by inclusion.

To apply Zorn's Lemma, let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots$ be an increasing sequence of elements of $\mathcal{X}$. Set $\mathcal{C} = \cup_{i \geq 1} \mathcal{C}_i$, so $\mathcal{C}$ is a set of circuits, all contained in X. Moreover, any two circuits in $\mathcal{C}$ are, for some $i \geq 1$, both in $\mathcal{C}_i$ and, therefore, are disjoint. Thus, $\mathcal{C} \in \mathcal{X}$, so Zorn's Lemma implies $\mathcal{X}$ has a maximal element, which we also denote by $\mathcal{C}$.

Set $X' = X \setminus (\cup_{C \in \mathcal{C}} C)$ and observe that the maximality of $\mathcal{C}$ shows X' does not contain any circuit of M, and so is independent. On the other hand, X' has even intersection with every cocircuit of M. Unless $X' = \varnothing$, the preceding two sentences are contradictory, as every non-empty independent set meets some cocircuit in just one element. ■

The following is an immediate corollary of Lemmas 3.4 and 3.6.

ryUnionCircuits

Corollary 3.7 *Let $M = (S, \mathcal{I})$ be a cofinitary, binary matroid.*

it:circInCtr

(3.7.1) If F is a subset of S and C is a circuit of M, then $C \setminus F$ is a disjoint union of circuits of M/F.

disjtUnionCircs

(3.7.2) If C_1 and C_2 are disjoint unions of circuits of M, then $C_1 \triangle C_2$ is a disjoint union of circuits of M. ■

As introduced earlier, the cycle space $\mathcal{Z}(M)$ of a binary matroid M is the set of all edge-disjoint unions of cycles. Let $\mathcal{A}$ be a set of sets of elements of M. Then $\mathcal{A}$ is *thin* if, for every element e of M, e is in only finitely many elements of $\mathcal{A}$. A principal property of a thin family is that the symmetric difference of its elements is well-defined. The following fact is central to the cycle space theory for infinite graphs and generalizes 3.7.2.

lm:thin

Lemma 3.8 *Let M be a cofinitary, binary matroid and let $\mathcal{A}$ be a thin set of elements of $\mathcal{Z}(M)$. Then the symmetric difference of the elements of $\mathcal{A}$ is in $\mathcal{Z}(M)$.*

Proof. Let K be any cocircuit of M. Since M is cofinitary, K is finite. Because $\mathcal{A}$ is thin, only finitely many elements of $\mathcal{A}$ have non-empty intersection with K. Letting $\bigoplus \mathcal{A}$ denote the symmetric difference of all the elements of $\mathcal{A}$, $K \triangle (\bigoplus \mathcal{A})$ is actually the symmetric difference of K with finitely many disjoint unions of circuits.

It follows from Lemma 3.4 (2) that $K \triangle (\bigoplus \mathcal{A})$ is even. Now Lemma 3.6 implies $\bigoplus \mathcal{A}$ is a disjoint union of circuits, as required. ■

4 Bridges in a matroid

sc:bridges

In this section we present the elements we require about bridges in a matroid. The main point of this section is the following result, proved at the end of the section, that is quite familiar for graphs.

th:bridgeGrow

Theorem 4.1 *Let M be a 3-connected, binary, cofinitary matroid, let C be a non-peripheral circuit of M, and let B be a C-bridge. Then either:*

it:twoCircuits

1. *there exist circuits C_1 and C_2 in M such that $C = C_1 \triangle C_2$ and, for $i \in \{1, 2\}$, there is a C_i-bridge B_i that properly contains B; or*

t:threeCircuits

2. *there exist circuits C_1, C_2, and C_3 in M so that $C = C_1 \triangle C_2 \triangle C_3$ and, for $i \in \{1, 2, 3\}$, there is a C_i-bridge B_i that properly contains B.*

For a circuit C in a finite graph G, a *residual arc of a C-bridge B* is one of the paths in C joining cyclically consecutive attachments of B. In the case of infinite circuits, this is not quite so easily defined. Here is a definition for matroids.

df:segments **Definition 4.2** *Let C be a circuit in a matroid M and let B be a C-bridge.*

1. *A B-*segment *is a series class of the restriction $M|_{C\cup B}$ that is contained in C.*

2. *If B' is a second C-bridge, then B and B'* avoid *each other if C is the union of a B-segment and a B'-segment. Otherwise, B and B'* overlap.

3. *The* overlap diagram *$\mathcal{O}(C)$ is the graph that has as its vertices the C-bridges, with distinct C-bridges adjacent if they overlap.*

4. *A* primary arc *in B is a circuit of M/C contained in B and not a circuit of M. If A is a primary arc of B, then a* primary segment for *A is a subset D of C such that $A \cup D$ is a circuit of M.*

The following technical fact will be helpful for showing that the overlap diagram of a circuit in a 3-connected, binary, cofinitary matroid is connected.

lm:2separation **Lemma 4.3** *Let C be a circuit in a matroid M and let B be a C-bridge.*

t:unionSegments
1. *If D is a primary segment for a primary arc A in B, then D is a union of B-segments.*

t:not3connected
2. *Suppose C is the disjoint union of the non-empty sets S_1 and S_2 such that, for every C-bridge B, there is a B-segment that contains one of S_1 and S_2. If M has at least 4 elements and, for each $i = 1, 2$, either $|S_i| \geq 2$ or there is a C-bridge B such that S_{3-i} is not contained in a B-segment, then M is not 3-connected.*

Proof. For (1), $A \cup D$ is a circuit in the restriction of M to $B \cup C$, so every series class in $B \cup C$ is either contained in $A \cup D$ or is disjoint from $A \cup D$.

For (2), if $|C| \leq 2$, then, for E the ground set of M, $(C, E \setminus C)$ is a $|C|$-separation of M. Therefore, we may assume $|C| \geq 3$.

Let $\mathcal{B}_1$ be the set of C-bridges B so that some B-segment contains S_1 and let $\mathcal{B}_2$ be the set of the remaining C-bridges. Then each $B \in \mathcal{B}_2$ has a B-segment containing S_2. Let X_1 denote the union of S_2 and all the C-bridges in $\mathcal{B}_1$ and let X_2 denote the union of S_1 and all the C-bridges in $\mathcal{B}_2$. Extend S_2 to a base J_1 of X_1 and S_1 to a base J_2 of X_2. We show that the only circuit contained in $J_1 \cup J_2$ is $C = S_1 \cup S_2$. The final hypothesis implies that, for $i = 1, 2$, $|X_i| \geq 2$, so (X_1, X_2) is a 2-separation, as required.

Suppose by way of contradiction that C' is another circuit contained in $J_1 \cup J_2$. Then $C' \setminus C$ is a union of circuits in M/C; let C'' be one of these and let B'' be the C-bridge in M containing C''. Deleting any of the series classes of $C'' \cup C$ yields a circuit of M, so there is a partition $\mathcal{P}$ of C such that, for each $P \in \mathcal{P}$, $C'' \cup (C \setminus P)$ is a circuit of M. Since the circuit C'' is not contained in the independent set J_1, $k \geq 2$.

We suppose that $B'' \in \mathcal{B}_1$ (the alternative is that $B'' \in \mathcal{B}_2$ as the argument is identical). For every B''-segment, there is a $P \in \mathcal{P}$ such that B” $\subseteq P$. Thus, there exists $P \in \mathcal{P}$ such that S_1 is contained in a B''-segment S'' such that $S'' \subseteq P$. Observe that $C \setminus P \subseteq C \setminus S_1 = S_2$ and $C'' \subseteq B'' \in \mathcal{B}_1$. Thus, $C'' \cap X_2 = \varnothing$. Since $C'' \subseteq J_1 \cup J_2$, $C'' \subseteq J_1$. Since $C \setminus P \subseteq J_1$, we have the contradiction that the circuit $C'' \cup (C \setminus P)$ is a circuit in the independent set J_1. ■

For binary, cofinitary matroids, we can expand on Lemma 4.3 (1).

:primaryCircuit **Lemma 4.4** *Let M be a 3-connected, binary, cofinitary matroid, let C be a circuit in M, and let B be a C-bridge. If D is a primary segment for a primary arc A in B, then:*

rPrimarySegment *1. $C \setminus D$ is a primary segment for A;*

rPrimarySegment *2. no other subset of C is a primary segment for A; and*

lyThreeCircuits *3. C, $A \cup D$, and $A \cup (C \setminus D)$ are the only circuits in $C \cup A$.*

Proof. For (1), we note that $A \cup (C \setminus D) = (A \cup D) \triangle C$, so Corollary 3.7.2 shows $A \cup (C \setminus D)$ is the disjoint union of circuits. Since A is a circuit in M/C, no circuit contained in $A \cup C$ other than C can contain a proper subset of A. Therefore, $A \cup (C \setminus D)$ is a circuit, as required.

For (2), if $D' \subseteq C$ is such that $A \cup D'$ is a circuit, then $(A \cup D) \triangle (A \cup D')$ is a disjoint union of circuits contained in C. Therefore, $D \triangle D'$ is either empty or C, so either $D' = D$ or $D' = C \setminus D$, as required.

Item (3) is an immediate consequence of the two preceding items. ■

We can now show that overlap diagrams are connected, a fact well known for graphs.

:overlapDiagram **Lemma 4.5** *If M is a 3-connected matroid and C is a circuit of M, then $\mathcal{O}(C)$ is connected.*

Proof. Let $\mathcal{B}$ be a component of $\mathcal{O}(C)$ and suppose by way of contradiction that there is a C-bridge B_0 not in $\mathcal{B}$.

cl:oneSegment **Claim 1** *There is a B_0-segment S_0 such that, for each C-bridge B in $\mathcal{B}$, there is a B-segment $S(B)$ with $C = S_0 \cup S(B)$.*

Proof. Let B_1 and B_2 be overlapping C-bridges in $\mathcal{B}$. Since B_0 does not overlap either B_1 or B_2, for $i = 1, 2$, there is a B_0-segment S_0^i and a B_i-segment S_i such that $C = S_0^i \cup S_i$. If $S_0^1 \neq S_0^2$, then $S_0^2 \subseteq S_1$. But then $C = S_2 \cup S_0^2 \subseteq S_2 \cup S_1$, implying the contradiction that B_1 and B_2 do not overlap. The connection of $\mathcal{B}$ completes the proof. □

Let $S_* = \bigcap_{B \in V(\mathcal{B})} S(B)$ and $S^* = C \setminus S_*$.

cl:S*S* **Claim 2** *If B is a C-bridge, then there is a B-segment S so that either $S_* \subseteq S$ or $S^* \subseteq S$.*

Proof. If B is in $\mathcal{B}$, then $S_* \subseteq S(B)$, as required. Thus, we may suppose B is not in $\mathcal{B}$. By Claim 1, there is a single B-segment S so that, for every C-bridge B' in $\mathcal{B}$, there is a B'-segment $S'(B')$ so that $C = S \cup S'(B')$.

Suppose that $S_* \setminus S \neq \varnothing$. Then $S_* \cap S'(B') \neq \varnothing$. Since $S_* \subseteq S(B')$ (as in Claim 1 for B_0), we deduce that $S(B') = S'(B')$. Since, for each $x \in S^*$, there is a B' in $\mathcal{B}$ so that $x \notin S(B')$, it must be that $x \in S$. Thus, $S^* \subseteq S$, as claimed. □

In particular, $S^* \subseteq S_0$. Thus, Claim 2 and Lemma 4.3 yield the contradiction that M is not 3-connected. ■

One more small observation is required before we can give a combinatorial characterization of overlapping C-bridges.

gmentsToPrimary **Lemma 4.6** *Let M be a matroid, let C be a circuit in M, and let B be a C-bridge. If S_1 and S_2 are distinct B-segments, then there is a primary arc A of B and a primary segment of A that contains S_1 but not S_2.*

Proof. For $i = 1, 2$, let $x_i \in S_i$. Extend $C - x_2$ to a base I of $C \cup B$. Let D be the fundamental cocircuit of $C \cup B$ that contains x_1. Because x_1 and x_2 are in different B-segments, $\{x_1, x_2\}$ is coindependent. Therefore, $D \not\subseteq C$; let $z \in D \setminus C$. Then the fundamental circuit in $I \cup \{z\}$ is the required primary arc. ■

The following is an immediate consequence of Lemma 4.6.

PrimarySegments **Corollary 4.7** *Let M be a matroid, let C be a circuit in M, and let B be a C-bridge. Then every B-segment is the intersection of primary segments of primary arcs in B.*

One more definition brings us to a principal intermediate result.

Definition 4.8 *Let M be a binary, cofinitary matroid, let C be a circuit in M, and let B_1 and B_2 be distinct C-bridges.*

1. *The C-bridges B_1 and B_2 are* skew *if, for $i = 1, 2$, there is a primary B_i-segment S_i so that the four sets $S_1 \cap S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$, and $C \setminus (S_1 \cup S_2)$ are all non-empty.*

2. *For a positive integer k, the C-bridges B_1 and B_2 are k-*equipartite *if they both partition C into the same k segments.*

haracterization **Theorem 4.9** *Let M be a cofinitary, binary matroid, let C be a circuit of M, and let B_1 and B_2 be overlapping C-bridges. Then B_1 and B_2 are either skew or 3-equipartite.*

For finite binary matroids, this theorem is proved by Tutte [13, 8.44, p. 35]. We shall deduce it from that result.

Proof. For $i = 1, 2$, let f_i be an element of a primary arc of B_i. In particular, f_i is in the same component of M as C. We begin by observing that f_1 and f_2 are not parallel either to each other or to any element of C. If they were parallel, then, since they are in different C-bridges, they are both loops of M/C. In particular, for $i = 1, 2$, $\{f_i\} = B_i$ and there are precisely two B_i-segments. Because f_1 and f_2 are parallel, these segments are the same and C is the union of them, showing B_1 and B_2 avoid each other, a contradiction.

If, say, f_1 were parallel to $e \in C$, then again $\{f_1\} = B_1$. Now the two B_1-segments are $\{e\}$ and $C - \{e\}$. Letting S be the B_2-segment containing e, $C = (C - \{e\}) \cup S$ and again B_1 and B_2 avoid each other, as required.

Let F be any base of M such that $|C \setminus F| = 1$ and let F' be a finite subset of F. Set $N = M/(F \setminus F')$ and let C^N denote the set of elements of C in N.

If I is a proper subset of C^N, then $I \cup (F \setminus F')$ is independent in M. Thus, I is independent in N. Since $C^N \cup (F \setminus F')$ contains C, it is dependent in M. Thus, C^N is a circuit in N.

Let C' be a circuit in M containing f_1, f_2. Since f_1 and f_2 are not parallel in M, there is an element e of $C' \setminus \{f_1, f_2\}$. Let K_1, K_2 be cocircuits of M such that, for $i = 1, 2$, $K_i \cap C' = \{e, f_i\}$. Since $K_1 \cup K_2$ is finite, there is a finite subset F_0 of F such that

$F \cap (K_1 \cup K_2) \subseteq F_0$ and $|F_0 \cap C| \geq 3$. Let F' be a finite set such that $F_0 \subseteq F' \subseteq F$. Then $K_1 \cup K_2$ is disjoint from $F \setminus F'$.

Let $N_0 = M/(F \setminus F_0)$. Suppose that, for some $i \in \{1, 2\}$, f_i is parallel in N_0 to $e_i \in C$. Then there is a circuit C_i in M containing e_i and f_i. As e_i, f_i are not parallel in M, there is a third element e'_i of C_i. Let K'_i be a cocircuit in M such that $K'_i \cap C_i = \{e'_i, f_i\}$. Now set $F^* = F \cap (K_1 \cup K_2 \cup K'_1 \cup K'_2)$ (using K'_1 and K'_2 only when they exist).

For any finite subset F' of F such that $F^* \subseteq F'$, set $N_{F'} = M/(F \setminus F')$. Then $C'/(F \setminus F')$ is a disjoint union of circuits in $N_{F'}$; let C'' be the one containing e. For $i = 1, 2$, $e \in (C''/(F \setminus F')) \cap K_i \subseteq C' \cap K_i = \{e, f_i\}$. Since K_i is a cocircuit of $N_{F'}$, $|K_i \cap C''|$ is even, so $K_i \cap (C''/(F \setminus F') = \{e, f_i\}$. In particular, e, f_1, f_2 are all in C'', so f_1 and f_2 are not parallel in $N_{F'}$.

Now suppose that, for some $i \in \{1, 2\}$, f_i is parallel in $N_{F'}$ to some element f'_i of $C_{N_{F'}}$. The cycle $\{f_i, f'_i\}$ contracts to a cycle in N_{F_0}, in which case either f_i and f'_i are parallel in N_{F_0} or f_i is a loop in N_{F_0}. The latter does not happen, as $|K_i \cap \{f_i\}|$ is not even. Thus, in N_{F_0}, f'_i is parallel to f_i, which in turn is parallel to e_i in N_{F_0}.

The only possibility is that $f'_i = e_i$. By definition, $\{f_i, e_i\}$ is a cycle in $N_{F'}$ and K'_i is a cocircuit in $N_{F'}$. However, $K'_i \cap \{e_i, f_i\} = \{f_i\}$, the final contradiction that shows neither f_1 nor f_2 is parallel in $N_{F'}$ to any element of $C^{N_{F'}}$.

Let $\mathcal{F}$ denote the set of all minors of M of the form $si(M/(F \setminus F'))$, where F' is a finite subset of F that contains F^*. Lemma 3.2 shows that every matroid in $\mathcal{F}$ is finite. The preceding discussion shows that f_1 and f_2 may be chosen as the representatives of their parallel classes in $M/(F \setminus F')$. For $N \in \mathcal{F}$, N/C^N is a minor of M/C, so f_1 and f_2 are in distinct C^N-bridges B_1^N and B_2^N, respectively.

Let $N, N' \in \mathcal{F}$ be such that N is a minor of N'. If $B_1^{N'}$ and $B_2^{N'}$ avoid one another, then it is a routine verification that B_1^N and B_2^N avoid one another.

In view of Theorem 4.9 for finite matroids, one of the following holds for $\mathcal{F}$:

1. for all $N \in \mathcal{F}$, B_1^N and B_2^N avoid each other;

2. for some $N \in \mathcal{F}$, B_1^N and B_2^N are skew to each other; and

3. for every $N \in \mathcal{F}$, there is an $N' \in \mathcal{F}$ such that N is a minor of N' and $B_1^{N'}$ and $B_2^{N'}$ are 3-equipartite.

In the next four claims, we show that these imply the corresponding result for M. We start with an important observation.

cl:segments

Claim 1 *For $j = 1, 2$, the elements x, y of C are in the same B_j-segment if and only if, for every $N \in \mathcal{F}$, there is an $N' \in \mathcal{F}$ having N as a minor and such that x, y are in the same $B_j^{N'}$-segment.*

Proof. The only if direction is trivial. Conversely, suppose by way of contradiction that x and y are in different B_i-segments in C. Lemma 4.6 and Corollary 4.7 imply there is a primary B_i-segment S containing x but not y. By definition, there is a primary arc A in B_j such that $S \cup A$ is a circuit of M. Let K be a cocircuit of M such that $C \cap K = \{x, y\}$.

Since $|K \cap (S \cup A)|$ is even and $|K \cap S| = 1$, we see that $|K \cap A|$ is odd. Since K is finite, there is an $N \in \mathcal{F}$ such that the finite set $K \cap A$ is contained in B_j^N and K is a cocircuit of N.

Suppose by way of contradiction that x and y are in the same B_j^N-segment. Then $\{x, y\}$ is a cocircuit of $C^N \cup B_j^N$. Consequently, N has a cocircuit K' such that $K' \cap (C^N \cup B_j^N) = \{x, y\}$. In particular, K' is disjoint from B_j^N.

Since N is a finite binary matroid, $K \triangle K'$ is a disjoint union of cocircuits of N and is evidently disjoint from C^N. Since $K \cap A$ has an odd number of elements, there is an element w of $K \cap A$. By choice of N, $w \in B_j^N$. Since $K' \cap B_j^N = \varnothing$, $w \notin K'$, so there is a cocircuit $K'' \subseteq K \triangle K'$ such that $w \in K''$.

We know that $(K \triangle K') \cap C^N = \varnothing$, so $K'' \cap C^N = \varnothing$. Since K does not properly contain another cocircuit, $K'' \not\subseteq K$. Therefore, there is a $z \in K'' \cap (K' \setminus \{x, y\})$. Since $K' \setminus \{x, y\}$ is disjoint from $C^N \cup B_j^N$, $z \notin C^N \cup B_j^N$.

But K'' is a cocircuit of N that is disjoint from C^N, so K'' is contained in a component of N/C^N. However, w is in the component B_j^N of N/C^N, while z is not. This is the desired contradiction. □

The next claim treats the case that the C^N-bridges are "eventually" k-equipartite.

l:k-equipartite **Claim 2** *Suppose k is a positive integer such that, for every $N \in \mathcal{F}$, there is an $N' \in \mathcal{F}$ such that N is a minor of N' and $B_1^{N'}$ and $B_2^{N'}$ are k-equipartite in N'. Then B_1 and B_2 are k-equipartite.*

Proof. For each $N \in \mathcal{F}$ such that B_1^N and B_2^N are k-equipartite, let $S_1^N, S_2^N, \ldots, S_k^N$ be the k distinct B_1^N-segments. (Of course, they are also the k distinct B_2^N-segments.) Let N_0 be a particular element of $\mathcal{F}$ such that $B_1^{N_0}$ and $B_2^{N_0}$ are k-equipartite. For $i = 1, 2, \ldots, k$, let e_i^0 be an arbitrary element of $S_i^{N_0}$. Finally, let $\mathcal{F}_0$ consist of those $N \in \mathcal{F}$ having N_0 as a minor and such that B_1^N and B_2^N are k-equipartite.

Let $N \in \mathcal{F}_0$. If two elements of N_0 are in the same B_1^N-segment, then evidently they are in the same $B_1^{N_0}$-segment. In particular, no two of the e_i^0 are in the same B_1^N-segment. Thus, we may choose the labelling of these segments so that $e_i^0 \in S_i^N$.

Let $N, N' \in \mathcal{F}_0$ both contain $e \in C$. We claim that if $e \in S_i^N$, then $e \in S_i^{N'}$. There is a j such that $e \in S_j^{N'}$ and there is an $N'' \in \mathcal{F}_0$ having both N and N' as minors. Let ℓ be such that $e \in S_\ell^{N''}$. Then $e, e_\ell \in S_\ell^{N''}$, and so e, e_ℓ are in the same B_1^N- and $B_1^{N'}$-segments. Hence ℓ is equal to both i and j; in particular, $i = j$.

It follows that the sets $S_i = \bigcup_{N \in \mathcal{F}_0} S_i^N$, for $i = 1, 2, \ldots, k$, form a partition of C into k non-empty sets. Claim 1 shows that each S_i is a B_1-segment. □

The following determines when B_1 and B_2 are skew.

cl:skew **Claim 3** *Suppose there is an $N \in \mathcal{F}$ such that B_1^N and B_2^N are skew. Then B_1 and B_2 are skew.*

Proof. For $j = 1, 2$, let A_j^N be a primary arc in B_j^N and let S_j^N be a primary segment for A_j^N such that all of $S_1^N \cap S_2^N$, $S_1^N \setminus S_2^N$, $S_2^N \setminus S_1^N$, and $C \setminus (S_1^N \cup S_2^N)$ are non-empty.

For $j = 1, 2$, $A_j^N \cup S_j^N$ is a circuit in N, so there is a subset F_j of F such that $A_j^N \cup S_j^N \cup F_j$ is a circuit of M. Corollary 3.7.1 implies $(A_j^N \cup S_j^N \cup F_j) \setminus C$ is the disjoint union of circuits of M/C. Each of these circuits is contained in $A_j^N \cup F_j$; let A_j be one that contains an element of A_j^N. Since B_j is a component of M/C, $A_j \subseteq B_j$. It follows that A_j is a primary arc of B_j.

Let S_j be a primary segment for A_j. Applying Corollary 3.7.1 again, the elements of $A_1 \cup S_1$ that are also in N is a disjoint union of circuits of N. These circuits are contained in $C^N \cup B_j^N$ and, therefore, each consists of a primary arc and corresponding primary segment, or else is the whole of C^N.

The same argument applies with $C \setminus S_j$ in place of S_j, so each of the primary segments with respect to A_1 contains one of S_j^N and $C^N \setminus S_j^N$. In particular, each primary segment for A_1 has non-empty intersection with each primary segment for A_2, as required. □

Finally, we deal with the case B_1 and B_2 avoid each other.

cl:avoid **Claim 4** *Suppose, for every $N \in \mathcal{F}$, B_1^N and B_2^N avoid each other on C^N. Then B_1 and B_2 avoid each other on C.*

Proof. By way of contradiction, we suppose that B_1 and B_2 overlap. That is:

it:notTwoUnion (i) for any B_1- and B_2-segments S_1 and S_2, respectively, $C \neq S_1 \cup S_2$.

Let $e \in C$ and, for $i = 1, 2$, let S_i^e be the B_i-segment containing e. By (i), there is some $f \in C \setminus (S_1^e \cup S_2^e)$. Let S_2^f be the B_2-segment containing f. Repeating with $C \neq S_1^e \cup S_2^f$, there is a $g \in C \setminus (S_1^e \cup S_2^f)$. For $N \in \mathcal{F}$ containing e, f, let S_1^N and S_2^N be the B_1^N- and B_2^N-segments containing e and f, respectively.

Let $N_0 \in \mathcal{F}$ contain e, f, g such that neither f nor g is in $S_1^{N_0}$ and neither e nor g is in $S_2^{N_0}$. Let $\mathcal{F}_0$ consist of those elements of $\mathcal{F}$ having N_0 as a minor.

For each $N \in \mathcal{F}_0$, neither f nor g is in S_1^N and neither e nor g is in S_2^N. By hypothesis, there are B_1^N- and B_2^N-segments T_1^N and T_2^N, respectively, such that $C^N = T_1^N \cup T_2^N$. If $e \in T_1^N$, then $f, g \in T_2^N$, a contradiction. Thus, $e \in T_2^N$ and, likewise, $f \in T_1^N$. Evidently,

$$C = \left(\bigcup_{N \in \mathcal{F}_0} T_1^N \right) \cup \left(\bigcup_{N \in \mathcal{F}_0} T_2^N \right) .$$

Let $N, N' \in \mathcal{F}_0$. Then there is an $N'' \in \mathcal{F}_0$ having both N and N' as minors. Thus, for $i = 1, 2$, $T_i^N \cup T_i^{N'} \subseteq T_i^{N''}$. Thus, Claim 1 shows $\bigcup_{N \in \mathcal{F}_0} T_1^N$ and $\bigcup_{N \in \mathcal{F}_0} T_1^N$ are B_1- and B_2-segments. In particular, B_1 and B_2 avoid each other, a contradiction. □

As we mentioned just before Claim 1, one of three possibilities occurs: for all i, B_1^i and B_2^i do not overlap; or, there exists an N such that B_1^N and B_2^N are skew; or, for every $N \in \mathcal{F}$, there is an $N' \in \mathcal{F}$ such that N is a minor of N' and $B_1^{N'}$ and $B_2^{N'}$ are 3-equipartite. In order, these imply: B_1 and B_2 do not overlap (Claim 4); B_1 and B_2 are skew (Claim 3); or B_1 and B_2 are 3-equipartite (Claim 2). ■

We are now set for the proof of the main result in this section.

Proof of Theorem 4.1. Lemma 4.5 implies that, if C is not peripheral, then there is a C-bridge that overlaps B. We begin with two claims.

verlapNewBridge **Claim 1** *Let B' be a C-bridge different from B, let A' be a primary arc in B' and let S' be a primary segment for A'. Then $C' = A' \cup S'$ is a circuit, there is a C'-bridge B'' such that $B \subseteq B''$, and $C \setminus S'$ is a circuit of M/C'.*

Proof. Definition 4.2 shows C' is a circuit. Evidently, $B \cap B' = \varnothing$, so $B \cap A' = \varnothing$. Also, $B \cap C = \varnothing$, so $B \cap S' = \varnothing$. Lemma 3.1 implies that there is a C'-bridge B'' such that $B \subseteq B''$.

Let $C'' = A' \cup (C \setminus S')$. Corollary 3.7.1 implies $C \setminus S'$ is the disjoint union of circuits of M/C'. Pick arbitrarily $x \in C \setminus S'$ and let $C_x \subseteq C \setminus S'$ be a circuit of M/C' containing x. Let $Y_x \subseteq C'$ be such that $C_x \cup Y_x$ is a circuit of M. Lemma 4.4 (3) implies that $C_x \cup Y_x$ is one of C, C', and C''. In particular, since $x \in (C \setminus S') \cap C_x$, $S'' \subseteq C_x \cup Y_x$. Thus, $S'' \subseteq C_x$. It follows that $S'' = C_x$, as required. □

cl:noSkew **Claim 2** *If C is not peripheral and no C-bridge is skew to B, then there are precisely three B-segments, each with size 1. In particular, $|C| = 3$.*

Proof. Lemma 4.5 and Theorem 4.9 imply that C has precisely three B-segments S_1, S_2, S_3. By way of contradiction, suppose S_1 has more than one element.

Let $\mathcal{B}_1$ be the set of C-bridges having a segment containing $C \setminus S_1$ and let $\mathcal{B}_2$ be the set of all C-bridges that have a segment containing S_1. Every C-bridge 3-equipartite with B is in $\mathcal{B}_2$. On the other hand, if B' is a C-bridge that avoids B, then there is a B'-segment S' such that, for some $j \in \{1, 2, 3\}$, $S' \cup S_j = C$. If $j = 1$, then $C \setminus S_1 \subseteq S'$ and $B' \in \mathcal{B}_1$. If $j \neq 1$, then $S_1 \subseteq S'$ and $B' \in \mathcal{B}_2$. Since no C-bridge is skew to B, we conclude that every C-bridge is in $\mathcal{B}_1 \cup \mathcal{B}_2$.

Since $|S_1| \geq 2$ and $|S_2 \cup S_3| \geq 2$, Lemma 4.3 implies the contradiction that M is not 3-connected. This shows that, for $i = 1, 2, 3$, $|S_i| = 1$. □

Case 1: *there is a C-bridge skew to B.*

Let B' be a C-bridge skew to B. There are primary arcs A and A' in B and B', respectively, with primary segments S and S', for A and A', respectively, such that $S \cap S'$, $S \setminus S'$, $S' \setminus S$ and $C \setminus (S \cup S')$ are all non-empty. Letting $C' = A' \cup S'$, Claim 1 shows that there is a C'-bridge B'' containing B.

In order to show $B \neq B''$, it suffices to show that there is a circuit of M/C' that intersects both B and $C \setminus S'$. Let $x \in S \setminus S'$. Corollary 3.7.1 implies $(A \cup S) \setminus C'$ is a disjoint union of circuits of M/C'; let C'_x be a circuit of M/C' containing x.

Skewness implies $S \setminus S'$ is a proper subset of $C \setminus S'$. By Claim 1, $C \setminus S'$ is a circuit of M/C'. Thus, $S \setminus S'$ is independent in M/C'. Therefore, $C'_x \cap A$ is not empty. That is, C'_x intersects both B and $C \setminus S'$, so $S' \subseteq B'' \setminus B$.

The preceding argument applies equally well to the primary segment $C \setminus S'$ for A'. Therefore, the two circuits $A' \cup S'$ and $A' \cup (C \setminus S')$ show that Conclusion 1 holds, as required.

Case 2: *no C-bridge is skew to B.*

Claim 2 shows that $|C| = 3$. As above, we let S_1, S_2, and S_3 be the three B-segments. For any C-bridge B', there is at least one primary arc for B', so some S_i is a primary segment; choose the labelling so that $i = 1$. Lemma 4.6 shows that there is a primary segment containing S_2 and not containing S_3. Therefore, at least one of S_2 and S_3 is also a primary segment for B'. Thus, at least two of S_1, S_2, S_3 are primary segments for B'.

Since each of B and B' has at least two of the S_i as primary segments, there is at least one S_i that is a primary segment for both B and B'; choose the labelling so that S_1 is one such.

Let A_1 be a primary arc in B' with primary segment S_1, so $S_2 \cup S_3$ is also a primary segment for A_1. Let $C_1 = A_1 \cup S_1$ and $C_2 = A_1 \cup S_2 \cup S_3$. By Claim 1, there is a C_1-bridge B_1 such that $B \subseteq B_1$ and $S_2 \cup S_3$ is a circuit of M/C_1.

Choose the labelling so that S_2 is also a primary segment for B, corresponding to the primary arc $A_2 \subseteq B$. Then $A_2 \cup S_2$ is a circuit in M, so Corollary 3.7 implies $A_2 \cup S_2$ is a disjoint union of circuits in M/C_1. If $x \in S_2$, then there is a circuit C_x of M/C_1 containing x and contained in $A_2 \cup S_2$.

Since $S_2 \cup S_3$ is a circuit of M/C_1, S_2 is not a circuit of M/C_1. Therefore, $C_x \neq S_2$, so $C_x \cap A_2 \neq \emptyset$. Therefore, $x \in B_1 \setminus B$, as required for the first pair C_1 and B_1.

By Lemma 3.1, there is a C_2-bridge B_2 such that $B \subseteq B_2$. Since M is 3-connected, $|A_1 \cup S_1| \geq 3$. As $|S_1| = 1$, $|A_1| \geq 2$ and, therefore, $|C_2| \geq 4$. There are two possibilities that arise from Claim 2.

If C_2 is peripheral, then S_1 is a subset of the unique C_2-bridge B_2, so Conclusion 1 applies with the pairs (C_1, B_1) and (C_2, B_2), as required.

Otherwise, there is a C_2-bridge B_3 that is skew to B_2. We can apply Case 1 to C_2 and B_2 to obtain cycles C_2' and C_3', with C_2'- and C_3'-bridges B_2' and B_3', respectively, properly containing B_2 (and therefore B). Now the three pairs (C_1, B_1), (C_2', B_2'), and (C_3', B_3') satisfy Conclusion 2, as required. ∎

We need one other tool to combine with Theorem 4.1 in order to prove our main result Theorem 2.3. Here we use $E(M)$ to denote the ground set of the matroid M.

m:maximalBridge **Lemma 4.10** *Let M be a cofinitary, binary matroid, let F be any finite subset of $E(M)$, and let $X \subseteq F$.*

t:maximalBridge *(1) Let $\mathcal{B}(F, X)$ denote the set of all subsets B of $E(M)$ such that, for some $z \in \mathcal{Z}(M)$ with $z \cap F = X$, there is a z-bridge containing B. For every $B \in \mathcal{B}(F, X)$, there is a maximal element B' of $\mathcal{B}(F, X)$ containing B.*

malCycleElement *(2) Let $B \subseteq E(M) \setminus X$. Let $\mathcal{X}(F, X, B)$ be the set of those z in $\mathcal{Z}(M)$ such that $z \cap F = X$ and there is a z-bridge containing B. If $\mathcal{X}(F, X, B) \neq \varnothing$, then there is a minimal element of $\mathcal{X}(F, X, B)$.*

In particular, for each pair (z, B) consisting of $z \in \mathcal{Z}(M)$ with $z \cap F = X$ and a z-bridge B, there is a pair (z^*, B^*) such that B^* is a maximal element of $\mathcal{B}(F, X)$ containing B, z^* is a minimal element of $\mathcal{X}(F, X, B^*)$, B^* is a z^*-bridge, and $z^* \cap F = X$. Such a z^* is necessarily a finite disjoint union of circuits.

Let (F, X, B) be a triple such that: F is a finite subset of $E(M)$; $X \subseteq F$; and $B \subseteq E(M) \setminus X$ is such that there is a $z \in \mathcal{Z}(M)$ such that $z \cap F = X$ and B is contained in a z-bridge. A *minimax pair for* (F, X, B) is any pair (z^*, B^*) from the "in particular" statement of Lemma 4.10. Thus, B^* is a maximal element of $\mathcal{B}(F, X)$ that contains B, $z^* \in \mathcal{Z}(M)$, $z^* \cap F = X$, B^* is a z^*-bridge; and z^* is minimal with respect to all these properties.

Proof. For (1), let $\mathcal{C}$ be a non-empty chain in $\mathcal{B}(F, X)$ such that, for each $C \in \mathcal{C}$, $B \subseteq C$. For each $C \in \mathcal{C}$, let $z_C \in \mathcal{Z}(M)$ be such that $z_C \cap F = X$ and C is a z_C-bridge. We prove that there is a $C^* \in \mathcal{B}(F, X)$ such that, for all $C \in \mathcal{C}$, $C \subseteq C^*$. The result then follows immediately from Zorn's Lemma. Obviously, we may assume $\mathcal{C}$ has no maximal element.

Let A denote the set of elements e of M such that, for every $C \in \mathcal{C}$, there is a $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $e \in z_{C'}$. Since, for every $C \in \mathcal{C}$, $z_C \cap F = X$, we see that $F \cap A = X$. Furthermore, for each $C \in \mathcal{C}$, let $e \in C$. Then, for all $C' \in \mathcal{C}$ such that $C \subseteq C'$, $e \in C'$. For all such C', $e \notin z_{C'}$, so $e \notin A$. That is, for all $C \in \mathcal{C}$, $C \cap A = \varnothing$.

We claim there is a $z^* \in \mathcal{Z}(A)$ for which $z \cap F = X$ and there is a z^*-bridge C^* that is an upper bound for $\mathcal{C}$. We proceed by induction on $|X|$. The base case $|X| = 0$ is trivial: we may choose $z^* = \varnothing$. Since every $C \in \mathcal{C}$ is contained in the same component N of M, N is a z^*-bridge and contains C. Moreover, $z^* \cap F = \varnothing = X$, as required.

For the induction step, let $x \in X$ and set $Y = X \setminus \{x\}$. There is an element z_Y of $\mathcal{Z}(A)$ such that $(F \setminus \{x\}) \cap z_Y = Y$ and there is a z_Y-bridge that is an upper bound for $\mathcal{C}$. If $x \in z_Y$, then we are done. As a second simple case, if there is a circuit z in $A \setminus Y$ that contains x, then $z \triangle z_Y$ is the desired element of $\mathcal{Z}(A)$. To see this, note that: $F \cap z = \{x\}$; $F \cap z_Y = Y$; $z \triangle z_Y \subseteq A$ is disjoint from each element C of $\mathcal{C}$; Lemma 3.1 implies C is contained in some $(z \triangle z_Y)$-bridge; and the fact that $\mathcal{C}$ is a chain implies it is the same $(z \triangle z_Y)$-bridge for all elements of $\mathcal{C}$.

Thus, we may assume that $x \notin z_Y$ and there is no circuit in $A \setminus Y$ containing x; that is, x is a coloop in $A \setminus Y$. Consequently, there is a cocircuit K of M such that $x \in K \subseteq X \cup (E(M) \setminus A)$. Since K is finite, the definition of A implies that there is a $C_0 \in \mathcal{C}$ such that, for all $C \in \mathcal{C}$ such that $C_0 \subseteq C$, $K \setminus A$ is disjoint from z_C. Henceforth, we redefine $\mathcal{C}$ to be the subchain consisting of all those $C \in \mathcal{C}$ such that $C_0 \subseteq C$.

By definition, for each $C \in \mathcal{C}$, $X \subseteq z_C$ and, from the preceding paragraph, $K \cap z_C \subseteq A$. Since $K \subseteq X \cup (E(M) \setminus A)$, we conclude that $K \cap z_C \subseteq X$. On the other hand the induction gave us $z_Y \subseteq A$, so also $K \cap z_Y \subseteq X$.

However, $X \cap z_C = X$ and $z_Y \cap X = Y$, so $X \cap (z_C \triangle z_Y) = \{x\}$. Since $x \in K$ and $K \cap (z_C \triangle z_Y) \subseteq X$, we conclude that $K \cap (z_C \triangle z_Y) = \{x\}$. However, this contradicts Lemma 3.4 (2), proving (1).

For (2), let $\mathcal{C}$ be a chain of elements of $\mathcal{X}(F, X, B)$. We claim $z_* = \bigcap_{z \in \mathcal{C}} z \in \mathcal{X}(F, X, B)$, providing the desired lower bound. This is obvious if $\mathcal{C}$ has a minimal element, so we assume it does not.

Let K be any cocircuit of M. Then Lemma 3.4 (2) implies that, for every $z \in \mathcal{C}$, $|z \cap K|$ is even (recall K is finite). Since $\mathcal{C}$ is a chain, $\{z \cap K \mid z \in \mathcal{C}\}$ is also a chain. As K is finite, this new chain has a lower bound. In particular, there is a $z \in \mathcal{C}$ such that, for every $z' \in \mathcal{C}$, $z \cap K \subseteq z' \cap K$. It follows that $z^* \cap K = z \cap K$, so $|z^* \cap K|$ is even. Since this holds for every cocircuit K, Lemma 3.6 implies $z^* \in \mathcal{Z}(M)$.

Evidently, $F \cap z^* = X$. For each $z \in \mathcal{C}$, there is a z-bridge B_z containing B. As $z^* \subseteq z$, Lemma 3.1 shows there is a z^*-bridge containing B_z and, therefore, B. Consequently, $z^* \in \mathcal{X}$, as required.

For the "in particular" statement, (1) implies there is a subset B^*_C of M and a $z^1_C \in \mathcal{Z}(M)$ such that:

it:sameK (i) $z^1_C \cap F = C \cap K$;

it:containsB (ii) B^*_C is a z^1_C-bridge containing B_C; and

(iii) over all z^1_C satisfying (i) and (ii), B^*_C is maximal.

On the other hand, (2) implies there is an element $z^*_C \in \mathcal{Z}(M)$ such that:

it:sameKint (i) $z^*_C \cap F = C \cap K$;

it:sameB (ii) B_C^* is contained in a z_C^*-bridge B_z; and

(iii) over all z_C^* satisfying (i) and (ii), z_C^* is minimal.

Because the z_C^*-bridge B_z is a candidate for the maximal B_C^*, $B_z = B_C^*$, so B_C^* is a z_C^*-bridge.

Finally, z_C^* is a disjoint union of circuits. Only finitely many of those circuits have non-empty intersection with F; let these be $C_1, C_2, \dots, C_k$ and let $z = \bigcup_{i=1}^k C_i$. Evidently, $z \subseteq z_C^*$ and $z \cap F = z_C^* \cap F = X$. Lemma 3.1 implies there is a z-bridge B containing B_C^*. Maximality of B_C^* implies $B = B_C^*$ and minimality of z_C^* implies $z = z_C^*$. ■

5 Peripheral circuits span the cycle space

sc:proofMain

In this section we prove Theorem 2.3. The first part is to prove that, given any two elements e and f of M, there is a peripheral circuit containing e and not containing f. The second part is to prove that, when M is countable, the cycle space is generated by all the peripheral cycles.

As it is helpful for the next section, we provide a slightly more detailed version of Theorem 2.3 (1).

Theorem 2.3 (1) *Let e, f be distinct elements in a 3-connected, cofinitary, binary matroid M. Then:*

1. *there is a circuit C in M containing e but not f; and*

2. *if C is any circuit containing e but not f, then, letting B be the C-bridge containing f, there is a peripheral circuit C' in M containing e such that the unique C'-bridge contains B.*

Proof. Since M is 3-connected, $M \setminus f$ is connected and, therefore, there is a circuit C_0 of $M \setminus f$ containing e; let B_0 be the C_0-bridge in M containing f. Lemma 4.10 implies there is a minimax pair (z^*, B^*) for $(\{e, f\}, \{e\}, B_0)$. We claim that z^* is a peripheral circuit containing e but not f.

By definition of $Z(M)$, z^* is the disjoint union of circuits; let C be the one containing e. Lemma 3.1, there is a C-bridge B containing B^*. Thus, $e \in C$ and $f \in B$, so maximality of B^* tells us that $B = B^*$. On the other hand, $C \subseteq z^*$, so minimality of z^* implies $C = z^*$. In particular, z^* is a circuit.

If z^* is not peripheral, then Theorem 4.1 implies that there a set $\mathcal{C}$ of two or three circuits, with z^* as their symmetric difference, and for each $C \in \mathcal{C}$, there is a C-bridge B_C that properly contains B^*. Choosing $C \in \mathcal{C}$ to be the one containing e, we see that B_C violates the maximality of B^*. ■

What remains is to show that, when M is countable, the peripheral cycles generate the cycle space. In outline, the proof follows the same pattern as Bruhn's proof for the Freudenthal compactification of a locally finite graph.

Theorem 2.3 (2) *If M is a countable, 3-connected, cofinitary, binary matroid, then the peripheral circuits generate the cycle space of M.*

Proof. Let $z_0 \in \mathcal{Z}(M)$ and let $e_1, e_2, \dots$ be an enumeration of the elements of M. Starting with an arbitrary z_0-bridge B_0, for each $i \geq 0$, we will determine $z_i \in \mathcal{Z}(M)$ and a z_i-bridge B_i such that $e_1, \dots, e_i \in B_i$ and, for $i \geq 1$, there are peripheral circuits $P_i^1, \dots, P_i^{k_i}$, all disjoint from B_{i-1}, and $z_i = z_{i-1} \triangle P_i^1 \triangle \cdots \triangle P_i^{k_i}$.

It is important to note that, for each i, j with $j > i$, e_i is not in any P_j^ℓ, so that the set of all P_j^ℓ is thin and, moreover, the symmetric difference of all the P_j^ℓ is z_0.

While the intent at each iteration is to grow the bridge, the peripheral circuits need to be more carefully chosen than has previously been the case. Bruhn introduced the notion of an extension tree to deal with this and we shall work with a small variation of his notion.

To be sure that e_i is in B_i, we begin with a cocircuit K such that $e_i \in K$ and $K \cap B_{i-1} \neq \varnothing$. (To see that K exists, let $f \in B_{i-1}$. Since M is connected, there is a circuit C in M containing e_i and f. There is a cocircuit K of M such that $K \cap C = \{e_i, f\}$.) We will use "extension trees" to determine z_i and B_i.

An *extension tree* with respect to K is a rooted tree T whose vertices are finite sequences of subsets of K, together with, for each vertex t of T, a label (C_t^T, B_t^T), such that:

it:exTrRoot (ET1) for some $z \in \mathcal{Z}(M)$, $K_0 = z \cap K$, and a z-bridge B, the root r is a sequence (K_0) of length one with label a minimax pair for (K, K_0, B);

it:exTrLabel (ET2) for $\ell > 0$, the vertex $t = (K_\ell, K_{\ell-1}, \dots, K_0)$ of T has label (z_t^T, B_t^T) that is a minimax pair for (K, K_ℓ, B);

xTrIntermediary (ET3) if t' is a vertex on the path in T from r to t and $t' \neq t$, then $B_{t'}^T$ is a proper subset of B_t^T;

TrChildSequence (ET4) each vertex $(K_\ell, K_{\ell-1}, \dots, K_0)$ that is not a leaf has at least two, but only finitely many, children and each child is of the form $(K_{\ell+1}^i, K_\ell, K_{\ell-1}, \dots, K_0)$; and

TrChildProperty (ET5) if $(K_\ell, K_{\ell-1}, \dots, K_0)$ has children $(K_{\ell+1}^1, K_\ell, K_{\ell-1}, \dots, K_0)$, $(K_{\ell+1}^2, K_\ell, K_{\ell-1}, \dots, K_0)$, $\dots$, $(K_{\ell+1}^k, K_\ell, K_{\ell-1}, \dots, K_0)$, then $K_{\ell+1}^1, K_{\ell+1}^2, \dots, K_{\ell+1}^k$ are distinct, non-empty, their symmetric difference is K_ℓ, and no proper subset of the $K_{\ell+1}^i$ has symmetric difference K_ℓ.

Suppose that $(K_\ell, K_{\ell-1}, \dots, K_0)$ is a vertex of an extension tree. For $0 \leq i < j \leq \ell$, (z_i, B_i) and (z_j, B_j) are minimax pairs for (K, K_i, B) and (K, K_j, B), respectively. As $B_i \subsetneq B_j$, it follows that $K_i \neq K_j$. That is, the sets K_i, $i = 0, 1, 2, \dots, \ell$, are distinct.

Thus, each vertex of an extension tree has at most $2^{|K|}$ children and each path starting from the root has at most $2^{|K|}$ vertices. Therefore, an extension tree has size bounded by a function of $|K|$. This is a slight simplification of Bruhn's extension trees.

Let $t_1, \dots, t_n$ be the leaves of an extension tree T with respect to K. If each $t_i = (K^i, \dots)$, then K_0 (the root is (K_0)) is the symmetric difference $K^1 \triangle \cdots \triangle K^n$. If t_i is distance ℓ_i from the root r of T, then t_i is a sequence of length $\ell_i + 1$, and all the intermediate vertices on the rt_i-path in T are the non-empty tails of the sequence t_i.

The following claim shows that only finitely many peripheral circuits are required to make K disjoint from the symmetric difference of $\mathcal{Z}$ with the peripheral circuits.

cl:clearK **Claim 1** *Let $z \in \mathcal{Z}(M)$, B a z-bridge, and K a cocircuit. Then there are peripheral circuits $P_1, \dots, P_k$ (depending on z, B, and K), each disjoint from B, such that $(P_1 \triangle \cdots \triangle P_k) \cap K = z \cap K$ and there is a $(z \triangle P_1 \triangle \cdots \triangle P_k)$-bridge containing B.*

Proof. We obtain a sequence $\{T_i\}$ of extension trees with respect to K such that, for every i, $|V(T_{i+1})| > |V(T_i)|$. As remarked above, $|V(T_i)|$ is bounded by a function of $|K|$, so the sequence $\{T_i\}$ is necessarily finite. The induction below that creates the T_i shows that, for each leaf ℓ of the last tree in the sequence, the label of ℓ consists of a peripheral circuit C and the unique C-bridge.

Let T_0 have the single node $(z \cap K)$ and give it as label a minimax pair for $(K, z \cap K, B)$. We obtain the sequence $\{T_i\}$ of extension trees with respect to K as follows. As long as some leaf t of T_i has label (z_t, B_t) such that z_t is not a peripheral circuit, we construct an extension tree T_{i+1} such that $|V(T_{i+1})| > |V(T_i)|$.

Because (z_t, B_t) is a minimax pair for $(K, z_t \cap K, B_t)$, z_t is a finite disjoint union of circuits $C_1, C_2, \dots, C_k$. The minimality of z_t also shows that, for each proper subset J of $\{1, 2, \dots, k\}$, $(\bigcup_{j \in J} C_j) \cap K \neq z_t \cap K$, while $(\bigcup_{j=1}^{k} C_j) \cap K = z_t \cap K$. Thus, each C_j has non-empty intersection with K. For each $j = 1, 2, \dots, k$, let B_j be the C_j-bridge containing B_t.

By way of contradiction, suppose that, for some $j \in \{1, 2, \dots, k\}$, both C_j is peripheral and $B_j = B_t$. Then $E(M)$ is the disjoint union of C_j and B_j and, therefore, $z_t \subseteq C_j$, yielding the contradiction that $z_t = C_j$.

For those C_j that are not peripheral, we apply Theorem 4.1 to obtain a set $\mathcal{C}_j$ of two or three circuits such that $C_j = \triangle_{C \in \mathcal{C}_j} C$ and, for each $C \in \mathcal{C}_j$, there is a C-bridge B_C that properly contains B_j. We note that no $C \in \mathcal{C}_j$ can have $C \cap K = z_t \cap K$, as B_C properly contains B_t.

Let $\mathcal{C}$ consist of all the C_j that are peripheral and, for those C_j that are not peripheral, a minimal subset $\mathcal{C}'_j$ of $\mathcal{C}_j$ such that $C_j \cap K$ is the symmetric difference of the elements of $\mathcal{C}'_j$ with K. This ensures (ET5).

We note that, for $C \in \mathcal{C}$, $C \cap K \neq z_t \cap K$. Thus, there are at least two, and only finitely many, elements C of $\mathcal{C}$, each $C \in \mathcal{C}$ has non-empty intersection with K. This is (ET4).

For each $C \in \mathcal{C}$, let B_C be the C-bridge containing B. Note that, if $C = C_j$ is peripheral, then B_C properly contains B_t, while if $C \in \mathcal{C}_j$, then B_C properly contains B_j, which in turn contains B_t. We construct T_{i+1} from T_i by adding, for each $C \in \mathcal{C}$, a child of t. The new coordinate in the vertex term is $C \cap K$ and its label is any minimax pair (z_C, B'_C) for $(K, C \cap K, B_C)$. This gives (ET3).

Evidently, T_{i+1} satisfies (ET1), (ET2). Thus, T_{i+1} is an extension tree.

Let T be an extension tree with respect to K such that its leaves have labels (z_1, B_1) ,(z_2, B_2), $\dots$, (z_k, B_k). As remarked earlier, $z \cap K = \triangle_{i=1}^{k} (z_i \cap K)$, as claimed. In particular, this holds for an extension tree with respect to K such that the corresponding z_i are all peripheral cycles, as required.

Finally, each B_i contains B, so B is disjoint from each of z, z_1, $\dots$, z_k. Lemma 3.1 implies B is contained in a $(z \triangle z_1 \triangle \cdots \triangle z_k)$-bridge, as required. □

Recall that we start with $z_0 \in \mathcal{Z}$ and an enumeration $e_1, e_2, \dots$ of the ground set of M. We begin with an arbitrary z_0-bridge B_0.

Let $i \geq 0$ be such that (z_i, B_i) satisfies $\{e_1, \dots, e_i\} \subseteq B_i$. We show there exist peripheral circuits $P_i^1, \dots, P_i^{k_i}$ such that: for $z_{i+1} = z_i \triangle P_i^1 \triangle \cdots \triangle P_i^{k_i}$, there is a z_{i+1}-bridge B_{i+1} with $B_i \subseteq B_{i+1}$ and $e_{i+1} \in B_{i+1}$. A trivial first case is if $e_{i+1} \in B_i$: set $z_{i+1} = z_i$ and $B_{i+1} = B_i$. Thus, we may assume $e_{i+1} \notin B_i$.

As mentioned at the beginning of the proof, there is a cocircuit K_i of M such that $e_{i+1} \in K_i$ and $K_i \cap B_i \neq \varnothing$. Claim 1 shows there are peripheral circuits $P_i^1, \dots, P_i^{k_i}$ such that each P_i^j is disjoint from B_i and $(P_i^1 \triangle \cdots \triangle P_i^{k_i}) \cap K_i = z_i \cap K_i$. We set $z_{i+1} = z_i \triangle P_i^1 \triangle \cdots \triangle P_i^{k_i}$.

Since z_{i+1} is disjoint from B_i, Lemma 3.1 implies there is a z_{i+1}-bridge B_{i+1} that contains B_i. The choice of the P_i^j implies $K_i \cap z_{i+1} = \varnothing$. Thus, K_i is a cocircuit of M/z_{i+1}. Since $K_i \cap B_i \neq \varnothing$ and $B_i \subseteq B_{i+1}$, $K_i \cap B_{i+1} \neq \varnothing$. Thus, $K_i \subseteq B_{i+1}$, so $e_{i+1} \in B_{i+1}$, as required. ■

6 Graph-like continua

sc:continua

A *graph-like continuum* is a connected, compact topological space G having a totally disconnected subspace V such that every component of $G - V$ is homeomorphic to $\mathbb{R}$. In this section, we show that Bruhn's results extend perfectly to graph-like continua: in a 3-connected graph-like continuum, every edge is in at least two peripheral cycles and the peripheral cycles span the cycle space.

In this context, a *cycle* in a graph-like continuum G is a homeomorph of a unit circle in the Euclidean plane. A *spanning tree* of G is a connected subspace containing V and not containing a cycle; these exist and form the bases of a binary matroid [14, Cor. 4].

:(i)ForContinua

Theorem 6.1 *Let e be an edge of a 3-connected graph-like continuum G. Then G contains two peripheral cycles whose intersection is e and its ends.*

Proof. To prove the existence of a peripheral cycle, it suffices to show that the peripheral circuits of the cycle matroid $\mathcal{M}(G)$ are peripheral cycles of G. If C is peripheral in $\mathcal{M}(G)$, then there is only one C-bridge; that is $\mathcal{M}(G)/C$ is a connected matroid. Since $\mathcal{M}(G)/C = \mathcal{M}(G/C)$, any two edges in G/C are in a cycle together. Letting c be the vertex of G/C to which C contracts, this implies that $(G/C) - c$ is an open, connected, locally connected subset of G/C. In particular, for any two points x, y in G that are not points of C (including in edges not in C but incident with a vertex of C), there is an xy-arc in $G - C$ joining x and y. Thus, $(G/C) - c$ is the unique C-bridge in G, as required.

Let C be a peripheral cycle containing e with unique C-bridge B. The set $\mathrm{att}(B)$ is, by definition, closed in C. Therefore, every point of $C - \mathrm{att}(B)$ is in the interior of an open arc in C. If $v \in C - \mathrm{att}(B)$ is not in the interior of an edge, then let I be the open arc in C containing v. Clearly $I - v$ has two components; each either contains a vertex or is contained in an edge of G. Either way, we get a 2-cut in G with v on one side, contradicting the assumption that G is 3-connected.

It follows that each vertex of G incident with e is an attachment of B, so there is a C-avoiding arc $A \subseteq B$ joining these vertices. Therefore, $A + e$ is a cycle C' such that $C - e$ is contained in a C'-bridge B'.

For the second peripheral cycle, we apply the revised version of Theorem 2.3 (1) as stated in Section 5 to C' and B'. Thus, there is a peripheral cycle C'' in G such that B', and therefore $C - e$, is contained in the unique C''-bridge. ■

We conclude with the generalization of Tutte's cycle space theorem to graph-like continua.

(ii)ForContinua **Theorem 6.2** *Let G be a 3-connected graph-like continuum. Then the peripheral cycles of G generate the cycle space of G.*

For this theorem, there is actually nothing to prove. Thomassen and Vella [11] prove there are only countably many edges in a graph-like continuum. The peripheral circuits of $\mathcal{M}(G)$ are peripheral cycles of G. Since the two cycle spaces are the same and the peripheral circuits of $\mathcal{M}(G)$ span the cycle space, the peripheral cycles of G span the cycle space.

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