Packing countably many branchings with prescribed root-sets in infinite digraphs

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Abstract

We generalize an unpublished result of C. Thomassen. Let $D = (V, A)$ be a digraph and let $\{V_i\}_{i\in\mathbb{N}}$ be a multiset of subsets of V in such a way that any backward-infinite path in D meets all the sets V_i. We show that if all $v \in V$ is simultaneously reachable from the sets V_i by edge-disjoint paths, then there exists a system of edge-disjoint spanning branchings $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ in D where the root-set of $\mathcal{B}_i$ is V_i.

1 Notations and background

The digraphs considered here may have multiple edges and arbitrary size. For $X \subseteq V$, let $\mathbf{in}_{\boldsymbol{D}}(\boldsymbol{X})$ and $\mathbf{out}_{\boldsymbol{D}}(\boldsymbol{X})$ be the set of ingoing and outgoing edges respectively of X in D, and let $\boldsymbol{\varrho_D(X)}$, $\boldsymbol{\delta_D(X)}$ be their respective cardinalities. Let $\mathbf{span}_{\boldsymbol{D}}(\boldsymbol{X})$ be the set of those edges e of D for which $\mathsf{start}(e), \mathsf{end}(e) \in X$. The paths in this paper are directed, finite, simple paths. We say that the path $\boldsymbol{P}$ **goes from** $\boldsymbol{X}$ **to** $\boldsymbol{Y}$ if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$ (we allow $\mathsf{start}(P) = \mathsf{end}(P)$).

Let $\mathcal{V} = \{V_i\}_{i\in I}$ be a multiset of subsets of V and let $D = (V, A)$ be a digraph. We say that $v \in V$ is **simultaneously reachable** from $\mathcal{V}$ in D if there is a system of edge-disjoint paths $\{P_i\}_{i\in I}$ in D such that P_i goes from V_i to v. The system $\mathcal{V}$ satisfies the **path condition** in D (alternatively the pair $(D, \mathcal{V})$ satisfies the path condition) if all $v \in V$ is simultaneously reachable from $\mathcal{V}$. A directed forest $\mathcal{B}_0 = (U, E)$ is a **branching** if for all $u \in U$ there is a unique path from the **root-set** $\{w \in U : \varrho_{\mathcal{B}_0}(w) = 0\}$ to u in $\mathcal{B}_0$. We call $\mathcal{B}$ a **branching packing** with respect to D and $\mathcal{V}$ if $\mathcal{B} = \{\mathcal{B}_i\}_{i\in I}$, where $\mathcal{B}_i$ are edge-disjoint branchings in D and the root-set of $\mathcal{B}_i$ is V_i. If in addition all the $\mathcal{B}_i$'s are spanning branchings of D (i.e. their vertex-set is V), then we call $\mathcal{B}$ a **spanning branching packing** with respect to $\mathcal{V}$ in D. If there is a spanning branching packing $\mathcal{B}$ with respect to $(D, \mathcal{V})$, then the system $\mathcal{V}$ obviously satisfies the path condition in D, since if $v \in V$, then $\mathcal{B}_i$ contains a path P_i from V_i to v ($i \in I$), and these paths are pairwise edge-disjoint because

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the branchings are pairwise edge-disjoint. If I is finite, then by Menger's theorem one can formulate the path condition in the following equivalent form:

$$\forall X \subseteq V \left(X \neq \varnothing \Longrightarrow \varrho_D(X) \geq |\{i \in I : V_i \cap X = \varnothing\}| \right). \tag{1}$$

The (strong form of) Edmonds' branching theorem (see [2] p. 349 Theorem 10.2.1) states that in the finite case (I and D are finite) condition (1) is enough to assure the existence of a spanning branching packing. R. Aharoni and C. Thomassen proved by a construction (see [1]) that this theorem fails for infinite digraphs. Even so, one can relax the finiteness condition for D in Edmonds' branching theorem. It is enough to assume that D does not contain backward-infinite paths as showed by C. Thomassen.

Theorem 1 (C. Thomassen (unpublished))**.** *Let $D = (V, A)$ be a digraph of arbitrary size that does not contain backward-infinite paths. Let $\varnothing \neq V_i \subseteq V$ for $i \in I$ where I is a finite index set. Then there are edge-disjoint spanning branchings $\{\mathcal{B}_i\}_{i \in I}$ in D (where the root-set of $\mathcal{B}_i$ is V_i) if and only if (1) holds.*

The main idea of Thomassen's proof is the following: construct first a spanning subgraph $D' = (V, A')$ of D such that D' also satisfies condition (1) and all vertices of D' have finite in-degrees. After that, one can build the desired spanning branching packing in D' using Edmonds' branching theorem and compactness arguments.

The exclusion of backward-infinite paths in Theorem 1 can be replaced by exclusion of forward-infinite paths as we proved in [4]. In this paper we are focusing on packing countably many spanning branchings with prescribed root-sets hence from now on our index set I will be $\mathbb{N}$. Let us introduce a slightly weaker assumption than excluding backward-infinite paths.

Condition 2. *Any backward-infinite path in D meets all the sets $\{V_i\}_{i \in \mathbb{N}}$.*

In this paper we prove the strengthening of Thomassen's result to countably many branchings, namely:

Theorem 3. *Suppose that the pair $(D, \{V_i\}_{i \in \mathbb{N}})$ satisfies Condition 2. Then there is a spanning branching packing in D with respect to $\{V_i\}_{i \in \mathbb{N}}$ (i.e. there is a system of edge-disjoint spanning branchings $\{\mathcal{B}_i\}_{i \in \mathbb{N}}$ in D such that the root-set of $\mathcal{B}_i$ is V_i) if and only if the path condition holds for $(D, \{V_i\}_{i \in \mathbb{N}})$.*

2 Proof of the main Theorem

The necessity of the path condition is obvious hence we will prove only its sufficiency. The key of the proof is the following lemma.

Lemma 4. *If the pair $(D, \{V_i\}_{i \in \mathbb{N}})$ satisfies Condition 2 and the path condition, then for any $j \in \mathbb{N}$ and $v \in V$ there is a path P from V_j to v in D such that the path condition holds for $\{V_i'\}_{i \in \mathbb{N}}$ in $D - A(P)$ where $V_i' = \begin{cases} V_i \cup V(P) & \text{if } i = j \\ V_i & \text{otherwise.} \end{cases}$*

We show first how Theorem 3 follows from Lemma 4. If $\mathcal{B}$ is a branching packing in D with respect to $\{V_i\}_{i \in \mathbb{N}}$, then let $D \setminus\!\!\setminus \mathcal{B} = (V, A \setminus \cup_{i \in \mathbb{N}} A(\mathcal{B}_i))$. We say that the branching packing $\mathcal{B}$

satisfies the path condition in D if the path condition holds with respect to $D \setminus\!\!\setminus \mathcal{B}$ and $\{V(\mathcal{B}_i)\}_{i\in\mathbb{N}}$. If $\mathcal{B}$ satisfies the path condition, then Lemma 4 makes possible to extend a prescribed $\mathcal{B}_j$ with a path in such a way that $\mathcal{B}_j$ reaches a prescribed vertex $v \in V \setminus V(\mathcal{B}_j)$ and the new branching packing still satisfies the path condition.

If V is just countable ($V = \{v_k\}_{k\in\mathbb{N}}$), then let $\{p_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{N}^2$. We build the branchings by recursion starting with the branching packing $\{(V_i, \varnothing)\}_{i\in\mathbb{N}}$. In the n-th step, we take $p_n = (j, k)$ and apply Lemma 4 with j and v_k for the actual branching packing. This process clearly builds the desired spanning branching packing.

For uncountable V, we apply transfinite recursion. We use Lemma 4 in every successor step and taking union in limit steps. The only arising problem is that we may violate the path condition at limit steps if we do the transfinite recursion in the "greedy way" as we did in the countable case. To handle the problem we organize the transfinite recursion in the following way. If a branching reaches a vertex v in some step, then before the next limit step we added v to all the branchings. It is doable since we have just countably many branchings. We claim that this ensures the path condition after the limit steps as well.

Indeed, let $\mathcal{B}$ be the branching packing that we have after some limit step and let $u \in V$ be arbitrary. We may fix a system of edge-disjoint paths $\{P_i\}_{i\in\mathbb{N}}$ in D such that P_i goes from V_i to u. Let v_i be the first vertex on P_i for which the terminal segment P_i' of P_i that starts at v_i is still a path in $D \setminus\!\!\setminus \mathcal{B}$. It is enough to show that $v_i \in V(\mathcal{B}_i)$. If $v_i = \mathsf{start}(P_i)$, then it is clear since $\mathsf{start}(P_i) \in V_i \subseteq V(\mathcal{B}_i)$. If $v_i \neq \mathsf{start}(P_i)$, then by choice of v_i there is a successor step in which some branching reaches v_i, but then before the next limit step we added v_i to all the branchings thus $v_i \in V(\mathcal{B}_i)$. ■

It worth to mention that in the case of infinitely many V_i the path condition and the exclusion of forward-infinite paths are no more enough to guarantee the existence of a spanning branching packing. Let for example $V = \{t\} \cup \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}$ and let A consists of the following edges (see Figure 1)

1. infinitely many parallel edges from $(m, n + 1)$ to (m, n),
2. edge from (m, n) to $(m + 1, n)$,
3. edge from $(2m + 2, n)$ to $(2m, n)$,
4. edge from (m, m) to t,
5. edge from t to $(2m + 1, n)$ (*not in the figure*!).

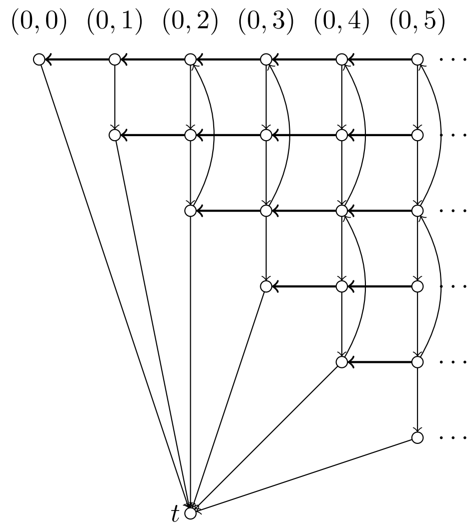

Figure 1: The outgoing edges of t (a single edge to each vertex in an odd row) are not on the figure because of transparency reasons. The thick horizontal edges stand for infinitely many parallel edges. Furthermore let $V_n = \{(0,n)\}$.

Observe that after the deletion of t just finitely many vertices are reachable from any vertex which shows that there is no forward-infinite path in $D := (V, A)$. Let $V_n = \{(0,n)\}$. It is easy to check (using Figure 1) that path condition holds. Suppose to the contrary that there is a $\mathcal{B} = \{\mathcal{B}_n\}_{n\in\mathbb{N}}$ spanning branching packing. For $\mathcal{B}_0$ the only possibility to reach t is to use the single edge from $(0,0)$ to t. Suppose that we already know for some $0 < N$ that $\mathcal{B}_n$ contains the path $P_n := (0,n), (1,n), \dots, (n,n), t$ whenever $n < N$. By using just the remaining edges, t is no more reachable from columns $0, \dots, N-1$. Hence for $\mathcal{B}_N$ the path $(0,N), (1,N), \dots, (N,N), t$ is the only possible option to reach t (see Figure 1). On the other hand after the deletion of the edges of paths P_n for all n the vertices $\{(0,n) : 1 \leq n \in \mathbb{N}\}$ are no longer reachable from $\{(0,0), t\}$. This prevents $\mathcal{B}_0$ to be a spanning branching rooted at $(0,0)$ which is a contradiction.

By symmetry, it is enough to prove Lemma 4 for $j = 0$. Before the proof we have to generalize some phenomena that are well-known from finite branching-packing theorems.

3 Generalization of tight and dangerous sets

In the context of Edmonds' branching theorem a set $X \subseteq V$ is usually called tight if $\varrho_D(X) = |\{i : V_i \cap X = \varnothing\}|$. In the presence of infinitely many V_i this definition of tightness is no more useful. In this section we give another definition of tightness which is equivalent with the original for finitely many V_i and keeps all the nice properties of tight sets that are known from the finite case.

From now on let $D = (V, A)$ and $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ be fixed in such a way that path condition and Condition 2 holds. We call a set $\varnothing \neq X \subseteq V$ **tight** (with respect to $\mathcal{V}$ and D) if whenever $\{P_i\}_{i\in\mathbb{N}}$ is a system of edge-disjoint paths in D such that P_i goes from V_i to X, then the paths P_i necessarily use all the ingoing edges of X i.e. $\mathsf{in}_D(X) \subseteq \cup_{i\in\mathbb{N}} A(P_i)$. (If $V_i \cap X \neq \varnothing$ then one may choose P_i as a path consisting of just a single vertex, thus the definition is really about those i for which $X \cap V_i = \varnothing$.)

Proposition 5. *Let B be tight, $v \in B$ arbitrary and let $\{P_i\}_{i\in\mathbb{N}}$ be a system of edge-disjoint paths in D such that P_i goes from V_i to v. Then $V(P_i) \subseteq B$ if $V_i \cap B \neq \varnothing$, and P_i uses exactly one ingoing edge of B i.e. $|A(P_i) \cap \mathsf{in}_D(B)| = 1$ if $V_i \cap B = \varnothing$.*

Proof: Assume, to the contrary, $V(P_i) \nsubseteq B$ and $V_i \cap B \neq \varnothing$ for some i. Then P_i uses some edge $e \in \mathsf{in}_D(B)$. Replace P_i by a path that consists of a single vertex $u \in V_0 \cap B$. The modified path-system no more uses e but all of its members have a vertex in B. Hence by taking the appropriate initial segments of the paths we get a contradiction with the tightness of B.

For the second part of the proposition, if $|A(P_i) \cap \mathsf{in}_D(B)| > 1$ holds, then by replacing P_i by its appropriate initial segment we get a contradiction in similar way. ●

A $B \subseteq V$ is **dangerous** if it is tight and $V_0 \cap B \neq \varnothing$.

Proposition 6. *If $B_0, B_1 \subseteq V$ are dangerous sets with nonempty intersection, then $B_0 \cap B_1$ is also dangerous.*

Proof: Let $\{P_i\}_{i\in\mathbb{N}}$ be a system of edge-disjoint paths in D such that P_i goes from V_i to $B_0 \cap B_1$. Suppose, to the contrary, that there is an edge $e \in \mathsf{in}_D(B_0 \cap B_1)$ which is not used by any P_i. By symmetry, we may assume that $e \in \mathsf{in}_D(B_0)$. By taking appropriate initial segments of the paths, we get a contradiction with the tightness of B_0. Therefore $B_0 \cap B_1$ is tight.

To prove the dangerousness let $v \in B_0 \cap B_1$ and let $\{Q_i\}_{i\in\mathbb{N}}$ be a system of edge-disjoint paths in D such that Q_i goes from V_i to v. By using Proposition 5 for $i = 0$ with B_0 and with B_1 separately, we get $V(Q_0) \subseteq B_0 \cap B_1$ hence $\mathsf{start}(Q_0) \in V_0 \cap B_0 \cap B_1$ thus $B_0 \cap B_1$ is dangerous.●

Remark 7. It is not too hard to show that $B_0 \cup B_1$ is also dangerous and there are no edges between $B_0 \setminus B_1$ and $B_1 \setminus B_0$ in any direction as it was the expectation from the finite case but in this paper we do not need these facts.

For multisets $\mathcal{V}$ and $\mathcal{T}$, we denote by $\mathcal{V} \uplus \mathcal{T}$ the multiset where the multiplicity of an element is the sum of its multiplicities in $\mathcal{V}$ and $\mathcal{T}$. For $X \subseteq V$, let

$$\boldsymbol{\mathcal{V}[X]} \stackrel{\text{def}}{=} \{\{\mathsf{end}(e)\} : e \in \mathsf{in}_D(X)\} \uplus \{V_i \cap X : i \in \mathbb{N},\ V_i \cap X \neq \varnothing\}$$

(Here we consider $\{\{\mathsf{end}(e)\} : e \in \mathsf{in}_D(X)\}$ as a multiset, one singleton for each edge.)

Proposition 8. *For a tight B, the multiset $\mathcal{V}[B]$ satisfies the path condition in $D[B]$. Furthermore a set $X \subseteq B$ is dangerous with respect to $(D, \mathcal{V})$ iff X is dangerous with respect to $(D[B], \mathcal{V}[B])$.*

Proof: Let $v \in B$ be arbitrary. The system $\mathcal{V}$ satisfies the path condition in D thus we may fix a system of edge-disjoint paths $\{P_i\}_{i\in\mathbb{N}}$ in D such that P_i goes from V_i to v. The definition of tightness and Proposition 5 shows that the terminal segments of paths $\{P_i\}_{i\in\mathbb{N}}$ from the first vertex in B certify that v is simultaneously reachable from $\mathcal{V}[B]$ in $D[B]$.

Assume that $X \subseteq B$ is not dangerous with respect to $(D, \mathcal{V})$. Pick a path-system $\{Q_i\}_{i\in\mathbb{N}}$ such that Q_i goes from V_i to X and some $f \in \mathsf{in}_D(X)$ is unused by the paths Q_i. Necessarily $f \in \mathsf{in}_{D[B]}(X)$ since paths Q_i have to use all the edges $\mathsf{in}_D(B)$ because the tightness of B. Cut the initial segments of the Q_i's that are not in B. The resulting system shows that X is not dangerous with respect to $D[B]$ and $\mathcal{V}[B]$. The other direction is similar. ●

Proposition 9. *If B is a tight set with $\mathsf{in}_D(B) \neq \varnothing$, then $D[B]$ does not contain backward-infinite paths.*

Proof: Such a B must be disjoint from at least one V_i hence the proposition follows directly from Condition 2. ●

4 Proof of the key-Lemma

For an $e \in \mathsf{out}_D(V_0)$, the **single-edge extension** of $(D, \mathcal{V})$ corresponds to e is the pair $(D-e, \mathcal{V}^+)$, where $\mathcal{V}^+ = \{V_i^+\}_{i\in\mathbb{N}}$ such that $V_i^+ = \begin{cases} V_i \cup \{\mathsf{end}(e)\} & \text{if } i = 0 \\ V_i & \text{otherwise.} \end{cases}$ Note that Condition 2 remains true for $(D-e, \mathcal{V}^+)$ automatically. The single-edge extension is called **feasible** if the path condition remains true as well. Without Condition 2 one cannot guarantee the existence a feasible single-edge extension. Not even if there are just two V_i (see Figure 2).

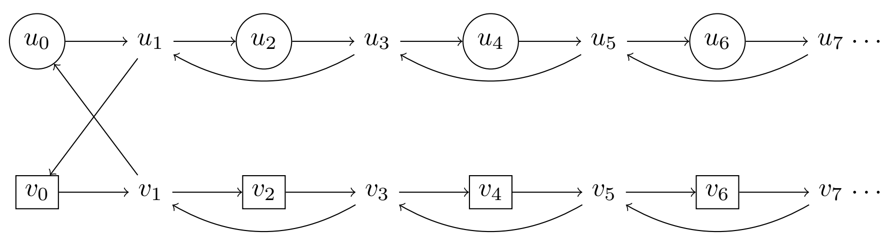

Figure 2: Path condition holds but there is no feasible single-edge extension. Elements of V_0 are circled and vertices in V_1 are in rectangle.

Remark 10. If V_0 is finite, then one can show that the path condition is enough to ensure the existence of a feasible single-edge extension. Even so it is not enough to guarantee the existence of a spanning branching packing. Indeed, pick a 2-edge-connected digraph D that contains vertices u, v such that there is no edge-disjoint back and forth paths between u and v. (Such a digraph exists, even with arbitrary large finite edge-connectivity as we have shown in [5].) Let $V_0 = \{u\}$ and $V_1 = \{v\}$. On the one hand, the 2-edge-connectivity it follows that every vertex can be reached simultaneously from u and v by edge-disjoint paths thus the path condition holds. On the other hand, a hypothetical spanning branching packing should contain back and forth paths between u and v which does not exist in D.

Proposition 11. *If for the edge $e \in \mathsf{out}_D(V_0)$ there is a dangerous set B such that $e \in \mathsf{in}_D(B)$, then set B is no more simultaneously reachable with respect to $(D-e, \mathcal{V}^+)$ and hence the single-edge extension corresponds to e is infeasible.*

Proof: The family of those sets V_i that are disjoint from B remains the same with respect to $\mathcal{V}^+$. When these sets reach simultaneously B in D they need to use all the edges in $\mathsf{in}_D(B)$ including e. But edge e is no more available with respect to $(D-e, \mathcal{V}^+)$. ●

The reverse implication is also true (without even assuming Condition 2).

Claim 12. *If the single-edge extension corresponds to $e \in \mathsf{out}_D(V_0)$ is infeasible, then e enters into some dangerous set B.*

If $V_i = V$ for all large enough i (i.e. we have essentially just finitely many root-sets), then the proof of Claim 12 is easy. Indeed, we may use the equivalent formulation of the path condition (namely condition (1)). In this case, tightness of a set X means that X satisfies the inequality at (1) with equality. Assume that the single-edge extension $(D - e, \mathcal{V}^+)$ corresponds to the edge $e \in \mathsf{out}_D(V_0)$ violates (1) and let B be a witness of it. Then necessarily $\varrho_{D-e}(B) < \varrho_D(B)$ (thus $e \in \mathsf{in}_D(B)$) and $|\{i \in \mathbb{N} : V_i \cap B = \varnothing\}| = \left|\{i \in \mathbb{N} : V_i^+ \cap B = \varnothing\}\right|$ (hence B intersects V_0). Finally $\varrho_D(B) = |\{i \in \mathbb{N} : V_i \cap B = \varnothing\}|$ (because the extension can worsen the inequality (1) at most by one) which implies the dangerousness of B. In the general case, the proof of Claim 12 is less trivial. We present the proof in the last section.

A pair $(D', \mathcal{V}')$ (where $\mathcal{V}' = \{V_i'\}_{i \in \mathbb{N}}$) is a **finitary extension** of $(D, \mathcal{V})$ if one can obtain it from $(D, \mathcal{V})$ as a finite sequence of consecutive feasible single-edge extensions. Note that for any $v \in V_0'$ there is a unique path in D from V_0 to v that consists of edges from $A(D) \setminus A(D')$.

Lemma 13. *For any $u_0 \in V$, there is a finitary extension $(D', \mathcal{V}')$ of $(D, \mathcal{V})$ for which $u_0 \in V_0'$.*

We claim that to prove Lemma 4 it is enough to show the lemma above. Indeed, suppose that such finitary extension $(D', \mathcal{V}')$ exists. Let P be the unique path from V_0 to u_0 in D that consists of edges from $A(D) \setminus A(D')$. We need to show that path condition holds for $(D - A(P), \{V_0 \cup V(P)\} \cup \{V_i\}_{1 \leq i \in \mathbb{N}})$. Let $x \in V$ be arbitrary. Since the path condition holds for $(D', \mathcal{V}')$, we can fix a system of edge-disjoint paths $\{P_i'\}_{i \in \mathbb{N}}$ in D' such that P_i' goes from V_i' to x. For $i \neq 0$, let $P_i = P_i'$. Consider the unique path Q that goes from V_0 to $\mathsf{start}(P_0')$ in D and for which $A(Q) \subseteq A(D) \setminus A(D')$. From Q and P_0' we can obtain a $V_0 \cup V(P) \to x$ path P_0 in $D - A(P)$ which is disjoint from the paths $\{P_i\}_{0 \neq i \in \mathbb{N}}$. The path-system $\{P_i\}_{i \in \mathbb{N}}$ shows that x is simultaneously reachable from $\{V_0 \cup V(P)\} \cup \{V_i\}_{0 \neq i \in \mathbb{N}}$ in $D - A(P)$.

To prove Lemma 13 assume, seeking contradiction, that there is no finitary extension $(D', \mathcal{V}')$ of $(D, \mathcal{V})$ for which $u_0 \in V_0'$. Pick a finitary extension $(D_0, \mathcal{V}_0)$ of $(D, \mathcal{V})$ and a system $\{P_i^0\}_{i \in \mathbb{N}}$ of edge-disjoint paths in D_0 (where P_i^0 goes from V_i^0 to u_0) for which $\left|A(P_0^0)\right|$ as small as possible. By the indirect assumption, it cannot be 0. Consider the first edge e_1 of P_0^0. The single-edge extension of $(D_0, \mathcal{V}_0)$ corresponds to e_1 may not be feasible because of the minimality of $\left|A(P_0^0)\right|$. By Claim 12, edge e_1 enters into a set B_1 which is dangerous with respect to $(D_0, \mathcal{V}_0)$. We claim $u_0 \notin B_1$. Indeed, if $u_0 \in B_1$, then all the paths P_i^0 meet B_1. Hence the trivial path consists of $\mathsf{end}(e_1)$ and the paths $\{P_i^0\}_{0 \neq i \in \mathbb{N}}$ shows that B_1 is simultaneously reachable with respect to the single-edge extension of $(D_0, \mathcal{V}_0)$ corresponds to e_1 which contradicts to Proposition 11. Let u_1 be the last vertex of P_0^0 in B_1 and let $\boldsymbol{Q_0}$ be the terminal segment of P_0^0 starting at u_1. Let us denote $(D_0[B_1], \mathcal{V}_0[B_1])$ by $\boldsymbol{(G_0, \mathcal{U}_0)}$. Let $\{U_i^0\}_{i \in \mathbb{N}}$ be an enumeration of $\mathcal{U}_0$ where $U_0^0 = V_0^0 \cap B_1$.

Proposition 14. *$(G_0, \mathcal{U}_0)$ and u_1 is a counterexample to Lemma 13 i.e. there is no finitary extension $(G_1, \mathcal{U}_1)$ of $(G_0, \mathcal{U}_0)$ for which $u_1 \in U_0^1$.*

First we show how Lemma 13 follows applying the Proposition above. We know that for an arbitrary counterexample $(D, \mathcal{V}), u_0$ we can find a finitary extension $(D_0, \mathcal{V}_0)$ of $(D, \mathcal{V})$ and a vertex set $B_1 \not\ni u_0$ such that there is an $u_1 \in B_1$ for which $(D_0[B_1], \mathcal{V}_0[B_1])$ and u_1 form a counterexample again (see Figure 3). Furthermore there is a path Q_0 from u_1 to u_0 for which $B_1 \cap V(Q_0) = \{u_1\}$. We may apply this fact to the new counterexample as well and all to the further counterexamples recursively. Let $B_0 = V$. During the process we obtain

- a nested sequence of vertex sets $(B_n)_{n\in\mathbb{N}}$,
- vertices $(u_n)_{n\in\mathbb{N}}$ where $u_n \in B_n \setminus B_{n+1}$,
- paths $(Q_n)_{n\in\mathbb{N}}$ where Q_n is a path from u_{n+1} to u_n in $D_0[B_n]$ with $B_{n+1} \cap V(Q_n) = \{u_{n+1}\}$.

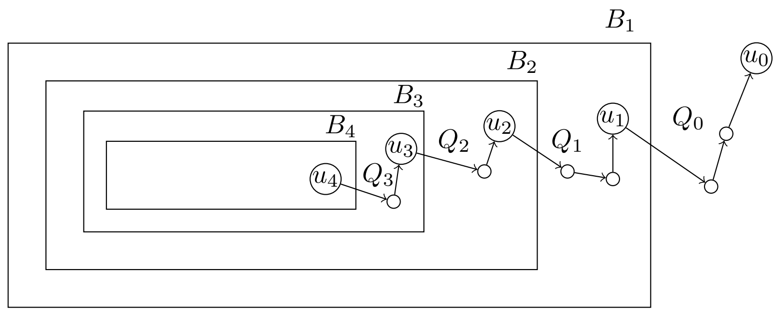

Figure 3: The construction of the backward-infinite path in $D_0[B_1]$.

By uniting the paths $\{Q_{n+1}\}_{n\in\mathbb{N}}$, we obtain a backward-infinite path in $D_0[B_1]$. Since B_1 is dangerous with respect to $(D_0, \mathcal{V}_0)$, it contradicts to Proposition 9 which proves Lemma 13.

To prove Proposition 14 observe that if some e determines a feasible single-edge extension of $(G_0, \mathcal{U}_0)$, then it determines such an extension of $(D_0, \mathcal{V}_0)$ as well. Indeed, if $e \in \mathsf{out}_{D_0[B_1]}(V_0^0 \cap B_1)$ enters into a set X which is dangerous with respect to $(D_0, \mathcal{V}_0)$, then $B_1 \cap X$ is also dangerous with respect to $(D_0, \mathcal{V}_0)$ by Proposition 6. But then by Proposition 8 it is dangerous with respect to $(G_0, \mathcal{U}_0)$ as well. Thus $X \cap B_1$ shows that the single-edge extension of $(G_0, \mathcal{U}_0)$ corresponds to e is not feasible. It follows that any finitary extension $(G_1, \mathcal{U}_1)$ of $(G_0, \mathcal{U}_0)$ determines a unique finitary extension $(D_1, \mathcal{V}_1)$ of $(D_0, \mathcal{V}_0)$.

Assume, to the contrary, that Proposition 14 is false. Then by the arguments above there is a finitary extension $(D_1, \mathcal{V}_1)$ of $(D_0, \mathcal{V}_0)$ such that $u_1 \in V_0^1$ and $A(D_0) \setminus A(D_1) \subseteq \mathsf{span}_{D_0}(B_1)$. Let $\boldsymbol{P_0^1} = Q_0$. It is enough to show the we are able to extend the singleton $\{P_0^1\}$ to a set of edge-disjoint paths $\{P_i^1\}_{i\in\mathbb{N}}$ such that P_i^1 goes from V_i^1 to u_0. Indeed, $(D_1, \mathcal{V}_1)$ is a finitary extension of $(D, \mathcal{V})$ as well and $\left|A(P_0^1)\right| < \left|A(P_0^0)\right|$ contradicts to the minimality of $\left|A(P_0^0)\right|$.

If none of the paths $\{P_i^0\}_{0\neq i\in\mathbb{N}}$ contains an edge from $A(D_0) \setminus A(D_1)$, then $P_i^1 := P_i^0$ for $i > 0$ is an appropriate choice. In general, the deletion of edges $A(D_0) \setminus A(D_1)$ ruins finitely many of the paths $\{P_i^0\}_{0\neq i\in\mathbb{N}}$. Our plan is to fix these paths inside B_1 applying the fact that $(G_1, \mathcal{U}_1)$ satisfies the path condition. To do so we need the following version of the well-known augmenting path technique developed by L. R Ford and D. R Fulkerson (see [3]).

Proposition 15. *Let $\{P_i\}_{i\in I}$ be a system of edge-disjoint $s \to t$ paths in a digraph H (where $s \neq t \in V(H)$) and denote the first and the last edge of P_i by e_i and by f_i respectively. Let $\overleftarrow{H}$ be the digraph that we obtain by changing the direction of edges $\cup_{i\in I} A(P_i)$ in H. We call these edges* ***backward edges*** *of $\overleftarrow{H}$ and we call* ***forward edges*** *the others. Denote by $\boldsymbol{U}$ the set of the vertices in $V(H)$ that are unreachable in $\overleftarrow{H}$ from s. If $t \notin U$ and path R certifies it, where the first edge of R is e and the last edge of R is f, then there is a system of edge-disjoint $s \to t$ paths $\{Q_j\}_{j\in J}$ in H such that the set of the first and the set of the last edges of paths $\{Q_j\}_{j\in J}$ are $\{e_i\}_{i\in I} \cup \{e\}$ and $\{f_i\}_{i\in I} \cup \{f\}$ respectively.*

If $t \in U$, then the paths $\{P_i\}_{i\in I}$ use all the edges in $\mathsf{in}_H(U)$ and each P_i uses exactly one such an edge.

Proof: Suppose first that $t \notin U$. Consider a finite subgraph H' of H that contains R and those from the paths P_i, say $P_1, \dots, P_k$, that give a backward-edge to R. The $P_1, \dots, P_k$ paths determinate a flow of amount k with respect to the constant 1 upper bound on the edges in H'. By the technique of Ford and Fulkerson, one can get by using R a flow of amount $k+1$ in H'. By decomposing this flow to $k+1$ edge-disjoint $s \to t$ paths (applying the greedy method) and keeping the untouched P_i's, we get the desired system $\{Q_j\}_{j\in J}$. The second part of the Proposition 15 follows directly from the construction of $\overleftarrow{H}$ and from the definition of U.●

If $A(P_i^0) \cap \mathsf{span}_{D_0}(B_1) = \varnothing$ for some $0 \neq i \in \mathbb{N}$, then let $P_i^1 = P_i^0$. Consider now $I := \{0 \neq i \in \mathbb{N} : A(P_i^0) \cap \mathsf{span}_{D_0}(B_1) \neq \varnothing\}$. All but finitely many from the paths $\{P_i^0\}_{i\in I}$ are still paths in D_1. To simplify the notation we assume that the problematic paths (i.e. paths that has edge in $A(D_0) \setminus A(D_1)$) are $P_1^0, \dots, P_k^0$. For $i \in I$, let us denote by u_i and v_i the first and the last intersection of P_i^0 with B_1. We construct a digraph H starting with $G_1 = D_1[B_1]$ (see figure 4). If for some $w, z \in B_1$ there is an $i \in I$ such that P_i^0 has a $w \to z$ segment for which the interior vertices of the segment are not in B_1, then draw a new e_{iwz} edge from w to z. Pick a new vertex t and draw an edge f_i from v_i to t $(i \in I)$. For all $i \in I$, pick a new vertex s_i. If path P_i^0 starts inside B_1, then draw a single edge from s_i to each element of $V_i^1 \cap B_1$, otherwise draw an edge from s_i to u_i. Finally pick a new vertex s and draw the edges ss_i. Construction of H is complete.

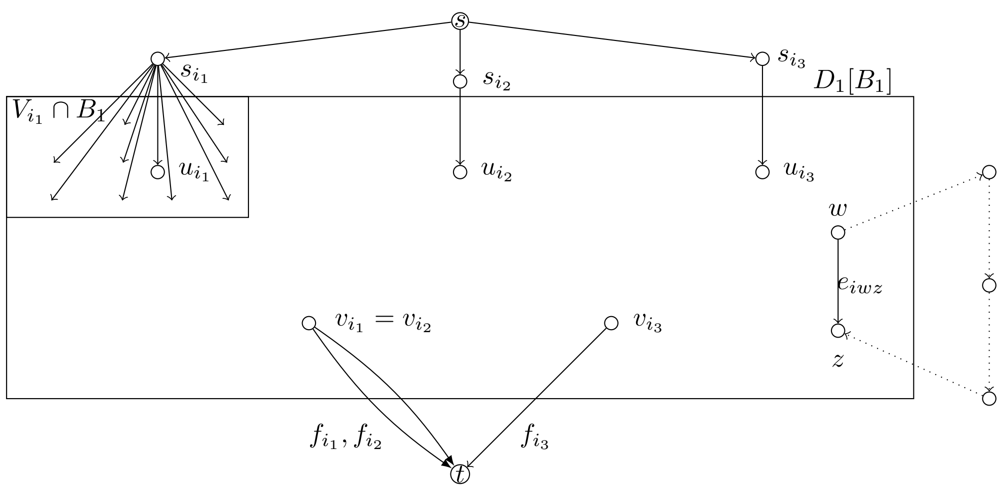

Figure 4: Construction of H from $D_1[B_1]$. Here $B_1 \cap V_{i_1} \neq \varnothing$ but $B_1 \cap V_{i_2} = B \cap V_{i_3} = \varnothing$.

It is enough to show that there is a system of edge-disjoint $s \to t$ paths $\{Q_i\}_{i\in I}$ in H such that the first edge of path Q_i is ss_i. Indeed, then we can construct the paths $\{P_i^1\}_{i\in I}$ in the following way. Let $i \in I$ arbitrary and assume first that P_i^0 starts in $B_1 \cap V_i$. Let the last edge of Q_i be f_j for some $j \in I$. Denote by Q_i' the path that we obtain from Q_i by deleting the first two vertices (s and s_i) and the last vertex (t) of it and replace the edges in the form e_{lwz} with the $w \to z$ segment of P_l^0. We get P_i^1 by uniting Q_i' with the terminal segment of P_j^0 that starts at v_j. If P_i^0 starts outside B_1 we do the same except we have to use also the initial segment of P_i^0 that ends in u_i to the construction of P_i^1. Finally $(D_1, \mathcal{V}_1)$ and $\{P_i^1\}_{i\in\mathbb{N}}$ will contradict to the choice of $(D_0, \mathcal{V}_0)$ and $\{P_i^0\}_{i\in\mathbb{N}}$ which justifies Lemma 13 as we have already mentioned.

For $k < i \in I$, we can construct from P_i^0 an $s \to t$ path R_i in H in a natural way. Indeed, take the segment of P_i^0 between u_i ad v_i and replace the segments of it that leave B_1 with the corresponding edges e_{lwz}, finally give the two new initial vertices s, s_i and the new last edge f_i to it. Then $\{R_i\}_{k<i\in I}$ is a system of edge-disjoint $s \to t$ paths in H such that the first edge of R_i is ss_i and edges $f_1, \dots, f_k$ are unused by the path-system. Try to extend this system applying augmentation path method (Proposition 15). If it succeeds, then iterate this with the resulting path-system. Assume that it does not. Then $\mathsf{start}(f_1) \in U$ (we use here the notation of Proposition 15) since f_1 is a forward edge of $\overleftarrow{H}$. The vertex $\mathsf{start}(f_1)$ is simultaneously reachable from $\mathcal{V}_1[B_1]$ in $D_1[B_1]$ which implies by construction of H that there is a system of edge-disjoint $s \to \mathsf{start}(f_1)$ paths in H that uses all the outgoing edges of s (namely $\{ss_i\}_{i\in I}$). Take the initial segments of these paths that goes from s to U and extend them by using the terminal segments of the paths $\{R_i\}_{k<i\in I}$ from the first vertex in U (see the second part of Proposition 15) to a system of edge-disjoint $s \to t$ paths that uses all the outgoing edges of s.

5 Characterization of infeasible single-edge extensions

It only remains to prove Claim 12. Let $D = (V, A)$ be a digraph and $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ where $V_i \subseteq V$ (we do not assume the path condition or Condition 2). Fore $X \subseteq V$ let us denote $\{i \in \mathbb{N} : V_i \cap X \neq \varnothing\}$ by $\boldsymbol{I_{\mathcal{V}}(X)}$ and let $\boldsymbol{O_{\mathcal{V}}(X)} = \mathbb{N} \setminus I_{\mathcal{V}}(X)$. Let $t \in V$ be arbitrary. A set $X \subseteq V$ is called ***t*-good** (with respect to $(D, \mathcal{V})$) if $t \in X$ and t is simultaneously reachable from $\mathcal{V}[X]$ in $D[X]$ i.e. there is a system $\{P_i\}_{i\in I_{\mathcal{V}}(X)} \cup \{P_e\}_{e\in \mathsf{in}_D(X)}$ of edge-disjoint paths in $D[X]$ such that P_i goes from $V_i \cap X$ to t and P_e goes from $\mathsf{end}(e)$ to t.

Observation 16. *There is a $\subseteq$-smallest t-good set, namely $\{t\}$.*

Proposition 17. *If X and Y are t-good sets, then $X \cup Y$ is a t-good set as well.*

Proof: Let $\{P_i\}_{i\in I_{\mathcal{V}}(X)} \cup \{P_e\}_{e\in \mathsf{in}_D(X)}$ and $\{Q_i\}_{i\in I_{\mathcal{V}}(Y)} \cup \{Q_e\}_{e\in \mathsf{in}_D(Y)}$ be path-systems that show the t-goodness of X and Y respectively. Note that all the common edges of the two path-systems are in $\mathsf{span}_D(X \cap Y)$. For

$$s \in (I_{\mathcal{V}}(Y) \setminus I_{\mathcal{V}}(X)) \cup [(\mathsf{in}_D(Y) \cap \mathsf{in}_D(X \cup Y)) \setminus \mathsf{in}_D(X)] =: S,$$

let R_s be the path that we obtain by taking the initial segment of Q_s up to the first vertex in X and join it with P_e, where e is the last edge of this initial segment. The path-system

$$\{P_s : s \in I_{\mathcal{V}}(X) \cup (\mathsf{in}_D(X) \cap \mathsf{in}_D(X \cup Y))\} \cup \{R_s : s \in S\}$$

shows that $X \cup Y$ is t-good. ●

Proposition 18. *For any $\subseteq$-increasing nonempty chain $\langle X_\beta : \beta < \alpha\rangle$ of t-good sets, $X := \bigcup_{\beta<\alpha} X_\beta$ is t-good.*

Proof: For $\beta \leq \alpha$, we define a path-system $\{P_s^\beta : s \in I_{\mathcal{V}}(X_\beta) \cup \mathsf{in}_D(X_\beta)\} =: \mathcal{P}_\beta$ by transfinite recursion such that

- $\mathcal{P}_\beta$ witnesses the t-goodness of X_β,

- $P_s^\beta = P_s^\gamma$ if $\gamma < \beta$ and $s \in I_\mathcal{V}(X_\gamma) \cup (\mathsf{in}_D(X_\beta) \cap \mathsf{in}_D(X_\gamma))$,
- if $\gamma < \beta$ and $s \in (I_\mathcal{V}(X_\beta) \setminus I_\mathcal{V}(X_\gamma)) \cup (\mathsf{in}_D(X_\beta) \setminus \mathsf{in}_D(X_\gamma))$, then there is unique $e \in \mathsf{in}_D(X_\gamma)$ such that $e \in A(P_s^\beta)$ and the terminal segment of P_s^β from $\mathsf{end}(e)$ is P_e^γ.

Let $\mathcal{P}_0$ be arbitrary that shows the t-goodness of X_0. To construct $\mathcal{P}_{\beta+1}$ pick first an arbitrary $\mathcal{Q} = \{Q_s : s \in I_\mathcal{V}(X_{\beta+1}) \cup \mathsf{in}_D(X_{\beta+1})\}$ that shows the t-goodness of $X_{\beta+1}$. For $s \in (I_\mathcal{V}(X_{\beta+1}) \setminus I_\mathcal{V}(X_\beta)) \cup (\mathsf{in}_D(X_{\beta+1}) \setminus \mathsf{in}_D(X_\beta))$, take the first edge e_s of Q_s that enters into X_β and join the initial segment of Q_s up to $\mathsf{end}(e_s)$ with $P_{e_s}^\beta$ to obtain $P_s^{\beta+1}$. For $s \in I_\mathcal{V}(X_\beta) \cup (\mathsf{in}_D(X_{\beta+1}) \cap \mathsf{in}_D(X_\beta))$, let $P_s^{\beta+1} = P_s^\beta$.

If β is a limit ordinal and $s \in I_\mathcal{V}(X_\beta) \cup \mathsf{in}_D(X_\beta)$, then consider the smallest $\gamma < \beta$ for which $s \in I_\mathcal{V}(X_\gamma) \cup \mathsf{in}_D(X_\gamma)$ and let $P_s^\beta = P_s^\gamma$. ●

Observation 16 and Proposition 17 and 18 imply the following.

Corollary 19. *There exists a $\subseteq$-largest t-good set, namely the union of all the t-good sets.*

Assume now that the path condition holds for $(D, \mathcal{V})$. Let $e_0 \in \mathsf{out}_D(V_0)$ and suppose that $t \in V$ shows that the path condition does not hold for the single-edge extension $(D - e_0, \mathcal{V}^+)$ of $(D, \mathcal{V})$ that corresponds to e_0. Let us denote by B the $\subseteq$-largest t-good set with respect to $(D - e_0, \mathcal{V}^+)$. We will show that B is dangerous with respect to $(D, \mathcal{V})$ and $e_0 \in \mathsf{in}_D(B)$.

Let $\{P_i\}_{i \in O_\mathcal{V}(B)}$ be an arbitrary system of edge-disjoint paths in D where P_i goes from V_i to B. Let $\{Q_i\}_{i \in I_{\mathcal{V}^+}(B)} \cup \{P_e\}_{e \in \mathsf{in}_{D-e_0}(B)}$ be a path-system which shows that B is t-good with respect to $(D - e_0, \mathcal{V}^+)$. There is some $i_0 \in O_\mathcal{V}(B)$ for which $e_0 \in A(P_{i_0})$ otherwise for $i \in O_\mathcal{V}(B)^+ \subseteq O_\mathcal{V}(B)$ we join path P_i with the path P_e (where e is the last edge of P_i) to obtain Q_i and then the path-system $\{Q_i\}_{i \in \mathbb{N}}$ contradicts to the choice of t. Clearly $i_0 \neq 0$ otherwise we may replace P_{i_0} by its own terminal segment starting at $\mathsf{end}(e_0) \in V_0^+$ and get contradiction in the same way. We are able define the path-system $\{Q_i\}_{i_0 \neq i \in \mathbb{N}}$ in $D - e_0$ as described above.

Extend $D - e_0$ with the new vertices s and $\{s_i\}_{i \in \mathbb{N}}$ and the new edges $\{ss_i\}_{i \in \mathbb{N}}$ and $\{s_i v : i \in \mathbb{N} \wedge v \in V_i^+\}$ to obtain H and for $i_0 \neq i \in \mathbb{N}$ extend Q_i with the new initial vertices s and s_i to get the path Q_i^+. Then $\{Q_i^+\}_{i_0 \neq i \in \mathbb{N}}$ is a system of edge-disjoint $s \to t$ paths in H. It uses all the outgoing edges of s except ss_{i_0} and by the choice of t there is no edge-disjoint system of $s \to t$ paths which uses all the edges $\{ss_i\}_{i \in \mathbb{N}}$. We apply the augmentation path method (Proposition 15). Let $\overleftarrow{H}$ be the digraph that we obtain from H by changing the direction of edges $\cup_{i_0 \neq i \in \mathbb{N}} A(Q_i^+)$ and let U be the set of vertices that are unreachable from s in $\overleftarrow{H}$. We know that $t \in U$ since augmentation is impossible.

Proposition 20. $U \cap V \subseteq B$.

Proof: It is enough to show that $U \cap V$ is t-good with respect to $(D - e_0, \mathcal{V}^+)$. By Proposition 15, the paths $\{Q_i^+\}_{i_0 \neq i \in \mathbb{N}}$ uses all the edges $\mathsf{in}_H(U)$ and each of them uses exactly one. For $e \in \mathsf{in}_{D-e_0}(U \cap V)$, let Q_e be the terminal segment (starting from $\mathsf{end}(e)$) of the unique Q_i^+ for which $e \in A(Q_i^+)$.

Assume that $i \in I_{\mathcal{V}^+}(U \cap V)$. We claim that $ss_i \in \mathsf{in}_H(U)$. Suppose that $s_i \notin U$. If $i = i_0$, then the forward edges $\{s_{i_0} v : v \in V_{i_0}^+\}$ ensures $V_{i_0}^+ \cap U = \varnothing$ which contradicts to $i \in I_{\mathcal{V}^+}(U \cap V)$. If $i \neq i_0$, then Q_i is defined and $\mathsf{start}(Q_i) \notin U$ because the only ingoing edge of s_i in $\overleftarrow{H}$ comes from $\mathsf{start}(Q_i)$. Furthermore $\{s_i v : v \in V_i^+ \setminus \{\mathsf{start}(Q_i)\}\}$ are forward edges of $\overleftarrow{H}$ hence $V_i^+ \cap U = \varnothing$ which is a contradiction again.

The forward edge ss_{i_0} shows $s_{i_0} \notin U$ and hence $i_0 \notin I_{\mathcal{V}^+}(U \cap V)$. Therefore Q_i is defined for $i \in I_{\mathcal{V}^+}(U \cap V)$. Furthermore it lies in $V \cap U$ since Q_i^+ enters into U by the edge ss_i. The path-system $\{Q_e : e \in \mathsf{in}_{D-e_0}(U \cap V)\} \cup \{Q_i : i \in I_{\mathcal{V}^+}(U \cap V)\}$ justifies the t-goodness of $U \cap V$ with respect to $(D - e, \mathcal{V}^+)$. ●

If some edge $f \in \mathsf{in}_D(B)$ is unused by the paths $\{P_i\}_{i \in O_{\mathcal{V}}(B)}$ (which implies $f \neq e_0$), then we did not use path P_f in the construction of paths $\{Q_i\}_{i_0 \neq i \in \mathbb{N}}$ and hence P_f is edge-disjoint from the paths $\{Q_i^+\}_{i_0 \neq i \in \mathbb{N}}$. Then necessarily $\mathsf{start}(f) \in U$ because otherwise P_f would make t reachable from s in $\overleftarrow{H}$. Therefore $\mathsf{start}(f) \in (U \cap V) \setminus B$ hence the existence of such an f contradicts to Proposition 20. Since $\{P_i\}_{i \in O_{\mathcal{V}}(B)}$ was arbitrary, it justifies the tightness of B with respect to $(D, \mathcal{V})$.

If $e_0 \notin \mathsf{in}_D(B)$, then the last edge f of P_{i_0} is distinct from e_0 and leads to contradiction in the same way as the f of the previous paragraph.

Finally suppose that $V_0 \cap B = \varnothing$ i.e. $0 \in O_{\mathcal{V}}(B)$. By the previous paragraph $\mathsf{end}(e_0) \in B$, and hence $0 \notin O_{\mathcal{V}^+}(B)$ (since $\mathsf{end}(e_0) \in V_0^+$). Then P_0 is defined and its last edge is some $f \neq e_0$ (since $0 \neq i_0$). The path P_f is not used in the construction of paths $\{Q_i\}_{i_0 \neq i \in \mathbb{N}}$ since $0 \notin O_{\mathcal{V}^+}(B)$. Existence of this f contradicts to the Proposition 20 as in the last two paragraphs.

References

[1] Aharoni, R., and Thomassen, C. Infinite, highly connected digraphs with no two arc-disjoint spanning trees. *Journal of graph theory 13*, 1 (1989), 71–74.

[2] Frank, A. *Connections in combinatorial optimization*, vol. 38. OUP Oxford, 2011.

[3] Fulkerson, D. Flow networks and combinatorial operations research. *American Mathematical Monthly 73*.

[4] Joó, A. Edmonds' branching theorem in digraphs without forward-infinite paths. *Journal of Graph Theory 83*, 3 (2016), 303–311.

[5] Joó, A. Highly connected infinite digraphs without edge-disjoint back and forth paths between a certain vertex pair. *Journal of Graph Theory 85*, 1 (2017), 51–55.

[6] Joó, A. Packing countably many branchings with prescribed root-sets in infinite digraphs. *Journal of Graph Theory* (2017), n/a–n/a.