On the fundamental group of the Freudenthal compactification of CW complexes

Fachbereich Mathematik
der Universität Hamburg

Abschlussarbeit

zur Erlangung des akademischen Grades
Master of Science

vorgelegt von

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geboren am 06.10.1992 in Minden

im September 2017

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Abstract

The Freudenthal compactification provides a functor from the category of locally finite CW complexes and proper maps into the category of compact spaces and continuous maps. It is natural to ask how a topological invariant such as simple–connectedness behaves under such a functor. In this thesis, we prove that the Freudenthal compactification of a simply–connected space is not necessary simply–connected. Moreover, we will introduce an algebraic property which ensures for CW complexes that simple–connectedness is preserved under the Freudenthal functor.

Contents

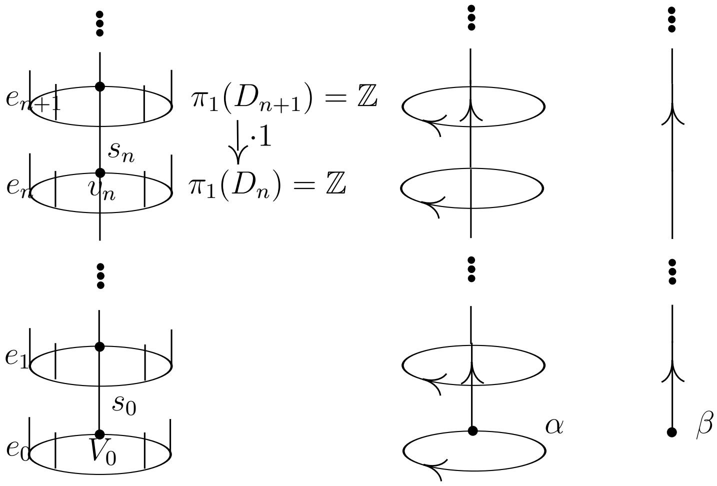
1 Introduction

In infinite graph theory, it is quite common not only to consider the actual infinite graph G but its Freudenthal compactification $|G|$. Indeed, many theorems of finite graph theory do not verbatim generalize to infinite graphs but to their appropriate analogues for $|G|$; see for example [1], [2]. It is not hard to see that the Freudenthal compactification of a tree, i.e. a simply–connected graph, is still simply–connected (see Chapter 4 below for a rigorous proof). So Diestel asked (personal communication) whether this holds for higher–dimensional complexes in general. Let us phrase the question in a slightly different manner: One can prove that every proper continuous map, i.e. every fiber is compact, $f\colon X \to Y$ between connected locally compact CW complexes has a unique extension to a continuous map $\widetilde{f}\colon F(X) \to F(Y)$ between the Freudenthal compactification $F(X)$ of X and $F(Y)$ of Y. In fact, this gives a functor from the category of connected locally compact CW complexes and proper maps into the category of compact spaces and continuous maps, [3]. Then it is a natural question in algebraic topology whether the Freudenthal functor preserves simple–connectedness. As it turns out, there are two–dimensional simply–connected CW complexes such that the fundamental group of their Freudenthal compactification is uncountable [4]. One aim of this thesis is to give such an example.

Since the general answer to Diestel's question is in the negative, one can hope to find a computable property which characterises the simply–connected CW complexes with a simply–connected Freudenthal compactification. The main aim of this thesis is to give an algebraic property for simply–connected CW complexes which ensures that their Freudenthal compactification is still simply–connected. We have reason to presume that our property is indeed a characterisation.

If such a property is of any use in practice, it should be phrased in terms of X and its homotopy type rather than using information or a deep insight of the homotopy type of $F(X)$. However, ends are defined in terms of components of $X \setminus K$ for a compact K but the homotopy type of such a component can be quite hard to understand. Our first step will be to establish the notation of a dummy. A *dummy* will be a maximal CW subcomplex of a component. Its homotopy type in particular will be much easier to understand since it is, by definition, a CW complex.

To give a brief insight on how our property works, consider the one-way infinite cylinder with bottom disk $X = \mathbb{D}^1 \cup \mathbb{S}^1 \times \mathbb{R}$. Then X is simply–connected and admits a canonical cell decomposition.



Fix the following exhausting sequence of X: Let K_n be the finite cylinder up to level n, i.e. $\mathbb{D}^1 \cup \mathbb{S}^1 \times [0, n]$. For every K_n, there is only one component C_n of $X \setminus K_n$ so X has exactly one end and its Freudenthal compactification is homeomorphic to a 2–sphere. The dummy D_n of C_n – the maximal subcomplex inside C_n– is just $\mathbb{S}^1 \times [n+1, \infty)$. Now observe that the fundamental group of every D_n is just $\mathbb{Z}$ and that there is a canonical map from $\pi_1(D_{n+1}, v_{n+1})$ to $\pi_1(D_n, v_n)$ by conjugation with s_n. Moreover, the induced morphism φ_n on the level of fundamental groups is just the identity since the generator of $\pi_1(D_{n+1}, v_{n+1})$ is mapped to the generator of $\pi_1(D_n, v_n)$. Having this in mind, consider the path $\alpha\colon I \to F(X)$ from v_0 to the one end ω in $F(X)$ which wraps exactly once around every e_n and the path $\beta\colon I \to F(X)$ from v_0 to ω which just runs down every s_n. Since $F(X) \cong \mathbb{S}^1$ is simply–connected, both paths are homotopic. Now, how do we prove this just by data of X? Clearly, every loop e_n is null–homotopic in X. However, every such homotopy uses a point in the bottom disk so there is no chance of fitting all these homotopies together to one (continuous) limit homomotopy. Alternatively, there is a smarter way: Since the morphisms φ_n are particularly surjective, we could start to deform e_1 into D_2 to obtain a loop e'_2 which wraps twice around the second level, and then deform e'_2 into D_3 to obtain a loop which wraps three times around the third level and so forth. The trick here is that all these little loops get sucked up into the end in the limit step; thus, we obtain a homotopy to β. Just by considering $\pi_1(D_n, v_n)$ and the φ_n, we were able to decide that α and β are homotopic in $F(X)$. The crucial property here was that every considered loop had a preimage in arbitrarily high dummies. This already gives a good idea on how our property worked. Let us phrase this more precisely:

For every n, there is an $N(n) = N$ such that for every $L \geq N$, we have $\qquad (*)$

$$(\varphi_1 \circ \cdots \circ \varphi_L)(\pi_1(D_L, v_L)) = (\varphi_1 \circ \cdots \circ \varphi_N)(\pi_1(D_N, v_N)).$$

Note that satisfying $(*)$ is to say that the inverse system $(\pi_1(D_n, v_n), \varphi_n)_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition. Due to this fact, we will name our property the *generalized Mittag–Leffler condition.* Accordingly, our main theorem will be:

Theorem. *Let X be a simply–connected strongly locally finite CW complex. If X satisfies the generalized Mittag-Leffler condition, then $F(X)$ is simply–connected.*

The thesis is organized as follows: In the next chapter we gather all standard definitions and theorems that we need in this thesis. In the third chapter, we show that there occurs no loss of information when using dummies instead of components. In chapter four, we will prove all tools and techniques that will be used before we phrase the actual condition and prove the main theorem in chapter five. On the contrary, we will discuss in chapter six what can be said if a CW complex does not satisfy the generalized Mittag–Leffler condition. In fact, we will see that if a CW complex does not satisfy the generalized Mittag–Leffler condition and in addition, a certain dummy has an abilian fundamental group, then the fundamental of $F(X)$ will be uncountable. This will show us a class of counterexamples of the initial question and leads us to the conjecture that the generalized Mittag–Leffler condition is a characterisation of the simply–connected CW complex with a simply–connected Freudenthal compactification.

2 Notation and basic facts

In the first chapter, we collect all the formal definitions and theorems that will be used later on in this thesis. All statements can be found in standard textbooks. As usual, we denote by I the unit interval and $\mathbb{S}^n$ for the n–dimensonal sphere. In the definition of a CW complex, we use $\mathbb{D}^n$ for the n–dimensional ball other than there the latter $\mathbb{D}$ reverse to dummies, see chapter 2 below. We use $\overline{A}$ for the closure of a subset and $\mathring{A}$ for its interior, the boundary ∂A of a set is $\overline{A} \setminus \mathring{A}$.

2.1 The Freudenthal compactification

For a connected, locally connected, locally compact Hausdorff space X, consider the set of all compact subsets $\mathcal{K}$ and denote by $\mathcal{C}(K)$ the set of all components of $X \setminus K$ for $K \in \mathcal{K}$. Then $(\mathcal{K}, \subseteq)$ is a directed partially ordered set. For any two compact subsets $K \subseteq K'$, there is a canonical map

$$f_{K' \to K} \colon \mathcal{C}(K') \to \mathcal{C}(K)$$

that maps every $C' \in \mathcal{C}(K')$ to the unique component $C \in \mathcal{C}(K)$ with $C' \subseteq C$. In fact, for every compact K'' with $K \subseteq K'' \subseteq K'$, we obtain a commutative diagram of sets.

$$\begin{tikzcd}
\mathcal{C}(K') \arrow[rr, "f_{K' \to K}"] \arrow[dr, "f_{K' \to K''}"'] & & \mathcal{C}(K) \\
& \mathcal{C}(K'') \arrow[ur, "f_{K'' \to K}"'] &
\end{tikzcd}$$

Therefore, $(\mathcal{C}(K))_{K \in \mathcal{K}}$ is a inverse system of sets. Its inverse limit is called the endspace and we will denote it by $\Omega(X) = \varprojlim (\mathcal{C}(K))_{K \in \mathcal{K}}$. An element of $\Omega(X)$ is called end. Given an end $\omega \in \Omega$ and a compact set K of X, then there is exactly one component C of $X \setminus K$ which were picked by ω, one says that ω lives in C and we will write $\hat{C}$ for the union of C together with all the ends that live in C. The Freudenthal compactification $F(X)$ of X is the space $X \cup \Omega(X)$ with the following base $\mathcal{B}$ of its topology:

$$\mathcal{B} = \{O \subseteq X \mid O \text{ open in } X\} \cup \{\hat{C} \mid C \in \mathcal{C}(K),\ K \in \mathcal{K}\}.$$

For a connected, locally connected, locally compact Hausdorff space X, $F(X)$ is indeed a Hausdorff compactification. See [3] for a detailed introduction of the Freudenthal compactification.

2.2 CW complexes

Every thing in this section can be easily found in the Appendix of [5].

Definition 2.2.1. For a topological space X, a *cell decomposition of* X a (set-theoretical) partition of X into subspaces $(X_i)_{i\in\Gamma}$, each of the (X_i) homeomorphic to $\mathbb{R}^{n(i)}$. The X_i are called $n(i)$–dimensional cells or just $n(i)$–cells.

Definition 2.2.2. A topological Hausdorff space X together with a cell decomposition of X is called *CW complex* if it satisfies the following properties:

(i) (Characteristic maps) For every n-cell σ of the cell decomposition of X, there is a continuous map

$$\Phi_\sigma\colon \mathbb{D}^n \to X$$

such that the restriction of Φ_σ to $\mathring{\mathbb{D}}$ is a homeomorphism $\Phi_\sigma\colon \mathring{\mathbb{D}} \to \sigma$ and Φ_σ maps $\mathbb{S}^{n-1} \cong \partial\mathbb{D}$ to the union of cells of dimension at most $n-1$.

(ii) (Closure finiteness) For every n–cell σ, the closure $\overline{\sigma} \subseteq X$ has a non-trivial intersection with only finitely many cells of X.

(iii) (Weak topology) A subset $A \subseteq X$ is closed if and only if $A\cap\overline{\sigma} \subseteq \overline{\sigma}$ is closed for all cells σ of X.

Remark 2.2.3. *Let X be a CW complex. If $\sigma \subseteq X$ is an n–cell of X and Φ_σ its attaching map, then $\Phi(\mathbb{S}^{n-1}) = \overline{\sigma} \setminus \sigma$.*

Straight from the definition we can conclude the following:

Lemma 2.2.4.

- *Every connected CW complex is path–connected.*
- *Every CW complex is locally path–connected.*
- *Every CW complex is locally contractible.*

Definition 2.2.5.

- A CW complex is called *finite* if its cell decomposition contains only finitely many cells.
- *The n–skeleton* X^n of a CW complex X is the union of all the cells of dimension at most n.
- If $X^n = X$ but the $(n-1)$–skeleton is a proper subset of X, then we say X is *n–dimensional.*

- A subset $Y \subseteq X$ of a CW complex X is called *subcomplex* of X if it is the union of cells of X and if for every cell $\sigma \subseteq Y$, its closure in X is contained in Y.
- A CW complex is called *locally finite* if every point is contained in a finite subcomplex.

The statements we will use in the early chapters are:

Lemma 2.2.6. *Let X be a CW complex and $Y \subseteq X$ a union of cells of X, then the following are equivalent:*

(i) *Y is a subcomplex.*

(ii) *Y is closed in X.*

(iii) *The inherited cell decomposition of Y endows Y with the structure of a CW complex.*

Lemma 2.2.7. *A CW complex is finite if and only if has only finitely many cells.*

Remark 2.2.8. *Thus, a CW complex is locally finite if and only if it is locally compact. In particular local–finiteness does not depend on the given cell decomposition.*

Lemma 2.2.9. *A locally finite CW complex is metrizable.*

The above lemma ensures that a locally finite CW complex is compactified by its end space. In fact:

Lemma 2.2.10. *If X is a locally finite CW complex and K a compact subcomplex, then $X \setminus K$ has only finitely many components.*

Proof. Suppose to the contrary that $X \setminus K$ has infinitely many components. Since the boundary of every component hits K and K has only finitely many cells, there is one cell $\sigma \subseteq K$ that is hit by the boundary of infinitely many components. As the image of a sphere $\overline{\sigma}$ is compact, there is one point $x \in \overline{\sigma}$ such that every neighbourhood of x is hit by the boundary of infinitely many components, contradicting that X is locally finite. □

In the later chapters we will use:

Lemma 2.2.11. *A CW complex is connected if and only if its 1–skeleton is connected.*

Lemma 2.2.12. *If X is a CW complex and $A \subseteq X$ a subcomplex, then X/A is a CW complex.*

2.3 The fundamental group

Given two continuous maps $f, g\colon X \to Y$ between topological spaces, then f is said to be *homotopic* to g, written $f \simeq g$, if there exists a continuous map

$$H\colon X \times I \to Y$$

with $H(\cdot, 0) = f(\cdot)$ and $H(\cdot, 1) = g(\cdot)$. Furthermore, if $f_{|A} = g_{|A}$ for a subset $A \subseteq X$, then f is homotopic to g relative A, written $f \simeq g\ rel\ A$, if f is homotopic to g throughout a homotopy H with $H(a,t) = g(a) = f(a)$ for every $a \in A$ and $t \in I$. Being homotopic and being homotopic relative A is clearly an equivalence relation and we write $[f]$ and $[f]_A$ for the respective equivalence classes of f and refer to them as the homotopy class of f (relative A). For a path–connected space X and a base point $x_0 \in X$, the set

$$\pi_1(X, x_0) = \{\, [\alpha]_{\{0,1\}} \,|\, \alpha\colon I \to X \text{ with } \alpha(0) = \alpha(1) = x_0\}$$

is called the fundamental group of X and has a natural structure of a group. To be more precise given two paths α, β based at x_0, we write $\alpha \cdot \beta$ for the following path:

$$\begin{aligned} \alpha \cdot \beta\colon I &\to X \\ t &\mapsto \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

It is not hard to see that $\alpha \cdot \beta$ is indeed a path, i.e. it is continuous. In fact, $(\,\cdot\,)$ induces a well–defined operation on the level of homotopy classes and equips $\pi_1(X, x_0)$ with the structure of a group. Moreover, $(\,\cdot\,)$ extends to all paths α and β with $\alpha(1) = \beta(0)$. It extends to all continuous maps $\alpha\colon I_1 \to X$ and $\beta\colon I_2 \to X$ from closed intervals $I_1, I_2 \subseteq \mathbb{R}$ with $\alpha(\max I_1) = \beta(\min I_2)$. If $\pi_0(X, x_0)$ is fixed, we will drop the index and just write $[\alpha]$ for a closed path at x_0. Furthermore, if $[\alpha] = [\beta]$, we will just say that α is homotopic to β, meaning throughout a homotopy relative $\{0, 1\}$. More generally, if we consider two paths with the same endpoints, then homotopic will always mean homotopic relative endpoints.

Every continuous map $f\colon X \to Y$ between path–connected spaces gives a group homomorphism $f_*\colon \pi_0(X,x_0) \to \pi_0(Y,f(x_0))$ by $[\alpha] \mapsto [f \circ \alpha]$. Notice even if f is injective or surjective, this does not hold for f_*. Every path f from x to y gives an isomorphism of groups $f^\sharp\colon \pi_0(X, x) \to \pi_0(X, y)$ by $[\alpha] \mapsto [f^{-1} \cdot \alpha \cdot f]$. More generally, if $A \subseteq X$, every path from $a \in A$ to $x \in X$ gives a group homomorphism $\overline{f} = \iota_* \circ f^\sharp\colon \pi_0(A, a) \to \pi_0(X, x)$. For simplicity, we will just write f instead of $\overline{f}$ and it will be clear from the context whether f denotes the continuous map or its assigned group homomorphism. Note that any two a–x paths induce the same group homomorphism if and only if they are homotopic relative $\{a, x\}$.

Definition 2.3.1.

- A closed path $\alpha\colon I \to X$ is called *null–homotopic* if it is homotopic to a constant map relative its endpoints.
- A path–connected space X is called *simply–connected* if every closed path is null–homotopic, i.e. $\pi_1(X, x) = 1$ for every $x \in X$.

Remark 2.3.2. *A path–connected space is simply–connected if and only if any two paths with same endpoints are homotopic.*

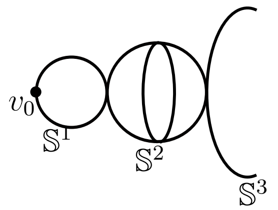
3 The dummy notation

In the first section of this chapter, we will introduce the concept of a dummy. So far, we have only mentioned locally finite CW complexes; however, to really apply combinatorial techniques, we need a little more structure on the cell decomposition of the CW complex. In fact, if we want to detect ends in terms of subcomplexes, we are somewhat forced to require more structure; see the upcoming examples. To be more precise, we will consider strongly locally finite CW complexes.

3.1 The dummy notation

Definition 3.1.1. A CW complex X is called *strongly locally finite* if there is a cover $(K_n)_{n\in\mathbb{N}}$ of finite subcomplexes such that every point in X is contained in only finitely many of the K_n.

Example 3.1.2. *Let X be the CW complex with one cell in every dimension and for the n–dimensional cell σ, let $\Phi_{\sigma|\mathbb{S}^{n-1}}$ be the projection to a point in the $(n-1)$–dimensional cell, see [3].*



Then X is not strongly locally finite since every subcomplex contains the only 0–cell, v_0. Since X is non–compact and its 1–skeleton is just v_0 we have no chance of understanding its end space by considering the "graph–part" of X.

However, X can be given a strongly locally finite cell decomposition. Hence, strongly locally finiteness depends on the given cell decomposition, unlike locally finiteness, see Remark 2.2.8. So the natural question arises whether every locally finite CW complex can be given a strongly locally finite cell decomposition. The answer to this question is in general unknown, see [6].

Definition 3.1.3. For a CW complex X, let V be a set of 0–cells. Then we write $X[V]$ for the following subcomplex:

- *The 0–skeleton of* $X[V]$ *is* V.
- *Given the* $(n–1)$ *skeleton* $(X[V])^{n-1}$ *of* $X[V]$, *its* n *skeleton is* $(X[V])^{n-1}$ *together with all* n*–dimensional cells of* X *whose boundary lies entirely in* $(X[V])^{n-1}$.

Indeed, $X[V]$ is a subcomplex of X and we will call it the *induced subcomplex (of* X*) by* V.

Lemma 3.1.4. *If* X *is a strongly locally finite CW complex and* V *is a finite set of* 0*–cells, then* $X[V]$ *is a finite subcomplex.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a strongly locally finite cover of finite subcomplexes. Suppose to the contrary that $X[V]$ is not a finite subcomplex. For every cell σ in $X[V]$, there is a $K_{n(\sigma)}$ that contains σ. By Lemma 2.2.6, every $K_{n(\sigma)}$ contains $\overline{\sigma}$. Since $X[V]$ and $K_{n(\sigma)}$ are subcomplexes, both need to contain every cell θ of X with $\theta \cap \overline{\sigma} \neq \emptyset$. By the definition of a CW complex, the boundary of a cell is contained in cells of lower dimension, recursively $K_{n(\sigma)}$ needs to contain at least one of the zero cells in V. By assumption, $X[V]$ contains infinitely many cells and V is finite, so there are infinitely many distinct cells $(\sigma_i)_{i\in\mathbb{N}}$ of $X[V]$ and a zero cell $v \in V$ such that $v \in K_{n(\sigma_i)}$ for every $i \in \mathbb{N}$. Due to the fact that every $K_{n(\sigma_i)}$ is a finite subcomplex, there are infinitely many distinct $K_{n(\sigma_i)}$. This, however, contradicts the fact that $(K_n)_{n\in\mathbb{N}}$ is strongly locally finite cover. □

Corollary 3.1.5. *Given a finite subcomplex* K *of a strongly locally finite CW complex, then every unbounded component* C *of* $X \setminus K$ *contains a zero cell.*

Proof. Every cell is path–connected so C has a cell decomposition by cells of X. Moreover, CW complexes are locally path–connected so $K \cup C$ is closed in X, i.e. $K \cup C$ is a subcomplex of X. If C contains no 0–cells, then $C \subseteq X[K^1]$, contradicting Lemma 3.1.4 since C contains infinitely many cells of X. □

Note that both statements are generally false if X is not strongly locally finite; see Example 3.1.2.

Definition 3.1.6. *Let* X *be a CW complex,* K *a finite subcomplex and* $(K_n)_{n\in\mathbb{N}}$ *an exhausting sequence of finite subcomplexes.*

- *A dummy of* $X \setminus K$ is maximal-connected subcomplex of X contained in $X \setminus K$, i.e. avoiding K.
- We will write $\mathbb{D}(K)$ for the set of all dummies of $X \setminus K$.
- We will write $\mathbb{D}((K_n)_{n\in\mathbb{N}})$ for the set of all dummies of $(K_n)_{n\in\mathbb{N}}$, i.e. $\mathbb{D}((K_n)_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \mathbb{D}(K_n)$.

Of course, every dummy of $X \setminus K$ is contained in exactly one component of $X \setminus K$ but it does not need to be a dummy in every component. However, Lemma 3.1.5 shows that if the component is unbounded, then there is at least one dummy contained in it.

Lemma 3.1.7. *For a locally finite CW complex X and a finite subcomplex K, there are only finitely many dummies of $X \setminus K$ contained in a component of $X \setminus K$, in particular $\mathbb{D}(K)$ is finite.*

Proof. Suppose to the contrary that there are infinitely many distinct dummies $(D_i)_{i\in\mathbb{N}}$ contained in a component C of $X \setminus K$. As mentioned in Chapter one, C is even path–connected so there is a path P_i in C from every D_i to D_1. This path is contained in finitely many cells. Denote by σ_i the first of these cells that is not contained in D_i. Then we have for every D_i that $\overline{\sigma_i} \cap D_i \neq \emptyset$ but $\sigma_i \not\subseteq D_i$. On the one hand, every dummy is maximal and connected. Thus, $\overline{\sigma_i} \cap K \neq \emptyset$, as otherwise $\overline{\sigma_i}$ would be contained in finitely many D_i, contradicting the maximality. On the other hand, the closure of every cell is contained in finitely many cells so there are infinitely many distinct $\overline{\sigma_i}$. This contradicts the fact that K is contains only finitely many cells and X is locally finite. As can be seen in the first chapter, $X \setminus K$ contains only finitely many components. □

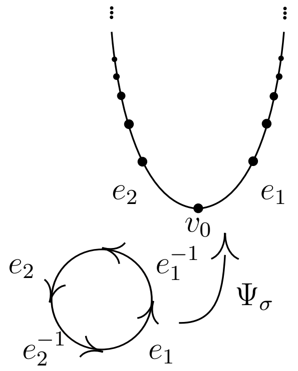
Corollary 3.1.8. *For a locally finite CW complex X and a finite subcomplex K, there are only finitely many cells that are not contained in one of the dummies of $X \setminus K$.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a strongly locally finite cover of X. Suppose to the contrary that there are infinitely many cells $(\sigma_i)_{i\in\mathbb{N}}$ that are not contained in one of the dummies of $X \setminus K$. Each cell σ_i is by definition contained in one of the $K_{n(i)}$. If these $K_{n(i)}$ would lie entirely in one of the components of $X \setminus K$, then σ_i would be contained in a dummy. However, K contains only finitely many cells so there are infinitely many distinct K_n containing the same cell of K. □

Similar to the definition of the Freudenthal compactification, there is a canonical map

$$f_{K' \to K} \colon \mathbb{D}(K') \to \mathbb{D}(K)$$

for any two compact sets $K \subseteq K'$ which maps every dummy in $\mathbb{D}(K')$ to the unique dummy in $\mathbb{D}(K)$ which includes it. These maps are compatible for every compact K'' with $K \subseteq K'' \subseteq K'$. Thus, we obtain an inverse system of sets $(\mathbb{D}(K))_{K\in\mathcal{K}}$. Moreover, there is a canonical map $\varphi_K \colon \mathbb{D}(K) \to \mathcal{C}(K)$ which maps every dummy of $X \setminus K$ to the component of $X \setminus K$ which includes it. These maps neither need to be surjective nor injective. On the one hand, a bounded component C does need to contain a 0–cell; thus there is no dummy contained in C. On the other hand, consider the following CW complex:



Deleting v_0 does not separate the two rays since some reminder of the 2–cell connects them. On the contrary, $X - v_0$ contains two dummies, namely the rays. However, the next theorem shows that both inverse limits define the same ends.

Theorem 3.1.9. *For a strongly locally finite CW complex X, there is a canonical bijection of inverse limits induced by the $(\varphi_K)_{K\in\mathcal{K}}$:*

$$\begin{aligned}\varphi\colon \varprojlim(\mathbb{D}(K))_{K\in\mathcal{K}} &\to \varprojlim(\mathcal{C}(K))_{K\in\mathcal{K}}\\ (D_K)_{K\in\mathcal{K}} &\mapsto (\varphi_K(D_K))_{K\in\mathcal{K}}\end{aligned}$$

Proof. Let $(D_K)_{K\in\mathcal{K}} \in \varprojlim(\mathbb{D}(K))_{K\in\mathcal{K}}$. Clearly, $\varphi(D_{K'}) \subseteq \varphi(D_K)$ whenever $K \subseteq K'$; thus, φ is well defined.

φ is surjective

Given an element $(C_K)_{K\in\mathcal{K}} \in \varprojlim(\mathcal{C}(K))_{K\in\mathcal{K}}$. Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of finite subcomplexes of X. Denote by W_n the set of all dummies in $\mathbb{D}(K_n)$ contained in C_{K_n}, for $n \in \mathbb{N}$. On the one hand, every W_n is non-empty since each C_K needs to be unbounded; see Corollary 3.1.5. On the other hand, every W_n is finite by Lemma 3.1.7. Moreover, every dummy in W_{n+1} is contained in one of the dummies in W_n. By Kőnig's infinity lemma, there is a sequence of dummies $D_1 \supseteq D_2 \supseteq \ldots$ with $D_n \subseteq C_{K_n}$. This sequence defines an element $(D_K)_{K\in\mathcal{K}} \in \varprojlim \mathbb{D}(K)$ with $\varphi((D_K)_{K\in\mathcal{K}}) = (C_K)_{K\in\mathcal{K}}$.

φ is injective

Suppose there are two distinct elements $(D_K)_{K\in\mathcal{K}}, (D'_K)_{K\in\mathcal{K}} \in \varprojlim(\mathbb{D}(K))_{K\in\mathcal{K}}$ with the same image. Then they need to differ in one component, say, $\mathbb{D}(K)$, i.e. D_K and D'_K are two distinct dummies in the same component of $X \setminus K$. There are only finitely many cells in C_K not contained in one of the dummies (Corollary 3.1.8) and every path from D_K to D'_K in C_K needs to hit one of these cells. Let $\tilde{K}$ be a compact set so large that it contains K and all the finitely many cells in C_K not contained in one of the dummies. Then $D_{\tilde{K}}$ and $D'_{\tilde{K}}$ are two distinct dummies of $C_{\tilde{K}}$ so there is a path from $D_{\tilde{K}}$ to $D'_{\tilde{K}}$ in $C_{\tilde{K}} \subseteq C_K$. Yet, this path avoids all the cells not contained in one of the dummies in C_K. $\square$

3.2 The Freudenthal compactification as a limit of finite complexes

It is known that the Freudenthal compactification of a space can be obtained as an inverse limit of compact spaces [7]. However, if the space is a strongly locally finite CW complex, we will show in this section that its Freudenthal compactification can be obtained by an inverse limit of finite CW complexes. Our preliminaries in the first section will enable us to imitate the techniques of infinite graph theory by Diestel [1].

Given a strongly locally finite CW complex, denote by $\mathcal{F}(X)$ the set of all finite partitions $F = \{V_1, \dots, V_n\}$ of X^0 such that only finitely many cells of X are not contained in one of the $X[V_i]$, $i \in \{1, \dots, n\}$. For two partitions F_1 and F_2 in $\mathcal{F}(X)$, F_1 is said to be *finer* than F_2 and we write $F_1 \preccurlyeq F_2$ if every partition class of F_1 is contained in one of the partition classes of F_2. This relation turns $\mathcal{F}(X)$ into a directed partially ordered set. Every partition $F = \{V_1, \dots, V_n\}$ in $\mathcal{F}(X)$ defines a finite CW complex $X[F]$, namely the CW complex that is obtained from X by contracting each of the $X[V_i]$, $i \in \{1, \dots, n\}$. Furthermore, if F_1 is finer than F_2, we obtain a commutative diagram of topological spaces

$$\begin{tikzcd}
 & X \arrow[dl, "q_{F_2}"'] \arrow[d, "q_{F_1}"] \\
X[F_2] & X[F_1] \arrow[l, "p_{F_1 \to F_2}"]
\end{tikzcd}$$

where q_{F_1} and q_{F_2} are the quotient projections and $p_{F_1 \to F_2}$ maps $[x]_{X[F_1]}$ to $[x]_{X[F_2]}$. The map $p_{F_1 \to F_2}$ is well defined since F_1 is finer than F_2 and it is continuous due to the universal property of the quotient topology. Moreover, for any $F_3 \in \mathcal{F}$ with $F_1 \succcurlyeq F_3 \succcurlyeq F_2$ the bonding maps are compatible i.e. $p_{F_1 \to F_2} = p_{F_1 \to F_3} \circ p_{F_3 \to F_2}$. This gives an inverse system of topological spaces $(X[F])_{F \in \mathcal{F}(X)}$ where the bonding maps $f_{F_1 \to F_2} \colon X[F_1] \to X[F_2]$ are the $p_{F_1 \to F_2}$.

On the one hand, every finite subcomplex K of X defines a finite CW complex $X[K]$ by contracting all the dummies of $X \setminus K$. On the other hand K defines a partition F_K in $\mathcal{F}(X)$. The partition classes of F_K are the 0–cells of every dummy of $X \setminus K$ and every 0–cell of K as a single partition class. In fact, $X[F_K] = X[K]$. Given two finite subcomplexes $K_1 \subseteq K_2$ then there is a projection map $q \colon X[K_2] \to X[K_1]$ and this map coincides with the bonding map of $X[F_{K_2}] = X[K_2]$ to $X[F_{K_1}] = X[K_2]$.

Theorem 3.2.1. *For a strongly locally finite CW complex X, its Freudenthal compactification is homeomorphic to the inverse limit of $(X[F])_{F \in \mathcal{F}(X)}$, i.e. $\varprojlim((X[F])_{F \in \mathcal{F}(X)}) \cong F(X)$.*

Proof. By Tychonoff's theorem, $\varprojlim((X[F])_{F\in\mathcal{F}(X)})$ is compact. Clearly, $F(X)$ is Hausdorff so it suffices to define a continuous bijective map:

$$\Psi\colon \varprojlim((X[F])_{F\in\mathcal{F}(X)}) \to F(X).$$

Given an element $(x_F)_{F\in\mathcal{F}(X)}$ in the inverse limit. If one of the x_F is a singleton equivalence class, i.e. $x_F = \{x\} \in X[F]$, then we define

$$\Psi\Big((x_F)_{F\in\mathcal{F}(X)}\Big) = x \in X \subseteq F(X).$$

To see that this is well defined, suppose that there are two singleton equivalence classes $x_{F_1} = \{x_1\}$ and $x_{F_2} = \{x_2\}$. Since $\mathcal{F}(X)$ is a directed ordered set, there is an $F_3 \in \mathcal{F}(X)$ which is finer than F_1 and F_2. By construction, $f_{F_3\to F_1}$ and $f_{F_3\to F_2}$ are projections so that x_{F_3} is a singleton equivalence class as well. Hence, $\{x_1\} = x_{F_3} = \{x_2\}$. If none of the x_F is a singleton partition class, consider the partitions $F[K]$ that are induced by the finite subcomplexes K of X. It can be assumed that every $x_{F[K]}$ is the set of 0–cells of a dummy D_K of $X \setminus K$. Since $(x_F)_{F\in\mathcal{F}(X)}$ is an element of the inverse limit, we have $D_{K'} \subseteq D_K$ for any compact $K' \subseteq K$. Moreover, $\varphi(D_{K'}) \subseteq \varphi(D_K)$. The set of finite subcomplexes is cofinal in the set of compact subsets so the $\varphi(D_K)$ define exactly one end $\omega \in \Omega(X)$ and we define

$$\Psi\Big((x_F)_{F\in\mathcal{F}(X)}\Big) = \omega \in \Omega(X) \subseteq F(X).$$

Ψ is surjective

Let $x \in X \subseteq F(X)$ be given. For every $F \in \mathcal{F}$, there is exactly one equivalence class x_F in $X[F]$ with $x \in x_F$ and $(x_F)_{F\in\mathcal{F}}$ is clearly an element of the inverse limit. Obviously, there are partitions in $F \in \mathcal{F}(X)$ with $x_F = \{x\}$ as a singleton partition class. By definition, $\Psi((x_F)_{F\in\mathcal{F}}) = x$.

Let $x = \omega \in \Omega(X)$ be given. Consider the partitions $F[K]$ that are induced by the finite subcomplexes K of X and $(D_K)_{K\in\mathcal{K}} := \varphi^{-1}(\omega)$. Let $x_{F[K]}$ be the set of zero cells of the dummy D_K. For an arbitrary partition $F \in \mathcal{F}$ chose K so large that $F[K]$ is finer that F and let $x_F := f_{F[K]\to F}(x_{F[K]})$. Then $(x_F)_{F\in\mathcal{F}}$ is well defined since the bonding maps are compatible. By the same argument, $(x_F)_{F\in\mathcal{F}}$ is an element of the inverse limit. As a consequence, $\Psi((x_F)_{F\in\mathcal{F}}) = \omega$ by definition.

Ψ is injective

If $(x_F)_{F\in\mathcal{F}}$ is an element of the inverse limit with $\Psi((x_F)_{F\in\mathcal{F}}) = x \in X$ then there is an x_F with $\{x\} = x_F$. This uniquely determines $(x_F)_{F\in\mathcal{F}}$. On the one hand, the partitions that are finer than F are cofinal in $\mathcal{F}(X)$; consequently, $x_F = \{x\} = x_{F'}$ for every finer partition by the well–definedness of Ψ. On the other hand, every other coordinate $x_{F''}$ is determined by the bonding map for an element finer than F and F''. Moreover, every end $\omega \in \Omega(X)$ is uniquely determined by the components C_K of $X \backslash K$ in which the end lives in. By Theorem 3.1.9 and the definition of Ψ, there can only be one element in the inverse limit that is mapped to ω.

Ψ is continuous

Let $x \in X$ and O be an open neighbourhoud of x in $F(X)$ without loss of generality $O \subseteq X$. By local finiteness, there is a finite subcomplex K with $O \subseteq K$. Clearly, O considered as a subset of $X[K]$ is open in $X[K]$. This shows that

$$\Psi^{-1}(O) = (O \times \prod_{\substack{F \in \mathcal{F}(X) \\ F \neq F_K}} X[F]) \cap \varprojlim((X[F])_{F \in \mathcal{F}(X)})$$

is open in $\varprojlim((X[F])_{F \in \mathcal{F}(X)})$.

Let $\omega = (C_K)_{K \in \mathcal{K}} \in \Omega(X)$ and $\hat{C}$ be a basic open set for ω, i.e. there is a compact set K such that C is the component of $X \setminus K$ the end ω lives in. Without loss of generality, we may assume that K is a finite subcomplex. Let $D_1, \ldots, D_n$ be all the dummies in $\mathbb{D}(K)$ with $D_i \subseteq C_K$. Then $C/D_1, \ldots, D_n$, considered as a subset of $X[K]$, is open in $X[K]$. This shows that

$$\Psi^{-1}(\hat{C}) = (C/D_1, \ldots, D_n \times \prod_{\substack{F \in \mathcal{F}(X) \\ F \neq F_K}} X[F]) \cap \varprojlim((X[F])_{F \in \mathcal{F}(X)})$$

is open in $\varprojlim((X[F])_{F \in \mathcal{F}(X)})$.

□

It is a basic theorem in topology that the countable product of metrizable spaces is metrizable, [8], so we obtain:

Corollary 3.2.2. *Let X be a strongly locally finite CW complex, then $F(X)$ is metrizable.* □

Every cofinal sequence of the underlying poset holds an induced inverse system and it is a basic fact that the inverse limit of the induced inverse system is homeomorphic to the original inverse limit, [9].

Corollary 3.2.3. *If X is a strongly locally finite CW complex and $(K_n)_{n \in \mathbb{N}}$ is an exhausting sequence of finite subcomplexes, then $\varprojlim(X[K_n]_{n \in \mathbb{N}}) \cong F(X)$.* □

Obviously, one can always find such an exhausting sequence, even for non-strongly locally finite complexes. However, Lemma 3.1.4 shows that for strongly locally finite complexes, one can choose an exhausting sequences in a favorable way: Choose an arbitrary enumeration $X^1 = \{x_1, x_2, \ldots\}$ of its 0–cells. Then $X[\{x_1, \ldots, x_n\}]$ is a finite subcomplex for every n. Thus, $K_n := X[\{x_1, \ldots, x_n\}]$ gives an exhausting sequence of finite subcomplexes.

4 Tools and techniques

As the title gives away, this chapter will introduce all the necessary auxiliary theorems we need in our proof of the main theorem. However, a good amount of the upcoming theorems is formulated in such a generality that they might be of independent interest.

4.1 Limits of maps

Our main theorem of this section gives a criterion when a sequence of continuous maps $f_n : A \to F(X)$ from a path–connected space A into $F(X)$ defines a continuous limit map. As we have seen in the introduction, it will be of use to construct a limit homotopy from a given sequence of homotopic paths and an important corollary of the main theorem of this section will tackle this problem.

Theorem 4.1.1. *Let X be a strongly locally finite CW complex and $(K_n)_{n\in\mathbb{N}}$ a cover of finite subcomplexes of X. For a path–connected Hausdorff space A, let $f_n\colon A \to F(X)$ be a sequence of continuous maps. Then the pointwise limit $f = \lim_{n\to\infty} f_n$ exists and is continuous if the following properties hold:*

(i) *There is at least one $a \in A$ such that $f_1(a) = f_n(a) \in X$ for every $n \in \mathbb{N}$.*

(ii) *For every $n \in \mathbb{N}$, there is an $N(n) \geq n$ such that for every $l \geq N(n)$, $f_{N(n)}^{-1}(K_n) = f_l^{-1}(K_n)$; in addition, $f_{N(n)}$ and f_l coincide on the preimage of K_n.*

Proof. Letting $\mathbb{K}_n := K_1 \cup \ldots \cup K_n$ gives another cover of finite subcomplexes $(\mathbb{K}_n)_{n\in\mathbb{N}}$. Then (i) and (ii) hold for $(\mathbb{K}_n)_{n\in\mathbb{N}}$ as well since (i) does not depend on the cover. For (ii) and a given n, consider $\max\{N(1), \ldots, N(n)\}$ so we may assume $K_1 \subseteq K_2 \subseteq \ldots$ and $f_1(a) \in K_1$.

To increase the readability of the proof, let us first show an auxiliary result:

For every compact set K of X with $f_1(a) \in K$ and every component C of $X \setminus K$, there is a index N such that $f_N^{-1}(\hat{C}) = f_l^{-1}(\hat{C})$ for every $l \geq N$. $(*)$

Given a compact set K of X, take n large enough that $K \subseteq \mathring{K}_n$ and then fix $N(n)$ of property (ii). We claim that $N = N(n)$ is the promised index of the auxiliary result. Suppose the contrary, fix an l such that f_l witnesses the failure of N. Then there are distinct components C_1, C_2 of $X \setminus K$ and $x \in A$ with

$f_N(x) \in \hat{C}_1$ and $f_l(x) \in \hat{C}_2$. Let α be a path from x to a in A and write W for $f_N^{-1}(K_n) = f_l^{-1}(K_n) \subseteq A$. Since f_N and f_l agree on W, we have $x \notin W$. Let $\tilde{t}$ be the infimum over all $t \in I$ with $\alpha(t) \in W$. Note that $f_N(\alpha(\tilde{t})) = f_l(\alpha(\tilde{t})) \in K_n$ but $\alpha([0,\tilde{t})) \cap W = \emptyset$. Now consider the two paths $f_N \circ \alpha$ and $f_l \circ \alpha$ in $F(X)$. On the one hand, $f_N(\alpha([0,\tilde{t}))) \subseteq \hat{C}_1$ and $f_l(\alpha([0,\tilde{t}))) \subseteq \hat{C}_2$. Since $\hat{C}_1$ and $\hat{C}_2$ are disjoint by the definition of $F(X)$, we have $f_N(\alpha(\tilde{t})) = f_l(\alpha(\tilde{t})) \in K$. On the other hand, $f_N(\alpha([0,\tilde{t}))) \subseteq \hat{C}_1 \setminus K_n$ and $f_l(\alpha([0,\tilde{t}))) \subseteq \hat{C}_2 \setminus K_n$ gives that $f_N(\alpha(\tilde{t})) = f_l(\alpha(\tilde{t})) \in K$ is a limit point of $\hat{C}_1 \setminus K_n$, contradicting the fact that $K \subseteq \mathring{K}_n$. This completes the proof of the auxiliary result.

Having $(*)$ at hand, let us prove the theorem.

$\lim\limits_{n\to\infty} f_n\,(x) := \lim\limits_{n\to\infty}(f_n(x))$ gives a well-defined map

Suppose first that $(f_n(x))_{n\in\mathbb{N}}$ is bounded in X, then there is an index n_0 such that $\{f_n(x) \,|\, n \in \mathbb{N}\} \subseteq K_{n_0}$. By property (ii), we can find an $N(n_0)$ with $f_{N(n_0)}(x) = f_l(x) \in K_{n_0}$ for every $l \geq N(n_0)$; thus, $(f_n(x))_{n\in\mathbb{N}}$ is eventually constant.

Suppose now that $(f_n(x))_{n\in\mathbb{N}}$ is unbounded in X. Then every subsequence $(f_{n_i}(x))_{i\in\mathbb{N}}$ is unbounded in X, too. If not, there is a K_n such that $\{f_{n_i}(x) \,|\, i \in \mathbb{N}\} \subseteq K_n$, but then for every $N \geq n$ there are $n_i, l \geq N$ with $f_{n_i}(x) \in K_n$ and $f_l(x) \notin K_n$. Consequently, property (ii) is violated for K_n. Furthermore, $F(X)$ is a compact metrizable space by Corollary 3.2.2; thus, $(f_n(x))_{n\in\mathbb{N}}$ contains a converging subsequence and its limit point needs to be an end, say $\omega \in \Omega(X)$. We claim that $\lim\limits_{n\to\infty} f_n(x) = \omega$. Given a basic open set $\hat{C}$ of ω for a component C of $X \setminus K$ without loss of generality, assume that $a \in K$. By our auxiliary result, there is an index N with $f_N^{-1}(\hat{C}) = f_l^{-1}(\hat{C})$ for every $l \geq N(n)$. Since the subsequence converges to ω, there is an index n_i with $f_{n_i}(x) \in \hat{C}$ and $n_i \geq N(n)$ for every $l \geq n_i \geq N$ $f_l(x) \in \hat{C}$.

$\lim\limits_{n\to\infty} f_n$ gives a continuous map

Given a basic open set O of X. We may assume O to be a bounded open set for X is locally compact. Fix K_n with $O \subseteq K_n$. By property (ii), $f_{N(n)}^{-1}(K_n) = \lim\limits_{n\to\infty} f_n{}^{-1}(K_n)$ and $f_{N(n)}^{-1}(O) = (\lim\limits_{n\to\infty} f_n)^{-1}(O)$ is open in A by the continuity of $f_{N(n)}$.

Given a basic open set $\hat{C}$ for a component C of $X \setminus K$ for compact K. We may assume without loss of generality that $a \in K$. Fix the index N of $(*)$, then $f_N^{-1}(\hat{C}) = (\lim\limits_{n\to\infty} f_n)^{-1}(\hat{C})$ is open in A by the continuity of f_N. □

Corollary 4.1.2. *Let X be a strongly locally finite CW complex and $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of finite subcomplexes of X. For a path–connected Hausdorff space A, let $\alpha_n\colon A \to F(X)$ be a sequence of continuous maps. Then $\lim\limits_{n\to\infty}\alpha_n$ exists, is continuous and $\alpha_1 \simeq \lim\limits_{n\to\infty}\alpha_n$ if the following properties hold:*

For every $n \in \mathbb{N}$, there are finitely many open disjoint $A_1,\dots,A_L \subseteq A$ such that:

(i) α_n and α_{n+1} coincide on $A \setminus (A_1 \cup \ldots \cup A_L)$.

(ii) The image of $\alpha_n(A_i)$ and $\alpha_{n+1}(A_i)$ is contained in $F(X) \setminus K_n$ for $i \in \{1,\dots,L\}$.

(iii) $\alpha_{n|\overline{A_i}} \simeq \alpha_{n+1|\overline{A_i}}$ relative ∂A_i in $F(X) \setminus K_n$ for $i \in \{1,\dots,L\}$.

Proof. In order to prove the corollary, we will first define the homotopies H_n from α_n to α_{n+1} then we fit the first n homotopies together to obtain $\mathcal{H}_n$. The sequence of the $\mathcal{H}_n$ will satisfy the requirements of Theorem 4.1.1 and their limit will prove the Corollary. Recall the following statement which one easily verifies.

Let $f\colon X \to Y$ be a map between topological spaces and $X_1 \cup \ldots \cup X_n = X$ a finite cover of closed subsets. Then f is continuous if and only if $f_{|X_i}$ is continuous for every $i \in \{1,\dots,n\}$. $(*)$

For a given n, let $A_1,\dots,A_L$ be the promised open sets and $H(A_i)$ be the homotopy from $\alpha_{n|\overline{A}_i}$ to $\alpha_{n+1|\overline{A}_i}$ of property (iii). Then define H_n as

$$\begin{aligned} H_n\colon A \times I &\to F(X) \\ (a,t) &\mapsto \begin{cases} H(A_i)(a,t) & \text{if } a \in \overline{A}_i \text{ for } i \in \{1,\dots,L\} \\ \alpha_n(a) & \text{if } a \in A \setminus (A_1 \cup \ldots \cup A_L). \end{cases} \end{aligned}$$

Since $H(A_i)$ is relative to ∂A_i, this map is well defined and it is continuous due to $(*)$. Note that $H_{n-1}(\cdot,1) = \alpha_n = H_n(\cdot,0)$. Now use order–preserving homeomorphisms $I \cong [\sum_{i=1}^{n-1}\frac{1}{2^i}, \sum_{i=1}^{n}\frac{1}{2^i}]$ to obtain the maps

$$H'_n\colon A \times [\sum_{i=1}^{n-1}\frac{1}{2^i}, \sum_{i=1}^{n}\frac{1}{2^i}] \to F(X).$$

We are now able to construct the $\mathcal{H}_n$. Let

$$\begin{aligned}\mathcal{H}_1\colon A\times I&\to F(X)\\ (a,t)&\mapsto\begin{cases}H_1'(a,t) & t\in[0,\frac{1}{2}]\\ H_1'(a,1) & t\in[\frac{1}{2},1]\end{cases}\end{aligned}\qquad\begin{aligned}\mathcal{H}_n\colon A\times I&\to F(X)\\ (a,t)&\mapsto\begin{cases}\mathcal{H}_{n-1}(a,t) & t\in[0,\sum_{i=1}^{n-1}\frac{1}{2^i}]\\ H_n'(a,t) & t\in[\sum_{i=1}^{n-1}\frac{1}{2^i},\sum_{i=1}^{n}\frac{1}{2^i}]\\ H_n'(a,1) & t\in[\sum_{i=1}^{n}\frac{1}{2^i},1].\end{cases}\end{aligned}$$

Every $\mathcal{H}_n$ is continuous due to $(*)$. The exhausting sequence $(K_n)_{n\in\mathbb{N}}$ especially is a cover of X. Since $\mathcal{H}_n(\cdot,0)=\alpha_1$, requirement (i) of Theorem 4.1.1 is satisfied. To see that the requirement (ii) of Theorem 4.1.1 is satisfied, consider a K_n. Any of the homotopies $H(A_i)$ took place in $F(X)\setminus K_n$; thus, letting $n+1=N$ ensures $H_N^{-1}(K_n)=H_L^{-1}(K_n)=: W$ and $H_{N|W}=H_{L|W}$ for every $L\geq N$. By the construction of the $\mathcal{H}_n$, this gives an index function for the requirement (ii). Now Theorem 4.1.1 gives a limit homotopy $\mathcal{H}=\lim\limits_{n\to\infty}\mathcal{H}_n$ and $\alpha_n=\mathcal{H}(\cdot,\sum_{i=1}^{n-1}(\frac{1}{2})^i)$ shows $\alpha_1\simeq\lim\limits_{n\to\infty}\alpha_n$. □

4.2 Normal forms of paths

In this section, we tackle the problem that a general path can be wild. Our first theorem gives us some global control, i.e. every path in $F(X)$ is homotopic to a path in $\overline{X^1}$. Then we will prove that one can even ensure for a path in $\overline{X^1}$ to be a local homeomorphism on every edge of X^1 which gives us some local control.

Theorem 4.2.1. *Let X be a strongly locally finite CW complex and let x and y be 0–cells. Then every x–y path in $F(X)$ is homotopic to an x–y path in $\overline{X^1}$.*

Proof. For this proof, call a cell σ of a CW complex *good* if no other cell is attached to σ, i.e. $\sigma\cap\overline{\theta}=\emptyset$ for every cell $\theta\neq\sigma$. Let us first prove:

$$\text{Every good cell } \sigma \text{ in a CW complex is open.}\tag{$*$}$$

Consider $X\setminus\sigma=\sigma^{\mathsf{c}}$. Let $\theta\neq\sigma$ be a cell. On the one hand, $\overline{\theta}\cap\sigma^{\mathsf{c}}=\overline{\theta}$ is closed. On the other hand, $\overline{\sigma}\cap\sigma^{\mathsf{c}}=\overline{\sigma}\setminus\sigma$ is closed since it is the image of a sphere. Consequently, σ^{c} is closed by the weak topology of X.

In order to prove the theorem, we need another auxiliary result:

$$\begin{gathered}\text{For every strongly locally finite CW complex } X\text{, there is an enumeration}\\ \sigma_1,\sigma_2,\dots \text{ of the cells of dimension greater than one, such that } \sigma_n \text{ is good in}\\ X\setminus\{\sigma_1,\dots,\sigma_{n-1}\} \text{ and } X\setminus\textstyle\bigcup_{n\in\mathbb{N}}\sigma_n=X^1.\end{gathered}\tag{$**$}$$

Note that by $(*)$ and Lemma 2.2.6, the space $X\setminus\sigma$ is a strongly locally finite CW complex for every good cell σ in X. Hence in the situation above, $X\setminus\{\sigma_1,\dots,\sigma_{n-1}\}$ is a strongly locally finite CW complex. In order to prove $(**)$, let us set up another definition. For a cell σ, denote by $W_1(\sigma)$ the set of cells in X

whose boundary intersects σ. More generally, let $W_i(\sigma)$ be the set of all cells of X whose boundary intersects one of the cells in $W_{i-1}(\sigma)$. Since X is particular locally finite, every $W_i(\sigma)$ is finite. We claim that $W_i(\sigma) = \emptyset$ eventually. If not then there is a sequence of cells $\sigma = \theta_0, \theta_1, \theta_2, \ldots$ with $\theta_n \cap \overline{\theta}_{n+1} \neq \emptyset$ by Kőnig's infinity lemma. This, however, contradicts the fact that X admits a strongly locally finite cover of finite subcomplexes since every subcomplex containing one of the θ_n needs to contain σ. Note that $W_i(\sigma)$ contains only cells of dimension greater than σ for every $i \geq 1$. Moreover, if σ is a cell in X and N is the largest index for which $V_N(\sigma)$ is non–empty then every cell in $W_N(\sigma)$ is good in X, in particular every cell in W_{N-1} is good in $X \setminus W_N(\sigma)$. Now we are able to prove $(**)$. Let $\theta_1, \theta_2, \ldots$ be an arbitrary enumeration of all the cells of dimension greater than one. For θ_i, let $n(i)$ be the greatest index such that $W_{n(i)}(\theta_i)$ is non–empty. Consider the sequence of sets:

$$W_{n(1)}(\theta_1), \ldots, W_1(\theta_1),\ W_{n(2)}(\theta_2), \ldots, W_1(\theta_2),\ W_{n(3)}(\theta_3), \ldots W_1(\theta_3),\ \ldots$$

Clearly, every cell of dimension greater than one is contained in one of the $W_i(\theta_j)$. Thus, enumerating the cells in $W_i(\theta_j)$ gives an enumeration of all the cells of dimension higher than one. Just keeping the first appearance of a cell gives a subsequence satisfying $(**)$.

Now let us prove the actual theorem. Let $\alpha\colon I \to F(X)$ be an x–y path. Let $\sigma_1, \sigma_2, \ldots$ be a sequence of all cells of dimension greater than one satisfying $(**)$. We will define a sequence of paths $\alpha = \alpha_1, \alpha_2, \ldots$ and a cover $(K_n)_{n\in\mathbb{N}}$ with the following properties:

(i) Every α_n is an x–y path in $X \setminus \{\sigma_1, \ldots, \sigma_n\}$.

(ii) The sequence $(\alpha_n)_{n\in\mathbb{N}}$ and the exhausting sequence $(K_n)_{n\in\mathbb{N}}$ satisfy the requirements of Corollary 4.1.2.

Let $\alpha_0 = \alpha$. Given α_n, by $(*)$ the cell σ_{n+1} is open in $X \setminus \{\sigma_1, \ldots, \sigma_n\}$. Choose a point $x \in \sigma_{n+1}$. By compactness of I, there are only finitely many open intervals of $\alpha^{-1}(\sigma_{n+1})$ that contain a point which is mapped to x. In these intervals, find finitely many closed subintervals (closed in I) such that the interior of their union contains $\alpha^{-1}(x)$, say $[a_1, b_1], \ldots, [a_L, b_L]$. Then $\alpha_{|[a_i,b_i]}$ is a path in σ_{n+1} ($\cong \mathbb{R}^m$ for $m \geq 2$). It is a standard result that every path in $\mathbb{R}^m$ for $m \geq 2$ is homotopic relative endpoints to a path that misses a certain point (except for the endpoints). So α_n is homotopic to a path α'_n which misses x. Moreover, α'_n coincides with α_n outside of the $(a_1, b_1), \ldots, (a_L, b_L)$ and $\alpha_{n|[a_i,b_i]}$ is homotopic to $\alpha'_{|[a_i,b_i]}$ by a homotopy in σ_{n+1}. Now α'_n is a path in $X \setminus \{\sigma_1, \ldots, \sigma_n, x\}$ but this space deformation retracts to $X \setminus \{\sigma_1, \ldots, \sigma_{n+1}, \}$ for every solid ball (of dimension higher than one) minus a point deformation retracts to its boundary.

In fact, α'_n is homotopic to a path α_{n+1} in $X\backslash\{\sigma_1,\dots,\sigma_{n+1}\}$ that coincides with α_n outside σ_n and all homotopies of the subintervals $[a_1,b_1],\dots,[a_L,b_L]$ took place in $\sigma_{n+1}\subseteq X\setminus\{\sigma_1,\dots,\sigma_n\}$. Choose an exhausting sequence of finite subcomplexes such that K_n contains none of the cells σ_i with $i\geq n$. Then $(\alpha_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ satisfy the requirements of Corollary 4.1.2. $\square$

Corollary 4.2.2. *Let X be a strongly locally finite CW complex and let x_0 be a 0–cell, then $\iota\colon \pi_1(F(X^1),x_0)\to\pi_1(F(X),x_0)$ is surjective.* $\square$

Definition 4.2.3. For a strongly locally finite subcomplex X, we say that a path $\alpha\colon I\to F(X)$ is *in normal form* if it satisfies the following two properties:

(i) The image of α is a subset of $F(X^1)$.

(ii) The path α is a local homeomorphism on every edge e of X^1, i.e. $\alpha^{-1}(e)$ is a union of finitely many disjoint open intervals $U_1,\dots,U_n$ such that $\alpha_{|U_i}\cong e$ for $i=1,\dots n$.

Lemma 4.2.4. *For a strongly locally finite CW complex X and 0–cells x and y, every x–y path is homotopic to a path in normal form.*

Proof. Given an x–y path $\alpha\colon I\to F(X)$, by Theorem 4.2.1 we may assume that (i) of Definition 4.2.3 holds for α. To find a homotopy to a path that also satisfies (ii) of Definition 4.2.3, we want to use Corollary 4.1.2. In fact, we will find one such homotopy in the 1–skeleton so let us assume $X^1=X$. Let $e_1,e_2,\dots$ be an enumeration of the edges of X. Furthermore, let $O(e_n)$ be the union of $\overline{e_n}$ and all the edges that are incident with e_n. Note that $O(e_n)$ is open in X and simply–connected. Let $\alpha_0:=\alpha$. Given α_n, consider all the path–connected components of $\alpha_n^{-1}(O(e_n))$ that contain a point mapped to $\overline{e_n}$. By compactness of I, as in the proof of the previous theorem, there are only finitely many such intervals. Let $(a_1,b_1),\dots,(a_L,b_L)\subseteq I$ be all of them. Then $\alpha_{n|[a_i,b_i]}$ is a path in $\overline{O(e_n)}$. Clearly, $\alpha_{|[a_i,b_i]}$ is homotopic relative $\{a_i,b_i\}$ to a path from a_i to b_i in $\overline{O(e_n)}$ that is a local homeomorphism on e_n. Moreover, these homotopies can be taken in $\overline{O(e_n)}$. Denote by H_i one such homotopy for $\alpha_{n|[a_i,b_i]}$. Let α_{n+1} be the path that arises from α_n by applying all the homotopies H_i on all the intervals $[a_i,b_i]$, $i=1,\dots,L$. Let K_n be the union of all the edges e_i with $i\leq n$ such that all the edges that are adjacent to e_i have an index less than n. By locally finiteness of X^1, this is an exhausting sequence of finite subcomplexes and by construction, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ satisfies the requirements of Corollary 4.1.2. $\square$

Corollary 4.2.5. *For a strongly locally finite CW complex X and a 0–cell x_0, every element of $\pi_1(X,x_0)$ contains a representative in normal form.* $\square$

4.3 Normal spanning trees

According to Diestel, "[n]ormal spanning trees are perhaps the most important single structural tool for analysing an infinite graph" [10] and with regard to Corollary 4.2.2, it might not be surprising that a crucial step in the proof of our main theorem will be to use normal spanning trees. We first transfer some basic graph theoretical notation into our topological setting. Then we show two lemmata in order to prove the main result of this section, namely that the closure of a normal spanning tree in a CW complex is contractible and thus particularly simply–connected.

A *tree* is a simply–connected one–dimensional CW complex. Fixing a 0–cell r called the root of a tree T induces a partial order on the set of 0–cells. For two 0–cells x and y we say that x is *lower* than y if x lies on the direct edge path from r to y. A *spanning tree* of a CW complex X is a subcomplex T of X which is a tree and contains every 0–cell of X. If X is a CW complex and T a spanning tree, then every 1–cell that is not contained in T is called a *chord* of T. A spanning tree T is called *normal* if the endpoints of every chord are compatible in the induced order of the 0–cells. It is a well-known fact that every locally finite graph contains a normal spanning tree [1].

Lemma 4.3.1. *Every locally finite CW complex has a normal spanning tree.*

□

Note that the end space of the CW complex and the end space of its normal spanning tree can differ; see Example 3.1.2. As can be seen below, this cannot happen if the CW complex is strongly locally finite.

Lemma 4.3.2. *For a strongly locally finite CW complex X and a normal spanning tree T of X, the inclusion ι extends to a continuous map:*

(i) $\bar{\iota}\colon F(X^1) \to F(X)$ *such that* $\bar{\iota}_{|\Omega(X^1)}\colon \Omega(X^1) \cong \Omega(X)$.

(ii) $\bar{\iota}\colon F(T) \to F(X^1)$ *such that* $\bar{\iota}_{|\Omega(T)}\colon \Omega(T) \cong \Omega(X^1)$.

Proof. Let us first prove (i). Let $X^0 = \{v_1, v_2, \ldots\}$ be an enumeration of the 0–cells of X and let $V_n := \{v_1, \ldots, v_n\}$. Then $(X^1[V_n])_{n\in\mathbb{N}}$ and $(X[V_n])_{n\in\mathbb{N}}$ are exhausting sequences of finite subcomplexes of X^1 and X respectively. Furthermore, the dummies of $X^1 \setminus X^1[V_n]$ and the dummies of $X \setminus X[V_n]$ are in one–to–one correspondence by inclusion; Lemma 2.2.11. So by Theorem 3.1.9, there is a canonical bijection $\tilde{\iota}$ of $\Omega(X^1)$ and $\Omega(X)$. Moreover, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in X^1 converging to an end $\omega \in \Omega(X^1)$, then $(\iota(x_n))_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ converges in $F(X)$ to $\tilde{\iota}(\omega)$. Since X^1 is dense in $F(X^1)$, the extension $\bar{\iota}$ of ι by $\tilde{\iota}$ is continuous. In particular $\tilde{\iota} = \bar{\iota}_{|\Omega(X^1)}$ is a continuous bijective map from a compact space to a Hausdorff space and therefore a homeomorphism. The statement of (ii) is a well–known fact in infinite graph theory, see [1]. □

Corollary 4.3.3. *For a strongly locally finite CW complex X and a normal spanning tree T of X, there are homeomorphisms:*

(*i*) $F(X^1) \cong \overline{X^1} \subseteq F(X)$ *that fixes* X^1.

(*ii*) $F(T) \cong \overline{T} \subseteq F(X)$ *that fixes* T.

Proof. Note that $\overline{X^1} = X^1 \dot\cup (\overline{X^1} \setminus X^1)$ and $\Omega(X) = \overline{X^1} \setminus X^1$ by Corollary 3.1.5. Thus, $\bar\iota\colon F(X^1) \to \overline{X^1}$ is a continuous bijective map from a compact space into a Hausdorff space by Lemma 4.3.2. This shows (i) and the very same argument proves (ii). □

Lemma 4.3.4. *Let T be a locally finite tree, then $F(T)$ is contractible.*

Proof. Let $T_{\leq n}$ be the tree up to and including the nth level of T. Then $T_{\leq n}$ is homeomorphic to the contraction space $T[T_{\leq n-1}]$ of Chapter 2. So we have continuous maps:

$$f_n\colon F(T) \to T_{\leq n} \subseteq F(T).$$

These maps are the identity on $T_{\leq n}$ and map every point $x \in F(T) \setminus T_{\leq n}$ to the dummy in $T \setminus T_{\leq n-1}$ which includes x. We claim that the f_n satisfy the requirements of Corollary 4.1.2. Let $T_{\leq n}$ be the exhausting sequence of T. For every n, the set $F(T) \setminus T_{\leq n}$ is open and $f_{n|T_{\leq n}} = id_{T_{\leq n}} = f_{n+1|T_{\leq n}}$ on $T_{\leq n}$ so (i) holds. To see that (ii) is satisfied, note that $T_{\leq n+1}$ deformation retracts onto $T_{\leq n}$ relative $T_{\leq n}$. These deformations give homotopies from f_n to f_{n+1} relative the vertices at level n, i.e. $\partial(F(T) \setminus T_{\leq n})$. □

Theorem 4.3.5. *Let X be a strongly locally finite CW complex and T a normal spanning tree of X, then $\overline{T}$ is contractible.*

Proof. By Lemma 4.3.4, $F(T)$ is contractible. Thus, Corollary 4.3.3 (ii) proves the theorem. □

4.4 Pruning of paths

As seen in the previous section, a path α in $F(X)$ can only fail to be null–homotopic if it leaves the closure of a normal spanning tree T of X. By Theorem 4.2.1, we may assume α to be a path in $\overline{X^1}$. Hence, in order to prove that a path is null–homotopic, we only need to deal with its behaviour on the chords of the tree. At first, it may seem appealing to deform every subpath of α that runs through a chord to a path in T. However, as seen in the example of the introduction, this attempt needs to fail. Our attempt will be a different one: We first start with a path α in normal form and then "cut out" all subpaths on which α "uses" a chord and "replace" them by paths in T to obtain a new path β which is clearly null–homotopic. Then we start two sequences of maps: One starting by

$\alpha \simeq \alpha_1 \simeq \ldots$ and the other starting by $\beta \simeq \beta_1 \simeq \ldots$. We will do this in such a way that their limit will be equal. The big advantage here is that we can suck up all twistings in the end space. In this section, we formalise "cut out", "replacing" and "uses" for a path. Moreover, as we describe, we want to alter α on infinitely many chords at once; thus, it is not clear whether β is continuous. The main thereom will ensure this for normal spanning trees.

Definition 4.4.1. Let X be a locally finite CW complex and T a normal spanning tree of X and let x_0 be a 0–cell as a base point, not necessarily the root of T.

- If x and y are two 0–zero cells, then we denote by $T_{x\to y}$ the path from x to y in normal form.
- For a chord e of T with end points e_1 and e_2 and a homeomorphism $\alpha_e \colon I \cong \overline{e}$, we call
$$[T_{x_0\to e_1} \cdot \alpha_e \cdot T_{e_2\to x_0}] \in \pi_1(F(X), x_0)$$
the *fundamental loop of e in $e_1 \to e_2$ direction based at x_0.*

Remark 4.4.2.

- *Clearly, $T_{x\to y}$ is not unique. However, these normal paths only differ by a parametrisation and any of them will suffice for our purposes but to overcome ambiguity, one can fix a $T_{x\to y}$ for any pair of 0–cells.*
- *Edge paths considered as subspaces of X^1 are simply–connected so the homotopy class of $T_{x_0\to e_1} \cdot \alpha_e \cdot T_{e_2\to x_0}$ does not depend on the individual paths. If the direction and the base point are clear from the context, we will not mention them. In fact, since we are only interested in the homotopy type, we will call $T_{x_0\to e_1} \cdot \alpha_e \cdot T_{e_2\to x_0}$ the fundamental loop of e, well-knowing that only $[T_{x_0\to e_1} \cdot \alpha_e \cdot T_{e_2\to x_0}]$ is unique.*

Definition 4.4.3. Let X be a strongly locally finite CW complex and T a fixed spanning tree of X.

- A *used chord for a path* $\alpha \colon I \to F(X)$ is a pair $(e, [a, b])$ consisting of a chord e of T together with a subinterval $[a, b] \subseteq I$ such that $\alpha_{|(a,b)} \cong e$.
- For a path $\alpha \colon I \to F(X)$ and a set of used chords $\mathcal{S}$ of α, the *pruned path* $\alpha{\downarrow}\mathcal{S}$ *of α by $\mathcal{S}$* is the map

$$\begin{aligned} \alpha{\downarrow}\mathcal{S} \colon I &\to F(X) \\ x &\mapsto \begin{cases} \alpha(x) & \text{if } x \notin \bigcup\limits_{(e,[a,b])\in\mathcal{S}} (a, b) \\ T_{\alpha(a)\to\alpha(b)}((1-x)a + xb) & \text{if } x \in [a, b],\ (e, [a, b]) \in \mathcal{S}. \end{cases} \end{aligned}$$

Remark 4.4.4. *If $\mathcal{S}$ is a finite set of used chords, $\alpha{\downarrow}\mathcal{S}$ is clearly continuous for it is the $(\,\cdot\,)$ sum over finitely many paths.*

Lemma 4.4.5. *Let X be a strongly locally finite CW complex and T a fixed spanning tree of X. For every path $\alpha\colon I \to F(X)$ and a set $\mathcal{S}$ of used chords of α, there are only finitely many used chords in $\mathcal{S}$ with the same chord.*

Proof. Suppose to the contrary that there are infinitely many distinct used chords in $\mathcal{S}$, say $(e, [a_i, b_i])_{i\in\mathbb{N}}$, with the same chord e. Let x be an inner point of e and let $x_i \in (a, b)$ be the point with $\alpha(x_i) = x$. Note that by definition of a used chord, none of the $\alpha(x_n)$ is an endpoint of e. Moreover, $(x_n)_{n\in\mathbb{N}}$ has a limit point $\tilde{x}$ in I. Since the length of the intervals $[a_i, b_i]$ needs to shrink, $\tilde{x}$ is a limit point of $(a_i)_{i\in\mathbb{N}}$, too. Yet, this contradicts the fact that α is continuous since $\alpha(a_i)$ is one of the endpoints of e and x is an inner point. □

Theorem 4.4.6. *Let X be a strongly locally finite CW complex and T a fixed normal spanning tree of X. For every path $\alpha\colon I \to F(X)$ and every set $\mathcal{S}$ of used chords of α, the pruned path is indeed a path, i.e. $\alpha{\downarrow}\mathcal{S}$ is continuous.*

Proof.

$$\text{Let } x \notin (X^1 \cup \Omega(X)).$$

Choose an open neighbourhood O of x that avoids $X^1 \cup \Omega(X) = \overline{X^1}$, then $\alpha{\downarrow}\mathcal{S}^{-1}(O) = \alpha^{-1}(O)$ is open.

$$\text{Let } x \in X^1.$$

If x additionally is an inner point of an edge in T, choose an open neighbourhood O of x that contains no 0–cell and denote by e_x the edge in T with $x \in e_x$. There are exactly two components of $T \setminus e_x$ and by normality and local finiteness, there are only finitely many chords of T that hit both components. By Lemma 4.4.5, there are only finitely many used chords in $\mathcal{S}$ with one of these chords, say $(e_1, [a_1, b_1]), \ldots, (e_n, [a_n, b_n])$. Then

$$\alpha{\downarrow}\mathcal{S}^{-1}(O) = \alpha{\downarrow}\{(e_1, [a_1, b_1]), \ldots, (e_n, [a_n, b_n])\}^{-1}(O)$$

is open. If x is an inner point of a chord, then the edge path in T from one endpoint of the chord to the other endpoint contains only finitely many edges. The very same argument as the above one shows that $\alpha{\downarrow}\mathcal{S}(O)^{-1}$ is open for a small O. If x is a 0–cell, let O be an open neighbourhood of x that contains no 0–cell. Fix for every chord or edge that O hits an inner point. Then the above arguments for the inner points show that $\alpha{\downarrow}\mathcal{S}(O)^{-1}$ is open.

$$\text{Let } x \in \Omega(X).$$

Consider the exhausting sequence given by $K_n := X[(T_n)^0]$. Using Kőnig's infinity lemma, we find a closed neighbourhood basis $(\overline{D}_n)_{n\in\mathbb{N}}$ of x such that D_n is a dummy of $X \setminus K_n$. By normality of T, an edge e is contained in D_n if and only if $T_{\alpha(a)\to\alpha(b)}$ is contained in D_n. Hence, $\alpha\downarrow\mathcal{S}^{-1}(D_n) = \alpha^{-1}(D_n)$ for every $n \in \mathbb{N}$. □

Remark 4.4.7. *The above statement is wrong in general if T is not normal.*

5 The main theorem

Finally, we come to the main body of this thesis. In section one, we will introduce the generalized Mittag-Leffler condition in detail. We will prove a couple of basic statements of the property and give two non–trivial examples. In the second section, we will prove our main theorem:

Theorem. *Let X be a simply–connected strongly locally finite CW complex. If X satisfies the generalized Mittag-Leffler condition, then $F(X)$ is simply–connected.*

5.1 The generalized Mittag-Leffler condition

For the rest of this section, let X be a connected strongly locally finite CW complex, not necessarily simply–connected. Consider a normal spanning tree T of X. Let $K_n := X[(T_{\leq n})^0]$ be the subcomplex that is induced by all the verices at level at most n. Clearly, $(K_n)_{n\in\mathbb{N}}$ is an exhausting sequence of connected finite subcomplexes. Fix for every dummy $D \in \mathbb{D} = \mathbb{D}((K_n)_{n\in\mathbb{N}})$ a 0–cell $x_D \in D$ as a base point, since T is spanning it contains x_D. Furthermore, for any two dummies $D, D' \in \mathbb{D}$ with $D' \subseteq D$ the path $T_{x_{D'}\to x_D}$, is a path in D by normality of T. Thus it induces a group homomorphism from $\pi_1(D', x_{D'})$ to $\pi_1(D, x_D)$. In fact the triple

$$((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (T_{D'\to D})_{D'\subseteq D})$$

holds an inverse system of fundamental groups: $(\pi_1(D, x_D))_{D\in\mathbb{D}}$ with bonding maps $T_{D'\to D}$. Notice that $\mathbb{D}$ is ordered by inclusion[1] and the bonding maps are indeed compatible since trees are by definition simply connected. More generally:

Definition 5.1.1. Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of connected finite subcomplexes of X and $(x_D)_{D\in\mathbb{D}}$ a system of base points with $x_D \in D$. If $(f_{D'\to D})_{D'\subseteq D}$ is a system of compatible paths i.e. $f_{D'\to D} \simeq f_{D'\to D''} \cdot f_{D''\to D}$, for any dummies $D' \subseteq D'' \subseteq D$, then the triple

$$((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$$

is called *an admissible triple for X.*

[1] Of course, $\mathbb{D}$ is not directed and not all authors require the underlying poset to be directed. We just want to point out the structure of a system of groups indexed by a poset with compatible bonding maps. We will never consider the limit of this system.

Remark 5.1.2. *For every strongly locally finite CW complex there exists an admissible triple. Indeed every normal spanning tree induces an admissible triple.*

In fact much more is true:

Lemma 5.1.3. *For a strongly locally finite CW complex every exhausting sequence of connected finite subcomplexes can be extended to an admissible triple.*

Proof.

We will prove a stronger statement:

For every strongly locally finite CW complex X and an arbitrary exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of connected finite subcomplexes, there is a spanning tree T of X such that $T \cap D$ is path–connected, for every dummy $D \in \mathbb{D}$. $(*)$

Then T defines an admissible triple with the given exhausting sequence by the same arguments as in the introduction of this chapter. Every finer exhausting sequence defines a larger set of dummies so we may assume that $(K_n)^0$ and $(K_{n+1})^0$ differ by at most one 0–cell. This defines an enumeration of all 0–cells $X^0 = \{v_1, v_2, \ldots\}$. Note that v_n is adjacent to one of its predecessor since K_n is connected. The proof of $(*)$ work by a common method in infinite graph theory to obtain topological spanning trees, see [1] for details. Let us define a sequence of trees $T_1 \subseteq T_2 \subseteq \ldots \subseteq X^1$ such that:

(*i*) T_n contains at $\{v_1, \ldots, v_n\}$.

(*ii*) T_n has a exactly one vertex in every dummy of $X \setminus K_n$, and this vertex has the lowest index under all 0–cells in his dummy.

(*iii*) T_n and T_{n+1} only differ by vertices and edges that are contained in dummies of $X \setminus K_n$.

For $n = 1$, consider $X \setminus v_1$. For a dummy $D \in \mathbb{D}(v_1)$, denote by v_D the 0–cell with the lowest index under all 0–cells in D. Then v_1 needs to be adjacent to v_D, for every $D \in \mathcal{D}(v_1)$. Let T_1 be the subgraph that is induced by v_1 and all the v_D. Clearly, T_1 is a tree that satisfies (*i*) and (*ii*), trivially it satisfies (*iii*). Given T_n by (*i*) and (*ii*) it contains v_{n+1}. Let D be the dummy of $X \setminus K_n$ with $v_{n+1} \in D$. Consider $D \setminus v_{n+1}$. Dummies are by definition CW complexes, hence by Lemma 3.1.7 there are only finitely many dummies of $D \setminus v_{n+1}$, say $D'_1, \ldots, D'_k$. Moreover, $\mathbb{D}(K_{n+1}) = \mathbb{D}(K_n) \setminus \{D\} \dot\cup \{D'_1, \ldots, D'_k\}$. Consequently the very same procedure as in the induction basis for v_{n+1} and D defines T_{n+1} and one easily verifies the properties (*i*) to (*iii*).

We claim that $T := \bigcup_{n\in\mathbb{N}} T_n$ proves $(*)$. Unions of CW complexes are CW complexes, so T is a CW complex. The image of a path is compact so it is contained in T_n eventually, hence T is path–connected and simply–connected, i.e. T is a tree. It is spanning by (i). Consider $T \cap D$ for an n and a dummy $D \in \mathbb{D}(K_n)$. Then $T \cap D$ is path–connected by (ii) and the fact that T is path–connected. □

We want to use the data of an admissible triple as a certificate to decide if $F(X)$ is simply–connected. For a given admissible triple, consider $\pi_1(D, x_D)$ for an $n \in \mathbb{N}$ and a dummy $D \in \mathbb{D}(K_n)$. For every $m \geq n$ and $D' \in \mathbb{D}(K_m)$ with $D' \subseteq D$, we obtain a subgroup $f_{D' \to D}(\pi_1(D', x_{D'}))$ of $\pi_1(D, x_D)$. Let us write $H^D(D')$ for the normal closure of $f_{D' \to D}(\pi_1(D', x_{D'}))$ in $\pi_1(D, x_D)$.

Definition 5.1.4. We say that X *satisfies the generalized Mittag–Leffler condition for the admissible triple* $((K_n)_{n \in \mathbb{N}}, (x_D)_{D \in \mathbb{D}}, (f_{D' \to D})_{D' \subseteq D})$ *for* n *and a dummy* $D \in \mathbb{D}(K_n)$ if the following condition holds:

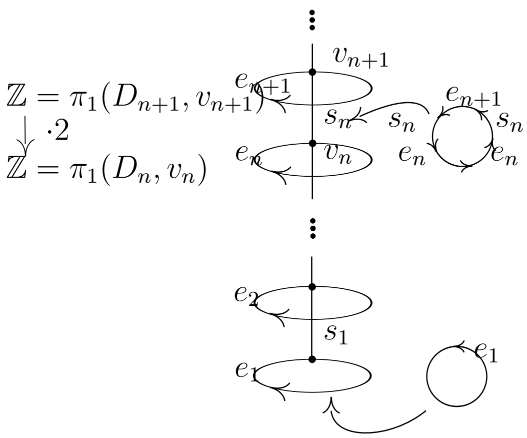
There is an $N \in \mathbb{N}$ such that for every $L \geq N$:

$$\Big\langle \bigcup_{\substack{D' \in \mathbb{D}(K_N) \\ D' \subseteq D}} H^D(D') \Big\rangle = \Big\langle \bigcup_{\substack{D' \in \mathbb{D}(K_L) \\ D' \subseteq D}} H^D(D') \Big\rangle.$$

We will say that X satisfies the generalized Mittag–Leffler condition for the admissible triple $((K_n)_{n \in \mathbb{N}}, (x_D)_{D \in \mathbb{D}}, (f_{D' \to D})_{D' \subseteq D})$ if X satisfies it for every $n \in \mathbb{N}$ and every $D \in \mathbb{D}(K_n)$.

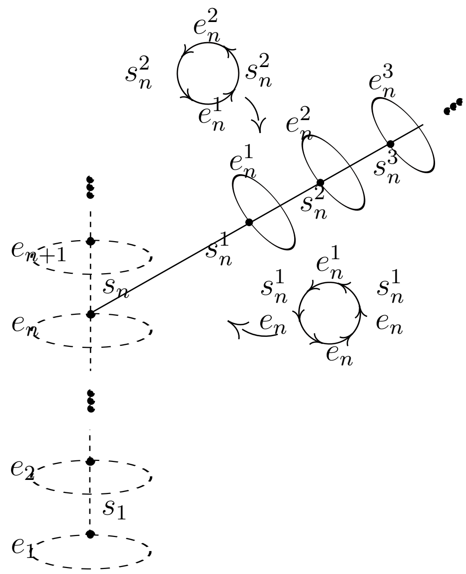
Note that in the above equality $\supseteq$ always holds. The point is to find an N such that the right–hand side is constant eventually.

Example 5.1.5. *Consider the following CW complex X with the indicated cell decomposition.*



Choosing the spine as a normal spanning tree gives an admissible triple. Note that $\pi_1(D_n, v_n) = \mathbb{Z}$ and the induced morphisms are, by definition of the attaching map of the 2–cells, just the multiplication by 2. Thus, the right–hand side of the defining equality of the generalized Mittag–Leffler condition is decreasing. Hence, X does not satisfy the generalized Mittag–Leffler condition.

Example 5.1.6. *Now let us extend X to Y by gluing on every level of X a cylinder as indicated.*



Choosing the union of all the spines as a normal spanning tree T gives an admissible triple. For every $X[T^1_{\leq n}] = K_n$, we obtain $(n+1)$–many copies of the regular cylinder as a dummy and one copy of Y without the bottom disk as a dummy D_n. Clearly, all the cylinder dummies satisfy the generalized Mittag–Leffler condition. To see that D_n satisfies the generalized Mittag–Leffler condition, note $\pi_1(D_n, v_n) = \mathbb{Z}$ and the induced morphisms are multiplications by 2 from above and multiplications by 3 from all the added cylinders. Since 2 and 3 are relative prime, the unit is always contained in the right–hand side of the defining equality of the generalized Mittag–Leffler condition, i.e. the right–hand side equals the entire group for every $L \geq n$. So Y satisfies the generalized Mittag–Leffler condition.

The goal for the rest of this section is to prove that a strongly locally finite CW complex satisfies the generalized Mittag–Leffler condition either for all admissible triples or for none, Theorem 5.1.8. This will justify the definition that X satisfies the generalized Mittag–Leffler condition if and only if it satisfies it for every admissible triple, Definition 5.1.9. In order to prove Theorem 5.1.8, let us first show an auxiliary result.

For an arbitrary admissible triple, every subsequence of the exhausting sequence induces another triple. Indeed, every subsequence of an exhausting sequence is again an exhausting sequence. Choosing all the x_D and $f_{D'\to D}$ for which D and D' are dummies of the subsequence gives a triple. This induced triple is admissible since it inherits all the necessary properties and we will call it *the induced admissible triple of the subsequence.*

Lemma 5.1.7. *X satisfies the generalized Mittag–Leffler condition for an admissible triple if and only if X satisfies the generalized Mittag–Leffler condition for one (equivalent all) induced admissible triple of a subsequence.*

Proof. The forward implication is immediate by definition. So, given an arbitrary admissible triple, say $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$. Let $(K_{n_i})_{i\in\mathbb{N}}$ be a subsequence of $(K_n)_{n\in\mathbb{N}}$ and let X satisfy the generalized Mittag–Leffler condition for the induced admissible triple $((K_{n_i})_{i\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}((K_{n_i})_{i\in\mathbb{N}})}, (f_{D'\to D})_{D'\subseteq D})$. Let $n\in\mathbb{N}$ and $D\in\mathbb{D}(K_n)$ and choose an index of the subsequence n_i with $n_i \geq n$. For n_i and every dummy $D'\in(K_{n_i})$ with $D'\subseteq D$, let $N_{D'}(n_i)$ be an index that witnesses that X satisfies the generalized Mittag–Leffler condition for the induced admissible triple. We claim that $N := \max\{N_{D'}(n_i) \mid D'\in\mathbb{D}(K_{n_i}),\ D'\subseteq D\}$ shows that X satisfies the generalized Mittag–Leffler condition for the admissible triple $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$ for n and D. Let:

$$[\alpha] \in \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_N)\\ D'\subseteq D}} H^D(D') \Big\rangle \subseteq \pi_1(D, x_D).$$

Fix an arbitrary $L \geq N$. Consider a summand of $[\alpha]$, say $g\cdot f_{D'\to D}([\beta])\cdot g^{-1}$ with $g\in\pi_1(D,x_D)$ and $[\beta]\in\pi_1(D',x_{D'})$ for some $D'\in\mathbb{D}(K_N)$. Then there is one dummy $D(n_i)\in\mathbb{D}(K_{n_i})$ with $D'\subseteq D(n_i)\subseteq D$. Furthermore, by definition of the index N:

$$\begin{gathered}
f_{D'\to D(n_i)}([\beta]) = \prod_{j=1}^{m} h_j\cdot[\gamma_j]\cdot h_j^{-1}\\
[\gamma_j]\in f_{D''_j\to D(n_i)}(\pi_1(D''_j, x_{D''_j})) \text{ for } D''_j\in\mathbb{D}(K_L) \text{ with } D''_j\subseteq D \qquad (*)\\
h_j\in\pi_1(D(n_i), x_{D(n_i)})
\end{gathered}$$

This defines a suitable representation of $[\alpha]$ for every summand:

$$\begin{aligned}
g\cdot f_{D'\to D}([\beta])\cdot g^{-1} &= g\cdot f_{D(n_i)\to D} f_{D'\to D(n_i)}([\beta])\cdot g^{-1}\\
&= g\cdot f_{D(n_i)\to D}\Big(\prod_{j=1}^{m} h_j[\gamma_j]h_j^{-1}\Big)\cdot g^{-1}\\
&= \Big(g\cdot f_{D(n_i)\to D}(h_1)\Big)\cdot f_{D(n_i)\to D}([\gamma_1])\cdot\Big(f_{D(n_i)\to D}(h_1^{-1})\cdot g^{-1}\Big)\\
&\quad\cdot\Big(g\cdot f_{D(n_i)\to D}(h_2)\Big)\cdot f_{D(n_i)\to D}([\gamma_2])\cdot\Big(f_{D(n_i)\to D}(h_2^{-1})\cdot g^{-1}\Big)\\
&\quad\vdots\\
&\quad\cdot\Big(g\cdot f_{D(n_i)\to D}(h_m)\Big)\cdot f_{D(n_i)\to D}([\gamma_m])\cdot\Big(f_{D(n_i)\to D}(h_m^{-1})\cdot g^{-1}\Big)\\
&\overset{(*)}{\in} \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_L)\\ D'\subseteq D}} H^D(D') \Big\rangle
\end{aligned}$$

□

Theorem 5.1.8. *If X satisfies the generalized Mittag-Leffler condition for one admissible triple, then it satisfies it for every admissible triple.*

Proof. Let $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$ be an admissible triple for which X satisfies the generalized Mittag–Leffler condition.

- Given an admissible triple with the same exhausting sequence and the same base points but different paths, say $(\tilde{f}_{D'\to D})_{D'\subseteq D}$. Let $n \in \mathbb{N}$ and $D \in \mathbb{D}(K_n)$. Then for every $m \geq n$ and every dummy $D' \in \mathbb{D}(K_m)$ with $D' \subseteq D$, the two paths $f_{D'\to D}$ and $\tilde{f}_{D'\to D}$ define a loop $\alpha := (f^{-1}_{D'\to D} \cdot \tilde{f}_{D'\to D})$ in D based at x_D. Then $[\alpha] \in \pi_1(D, x_D)$ and

$$\tilde{f}_{D'\to D}\Big(\pi_1(D', x_D)\Big) = [\alpha]^{-1} \cdot \Big(f_{D'\to D}(\pi_1(D', x_D))\Big) \cdot [\alpha],$$

  thus $H^D(D')$ does not depend on the set of morphisms.

- Given a triple with the same exhausting sequence but different base points, say $(\tilde{x}_D)_{D\in\mathbb{D}}$, and different morphisms, say $(\tilde{f}_{D'\to D})_{D'\subseteq D}$. Every dummy D of $(K_n)_{n\in\mathbb{N}}$ is path–connected, so fix a path P_D in D from x_D to $\tilde{x}_D$. Letting $g_{D'\to D} := P_{D'} \cdot \tilde{f}_{D'\to D} \cdot P_D^{-1}$ gives a system of compatible paths for $(K_n)_{n\in\mathbb{N}}$ and $(x_D)_{D\in\mathbb{D}}$. Indeed, for any dummies $D, D', D'' \in \mathbb{D}$ with $D' \subseteq D'' \subseteq D$:

$$\begin{aligned} g_{D'\to D} &= P_{D'} \cdot \tilde{f}_{D'\to D} \cdot P_D^{-1} \\ &\simeq P_{D'} \cdot (\tilde{f}_{D'\to D''} \cdot \tilde{f}_{D''\to D}) \cdot P_D^{-1} \\ &\simeq (P_{D'} \cdot \tilde{f}_{D'\to D''} \cdot P_{D''}^{-1}) \cdot (P_{D''} \cdot \tilde{f}_{D''\to D} \cdot P_D^{-1}) \\ &= g_{D'\to D''} \cdot g_{D''\to D} \end{aligned}$$

  By the previous point, X satisfies the generalized Mittag–Leffler condition for the admissible triple $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (g_{D'\to D})_{D'\subseteq D})$. Then we can conclude that X satisfies the generalized Mittag–Leffler condition for the admissible triple $((K_n)_{n\in\mathbb{N}}, (\tilde{x}_D)_{D\in\mathbb{D}}, (\tilde{f}_{D'\to D})_{D'\subseteq D})$ by applying the induced group homomorphism of the path P_D to the defining equality in the definition of generalized Mittag–Leffler condition. Notice that every P_D gives an isomorphism and for any two dummies $D, D' \in \mathbb{D}$ with $D' \subseteq D$ we have a commutative diagram of groups

$$\begin{tikzcd} \pi_1(D', x_{D'}) \arrow[r, "P_{D'}"] \arrow[d, "g_{D'\to D}"] & \pi_1(D', \tilde{x}_{D'}) \arrow[d, "\tilde{f}_{D'\to D}"] \\ \pi_1(D, x_D) \arrow[r, "P_D"] & \pi_1(D, \tilde{x}_D) \end{tikzcd}$$

- Given an arbitrary admissible triple, say $((\tilde{K}_n)_{n\in\mathbb{N}}, (\tilde{x}_D)_{D\in\tilde{\mathbb{D}}}, (\tilde{f}_{D'\to D})_{D'\subseteq D})$. By Lemma 5.1.7 we may assume $K_n \subseteq \tilde{K}_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$. By Lemma 5.1.3 we can extend the exhausting sequence $K_1 \subseteq \tilde{K}_1 \subseteq K_2 \subseteq$

... to an admissible triple $((K_n, \tilde{K}_n)_{n\in\mathbb{N}}, (y_D)_{D\in\mathbb{D}}, (g_{D'\to D})_{D'\subseteq D})$. Now X satisfies the generalized Mittag–Leffler condition:

$$
\begin{array}{ll}
\text{for } ((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D}) & \text{by assumption,} \\
\Rightarrow \text{ for } ((K_n)_{n\in\mathbb{N}}, (y_D)_{D\in\mathbb{D}}, (g_{D'\to D})_{D'\subseteq D}) & \text{by our previous point,} \\
\quad \Rightarrow \text{ for } ((K_n, \tilde{K})_{n\in\mathbb{N}}, (y_D)_{D\in\mathbb{D}\cup\tilde{\mathbb{D}}}, (g_{D'\to D})_{D'\subseteq D}) & \text{by Lemma 5.1.7,} \\
\qquad \Rightarrow \text{ for } ((\tilde{K})_{n\in\mathbb{N}}, (y_D)_{D\in\tilde{\mathbb{D}}}, (g_{D'\to D})_{D'\subseteq D}) & \text{by Lemma 5.1.7,} \\
\qquad\quad \Rightarrow \text{ for } ((\tilde{K}_n)_{n\in\mathbb{N}}, (\tilde{x}_D)_{D\in\tilde{\mathbb{D}}}, (\tilde{f}_{D'\to D})_{D'\subseteq D}) & \text{by our previous point.}
\end{array}
$$

□

The effort of this section culminates in the following definition:

Definition 5.1.9. We will say that X *satisfies the generalized Mittag–Leffler condition* if X satisfies the generalized Mittag–Leffler condition for one (equivalent every) admissible triple.

5.2 Proof of the main theorem

Theorem 5.2.1. *Let X be a simply–connected strongly locally finite CW complex. If X satisfies the generalized Mittag-Leffler condition, then $F(X)$ is simply–connected.*

Proof. For the rest of the proof, let T be a fixed normal spanning tree of X with root r. The exhausting sequence we refer to is given by $K_n := X[(T_{\leq n})^0]$, the finite subcomplex that is induced by the vertices of level at n.

The proof is organized as follows: Given an arbitrary element $[\alpha] \in \pi_1(F(X), r)$, by Corollary 4.2.5 we may assume that its representative α is in normal form. Consider the set $\mathcal{S}$ of all used chords of α and define $\beta := \alpha\downarrow\mathcal{S}$. Then β is a loop by Theorem 4.4.6 based at r. Furthermore, β lies entirely in $\overline{T}$, thus β is null–homotopic by Theorem 4.3.5. Starting with α and β we will simultaneously define two sequences $\alpha = \alpha_0 \simeq \alpha_1 \simeq \ldots$ and $\beta = \beta_0 \simeq \beta_1 \simeq \ldots$ of loops based at r by using the generalized Mittag–Leffler condition again and again. Both will satisfy the requirements of Corollary 4.1.2. However, we need X to be simply–connected to start the procedure. Moreover, we will make sure that

$$\alpha \simeq \alpha_1 \simeq \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n \simeq \beta_1 \simeq \beta$$

hence α is null–homotopic, i.e. $\pi_1(F(X), r) = 1$.

Let $\alpha\colon I \to F(X)$ be a loop in normal form based at the root r of T. We will define the α_n and β_n inductively and assign to every α_n a set $\mathcal{S}_n$ of used chords of α_n, arranging that:

1. Every edge of a used chord in $\mathcal{S}_n$ has height[2] at least n. Moreover, for every used chord $(e, [a, b]) \in \mathcal{S}_n$ and the unique dummy $D \in \mathbb{D}(K_n)$ with $e \subseteq D$,
$$[T_{x_D \to \alpha(a)} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to x_D}] \in \Big\langle \bigcup_{\substack{D' \in \mathbb{D}(K_N)\\ D' \subseteq D}} H^D(D')\Big\rangle,$$
for the index N that witnesses that X satisfies the generalized Mittag–Leffler condition for n and D.

2. $\alpha_n \simeq \alpha_{n+1}$ and $\beta_n = \alpha_n \downarrow \mathcal{S}_n \simeq \alpha_{n+1} \downarrow \mathcal{S}_{n+1} = \beta_{n+1}$ by homotopies that satisfy the requirements of Corollary 4.1.2.

$$(0 \to 1)$$

Let $\alpha_0 := \alpha$, $\mathcal{S}_0$ be the set of all used chords of α and $\beta_0 = \alpha_0 \downarrow \mathcal{S}_0$. For $n = 1$ and a dummy $D \in \mathbb{D}(K_1)$, denote by $N_D(1)$ the index that witnesses that X satisfies the generalized Mittag-Leffler condition for 1 and D and let $N = \max\{N_D(1) \mid D \in \mathbb{D}(K_1)\}$. Let U_0 be the set of all used chords $(e, [a, b])$ in $\mathcal{S}_0$ such that e is adjacent to r. Since X is in particular locally finite U_0 is finite by Lemma 4.4.5. Furthermore, for every $D \in \mathbb{D}(K_1)$ let V_D be the set of all used chords $(e, [a, b])$ in $\mathcal{S}_0$, such that e has height at most N. Every V_D is finite, since there are only finitely many chords with a height less than N. In fact there are only finitely many dummies in $\mathbb{D}(K_1)$, so $W_0 := \bigcup_{D \in \mathbb{D}(K_1)} V_D \cup U_0$ is finite. Note that the fundamental loop of a chord for a used chord in $\mathcal{S}_0 \setminus W_0$ lies in

$$\Big\langle \bigcup_{\substack{D' \in \mathbb{D}(K_N)\\ D' \subseteq D}} H^D(D')\Big\rangle \subseteq \pi_1(D, x_D), \text{ for some } D \in \mathbb{D}(K_1). \qquad (*)$$

For every used chord $(e, [a, b]) \in W_0$, let

$$P_{(e,[a,b])} := T_{\alpha(a)\to r} \cdot T^{-1}_{\alpha(a)\to r} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to r} \cdot T^{-1}_{\alpha(b)\to r}.$$

Let α_0' be the path that coincides with α_0, but for every interval $[a, b]$ of a used chord in W_0, plug in the path $P_{(e,[a,b])}$. This path is continuous since W_0 is finite and obviously homotopic to α_0. Moreover, $T^{-1}_{\alpha(a)\to r} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to r}$ defines the fundamental loop of e at r. Since X is simply–connected, the fundamental loop of e is homotopic to any fundamental loop of a chord with height at least N. Fix for every such $T^{-1}_{\alpha(a)\to r} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to r}$ a fundamental loop $g_{[a,b]}$ based at

[2]For us the height of an edge is the height of the lower endpoint of the edge. Note that the endpoints of a chord are in deed comparable by the definition of a normal tree.

r of a chord with height at least N. Define α_1 to be the path that coincides with α'_0, but for every $T^{-1}_{\alpha(a)\to r} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to r}$ plug in $g_{[a,b]}$. We may assume α_1 in normal form since all the used loops and paths could have been chosen in normal form. Let $\mathcal{S}_1$ be the union of $\mathcal{S}_0 \setminus W_0$ and all the used chords of all $g_{[a,b]}$. Then α_1 and $\mathcal{S}_1$ satisfy (1) by $(*)$ and the height of the $g_{|[a,b]}$. Define β_1 as the pruned path of α_1 by $\mathcal{S}_1$, which is continuous by Remark 4.4.4. Now β_0 and β_1 only differ on the finitely many intervals of the used chords in W_0. For such a used chord $(e, [a, b]) \in W_0$, the paths $\beta_{0|[a,b]}$ and $\beta_{1|[a,b]}$ have the same start and endpoint. Since X is simply–connected, they are homotopic. Clearly, any of the used homotopies avoid $K_0 = \emptyset$. Consequently 2. is satisfied.

The idea for the induction step is that (1) mimics the roll of the simple–connectedness in the induction basis. To be more precise:

$$(n \to n+1)$$

For $(n+1)$ and a dummy $D \in \mathbb{D}(K_{n+1})$, denote by $N_D(n+1)$ the index that witnesses that X satisfies the generalized Mittag-Leffler condition for $(n+1)$ and D and let $N = \max\{N_D(n+1) \mid D \in \mathbb{D}(K_{n+1})\}$. Let U_n be the set of all used chords $(e, [a, b])$ in $\mathcal{S}_n$ such that e is adjecent to a vertex in K_{n+1}. Then U_n is finite by the very same arguments as in the induction basis. Furthermore, for every $D \in \mathbb{D}(K_{n+1})$, let V_D be the set of all used chords $(e, [a, b])$ in $\mathcal{S}_n$, such that e has height of at most N. Again every V_D is finite and there are only finitely many dummies in $\mathbb{D}(K_{n+1})$ so $W_n := \bigcup_{D\in\mathbb{D}(K_{n+1})} V_D \cup U_n$ is finite. Note that the fundamental loop of a chord for a used chord in $\mathcal{S}_n \setminus W_n$ lies in

$$\Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_N)\\ D'\subseteq D}} H^D(D') \Big\rangle \subseteq \pi_1(D, x_D), \text{ for some } D \in \mathbb{D}(K_{n+1}). \qquad (**)$$

For every used chord $(e, [a, b]) \in W_n$, there is exactly one dummy $D \in \mathbb{D}(K_n)$ with $\alpha_n(a), \alpha_n(b) \in D$, since e has height at least n by 1. Let

$$P_{(e,[a,b])} := T_{\alpha_n(a)\to x_D} \cdot T^{-1}_{\alpha_n(a)\to x_D} \cdot \alpha_{|[a,b]} \cdot T_{\alpha_n(b)\to x_D} \cdot T^{-1}_{\alpha_n(b)\to x_D}.$$

Let α'_n be the path that coincides with α_n, but for every interval $[a, b]$ of a used chord in W_{n+1}, plug in the path $P_{(e,[a,b])}$. This path is continuous since W_{n+1} is finite and obviously homotopic to α_n. In fact, we alter α_n only on finitely many open intervals and the used homotopies take place in $X \setminus K_n$. Moreover, $(\dagger)$
$T^{-1}_{\alpha(a)\to x_D} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to x_D}$ defines a loop based at x_D. By property (1) for α_n, this

loop is homotopic in $D \subseteq X \setminus K_n$ to a loop of the form

$$\begin{aligned} T^{-1}_{\alpha(a)\to x_D} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to x_D} &\simeq \prod_{i=1}^{m} h_i^{-1} \cdot \Big(f_{D^i \to D}(g_i)\Big) \cdot h_i \\ &= \prod_{i=1}^{m} h_i^{-1} \cdot (T^{-1}_{x_{D^i}\to x_D} \cdot g_i \cdot T_{x_{D^i}\to x_D}) \cdot h_i \end{aligned}$$

with $[h_i] \in \pi_1(D, x_D)$ and $[g_i] \in \pi_1(D^i, x_{D^i})$ for dummies $D^i \in \mathbb{D}(K_N)$. Let α_{n+1} be the path that coincides with α'_n, but for every $T^{-1}_{\alpha(a)\to x_D} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to x_D}$, plug in the right hand side of the above homotopy. We may assume α_{n+1} in normal form, since we could have chosen all the used paths in normal form. Let $\mathcal{S}_{n+1}$ be the union of $\mathcal{S}_n \setminus W_n$ and all the used chords of all the g_i $i = 1, \ldots, m$ for every $T^{-1}_{\alpha(a)\to x_D} \cdot \alpha_{|[a,b]} \cdot T_{\alpha(b)\to x_D}$. Apparently, α_{n+1} and $\mathcal{S}_{n+1}$ satisfy property 1. because of $(**)$ and the height of the chords of the used chords of the g_i. Furthermore, α_n and α_{n+1} only differ on the finitely many intervals of the used chords in W_{n+1}, and all the used homotopies took place in $X \setminus K_n$, by (†) so (2) for α_n and α_{n+1} holds. Let $\beta_{n+1} = \alpha_{n+1} \downarrow \mathcal{S}_{n+1}$. Then, β_n and β_{n+1} only differ on the finitely many intervals of the used chords in W_{n+1}. For such a used chord $(e, [a,b]) \in W_{n+1}$, the paths $\beta_{n|[a,b]}$ and $\beta_{n+1|[a,b]}$ have the same start and endpoint. One the one hand $\beta_{n|[a,b]}$ is homotopic to

$$T_{\alpha_n(a)\to x_D} \cdot T^{-1}_{\alpha_n(a)\to x_D} \cdot \beta_{n|[a,b]} \cdot T_{\alpha_n(b)\to x_D} \cdot T^{-1}_{\alpha_n(b)\to x_D},$$

by a homotopy in $X \setminus K_n$. Clearly $T^{-1}_{\alpha_n(a)\to x_D} \cdot \beta_{n|[a,b]} \cdot T_{\alpha_n(b)\to x_D}$ is null–homotopic in D, since it is a loop in T. On the other hand $\alpha_{n+1|[a,b]}$ is a path of the form

$$\alpha_{n+1|[a,b]} = T_{\alpha_n(a)\to x_D} \cdot \Big(\prod_{i=1}^{m} h_i^{-1} \cdot (T^{-1}_{x_{D^i}\to x_D} \cdot g_i \cdot T_{x_{D^i}\to x_D}) \cdot h_i\Big) \cdot T^{-1}_{\alpha_n(b)\to x_D}.$$

By the definition of β_{n+1} the restriction to $[a,b]$ defines a path of the form:

$$\beta_{n+1|[a,b]} = T_{\alpha_n(a)\to x_D} \cdot \Big(\prod_{i=1}^{m} h_i^{-1} \cdot (T^{-1}_{x_{D^i}\to x_D} \cdot \gamma_i \cdot T_{x_{D^i}\to x_D}) \cdot h_i\Big) \cdot T^{-1}_{\alpha_n(b)\to x_D},$$

where γ_i is the pruned path of g_i by all its used chords. In particular γ_i is a path in T, hence $T^{-1}_{x_{D^i}\to x_D} \cdot \gamma_i \cdot T_{x_{D^i}\to x_D}$ is null homotopic. Consequently, the above sum defines a null homotopic loop, in D. Thus $\beta_{n|[a,b]}$ and $\beta_{n+1|[a,b]}$ are homotopic by a homotopy in $D \subseteq X \setminus K_n$, i.e. 2. holds for β_n and β_{n+1}.

Since $\alpha_0 = \alpha$ is in normal form, the pruned path of α by all its used chords is a path in $\overline{T}$, thus $\beta_0 = \alpha \downarrow \mathcal{S}_0$ is null homotopic. To see that $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n$ consider $\mathcal{S} := \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$.

Let $x \in I$ be a point that lies in only finitely many intervals of the used chords in $\mathcal{S}$. Then there exists an $N \in \mathbb{N}$, such that x lies in none of the intervals of the used chords in $\mathcal{S}_L$ for every $L \geq N$. Since we alter α_L only the intervals of $\mathcal{S}_L$ we have $\alpha_N(x) = \alpha_L(x)$, for every $L \geq N$. Consequently $\lim\limits_{n\to\infty} \alpha_n(x) = \alpha_N(x)$. By definition $\beta_n = \alpha_n \downarrow \mathcal{S}_n$, so $\alpha_L(x) = \beta_L(x)$ for every $L \geq N$. Consequently, $\lim\limits_{n\to\infty} \beta_n(x) = \lim\limits_{n\to\infty} \alpha_n(x)$.

Let $x \in I$ be a point that lies in infinitely many intervals of the used chords in $\mathcal{S}$. Then for every $n \in \mathbb{N}$ there is exactly one used chord $(e, [a, b]) \in \mathcal{S}_n$ with $x \in [a, b]$. Denote by D_n the dummy of $\mathbb{D}(K_n)$ with $\alpha_n(x) \in D_n$. The homotopie from $\alpha_{n|[a,b]}$ to $\alpha_{n+1|[a,b]}$ avoids K_n, so $D_n \supseteq D_{n+1}$. This gives a sequence of dummies $D_1 \subseteq D_2 \subseteq \ldots$ and there is exactly one end $\omega \in \Omega$, such that $\{\omega\} = \bigcap \overline{D_n}$. Consequently $\lim\limits_{n\to\infty} \alpha_n(x) = \omega$. Furthermore, $e_n \subseteq D_n$, so ω is the unique limit point of the sequence of start points of the e_n. Since T is normal the path in T between the start and end point for every e_n lies in D_n, thus $\beta_n(x) \in D_n$, so $\lim\limits_{n\to\infty} \beta_n(x) = \omega$. $\square$

6 Failing the generalized Mittag-Leffler condition

In this chapter, we want to understand what happens if a CW complex fails to satisfy the generalized Mittag–Leffler condition. The best possible outcome would be if we could prove that $\pi_(F(X))$ is non–trivial as soon as a simply–connected X fails to satisfy the generalized Mittag–Leffler condition for this would characterise all the simply–connected CW complexes with a simply–connected Freudenthal compactification. Unfortunately, we are neither able to do so nor find a counter–example, i.e. a simply–connected CW complex that fails to satisfy the generalized Mittag–Leffler condition for which $F(X)$ is still simply–connected. Instead, we will prove that if the fundamental group of a dummy witnessing the failing of the generalized Mittag–Leffler condition has an abelian fundamental group, then its Freudenthal compactification is not simply–connected. Unfortunately, the commutativity of this fundamental group will be a crucial part because it will guarantee us that its first homology group coincides with its fundamental group which makes it possible to apply a version of the Mayer–Vietoris sequence.

6.1 The Seifert–van Kampen theorem and the Mayer–Vietoris sequence for dummies

One advantage of CW complexes is that the Seifert–van Kampen theorem and the Mayer–Vietoris sequence apply to a cover of subcomplexes rather than just open covers. Our goal in this section is to prove the analogue in our context, i.e. for a cover of subcomplexes $A \cup B = X$, the Seifert–van Kampen theorem and the Mayer–Vietoris sequence apply to the cover $\overline{A} \cup \overline{B} = F(X)$. We will not be able to prove this for arbitrary subcomplexes A and B but for a large class of subcomplexes which, most importantly, include all dummies.

Definition 6.1.1. A *deformation retraction of a space X onto a subspace A* is a continuous map:

$$R\colon X \times I \to X$$

such that $R(\cdot, 0) = id_X$, $R(X, 1) \subseteq A$ and $R(a, t) = a$ for all $a \in A$ and $t \in I$. In the above situation we say that X deformation retracts onto A.

Remark 6.1.2. *A deformation retraction gives a homotopie equivalence, in particular $\pi_1(X) \cong \pi_1(A)$.*

A well–known fact of CW complex is the following, for details see [5]

Lemma 6.1.3. *For a CW complex X and a subcomplex A of X and an $\varepsilon < 1$, there is an open neighbourhood $N_\varepsilon(A)$ contained in the ε–neigboudhood of A such that:*

- *$N_\varepsilon(A)$ deformation retract onto A.*
- *If A and B are two subcomplexes, then $N_\varepsilon(A) \cap N_\varepsilon(B) = N_\varepsilon(A \cap B)$.*

Definition 6.1.4. If X is a locally finite CW complex, we call a subcomplex A *wide* if every end in $\overline{A} \subseteq F(X)$ has an open neighbourhood contained in $\overline{A}$.

Lemma 6.1.5.

(i) *For a strongly locally finite CW complex X and a finite subcomplex, K every dummy of $X \setminus K$ is wide.*

(ii) *Finite unions and finite intersections of wide complexes are wide.*

Proof. For the first statement, just note that there are only finitely many cells of X not contained in one of the dummies of $X \setminus K$. For the second statement, consider two wide complexes A and B. Then every accumulation point of $A \cup B$ is an accumulation point of A or B. Hence, $A \cup B$ is wide. Clearly, $A \cap B$ is wide. □

Lemma 6.1.6. *Let X be a locally finite CW complex and A a wide subcomplex. For every $\varepsilon < 1$ and the open neighbourhood $N_\varepsilon(A)$ of Lemma 6.1.3 the subspace $N_\varepsilon \cup \overline{A} \subseteq F(X)$ deformation retracts onto $\overline{A}$.*

Proof. Let $R\colon N_\varepsilon(A) \times I \to N_\varepsilon(A) \times \overline{A}$ be a strong deformation retraction onto A. Consider:

$$\begin{aligned} \overline{R}\colon N_\varepsilon(A) \cup \overline{A} \times I &\to N_\varepsilon(A) \\ (x,t) &\mapsto \begin{cases} R(x,t) & \text{if } x \in N_\varepsilon(A) \\ x & \text{if } x \in \overline{A} \setminus A \end{cases} . \end{aligned}$$

Obviously, we only need to show that $\overline{R}$ is indeed continuous. Let $A_1 := \overline{A}$ and $A_2 := \overline{N_\varepsilon(A) \setminus A}$ (the closure in $N_\varepsilon(A) \cup \overline{A}$). Both sets are closed in $N_\varepsilon(A) \cup \overline{A}$ and their union is $N_\varepsilon(A) \cup \overline{A}$. Moreover, $\overline{R}_{|A_1 \times I}$ is the projection onto the first component so it is continuous. Since A is wide, $\overline{N_\varepsilon(A) \setminus A}$ contains no ends. Consequently, $\overline{N_\varepsilon(A) \setminus A} \subseteq X$, so $\overline{R}_{|A_2 \times I} = R_{|A_2 \times I}$ is continuous. Then $\overline{R}$ is continuous by the auxiliary statement $(*)$ in the proof of Corollary 4.1.2. □

Theorem 6.1.7. *Let X be a strongly locally finite CW complex and A and B two wide subcomplexes such that $X = A \cup B$, then:*

(i) *(Seifert–van Kampen theorem) If, in addition, A and B are path–connected and $A \cap B$ is non–empty and path–connected, then $\overline{A}$, $\overline{B}$ and $\overline{A} \cap \overline{B}$ are path–connected and*

$$\pi_1(F(X)) = \pi_1(\overline{A}) \star_{\pi_1(\overline{A} \cap \overline{B})} \pi_1(\overline{B}).$$

(ii) *(Mayer–Vietoris sequence) There is a long exact sequence in homology:*

$$\ldots \to H_{n+1}(F(X)) \to H_n(\overline{A} \cap \overline{B}) \to H_n(\overline{A}) \oplus H_n(\overline{B}) \to H_n(F(X)) \to \ldots$$

Proof. Let A, B and X as in (i). Clearly, $\overline{A} \cap \overline{B}$ is non–empty and $\overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{X} = F(X)$. To see that $\overline{A}, \overline{B}$ and $\overline{A} \cap \overline{B} = \overline{A \cap B}$ are path–connected, we prove:

If C is a path–connected subcomplex, then $\overline{C}$ is path–connected.

For every end ω in $\overline{C}$, there is a sequence of 1–cells in C converting to ω. For instance, choose an exhausting sequence $(K_n)_{n \in \mathbb{N}}$ of X and let D_n be the dummy ω lives in. Then $D_n \cap C$ is a non–empty CW complex; hence, it contains a 1–cell. As a result, every end in $\overline{C}$ is in $\overline{C^1}$. By a theorem of Diestel [...] the closure of a connected subgraph in the Freudenthal compactification of a graph is path–connected. In particular, the closure of C^1 in $F(X^1)$ is path–connected. By Lemma 4.3.2, $\Omega(X^1) \cong \Omega(X)$. Consequently, there is a path in $\overline{C}$ from C to every end in $\overline{C}$.

For the actual statements in (i) and (ii), note that $N_\varepsilon(A) \cup \overline{A}$ and $N_\varepsilon(B) \cup \overline{B}$ are open in $F(X)$ since A and B are wide. The theorem then derives from the usual Seifert–van Kampen theorem/the Mayer–Vietoris sequence and Lemma 6.1.3. Note Remark 6.1.2. □

6.2 A class of counterexamples

In this section, we prove the main theorem of the chapter: If X fails to satisfy the generalized Mittag–Leffler condition and at least one of its dummies witnessing the failing has an abelian fundamental group, then the fundamental group of $F(X)$ is non–trivial. In fact we will prove more, namely that in the above situation, the first homology group of $F(X)$ is uncountable. This will give us a class of counterexamples for the overall question whether simple–connectedness is preserved under the Freudenthal functor. Note that for the above statement, we do not require X to be simply–connected. The first two lemmata that we will prove may look quite technical. However, it will be clear what they invoke.

Lemma 6.2.1. *Let X be a strongly locally finite CW complex that does not satisfy the generalized Mittag–Leffler condition and let $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$ be an arbitrary admissible triple. Then there is a sequence of natural numbers $N(1) < N(2) < \dots$ such that the following properties hold.*

(i) *There is a decreasing sequence of dummies $D_{N(1)} \subseteq D_{N(2)} \subseteq \dots$ with $D_{N(n)} \in \mathbb{D}(K_{N(n)})$ for every $n \in \mathbb{N}$.*

(ii) *Together with a sequence of loops $l_{N(1)}, l_{N(2)}, \dots$ such that every loop $l_{N(n)}$ is based at $x_{D_{N(n)}}$ and*

$$f_{N(n)\to N(1)}([l_{N(n)}]) \notin \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n+1)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle \subseteq \pi_1(D_{N(1)}, x_{D_{N(1)}}).$$

Proof. By Theorem 5.1.8, X fails to satisfy the generalized Mittag–Leffler condition for the admissible triple. Hence, there is an index $N(1)$, a dummy $D_{N(1)} \in \mathbb{D}(K_{N(1)})$ and a sequence of natural numbers $N(1) < N(2) < \dots$ with

$$\pi_1(D_{N(1)}, x_{D_{N(1)}}) \supsetneq \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(2)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle \supsetneq \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(3)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle \supsetneq \dots \qquad (*)$$

Consider $N(n)$ and denote for the moment by $D^1, \dots, D^k$ all the dummies in $\mathbb{D}(K_{N(n)})$ with $D^i \subseteq D_{N(1)}$. If

$$H^{D_{N(1)}}(D^i) \subseteq \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n+1)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle$$

for every D^i, $i = 1, \dots, k$, then

$$\Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle \subseteq \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n+1)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle,$$

contradicting $(*)$. Thus, there is one D^i with

$$H^{D_{N(1)}}(D^i) \not\subseteq \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n+1)}) \\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D')\Big\rangle.$$

In particular, there is an element $[l_i] \in \pi_1(D^i, x_{D^i})$ with

$$f_{N(n)\to N(1)}([l_i]) \notin \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{N(n+1)})\\ D'\subseteq D_{N(1)}}} H^{D_{N(1)}}(D') \Big\rangle$$

since the above right–hand side is normal. Let $D_{N(n)} := D^i$ and $l_{N(n)} := l_i$. Clearly, all properties of (i) and (ii) are satisfied except that $D_{N(n)}$ does not need to be a superset of $D_{N(n+1)}$. However, by a compactness argument, there is a subsequence $(N(n_i))_{i\in\mathbb{N}}$ such that the associated sequence of dummies is decreasing. Then the subsequence inherits all the other properties of its supersequence. □

Remark 6.2.2. *Starting with an arbitrary admissible triple, the above lemma gives an index function $N(n)$ with certain properties. Applying the lemma to the induced admissible triple of the $(K_{N(n)})_{n\in\mathbb{N}}$, we can ensure that $N(n) = n+1$ to avoid an overflow of indices. Using that I is compact, we can even ensure, by the same argument as above, that the image of every l_n does not hit K_{n+1}. Note that we did not change the dummy with the lowest index.*

Lemma 6.2.3. *Let X be a strongly locally finite CW complex. For an arbitrary admissible triple $((K_n)_{n\in\mathbb{N}}, (x_D)_{D\in\mathbb{D}}, (f_{D'\to D})_{D'\subseteq D})$, fix an $n\in\mathbb{N}$ and a dummy $D\in\mathbb{D}(K_n)$. Let $D^1,\dots,D^k \in \mathbb{D}(K_{n+1})$ be all the dummies with $D^i \subseteq D$. Suppose α is a loop based at x_D that hits none of the D^i and*

$$[\alpha] \notin \Big\langle \bigcup_{\substack{D'\in\mathbb{D}(K_{n+1})\\ D'\subseteq D}} H^D(D') \Big\rangle = \Big\langle \bigcup_{i=1,\dots,k} H^D(D^i) \Big\rangle.$$

Then α is not even null-homotopic in $D/D^1,\dots,D^k$, i.e.

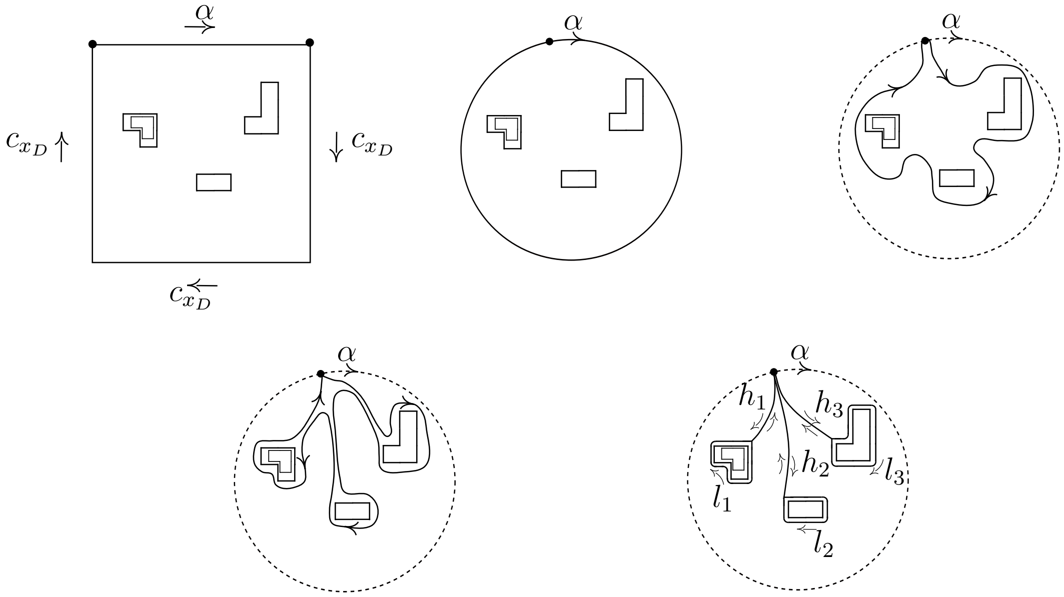
$$q([\alpha]) \neq 0 \in \pi_1(D/D^1,\dots,D^k, q(x_D)).$$

Proof. Suppose to the contrary that α is null-homotopic in $D/D^1,\dots,D^k$ and let $H\colon I\times I \to D/D^1,\dots,D^k$ witness $\alpha \simeq c_{x_D}$. Consider for every D^i the open neighbourhood retract $N_\varepsilon(D^i)$ by choosing ε small enough to ensure that the $N_\varepsilon(D^1),\dots,N_\varepsilon(D^k)$ are pairwise disjoint. Then D^i considered as a point in $D/D^1,\dots,D^k$ is closed and $N_\varepsilon(D^i)$ considered as a subset of $D/D^1,\dots,D^k$ is open. Consequently, $A := H^{-1}(D^1\cup\ldots\cup D^k)$ is closed in $I\times I$ and $B := H^{-1}(N_\varepsilon(D^1)\cup\ldots\cup N_\varepsilon(D^k))$ is open in $I\times I$. Hence, A and B^{c} are two closed disjoint subsets so their distance $dist(A,B) = r > 0$ is a positive real number. The map

$$\begin{aligned} dist(\cdot, A)\colon I\times I &\to \mathbb{R}_{\geq 0} \\ x &\mapsto dist(x,A) \end{aligned}$$

is continuous. Therefore, $C := dist^{-1}(\frac{r}{2}, A)$ is closed. Let $L\in\mathbb{N}$ be so large that the square of length $\frac{1}{L}$ has diameter less than $\frac{r}{2}$. Consider the partition of $I\times I$ that is given by L^2 many squares of length $\frac{1}{L}$. By construction, every square hits at most one of A, B^{c} or C. Note that avoiding B^{c} means lying entirely in B; for example, every square that hits C is a subset of B. Denote by Q_C the union of all the squares that hit C and let $Q_1, \ldots, Q_m$ be all the finitely many outer path–components [1] of Q_C. Let $l_i \colon I \to I \times I$ be a path that runs exactly once around the outer boundary of Q_i, $i = 1, \ldots, m$. Every point in $H(A^{\mathsf{c}})$ is a singleton equivalence class in $D/D^1, \ldots, D^k$ so $H_{|A^{\mathsf{c}}} \to D \setminus (D^1 \cup \ldots \cup D^k)$ is continuous. Moreover, $H_{|A^{\mathsf{c}}}$ defines a homotopy in D from α to a path based at x_D of the form

$$\alpha \simeq \prod_{i=1}^{m} (H_{|A^{\mathsf{c}}} \circ h_i) \cdot (H_{|A^{\mathsf{c}}} \circ l_i) \cdot (H_{|A^{\mathsf{c}}} \circ h_i^{-1}).$$



Note that $(H_{|A^{\mathsf{c}}} \circ l_i)$ is a path in one of the $N_\varepsilon(D^{j(i)})$ for they are path–connected and pairwise disjoint. Now fix for every $i \in \{1, \ldots, m\}$ a path P_i in $N_\varepsilon(D^{j(i)})$, $j(i) \in \{1, \ldots, k\}$ from the base point of $(H_{|A^{\mathsf{c}}} \circ l_i)$ to the base point of $D^{j(i)}$ for the unique $N_\varepsilon(D^{j(i)})$ the loop $(H_{|A^{\mathsf{c}}} \circ l_i)$ runs in. Then

$$\begin{aligned}\alpha &\simeq \prod_{i=1}^{m} (H_{|A^{\mathsf{c}}} \circ h_i) \cdot (H_{|A^{\mathsf{c}}} \circ l_i) \cdot (H_{|A^{\mathsf{c}}} \circ h_i^{-1}) \\ &\simeq \prod_{i=1}^{m} \Big((H_{|A^{\mathsf{c}}} \circ h_i) \cdot P_i\Big) \cdot \Big(P_i^{-1} \cdot (H_{|A^{\mathsf{c}}} \circ l_i) \cdot P_i\Big) \cdot \Big(P_i^{-1} \cdot (H_{|A^{\mathsf{c}}} \circ h_i^{-1})\Big).\end{aligned}$$

[1] A path–component of Q_C is an outer path–component if there is a path in $I \times I$ to a corner that avoids all other path–components of Q_C.

Note that $\Big((H_{|A^c} \circ h_i) \cdot P_i\Big)$ is a path from the base point x_D to the base point of the dummy $x_{D^{j(i)}}$. Consequently, $\tilde{h}_i := \Big((H_{|A^c} \circ h_i) \cdot P_i\Big) \cdot f_{x_{D^{j(i)}} \to x_D}$ is a loop based at x_D. Furthermore, $g := P_i^{-1} \cdot (l_i \circ H) \cdot P_i)$ is a loop in $N_\varepsilon(D^{j(i)})$ based at $x_D^{j(i)}$ so g_i is homotopic to a loop in $D^{j(i)}$ since $N_\varepsilon(D^{j(i)})$ deformation retracts onto $D^{j(i)}$. Then

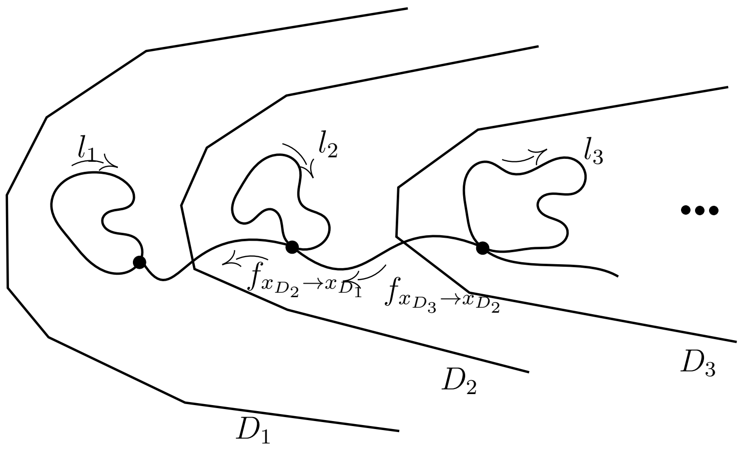
$$\begin{aligned}
\alpha &\simeq \prod_{i=1}^{m} (H_{|A^c} \circ h_i) \cdot (H_{|A^c} \circ l_i) \cdot (H_{|A^c} \circ h_i^{-1}) \\
&\simeq \prod_{i=1}^{m} \Big((H_{|A^c} \circ h_i) \cdot P_i\Big) \cdot \Big(P_i^{-1} \cdot (H_{|A^c} \circ l_i) \cdot P_i\Big) \cdot \Big(P_i^{-1} \cdot (H_{|A^c} \circ h_i^{-1})\Big) \\
&\simeq \prod_{i=1}^{m} (\tilde{h}_i) \cdot (f^{-1}_{x_{D^{j(i)}} \to x_D} \cdot g_i \cdot f_{x_{D^{j(i)}} \to x_D}) \cdot (\tilde{h}_i^{-1}) \\
&= \prod_{i=1}^{m} \tilde{h}_i \cdot \Big(f_{x_{D^{j(i)}} \to x_D}(g_i)\Big) \cdot \tilde{h}_i^{-1}
\end{aligned}$$

gives the desired contradiction

$$[\alpha] = \prod_{i=1}^{m} [\tilde{h}_i] \cdot \Big(f_{x_{D^{j(i)}} \to x_D}([g_i])\Big) \cdot [\tilde{h}_i]^{-1} \in \Big\langle \bigcup_{i=1,\dots,k} H^D(D^i) \Big\rangle.$$

□

If a strongly locally finite CW complex fails to satisfy the generalized Mittag–Leffler condition, we obtain the following picture by applying the two previous lemmata where every l_i satisfies the conclusion of Lemma 6.2.3.



Theorem 6.2.4. *Let X be a strongly locally finite CW complex that fails to satisfy the generalized Mittag–Leffler condition. If there is an admissible triple such that one of the dummies that witnesses the failing has an abelian fundamental group, then $\pi_1(F(X))$ is uncountable.*

Proof. In fact, we prove that the first homology group $H_1(F(X))$ is uncountable. By the Hurwicz theorem, this shows in particular that $\pi_1(F(X))$ is uncountable. Suppose to the contrary that $H_1(F(X))$ is countable. Let $n \in \mathbb{N}$ and $D \in \mathbb{D}(K_n)$ such that D witnesses the failing of the generalized Mittag–Leffler condition and $\pi_1(D)$ is abelian. We may assume $n = 1$ by an index shift of the exhausting sequence. Consider the subcomplex

$$A' := \bigcup_{\substack{D' \in \mathbb{D}(K_1) \\ D' \neq D}} D'.$$

Then A' is wide by Lemma 6.1.5. There are only finitely many cells not contained in one of the dummies of $X \setminus K_1$. Denote by K a finite subcomplex that contains all of them and let $A := A' \cup K$. Then $\overline{K} = K$, hence A is still wide. Moreover, $A \cup D = X$ so Theorem 6.1.7 gives a long exact sequence in homology:

$$\ldots \to H_2(F(X)) \to H_1(\overline{A} \cap \overline{D}) \to H_1(\overline{A}) \oplus H_1(\overline{D}) \to H_1(F(X)) \to \ldots$$

In fact, $\overline{A} \cap \overline{D} = A \cap D \subseteq K$ is a finite subcomplex so its homology group is finitely generated and in particular countable. By exactness, $H_1(\overline{D})$ is countable. Our goal now is to show that $H_1(\overline{D})$ is uncountable which will be the desired contradiction and this will work by a technique of [...]. For this purpose, let $D = D_1 \subseteq D_2 \subseteq \ldots$ be the sequence of dummies of Lemma 6.2.1 and $(l_n)_{n \in \mathbb{N}}$ the promised sequence of loops. Consider $\mathbb{S}^1$ as the union of two copies I_1 and I_2 of the unit interval glued together in their endpoints. For a sequence of zeroes and ones $P \in \mathbb{N}^{\{0,1\}}$, let $\alpha_P \colon \mathbb{S}^1 \to \overline{D}$ be the following map: On the intervals $[\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n} \frac{1}{2^i}] \subseteq I_1$, let

$$\alpha_{|[\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n} \frac{1}{2^i}]} = \begin{cases} f^{-1}_{D_{n+1} \to D_n} \cdot l_n & if\ P \text{ has a one on the } nth \text{ entry} \\ f^{-1}_{D_{n+1} \to D_n} & if\ P \text{ has a zero on the } nth \text{ entry} \end{cases}$$

and for the intervals $[\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n} \frac{1}{2^i}] \subseteq I_2$, let $\alpha_{|[\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n} \frac{1}{2^i}]} = f_{D_{n+1} \to D_n}$. This defines α_P on $\mathbb{S}^1 - 1$, letting $\alpha_P(1) = \omega$ for the unique end that is contained in all the $\overline{D_n}$, turns α_P into a continuous map since l_n is contained in D_n. Every continuous map induces a group homomorphism on the level of homology groups so that every α_P gives an associated group homomorphism

$$\alpha_P \colon H_1(\mathbb{S}^1) \to H_1(\overline{D}).$$

Every group homomorphism from $\mathbb{Z} = H_1(\mathbb{S}^1)$ is determined by the image of the unit. By assumption $H_1(\overline{D})$ is countable so there need to be two distinct sequences $P, Q \in \mathbb{N}^{\{0,1\}}$ which induce the same group homomorphism. Let

n be the first coordinate where they differ and let $D^1, \ldots, D^k$ be all the dummies in $\mathbb{D}(K_{n+1})$ with $D^i \subseteq D$. Denote by q the quotient projection $q\colon \overline{D} \to \overline{D}/D^1, \ldots, D^k$. Then $q \circ \alpha_P$ and $p \circ \alpha_Q$ still need to induce the same group homomorphism

$$H_1(\mathbb{S}^1) \overset{\alpha_P, \alpha_Q}{\to} H_1(\overline{D}) \overset{q}{\to} H_1(\overline{D}/D^1, \ldots, D^k).$$

Furthermore, $\overline{D}/\overline{D^1}, \ldots, \overline{D^k} = D/D^1, \ldots, D^k$. It is not hard to see that the quotient map $q\colon D \to D/D^1, \ldots, D^k$ gives a surjection on the level of fundamental groups; for example, use that any path in a CW complex is homotopic to a path in its one skeleton. We have required that $\pi_1(D)$ is abelian so in particular $\pi_1(D/D^1, \ldots, D^k)$ is abelian. Again by the Hurwicz theorem, we have

$$H_1(\overline{D}/\overline{D^1}, \ldots, \overline{D^k}) = H_1(D/D^1, \ldots, D^k) = \pi_1(D/D^1, \ldots, D^k).$$

This gives a contradiction by using Lemma 6.2.3:

$$\begin{aligned} 0 = q(\alpha_P(1_{\pi_1(\mathbb{S}^1)})) - q(\alpha_Q(1_{\pi_1(\mathbb{S}^1)})) &= [\sum_{\substack{i \leq n \\ P(i)=1}} f_{D_n \to D_1}[l_i]] - [\sum_{\substack{i \leq n \\ Q(i)=1}} f_{D_n \to D_1}([l_i])] \\ &= [f_{D_n \to D_1}([l_n])] \neq 0. \end{aligned}$$

□

Example 6.2.5. *Let X be the first CW complex of Example 5.1.5, then X fails to satisfy the generalized Mittag–Leffler condition. As seen in the example, $\pi_1(D) = \mathbb{Z}$ for a dummy D so Theorem 6.2.4 shows that the fundamental group of $F(X)$ is uncountable despite the fact that X is simply–connected.*

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Ort, Datum

Ruben Melcher