Investigations in infinite matroid theory

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1. Introduction

The definition of an infinite matroid on which this thesis is based is relatively new. The historic development of the research towards this definition is described in [9], which is the paper in which the axioms where introduced. It was published in 2013, but others already worked with the definition for some years before.
Several concepts of finite matroids were already generalised to infinite matroids. For this thesis the most important generalisations are those of representability [1] (which has already several equivalent definitions for tame matroids and will again be modified here) and finite separations [10].
But infinite matroid theory is not only about finding suitable infinite analogues of concepts which are important in finite matroid theory, but also about investigating phenomena which only (may) occur in infinite matroids. For example, a matroid can be finitary (containing only finite circuits, these matroids are in most cases a lot easier to handle than arbitrary matroids) or at least similar to a finitary matroid in the following way: the *finitarisation* of a matroid M is the matroid M_{fin} whose circuits are precisely the finite circuits of M. Each base of M is contained in a base of M_{fin}, and a matroid is called *nearly* finitary if every base of M can be extended to a base of M_{fin} by adding only finitely many edges. Nearly finitary matroids form one of the special cases for which it was proved that the union of two matroids is again a matroid ([3, Theorem 1.2]. If there is even a natural number l such that every base of M can be extended to a base of M_{fin} by adding at most l edges, then M is l-nearly finitary. Matroids which are l-nearly finitary for some l resemble finitary matroids even more than just nearly finitary matroids. In this context, the following question arises naturally:

Open Question 1.1. *Let M be a nearly finitary matroid. Is M also l-nearly finitary for some natural number l?*

Question 1.1 is confirmed for cofinitary matroids in Theorem 4.4. Surprisingly, if M is the algebraic cycle matroid of some locally finite connected graph G, then this is equivalent to Halin's theorem for G (and this equivalence is a lot more direct than the fact that both are true): For a family of vertex disjoint rays there are a base B of M and a base B_{fin} of the finitarisation of M such that $B \subset B_{\text{fin}}$ and the edges of the rays are all contained in B. Then the set of rays corresponds (possibly after deleting one ray) to a subset of $B_{\text{fin}} \backslash B$, so the latter is large if the family is large. If in the other direction bases B of M and B_{fin} of M_{fin} with $B \subset B_{\text{fin}}$ are given, then these are the edge sets of subgraphs of G. Let G_B be the graph which is induced by B, then $B_{\text{fin}} \backslash B$ corresponds to a subset of the components of G_B which all contain a ray. This link is made more plausible by the fact that the infinite circuits of M are the double rays of G, so connecting two rays to a double ray makes these rays somehow visible in the algebraic cycle matroid. The link between Halin's theorem and Question 1.1 will be established in Section 3.
Section 4 mainly consists of the proof of Theorem 4.4 and the lemmas which will be needed for this proof. In contrary to the proof of Halin's theorem which is given in [11], the proof of Theorem 4.4 first collects infinitely many pairwise disjoint finite sets S_i such $B_{\text{fin}} \backslash B$ will contain an edge of each S_i (thereby somehow determining where the rays have to start if M is the algebraic cycle matroid of some graph), and then in a second step extends this edge set to B_{fin} such that $B_{\text{fin}} \backslash F$ contains a base of M. This last step is a straightforward compactness argument. That this

approach works relies heavily on the assumption that M is cofinitary.
Section 2 (Preliminaries) already contains some new proofs in addition to the collection of the lemmas and definitions from other papers which will be needed later on: in Subsection 2.3 the definition of tame representable matroids given in [1] is modified to an equivalent one which is better suited for the purposes of this thesis. In the same Subsection there is also a generalisation of the axiom (C3) of infinite matroids. This generalisation can only be formulated for representable matroids. The adjusted definition will be a basic definition for Section 5 and the generalisation of the axiom will be used in a proof in the same section.
Several further results occurred during the earlier attempts to solve special cases of Theorem 4.4. They lost their importance for Question 1.1 now that Theorem 4.4 is proved, but the ones which are interesting in themselves are collected in Sections 5 and 6.
Section 5 is concerned with finite separations of representable matroids. Separations display a lot of information on the structure of matroids: If $E(M)$ is the ground set of some matroid M and $E(M) = P_1 \mathbin{\dot\cup} P_2$ is a small separation of M (compared to $\min(|P_1|, |P_2|)$), then what the matroid looks like on one P_i influences what M looks like on the other P_i less than would typically be possible. In an infinite matroid M, any finite separation where both sides P_i are infinite is a small separation and thus displays a lot of structural information about M. In Section 5 it is shown that if a representable matroid has such a finite separation, then the structure of M can be displayed even better by finding other representable matroids M_i on a set similar to P_i such that M is a nice sum of M_1 and M_2.
Waves and hindrances in graphs were invented in order to prove infinite analogues of Menger's theorem [2]. They were then translated to matroids in order to investigate matroid intersection for infinite matroids. Both applications suggest that a promising approach for proving an infinite version of Lemmas 4.2 and 4.3 would be to apply waves and hindrances to the assumptions of the lemmas. Section 6 contains some proofs of properties of hindrances.
Section 7 is about lattices of cyclic flats of matroids and is not connected to the other results of thesis in another way than that it is also about infinite matroids. In this section the main statement is that there is a correspondence between the lattices which are the lattice of cyclic flats of finite rank of a finitary matroid and the lattices in which every element has finite height and on which there is a submodular rank-function. It is also explained how the lattice of cyclic flats of finite rank and the lattice of all cyclic flats of a finitary matroid are linked. This enables us to construct new finitary matroids from submodular lattices and in particular yields an example of a matroid whose lattice of cyclic flats is not atomic.

2. Preliminaries

2.1. **Matroids.** The matroids used here are infinite. The axioms presented below coincide an a finite ground set with the usual matroid axioms and enable infinite analogues of important finite matroid concepts such as duality and minors. In the paper where the axioms are introduced ([9]), there are many equivalent axiom systems presented. All the objects on which these axiom systems are defined (circuits, bases, independent sets etc.) will be used, but of the axioms the circuit axioms will be most important. Usually a matroid is defined via the independence axioms:

Definition 2.1. [9, Subsection 1.1] Let E be a set. A set $\mathcal{I} \subset \mathcal{P}(E)$ is the set of independent sets of a matroid if it satisfies

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) $\mathcal{I}$ is closed under taking subsets.
- (I3) For every $I \in \mathcal{I}$ which is not maximal in $\mathcal{I}$ and every I' which is maximal in $\mathcal{I}$ there is $x \in I' \backslash I$ such that $I + x \in \mathcal{I}$.
- (IM) For all sets $J \in \mathcal{I}$, $X \subset E$ such that $J \subset X$ the set $\{I \in \mathcal{I} | J \subset I \subset X\}$ has a maximal element.

Definition 2.2. [9, definition in Subsection 1.4 and Theorem 4.3] Let E be a set. A set $\mathcal{C} \subset \mathcal{P}(E)$ is the set of circuits of a matroid if it satisfies

- (C1) $\emptyset \notin \mathcal{C}$
- (C2) No element of $\mathcal{C}$ is a subset of another.
- (C3) Whenever $X \subset C \in \mathcal{C}$ and $(C_x | x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \backslash \left(\bigcup_{x \in X} C_x\right)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subset \left(C \cup \bigcup_{x \in X} C_x\right) \backslash X$.
- (CM) The set $\mathcal{I}(M)$ of all subsets of E not containing an element of $\mathcal{C}$ satisfies that for all $I \in \mathcal{I}$ and for all $X \subset E(M)$ such that $I \subset X$ the set $\{J \in \mathcal{I} : I \subset J \subset X\}$ has a maximal element.

Let M be a matroid.

Definition 2.3. [4, Subsection 2.2][14] The ground set of M is denoted by $E(M)$. A *scrawl* of M is a union of circuits, a subset of $E(M)$ not containing a circuit is an independent set, a *base* is a maximal independent set. The set of scrawls is denoted by $\mathcal{S}(M)$, the set of independent sets by $\mathcal{I}(M)$ and the set of bases by $\mathcal{B}(M)$.

Remark 2.4. [7, between Lemmas 2.1 and 2.2] Let X be a subset of $E(M)$. The set $\{C \in \mathcal{C} | C \cap X = \emptyset\}$ is the set of circuits of a matroid on ground set $E(M) \backslash X$ denoted by $M \backslash X$ or $M\restriction_{E \backslash X}$. The minimal non-empty elements of the set $\{C \backslash X | C \in \mathcal{C}\}$ form the set of circuits of a matroid on ground set $E(M) \backslash X$. This matroid is denoted by M/X or $M.(E \backslash X)$.

Remark 2.5. [9, Lemma 3.7] Let B_1, B_2 be two bases of M. If $|B_1 \backslash B_2| < \infty$, then $|B_2 \backslash B_1| = |B_1 \backslash B_2|$.

Definition 2.6. [9, Theorem 3.1] Let $\mathcal{B}^* = \{E(M) \backslash B | B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*$ is the set of bases of a matroid, called the *dual matroid* M^* of M. The bases, circuits, scrawls etc. of M^* are called the *cobases, cocircuits, coscrawls* etc. of M. A matroid is *finitary* if every circuit is finite (Corollary 3.9).

Remark 2.7. [4, Lemma 2.6] A set $S \subset E(M)$ is a scrawl of M iff it meets no cocircuit of M in exactly one edge.

Lemma 2.8. [5, Lemma 2.4] *Let C be a circuit of M containing two different edges e and f. Then there is a cocircuit D of M such that $C \cap D = \{e, f\}$.*

Definition 2.9. [3, subsection 4.3] Let M be a matroid. The *finitarisation* M_{fin} of M is the matroid on the same ground set $E(M)$ as M with circuit set $\mathcal{C}(M_{\text{fin}}) = \{C \in \mathcal{C}(M) | C \text{ is finite}\}$.

Remark 2.10. [3, Proposition 4.11] The finitarisation of a matroid is a matroid.

Definition 2.11. [3, subsection 4.3] Let M be a matroid, M_{fin} its finitarisation. If for all bases B of M and all bases B_{fin} of M_{fin} such that $B \subset B_{\text{fin}}$ $B_{\text{fin}} \backslash B$ is finite, then M is called *nearly finitary*. If there is $k \in \mathbb{N}$ such that $B_{\text{fin}} \backslash B$ has at most size k for all bases $B \subset B_{\text{fin}}$ of M and M_{fin} respectively, then M is called *k-nearly finitary*.

Remark 2.12. [9, Lemma 3.5] Let M be a matroid and $X \subset E(M)$. For $B \subset X$, the following are equivalent:

- B is a base of $M.X$
- $B \cup B'$ is a base of M for every base B' of $M \backslash X$.

Definition 2.13. [10, section 4]Let B_i be a base of $M \restriction P_i$ and B a base of M such that $B \subset B_1 \cup B_2$. If $|(B_1 \cup B_2) \backslash B| \leq l-1 \in \mathbb{N}$ and $|P_i| \geq l$, then $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an *l-separation* of M. If there is an l, such that $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an l-separation, then it is a finite separation. If $|(B_1 \cup B_2) \backslash B| = l-1$, then $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an exact l-separation (even if the size of some P_i is $l-1$). If M has no l-separation with $l < m$, then M is *m-connected*. If M is 2-connected, then it is *connected*.

By Definition 2.13 there may be separations $E(M) = P_1 \mathbin{\dot\cup} P_2$ which are an exact l-separation for some l but are not an l-separation because one set P_i is not large enough. In this case the small P_i has to be independent and coindependent in M. This is an unusual convention but in Section 4 there will be a lot fewer special cases because of it.

Remark 2.14. [10, Lemma 14] Let B_1 be a base of $M \restriction_{P_1}$, B_1' a base of $M.P_1$ and B_2 a base of $M \restriction_{P_2}$. Then $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an l-separation of M iff $|B_1 \backslash B_1'| \leq l-1$.

2.2. **Matroids which are associated with graphs.** Facts and lemmas about infinite graphs and about matroids on infinite graphs will only appear in Section 3 and will only be relevant there. Nevertheless, they are important in the whole thesis because a good way to give examples of matroids is to specify them as the cycle matroid of some infinite graph. Because of this I will give two examples of matroids on infinite graphs here. The definitions for graphs are mainly taken from [11]. The only exception is the following: For a graph G there should be a subgraph which is not induced by an edge set but by a vertex set. As matroids associated with a graph have its edge set as ground set, it is then possible to consider subgraphs that correspond to edge sets in the matroid.

Definition 2.15. Let $S \subset E(G)$. The subgraph of G whose edge set is S and whose vertex set consists of the end vertices of edges of S is called the *graph induced by S*.

Paths and circles are officially subgraphs but often the distinction between these graphs and their edge sets is forgotten.

Definition 2.16. [8, section 3][12] Let G be a graph. The set of edge sets of finite circuits of G is the set of circuits of a matroid, called the *finite cycle matroid* of G. The set of edge sets of finite circuits of G together with the set of edge sets of double rays of G is the set of circuits of a matroid, the *algebraic cycle matroid*, if G does not contain a subdivision of the Bean graph.

Remark 2.17. [8] Let G be a locally finite connected graph. The Bean graph contains a vertex of infinite degree, so G cannot contain a subdivision of the Bean graph and thus the finite cycle matroid of G exists. Furthermore the algebraic cycle matroid of G is cofinitary, i.e. its dual is finitary.

2.3. **Representable matroids.**

Definition 2.18. The support of a function f is denoted by $\underline{f}$. For a set of functions V denote the set of supports of functions in V by $\underline{V}$. If f has a set A as its domain, then for a set $B \subset A$ denote the set $\{f(b)|b \in B\}$ by $f(B)$.
Let S be a set, k a field and S' a subset of S. Then the *characteristic function* of S' is the function $\chi_{S'} : S \to k$ which maps all elements of S' to 1 and all elements of $S \backslash S'$ to 0. If S' contains only one element e, then abbreviate $\chi_{\{e\}}$ by χ_e. Denote the projection from k^S to $k^{S'}$ by $p_{S'}$.

Definition 2.19. [1] A matroid is called *tame* if the intersection of any of its circuits with any of its cocircuits is finite.

A finite matroid M is called representable if there are a vector space V and a function $i : E(M) \to V$ such that a subset F of $E(M)$ is independent in M iff the family $(e_f)_{f \in F}$ of vectors is independent in V. In order to define what an infinite representable matroid is it is not a good idea to just drop the condition of finiteness in that definition, because then all representable matroids would be finitary which is a severe restriction. Instead, in [1] a notion of representability is introduced which coincides with the usual definition on finite sets but defines a much larger class of infinite matroids to be representable, namely that of thin sums matroids.
Let M be a tame matroid, let k be a field and denote $k - 0$ by k^*.

Definition 2.20. [1] Let E, A be sets and $f : E \to k^A$ a function. A *thin dependence* of f is a map $c : E \to k$ such that for all $a \in A$ there are only finitely many $e \in E$ with $f(e)c(e) \neq 0$ and $\sum_{e \in E} c(e)f(e)(a) = 0$. M is a *thin sums matroid over* k if there are a set A and a function $f : E(M) \to k^A$ such that $I \subset E(M)$ is independent iff for all thin dependencies $c : E(M) \to k$ such that $c \neq 0$ there is $e \in E$ with $c(e) \neq 0$ and $e \notin I$.

In [1] there is a second definition of a thin sums matroid which is equivalent to the first one for tame matroids:

Lemma 2.21. [1, Lemma 6.2] *M is a thin sums matroid over k iff there are for each circuit $C \in \mathcal{C}(M)$ and for each cocircuit $D \in \mathcal{C}(M^*)$ functions $f_C : C \to k^*$ and $g_D : D \to k^*$ such that for all circuits C' and cocircuits D' $\sum_{e \in C' \cap D'} f_{C'}(e)g_{D'}(e) = 0$.*

This definition can be modified further for tame matroids: in [6] there is directly after Definition 5.1 an equivalent definition for a tame thin sums matroid: A matroid M is a thin sums matroid over a field k iff there are orthogonal vector spaces $V, W \leq k^{E(M)}$ satisfying an extra condition such that the minimal non-empty elements of $\underline{V}$ are $\mathcal{C}(M)$ and the minimal non-empty sets of $\underline{W}$ are $\mathcal{C}(M^*)$. I will later want to modify a thin sums matroids slightly and show that the result is still a thin sums matroid, so modifying two vector spaces instead of two large families is a more suitable definition of a thin sums matroid. But it is even possible to get rid of the second vector space when putting more conditions on the first one which ensure that its orthogonal complement can take the role of the second vector space. The following three statements show this by methods which are used in [1].

Lemma 2.22. *A tame matroid M is a thin sums matroid over k iff there is a vector space $V \leq k^{E(M)}$ such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subset \underline{V^\perp} \subset \mathcal{S}(M^*)$.*

Proof. Let M be a thin sums matroid over k. By Theorem 2.21 there are for each circuit $C \in \mathcal{C}(M)$ and each cocircuit $D \in \mathcal{C}(M^*)$ functions $f_C : C \to k^*$ and $g_D : D \to k^*$ such that for all circuits C' and cocircuits D' $\sum_{v \in C' \cap D'} f_{C'}(e) g_{D'}(e) = 0$. Consider the functions f_C as elements of $k^{E(M)}$ by mapping all elements not in C to zero and let V be the linear span of the functions f_C over k. Then obviously $\mathcal{C}(M) \subset \underline{V}$. In order to show $\underline{V} \subset \mathcal{S}(M)$, let $v \in V$ and D be a cocircuit of M. Then v is of the form $v = \sum_{i=1}^n \lambda_i f_{C_i}$ for suitable circuits C_i and elements $\lambda_i \in k$. Hence

$$\langle v, g_D \rangle = \langle \sum_{i=1}^n \lambda_i f_{C_i}, w \rangle = \sum_{i=1}^n \lambda_i \langle f_{C_i}, g_D \rangle = \sum_{i=1}^n \lambda_i \cdot 0 = 0$$

so g_D is orthogonal to all vectors in V and thus an element of $V^\perp$. Thus the support of v cannot meet D in exactly one edge. As this is true for all cocircuits D, by Lemma 2.7 v is a scrawl, hence $\underline{V} \subset \mathcal{S}(M)$. The fact that each g_D is an element of $V^\perp$ also shows that $\mathcal{C}(M^*) \subset \underline{V^\perp}$. Let w be a vector of $V^\perp$. Then for all circuits C, w is orthogonal to f_C and thus C and $\operatorname{supp}(w)$ do not meet in exactly one edge. Again by Lemma 2.7 the support of w must be a scrawl, so $\underline{V^\perp} \subset \mathcal{S}(M^*)$.
Let $V \subset k^{E(M)}$ be a vector space such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subset \underline{V^\perp} \subset \mathcal{S}(M^*)$. Then for each circuit C of M there is a vector $f_C \in V$ such that $\underline{f_C} = C$ and for each cocircuit D of M there is a vector $g_D \in V^\perp$ such that $\underline{g_D} = D$. By restricting these functions to their respective supports, these are functions $f_C : C \to k^*$ and $g_D : D \to k^*$ such that $\sum_{e \in C \cap D} F_C(e) g_D(e) = 0$ as they are orthogonal vectors. □

Lemma 2.23. *Let $V \subset k^{E(M)}$ be a vector space such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and no element of $\underline{V}$ meets a cocircuit of M in an infinite set. Then $\mathcal{C}(M^*) \subset \underline{V^\perp} \subset \mathcal{S}(M^*)$.*

Proof. Let $w \in V^\perp$ and C a circuit of M. Then there is $v \in V$ such that $\underline{v} = C$. As $w \perp v$, $|\underline{w} \cap \underline{v}| \neq 1$. So $\underline{w}$ meets no circuit of M in exactly one edge and is thus by Remark 2.7 a scrawl of M^*.
Let $D \in \mathcal{C}(M^*)$ and $e \in D$. By the dual of Lemma 2.8, for all $f \in D - e$ there is a circuit C_f of M such that $C_f \cap D = \{e, f\}$. Then for each such f there also is and a vector $v_f \in V$ such that $\underline{v_f} = C_f$ and $v_f(f) = 1$. Define a vector $w \in k^E$ via $w(e) = 1$, $w(f) = -v_f(e)$ for $f \in D - e$ and 0 everywhere else. In order to show $w \in V^\perp$, let $v \in V$. Define $z = v - \sum_{f \in D - e} v(f) v_f$. This is well-defined and an element of V as the support of v meets D only in finitely many edges, so only for

finitely many edges $f \in D - e$, $v(f)v_f$ is non-zero. Then for $g \in (D \cap \underline{v}) - e$,

$$\begin{aligned} z(g) &= v(g) - \sum_{f \in D-e} v(f)v_f(g) \\ &= v(g) - \sum_{f \in D-e} v(f)\delta_{fg} = v(g) - v(g) = 0 \end{aligned}$$

and for $g \in (D \backslash \underline{v}) - e$

$$\begin{aligned} z(g) &= v(g) - \sum_{f \in D-e} v(f)v_f(g) \\ &= 0 - \sum_{f \in D-e} v(f)\delta_{fg} = 0 - v(g) = 0 - 0 = 0. \end{aligned}$$

Hence $\underline{z} \cap D$ contains at most e. But $z \in V$, so $\underline{z}$ is a scrawl and by Remark 2.7 does not meet D in exactly one edge. Thus $z(e) = 0$ and $v(e) = \sum_{f \in D-e} v(f)v_f(e)$. Now $\underline{v} \cap D$ is finite, hence

$$\begin{aligned} \langle v, w \rangle &= v(e)w(e) + \sum_{f \in (D \cap \underline{v})-e} v(f)w(f) \\ &= \sum_{f \in D-e} v(f)v_f(e) + \sum_{f \in D-e} v(f)(-v_f(e)) \\ &= \sum_{f \in D-e} (v(f)v_f(e) - v(f)v_f(e)) = 0. \end{aligned}$$

So $w \in V^{\perp}$. □

Corollary 2.24. *Let k be a field and M a tame matroid. Then M is a thin sums matroid over k iff there is a vector space such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and no support of a vector of V meets a cocircuit in an infinite set.*

Proof. Let M be a thin sums matroid over k. Then by Lemma 2.22 there is a vector space V such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subset \underline{V^{\perp}} \subset \mathcal{S}(M^*)$. Let v be a vector of V and D a cocircuit of M. As $\mathcal{C}(M^*) \subset \underline{V^{\perp}}$, there is a vector $w \in V^{\perp}$ whose support equals D. The fact that w is orthogonal to v implies that their supports only share finitely many edges, hence the support of v meets D only in a finite set.
For the other direction let $V \leq k^{E(M)}$ be a vector space such that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and the support of no vector of V meets a cocircuit of M in an infinite set. Then by Lemma 2.23 $\mathcal{C}(M^*) \subset \underline{V^{\perp}} \subset \mathcal{S}(M^*)$. Thus again by Lemma 2.22 M is a thin sums matroid over k. □

This last equivalent definition of a representable matroid is the one which will be used in this thesis (note that a matroid which is representable in this sense is necessarily tame):

Definition 2.25. Let $V \subset k^{E(M)}$ a vector space. A matroid M is said to be *represented by* V if $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$ and the support of every element of V meets every cocircuit of M in only finitely many edges. If there is a vector space $V \leq k^{E(M)}$ such that M is represented by V, then M is representable (over k).

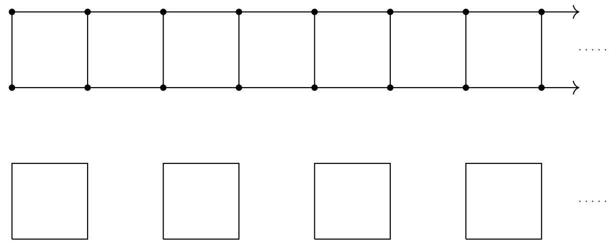

FIGURE 1. Above is an infinite graph and below is a subset of its edge set which is an infinite union of finite cycles.

If V is a vector space such that there is a matroid M which is represented by V, then M is uniquely determined by V. But not for every vector space there is a matroid which is represented by it, and for a representable matroid M the corresponding vector space is not uniquely determined by M.

Example 2.26. There are a field k, a set E and a vector space $V \subset k^E$ such that there is no matroid which is represented by V.

Consider the finite cycle matroid M of the graph in Figure 1. Let v be the characteristic function of the set in Figure 1 and $V =< \{\chi_C | C \in \mathcal{C}(M)\} \cup v >$. Then $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$, so if there is a matroid which is represented by V, then it is M. But if D is the cocircuit of M consisting of all the edges which are depicted vertically in the graph, then the support of v and D meet in infinitely many edges, so M is not represented by V. However, the problem here is that V simply contains too many vectors: If for a field k, a tame matroid M and a vector space $V \leq k^{E(M)}$ it is true that $\mathcal{C}(M) \subset \underline{V} \subset \mathcal{S}(M)$, then M is represented by the vector space V' which is spanned (as a vector space) by the vectors of V which have a circuit of M as support.

Example 2.27. V is not uniquely determined by M.
By Lemma 2.23 any matroid represented by a vector space is thinly representable. M is uniquely determined by V. On the other hand, V is not uniquely determined by M, not even when the vectors whose subsets are the circuits are already defined: Let M be the algebraic cycle matroid of the graph in Figure 1, $k = \mathcal{F}_2$ and let v be the characteristic function of the edge set shown below the graph. Let

$$V = \{\chi_F | \text{ every vertex is contained in a even number of elements of } F\}.$$

and let V' be the vector space which is spanned by the vectors of V whose supports are circuits of M. Every infinite circuit of M consists of a tail of the upper ray, a tail of the lower ray and an additional edge and hence the sum of two characteristic functions of infinite circuits is a finite circuit or zero. So a linear combination of vectors of which the support is a circuit of M has either finite support or is the sum of a vector corresponding to an infinite circuit and the characteristic function of a finite set, hence has a support which consists again of a tail of the upper ray, a tail of the lower ray and finitely many other edges. The support of v is not of this form, so v is not a vector of V' and thus $V' \neq V$. Nevertheless, M is represented by both.

Definition 2.28. Let M be a matroid, I an independent set and e an edge such that $I + e$ contains a circuit. Then there is exactly one circuit contained in $I + e$.

This is the *fundamental circuit* of e in I and is denoted by C_e^I. If M is represented by a vector space V, then there is exactly one vector $v_e^I \in V$ such that its support equals C_e^I and $v_e^I(e) = 1$. This is the *fundamental vector* of e in I.

If e is an element of I or $I + e$ is independent, then define $C_e^I = \emptyset$ and $v_e^I = 0$. If I is a base of M, then for all $e \in e(M)\backslash I$, $I + e$ contains a circuit.
The assumptions in the following lemma look relatively technical, but if Z contains only one edge e, then the support of v is a circuit as is the support of w and thus this lemma is just a reformulation of (C3). Hence this lemma is a generalisation of (C3) for representable matroids.

Lemma 2.29. *Let k be a field and M a matroid on ground set $E(M)$ which is represented by a vector space $V \subset k^{E(M)}$. Let $v \in V$, $X \subset \mathrm{supp}\,(v)$, $(C_x | x \in X)$ a family of circuits and $Z \subset \mathrm{supp}\,(v) \setminus \bigcup_{x \in X} C_x$ a finite set such that $\mathrm{supp}\,(v) \setminus Z$ is independent and for all $x \in X$, $C_x \cap \mathrm{supp}\,(v) = \{x\}$. Then there is $w \in V$ such that*
- $\mathrm{supp}\,(w) \subset \left(\mathrm{supp}\,(v) \cup \bigcup_{x \in X} C_x\right) \setminus X$
- $\mathrm{supp}\,(w) \setminus Z$ *is independent and*
- $p_Z(w) = p_Z(v)$.

Proof. Let $Y = \left(\mathrm{supp}\,(v) \cup \bigcup_{x \in X} C_x\right) \setminus (X \cup Z)$ and let B be a base of $M\restriction_Y$. As X is spanned by Y, B is also a base of $M\restriction_{Y \cup X}$. Let B' be a base of $M\restriction_{Y \cup X \cup Z}$ containing B. Define $w = \sum_{f \in Z \setminus B'} v(f) v_f^{B'}$ and $u = v - w$. Hence for all $f \in Z \setminus B'$, $u(f) = v(f) - v(f) v_f^{B'}(f) = 0$ and thus $\mathrm{supp}\,(u) \cap Z \subset B'$. Assume for a contradiction that there is an edge $z \in \mathrm{supp}\,(u) \cap Z$. As $u \in V$ there is a circuit C such that $z \in C \subset \mathrm{supp}\,(u)$. Let $U = C \setminus (Z \cup B')$, then $z \notin U$. Then by (C3) there is a circuit C' such that $z \in C' \subset \left(C \cup \bigcup_{e \in U} C_e^{B'}\right) \setminus U$ which is a subset of

$$\begin{aligned}(C \setminus U) \cup \left(\bigcup_{e \in U} C_e^{B'} \setminus U\right) &\subset (C \cap (Z \cup B')) \cup B' \\ &\subset (\mathrm{supp}\,(u)) \cap Z) \cup B' \subset B'\end{aligned}$$

which is the desired contradiction. Hence $\mathrm{supp}\,(u) \cap Z = \emptyset$, thus $p_Z(v) = p_Z(w)$. Also $\mathrm{supp}\,(w) \subset \bigcup_{f \in Z \setminus B'} \mathrm{supp}\,(v_f) \subset B' \cup Z$, so $\mathrm{supp}\,(w) \setminus Z$ is independent and $\mathrm{supp}\,(w) \subset Y \cup Z = (\mathrm{supp}\,(v) \cup \bigcup_{x \in X} C_x) \setminus X$. ∎

3. Halin's theorem for locally finite graphs

Remark 3.1. [11, Theorem 8.2.5, Halin] If an infinite graph G contains l (vertex-) disjoint rays for every $l \in \mathbb{N}$, then G contains infinitely many disjoint rays.

We want to generalise this theorem to matroids. The first problem to be solved is that there is no good analogue of rays in matroids. But as the infinite circuits of the algebraic cycle matroid of a graph are just its double rays, in this matroid circuits can be seen as analogues of double rays. So we wish to translate Halin's theorem to a statement about double rays. In a connected graph, that is easily done for two rays: for any two rays of a connected graph there is a double ray consisting of a finite number of edges and tails of the two rays. From such a double ray one can again obtain two rays by just deleting one edge. This But the concept of a ray cannot be translated to a matroid very well. What is the analogue of a family of pairwise rays in this context?

Definition 3.2. Let M be a matroid. A family of circuits $(C_i)_{i\in I}$ is *independent concerning finite circuits* if the edge set $E\left((C_i)_{i\in I}\right) = E\left(\bigcup_{i\in I} C_i\right)$ does not contain finite circuits and there is a set $F = (f_i)_{i\in I}$ such that $f_i \in C_j \Leftrightarrow i = j$.

That a matroid contains a large such family of circuits is equivalent to the statement that it is not l-nearly finitary for large l:

Remark 3.3. Let M be a matroid. For any $l \in \mathbb{N}$, M is not l-nearly finitary iff M contains a family of circuits $(C_i)_{i\in I}$ which is independent concerning finite circuits such that $|I| = l + 1$.

The following Lemma will show that in locally finite connected graphs it is possible to switch between families of rays and families of circuits easily. The statement will just give a family of the same size, but in the proof this family will be constructed such that the rays are subsets of the circuits. In this section, infinite sets have all the same size, forgetting about possibly different cardinalities.

Lemma 3.4. *Let G be a locally finite connected graph containing a ray and M its algebraic cycle matroid. Then G contains a family $(R_i)_{i\in I}$ of vertex disjoint rays iff there are bases B of M and B_{fin} of M_{fin} containing B such that $|B_{fin}\backslash B| \geq |I| - 1$.*

Proof. Let $(R_i)_{i\in I}$ be a family of vertex disjoint rays. Denote the set of edges contained in one of these rays by E_R. Let B_{fin} be a base of M and B a base of M such that $E_R \subset B \subset B_{\text{fin}}$. As G is connected, B_{fin} is the edge set of a spanning tree of G. Let $F = B_{\text{fin}}\backslash B$ and G_B be the graph which is induced by B. Then G_B is a forest. Each ray R_i is contained in G_B, as it does not use edges of F. As B is independent in M, it does not contain double rays, hence if $i \neq j$, then R_i and R_j are contained in different components of G_B, so G_B has at least as many components as I contains elements. As B_{fin} is the edge set of a spanning tree containing G_B, $|F| \geq |I| - 1$.
So let B and B_{fin} be bases of M and M_{fin} respectively such that $B \subset B_{\text{fin}}$ and $|B_{\text{fin}}\backslash B| \geq |I| - 1$ for some set I. If $F = B_{\text{fin}}\backslash B$ is empty, then by assumption there is a ray contained in G and the lemma holds. So assume that F contains at least one element. For each element f of F there is a fundamental circuit C_f of f in B with respect to M. As C_f is a subset of B_{fin} which does not contain finite circuits of M, it is an infinite circuit, hence the edge set of a double ray in the graph. $C_f - f$ is thus a union of two vertex disjoint rays which only use edges of B and

thus are contained in G_B. Let H be the graph whose vertex set is the set of those components of G_B and whose edge set is F such that each edge of F connects the two components of G_B in which its end vertices lie. This is well-defined: the end vertices of every edge f of F are starting edges of the two rays of $C_f - f$, thus every end vertex of f is contained in a component of G_B. Furthermore every component of G_B which contains an end vertex of an edge from F contains a ray. As $G_{B_{\mathrm{fin}}}$ is a spanning tree of a connected graph and thus connected, every component of G_B contains an end vertex of an edge of $F = B_{\mathrm{fin}} \backslash B$ and thus contains a ray. In particular H is connected. Any path in H can be extended to a walk in $G_{B_{\mathrm{fin}}}$ by inserting paths inside components of G_B and thus also every finite circuit in H can be extended to a finite circuit in $G_{B_{\mathrm{fin}}}$. As B_{fin} does not contain finite circuits, H also cannot contain finite circuits and is thus a tree. So the number of vertices of H is $|F| + 1 \geq |I| - 1 + 1 = |I|$. As each vertex of H corresponds to a component of G_B containing a ray, there is a family of vertex disjoint rays of G of size at least $|I|$. □

Of course this characterisation in terms of bases is the one we will work with in the context of matroids, but the intuition of what happens in Halin's theorem, namely that systems of something are increased, is nearer to the families of circuits defined in Definition 3.2.

Corollary 3.5. *Let G be a locally finite connected graph containing a ray and M its algebraic cycle matroid. Then G contains a family $(R_i)_{i \in I+j}$ of vertex disjoint rays such that j is not an element of I iff M contains a family of circuits $(C_i)_{i \in I}$ which is independent concerning finite circuits.*

So for a given locally finite graph the fact that Halin's theorem holds is equivalent to the fact that Question 1.1 holds for its algebraic cycle matroid, which will be shown in Section 4.

Lemma 3.6. *Let G be a locally finite graph. If there are arbitrarily large finite families of pairwise vertex disjoint rays of G, then there is an infinite such family.*

Proof. Let G have arbitrarily large finite families of pairwise vertex disjoint rays. If G has infinitely many components containing a ray, then taking one ray from each of these components gives rise to an infinite family of vertex disjoint rays and this Lemma holds for G. If G does not have infinitely many components containing a ray, then there is one component of G containing arbitrarily large finite families of vertex disjoint rays. So without loss of generality it may be assumed that G is connected.
Let M be the algebraic cycle matroid of G. As G is locally finite, M exists and is cofinitary. By Lemma 3.4 M is not l-nearly finitary for any natural number l, so by Lemma 4.4 there are bases B of M and B_{fin} of M_{fin} such that $B \subset B_{\mathrm{fin}}$ and $B_{\mathrm{fin}} \backslash B$ is infinite. Hence again by Lemma 3.4 this implies that G contains an infinite family of pairwise vertex disjoint rays. □

In the last proof we used that G is locally finite. But as Halin's theorem can be reduced to the case where G is locally finite, this re-proves Halin's theorem from Lemma 4.4.

4. The assertion of Question 1.1 for cofinitary matroids

The proof uses the following observation:

Lemma 4.1. *Let M be a matroid and B, B_{fin} bases of M and M_{fin} respectively such that $B \subset B_{fin}$. Let $F = B_{fin} \backslash B$ be a finite set and and $n = |F|$. Then there is no n-separation $E(M) = P_1 \uplus P_2$ of M such that P_1 is finite and contains F.*

Proof. Let $E(M) = P_1 \uplus P_2$ be a separation of M such that P_1 is finite and contains F. In order to show that $E(M) = P_1 \uplus P_2$ is not an n-separation of M, let B_1 be a base of $M\restriction_{P_1}$ containing $B_{\text{fin}} \cap P_1$. This is possible as P_1 is finite and B_{fin} does not contain finite circuits. As B is a base of M, every edge of F is spanned in M by B, so F is spanned in $M.P_1$ by $B \cap P_1$. Hence F is also spanned by $B_1 \backslash F$ in $M.P_1$, so $E(M) = P_1 \uplus P_2$ cannot be an n-separation of M. □

This leads to the question whether the other direction is true as well: If there is a set F of size n and there is no n-separation $E(M) = P_1 \uplus P_2$ of M such that P_1 is finite and contains F, are there then bases B of M and B_{fin} of M_{fin} such that $B_{\text{fin}} \backslash B$ contains F?
This is the same question as the one whether for every such F there is a family of circuits $(C_f)_{f \in F}$ which is independent concerning finite circuits such that $f \in C_g \Leftrightarrow f = g$. If F is finite and M is countable and cofinitary, then it is possible to extend F to such a family by adding finite pieces to the (future) circuits in a way such in no step a finite circuit emerges and the results are indeed circuits. If M has more than countably many edges, then it is necessary to complete the circuits by a compactness argument instead of adding countable many finite pieces. Lemmas 4.2 and 4.3 show that F can be extended suitably by finite pieces.

Lemma 4.2. *Let M be a matroid and $E(M) = P_1 \uplus Q_1 = Q_2 \uplus P_2$ two exact $n+1$-separations of M such that P_1 is a subset of Q_2 and $F := Q_2 \backslash P_1$ is finite. Assume further that there is no n-separation $E(M) = Z_1 \uplus Z_2$ of M such that $P_i \subset Z_i$. Then there is a set X which is a base of $M/P_1 \backslash P_2$ as well as of $M \backslash P_1/P_2$. The following two lemmas show the existence of pieces which can be added to finite parts of circuits.*

Proof. The proof is by induction on the size of F. Let B_i be a base of $M\restriction_{P_i}$ and B'_i a base of $M.P_i$ which is contained in B_i. This implies that $|B_i \backslash B'_i| = n$. If F is the empty set, then $X = \emptyset$ meets the requirements of this lemma. If F contains a unique element e, then by Remark 2.5 $B_1 \cup B'_2$ is a base of M iff $B'_1 \cup B_2$ is a base of M. In this case let $X = \emptyset$. Otherwise $B_1 \cup B'_2 + e$ and $B'_1 \cup B_2 + e$ are bases of M, let $X = \{e\}$.
So let $|F| \geq 2$ and pick an edge $e \in F$. If $M \backslash e$ has no n-separation $E(M-e) = Z_1 \uplus Z_2$ such that $P_i \subset Z_i$, then by the induction hypothesis there is a set $X' \subset F - e$ which is a base both of $(M-e) \backslash P_1/P_2$ and of $(M-e)/P_1 \backslash P_2$. By Remark 2.5 either $B_1 \cup X' \cup B'_2$ and $B'_1 \cup X' \cup B_2$ are bases of M or $B'_1 \cup (X'+e) \cup B_2$ and $B_1 \cup (X'+e) \cup B'_2$ are bases of M. In the first case let $X = X'$ and in the second case define $X' = X + e$.
If $M \backslash e$ has an n-separation $E(M-e) = Z_2 \uplus Z_2$ such that $P_i \subset Z_i$, then by $|F| \geq 2$ some $Z_i \backslash P_i$ is non-empty. Assume without loss of generality that $Z_1 \backslash P_1$ is non-empty. Then $E(M) = Z_1 \uplus (Z_2 + e)$ is an exact $n+1$-separation of M. By the induction hypothesis there are sets $X_i \subset Z_i \backslash P_i$ such that X_i is a base of

$M.Z_i \backslash P_i$ and of $M\restriction_{Z_i}/P_i$. Then $X = X_1 \cup X_2$ is a base of $M\backslash P_1/P_2$ as well as of $M/P_1\backslash P_2$. $\square$

Lemma 4.3. *Let M be a matroid, $E(M) = P_1 \uplus Q_1$ an exact n_1-separation of M and $E(M) = Q_2 \uplus P_2$ and exact n_2-separation of M such that $P_1 \subset Q_2$ and $|Q_2\backslash P_1|$ is finite. Assume further that $n_1 \leq n_2$ and there is no $n_1 - 1$-separation $E(M) = Z_1 \uplus Z_2$ of M such that $P_i \subset Z_i$. Then there are bases B_1, B_2 of $M/P_1\backslash P_2$ and $M\backslash P_1/P_2$ respectively such that $B_2 \subset B_1$ and $|B_1\backslash B_2| = n_2 - n_1$.*

Proof. The proof is by induction on the size of $F := E(M)\backslash(P_1 \cup P_2)$. If $n_1 = n_2$, then we are done by Lemma 4.2, so assume $n_2 > n_1$. If F contains exactly one edge e, then $n_1 + 1 = n_2$, so let $B_1 = \{e\}$ and $B_2 = \emptyset$. B_1 and B_2 meet the requirements of this lemma.
So let F contain at least two elements. If there is an exact n_1-separation $E(M) = Z_1 \uplus Z_2$ such that each P_i is a proper subset of Z_i then by Lemma 4.2 there is a set X which is a base of $M/P_1\backslash Z_2$ as well as of $M\backslash P_1/Z_2$. By the induction hypothesis there are bases B_1' of $M/Z_1\backslash P_2$ and B_2' of $M\backslash Z_1/P_2$ such that $B_2' \subset B_1'$ and $|B_1'\backslash B_2'| = n_2 - n_1$. Then $B_1 = B \cup B_1'$ is a base of $M/P_1\backslash P_2$ and $B_2 = B \cup B_2'$ is a base of $M\backslash P_1/P_2$. Also $B_2 \subset B_1$ and $|B_1\backslash B_2| = |B_1'\backslash B_2'| = n_2 - n_1$.
So assume that there is no such exact n_1-separation of M. Let e be an edge of F, then $E(M) = (P_1 + e) \uplus (Q_1 - e)$ is an exact $n_1 + 1$-separation of M. By the induction hypothesis e is a base of $M/P_1\backslash(Q_1 - e)$ and the empty set is a base of $M\backslash P_1/(Q_1 - e)$. Also by the induction hypothesis there are bases B_1' of $M/(P_1 + e)\backslash P_2$ and B_2' of $M\backslash(P_1 + e)/P_2$ such that $B_2' \subset B_1'$ and $|B_1'\backslash B_2'| = n_2 - (n_1 + 1)$. So $B_1 = B_1' + e$ is a base of $M/P_1\backslash P_2$, $B_2 = B_2'$ is a base of $M\backslash P_1/P_2$, $B_2 \subset B_1$ and $|B_1\backslash B_2| = |B_1'\backslash B_2'| + 1 = n_2 - n_1$. $\square$

The idea for the proof of Theorem 4.4 is first to collect edges $f_1, f_2, ...$ recursively such that every finite subset of $F = \{f_1, f_2, ...\}$ is as above; and then to show with compactness that there is a family $(C_f)_{f\in F}$ of circuits which is independent concerning finite circuits such that $f \in C_g \Leftrightarrow f = g$. It is possible to do so, but the proof becomes a lot shorter when not defining edges $f_1, f_2, ...$ but instead finite sets $S_1\backslash S_0, S_2\backslash S_1, ...$ such that each $S_{i+1}\backslash S_i$ contains possible candidates for f_i but not specifying yet which edge of $S_{i+1}\backslash S_i$ will be in F.

Theorem 4.4. *Let M be a cofinitary matroid. Then either M is l-nearly finitary for some natural number l or M is not nearly finitary at all.*

Proof. Assume that M is not l-nearly finitary for any natural number l. First we show the existence of sets $(S_i)_{i\in\mathbb{N}}$ such that the following conditions hold for all $i \geq 1$:

- $S_i \subset E(M)$ is finite and contains S_{i-1}.
- $E(M) = S_i \uplus (E(M)\backslash S_i)$ is an exact $i + 1$-separation of M.
- There is no exact i-separation $E(M) = Z_1 \uplus Z_2$ of M such that Z_1 is finite and contains S_i.

To do so let $S_0 = \emptyset$. Let $i \geq 1$. As M is not $i - 1$-nearly finitary, there are bases B of M and B_{fin} of M_{fin} such that $B \subset B_{\text{fin}}$ and $|B_{\text{fin}}\backslash B| \geq i$. Let F be a subset of $B_{\text{fin}}\backslash B$ of size i. By Lemma 4.1 there is no i-separation $E(M) = Z_1 \uplus Z_2$ of M such that Z_1 is finite and contains $S_{i-1} \cup F$. Let $G \subset F$ be a minimal non-empty set such that there is no i-separation $E(M) = Z_1 \uplus Z_2$ of M such that Z_1 is finite and contains $S_{i-1} \cup G$. Then G contains an edge e. Let $E(M) = Z_1 \uplus Z_2$ be an i-separation of M such that Z_1 is finite and contains $S_{i-1} \cup (G - e)$, possibly $Z_1 = S_{i-1}$ and define $S_i = S_{i-1} \cup Z_1 + e$. Then $E(M) = S_i \uplus (E(M) \backslash S_i)$ is an exact $i+1$-separation of M. As S_i contains $G \cup S_{i-1}$, there is no i-separation $E(M) = Z_1 \uplus Z_2$ of M such that Z_1 is finite and contains S_i.

Now we show that for all natural numbers j there are bases B of $M\restriction_{S_j}$ and B' of $M.S_j$ such that $B' \subset B$ and $B \backslash B'$ meets all the sets $S_{i+1} \backslash S_i$ for $i < j$. The proof is by induction on j. For $j = 0$ let $B = B' = \emptyset$. So let $j \geq 1$. By the induction hypothesis there are bases B_1, B_1' of $M\restriction_{S_{j-1}}$ and $M.S_{j-1}$ respectively such that $B_1' \subset B_1$ and $B_1 \backslash B_1'$ meets all sets $S_{i+1} \backslash S_i$ for $i < j-1$. Define $T = E(M) \backslash S_j$. Then by Lemma 4.3 there are bases B_2 of $M/S_{j-1} \backslash T$ and B_2' of $M \backslash S_{j-1}/T$ such that $B_2' \subset B_2$ and $|B_2 \backslash B_2'| = (j+1) - ((j-1)+1) = 1$. Then $B_1 \cup B_2$ is a base of $M\restriction_{S_j}$, $B_1' \cup B_2'$ is a base of $M.S_j$ and $(B_1 \cup B_2) \backslash (B_1' \cup B_2')$ meets all sets $S_{i+1} \backslash S_i$ for $i < j$.

Let $Y = \{0, 1, 2\}^{E(M)}$ be a topological space with the product topology where each component carries the discrete topology. Consider the following three types of closed subsets of Y:

$$Y_C = \bigcup_{e \in C} \{y \in Y | y(e) = 0\} \qquad \text{for a finite circuit } C \text{ of } M$$

$$Y_D = \bigcup_{e \in D} \{y \in Y | y(e) = 2\} \qquad \text{for a cocircuit } D \text{ of } M$$

$$Y_i = \bigcup_{e \in S_{i+1} \backslash S_i} \{y \in Y | y(e) = 1\} \qquad \text{for every natural number } i$$

If there is an $x \in Y$ which is contained in all of these sets, then $B = \{e \in E(M) | x(e) = 2\}$ is spanning in M (and hence contains a base of M), $B_{\text{fin}} = \{e \in E(M) | x(e) \neq 0\}$ is independent in M_{fin} (and is thus contained in a base of M_{fin})and $B_{\text{fin}} \backslash B = \{e \in E(M) | x(e) = 1\}$ is infinite. So B and B_{fin} witness that M is not nearly finitary. In order to show that there is such an x, it is enough to show that for each finite set $\mathcal{Y}$ of closed subsets of the form Y_C, Y_D or Y_i their intersection is non-empty.

So let $\mathcal{Y}$ be a finite set of closed subsets of the form Y_C, Y_D or Y_i. Let j be the largest number such that $Y_j \in \mathcal{Y}$ and let R be the union of S_j, the circuits C such that $Y_C \in \mathcal{Y}$ and the cocircuits D such that $Y_D \in \mathcal{Y}$. As shown before in this proof, there are bases B_1 of $M\restriction_{S_j}$ and B_1' of $M.S_j$ such that B_1' is a subset of B_1 and $B_1 \backslash B_1'$ meets all sets $S_{i+1} \backslash S_i$ for $i < j$. As there is no j-separation $E(M) = Z_1 \uplus Z_1$ of M such that Z_1 is finite and contains S_j, $E(M) = R \uplus (E(M) \backslash R)$ is an exact n-separation for some $n > j$. So by Lemma 4.3 there are bases B_2 of $M/S_j \backslash (E(M) \backslash R)$ and B_2' of $M \backslash S_j/(E(M) \backslash R)$ such that B_2' is a subset of B_2 and $|B_2 \backslash B_2'| = n - (j+1)$. Then $B_3 = B_1 \cup B_2$ is a base of $M\restriction_R/S_j$ and $B_3' = B_1' \cup B_2'$ is a base of $M.R \backslash S_j$ such that their difference meets all sets $S_{i+1} \backslash S_i$ for $i > j$. Thus every $x \in \mathcal{Y}$ with $x(e) = 0$ if $e \in R \backslash B_3$, $x(e) = 1$ if $e \in B_3 \backslash B_3'$ and $x(e) = 2$ if $e \in B_3'$ is contained in the intersection of the closed sets contained in $\mathcal{Y}$. $\square$

Corollary 4.5. *Let G be a locally finite connected graph and M its algebraic cycle matroid. Then M is l-nearly finitary for some natural number l or M is not nearly finitary at all.*

5. Full separations of glued represented matroids

5.1. **In general.** The following operation of gluing together two matroids in order to get a new matroid is taken from [5], where it is only defined for finite matroids. The definition makes sense for infinite matroids which are represented as defined in this thesis. The result of such an operation should also be a matroid if the parts are infinite, but this is not stated in [5] and will not be proved here. It does not matter here, because the leading question in this section is when a matroid M can be written as the sum of two smaller ones. So the structure resulting from the gluing operation is M and thus already a matroid by assumption.

Remark 5.1. [5, section 7] Let M_1, M_2 be two matroids which are thinly representable over the same field k and represented by V_1, V_2 respectively such that their edge sets intersect in a finite set K. Define $P = E(M_1)\backslash K$ and $Q = E(M_2)\backslash K$. Let $V_1 \oplus_K V_2 = \{(v_1, v_2) \in k^P \times k^Q | \exists v_1' \in V_1 \exists v_2' \in V_2 : p_P(v_1') = v_1 \text{ and } p_Q(v_2') = v_2 \text{ and } p_K(v_1') = p_K(v_2')\}$. If there is a matroid which is represented by $V_1 \oplus_K V_2$, then denote $M = M_1 \oplus_K M_2$.

Definition 5.2. [5] The matroids M_1, M_2 from Remark 5.1 are *glued together along* K. If there is a matroid represented by $V_1 \oplus_K V_2$, then denote it by $M_1 \oplus_K M_2$.

When a representable matroid is the sum of two others, then this sum displays a lot of structure of M. When $M_1 \restriction P = M \restriction P$ and $M_2 \restriction Q = M \restriction Q$ then this sum makes the structure of M even clearer. In Subsection 5.2 it will be shown that whenever $E(M) = P_1 \mathbin{\dot\cup} P_2$ is a finite separation of M, then there are matroids M_1, M_2 such that $E(M_1) \cap E(M_2) = K$ is a finite set, $M = M_1 \oplus_K M_2$, $M \restriction_{P_i} = M_i \restriction_{P_i}$ and this sum has several other properties. This is a very long proof, so it will be delayed until the end of this section and gets its own subsection. Lemma 5.3 establishes that $E(M) = P_1 \mathbin{\dot\cup} P_2$ is indeed a finite separation and characterises when this separation has nice properties. As this is possible for all finite separations of M, being able to write it as a sum along that separation does not give it extra structure, but the structure of M can be much better seen.
Let M_1, M_2 be two matroids represented by $V_i \leq k^{E(M_i)}$ such that $K = E(M_1) \cap E(M_2)$ is finite and $M = M_1 \oplus_K M_2$. Let $P_i = E(M_i)\backslash K$.

Lemma 5.3. *$E(M) = P_1 \mathbin{\dot\cup} P_2$ is an l-separation of M with $l \leq |K| + 1$. It is an exact $(|K| + 1)$-separation of M iff K is independent and coindependent in M_1 as well as in M_2.*

Proof. Let $B_M \in \mathcal{B}(M\restriction_{P_1})$, $B_M' \in \mathcal{B}(M.P_1)$, $B_1 \in \mathcal{B}(M_1\restriction_{P_1})$ and $B_1' \in \mathcal{B}(M_1.P_1)$ such that $B_1' \subset B_M' \subset B_M \subset B_1$. Then we have

$$|B_M \backslash B_M'| \leq |B_1 \backslash B_1'| \leq |K|$$

so $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an l-separation of M with $l \leq |K| + 1$.
"$\Rightarrow$" If $E(M) = P_1 \mathbin{\dot\cup} P_2$ is an exact $|K| + 1$-separation of M, then we have

$$|K| = |B_M \backslash B_M'| \leq |B_1 \backslash B_1'| \leq |K|$$

so we get that $|B_1 \backslash B_1'| = |K|$. Pick a base $B_K \in \mathcal{B}(M_1 \restriction K)$. Then $B_1' \cup K$ is a base of M_1 and $|(B_1 \cup B_K)\backslash B_1| \leq |K| = |B_1 \backslash (B_1' \cup B_K)|$, so B_1 is a base of M_1 (thus K is coindependent in M_1) and $B_K = K$, hence K is independent in M_1. Similarly K is independent and coindependent in M_2.

"$\Leftarrow$" B_1 is spanning in M_1 and thus a base as well as $B'_1 \cup K$. Hence $|B_1 \backslash (B'_1 \cup K)| = |(B'_1 \cup K) \backslash B_1| = |K|$. As K is independent in M_2, every circuit of M which is a subset of P_1 is also a circuit of M_1. So B_1 is independent in M and thus a base of $M\restriction_{P_1} \Rightarrow B_1 = B_M$. Dually, as K is coindependent in M_2, we get that B'_1 is spanning in $M.P_1$, thus $B'_1 \in \mathcal{B}(M.P_1)$ and therefore $B'_1 = B'_M$. So we get that

$$|B_M \backslash B'_M| = |B_1 \backslash B'_1| = |K|$$

so $E(M) = P_1 \uplus P_2$ is an exact $|K| + 1$-separation of M. $\square$

Definition 5.4. If $E(M) = P_1 \uplus P_2$ is an exact $|K| + 1$-separation, then M is *properly glued* along K.

Corollary 5.5. *If $M = M_1 \oplus_K M_2$ is properly glued along K, then the following things hold:*

- *$M\restriction_{P_i} = M_i\restriction_{P_i}$ and $M.P_i = M_i.P_i$.*
- *The projection $p_{P_i} : V(M_i) \to k^{P_i}$ is injective*
- *For every vector $v \in k^K$ there is a vector $v' \in V(M_i)$ such that $p_K(v') = v$*
- *For every vector $v \in V(M)$ there are unique vectors $v_i \in V(M_i)$ such that $p_{P_i}(v_i) = p_{P_i}(v)$ and $v = v_1 \oplus_K v_2$.*

Also any base of M can be split up into a base of M_1 and a base of M_2. This is shown in Lemma 5.9. The other statements before that are necessary for that Lemma.

Remark 5.6. Let B be a base of M and let B_i be a base of M_i such that $B \cap E_i \subset B_i \subset B \cup K$. Then $|B_1 \cap K| + |B_2 \cap K| = |K|$.

Lemma 5.7. *Let $k^E = V_1 \oplus V_2$ be a finite-dimensional vector space which is the direct sum of two subspaces. Then there is a partition $E = B_1 \cup B_2$ into possibly empty sets such that $V_i \cap k^{B_i} = \{0\}$.*

Proof. By induction on $|E|$. For $|E| = 0$ we can take $B_1 = B_2 = \emptyset$. So let $|E|$ be at least one, and let $e \in E$. Let p_e be the projection from k^E onto k^{E-e}. There are two cases:

$$\exists v_1 \in V_1 - 0 : p_e(v_1) \in V_2 \tag{1}$$

In this case it is not possible that there is also a $v_2 \in V_2 \backslash \{0\}$ with $p_e(v_2) \in V_1$. Assume for a contradiction that there is: so we have some $v_1 = v'_2 + v_1(e)\chi_e$ and some $v_2 = v'_1 + v_2(e)\chi_e$ such that $v_i \in V_i \backslash \{0\}$ and $v'_i \in V_i \cap k^{E-e}$. Then $v_i(e) \neq 0$ as $V_1 \cap V_2 = \{0\}$. Define $\lambda = \frac{v_1(e)}{v_2(e)}$. Then we have $v'_1(e) = 0$ and

$$v_1 = v'_2 + v_1(e)\chi_e \qquad \text{and} \qquad \lambda v_2 = \lambda v'_1 + v_1(e)\chi_e$$

which implies that

$$v_1 + \lambda v'_1 = \lambda v_2 + v'_2.$$

As $V_1 \cap V_2 = \{0\}$, it follows that $0 = 0(e) = v_1(e) + \lambda v'_1(e) = v_1(e)$, which is a contradiction to the choice of v_1. So in this case we get that there is no $v_2 \in V_2 \backslash \{0\}$ such that $p_e(v_2) \in V_1$. By swapping the numbers 1 and 2 we get into the next case:

$$\nexists v_1 \in V_1 \backslash \{0\} : p_e(v_1) \in V_2 \tag{2}$$

In this case, consider $p_e(V_1) + (V_2 \cap k^{E-e})$. Then 0 is the only element of V_1 which is mapped by p_2 to an element of V_2 and ker $p_e \cap V_1 = \{0\}$, so $\dim p_e(V_1) = \dim V_1$, hence $k^{E-e} = p_e(V_1) \oplus (V_2 \cap k^{E-e})$. As the size of $E - e$ is smaller than the size of

E, we get by induction hypothesis that there is a partition $E - e = B'_1 \cup B'_2$ such that $k^{B'_1} \cap p_e(V_1) = \{0\}$ and $k^{B'_2} \cap (V_2 \cap k^{E-e}) = \{0\}$. Define $B_1 := B'_1 + e$ and $B_2 := B'_2$. Then we immediately get that $k^{B_2} \cap V_2 = k^{B'_2} \cap (V_2 \cap k^{E-e}) = \{0\}$. Let $x \in k^{B_1} \cap V_1$. Then we have that $p_e(x) \in k^{B'_1} \cap p_e(V_1)$, so $p_e(x) = 0$. Again by assumption we have that $\ker p_e \cap V_1 = \{0\}$, so $x = 0$. Therefore we have that $k^{B_1} \cap V_1 = \{0\}$. □

Remark 5.8. Let B be a base of M. Let $V'_i := p_K(V_i \cap k^{B \cup K})$. This is a subspace of k^K and represents $M \backslash (P_i \backslash B)/(B \cap P_i)$.

Lemma 5.9. *If $E(M) = P_1 \uplus P_2$ is an exact $|K|+1$-separation of M, then there is a partition $K = B_1 \uplus B_2$ such that $(B \cap P_i) \cup B_i$ is a base of M_i.*

Proof. By Remark 5.6 we have that

$$\begin{aligned}
\dim V_1 + \dim V_2 &= |K| - rank_{M_1}(K) + |K| - rank_{M'_2}(K) \\
&= |K| + |K| - |K| \\
&= |K|
\end{aligned}$$

We also have that for $v \in V_1 \cap V_2 - 0$ there are $v_i \in V(M_i)$ with $\operatorname{supp}(()\, v_i) \subset B \cup K$ and $v_i = v$ on K. As B is a base, this implies that $v_i = v$ on all of P_i. As $E(M) = P_1 \uplus P_2$ is an exact $|K|+1$-separation of M, we have that K is independent in both M_i, so $v = 0$, which is a contradiction to the choice of v, so we have that $k^K = V_1 \oplus V_2$. By Lemma 5.7 we get that there is a partition $K = B_1 \uplus B_2$ such that $k^{B_i} \cap V_i = \{0\}$. So each B_i is independent in M'_i. As the size of each B_i equals the rank of M'_i, we have that B_i is a base of M'_i, hence $(B \cap P_i) \cup B_i$ is a base of M_i. □

5.2. **Dividing a matroid.** In this section it will be shown that for a given thinly representable matroid M represented over a vector space V, and a finite separation $E(M) = P_1 \uplus P_2$ of M there are two matroids M_1, M_2 such that $E(M_i) = P_i \uplus K$ for some finite set K and $M = M_1 \oplus_K M_2$. The matroids M_i will be constructed carefully to ensure that M is glued properly along K.
There are many ways to manipulate a matroid in a finite way and get a new matroid. When M is represented by a vector space V, then one of these ways is to apply an isomorphism to V which only changes something on a finite subset of $E(M)$. We will need this fact for the proof of Lemma 5.11.

Lemma 5.10. *Let M be a tame matroid on ground set E, which is represented by $V \subset k^E$. Let $L \subset E$ be an independent and coindependent finite set and $\phi_L : k^L \to k^L$ an isomorphism. Define $\phi : k^E \to k^E$ to be the unique linear function such that $p_{k^L} \circ \phi = \phi_L$ and $p_{k^{E \backslash L}} \circ \phi = id_{k^{E \backslash L}}$. Let $V' = \phi(V)$. Then there is a matroid M' represented by V'.*

Proof. Let

$$\begin{aligned}
\underline{V'} &= \{S \subset E \mid \exists v \in V' : S = \operatorname{supp}(v)\} \\
\mathcal{C}' &= \{C \in \underline{V'} \mid C \text{ is minimal non-empty}\} \\
\mathcal{S}' &= \{S \subset E \mid S \text{ is a union of elements of } \mathcal{C}\} \qquad \text{and} \\
\mathcal{I}' &= \{I \subset E \mid I \text{ contains no element of } \mathcal{C}\}
\end{aligned}$$

Claim 1. *For all $v' \in V'$, $\operatorname{supp}(v')$ is an element of $\mathcal{S}'$.*

Proof. The proof goes by induction on $|\mathrm{supp}\,(v') \cap L|$. If $|\mathrm{supp}\,(v') \cap L| = 0$, then $v' \in V$ and thus $\mathrm{supp}\,(v')$ is a union of circuits of M. For each of these circuits C there is a vector $v_C \in V$ such that $\mathrm{supp}\,(v_C) = C$. The support of v_C does not meet L, hence v_C is also a vector of V' with minimal non-empty support. Thus $\mathrm{supp}\,(v')$ is a union of elements of $\mathcal{C}'$.
So consider the case that $|\mathrm{supp}\,(v') \cap L| > 0$. Let B' be a base of $M\restriction_{\mathrm{supp}(v')\setminus L}$. Then B' does not contain an element of $\mathcal{C}'$ and spans $\mathrm{supp}\,(v)\setminus L$. Let $v = \phi^{-1}(v')$ and B a base of $M\restriction_{\mathrm{supp}(v)}$ containing B'. Define $w = v - \sum_{e\in L\cap(\mathrm{supp}(v)\setminus B)} v(e)v_e^B \in V$ which is a finite linear combination of fundamental vectors of B.
Assume for a contradiction that $\mathrm{supp}\,(w)$ contains edges from L. As $\mathrm{supp}\,(w)$ is a scrawl of M, it contains a circuit C containing an edge z of L. By the construction of w, $\mathrm{supp}\,(w) \subset (\mathrm{supp}\,(v)\setminus L) \cup (L \cap B)$. Hence $C \cap L \subset B$. Then by (C3) for M applied to $z \in C$, $C\setminus(B' \cup L)$ and $(C_e^{B'})_{e\in C\setminus(B'\cup L)}$, there is a circuit using z which is a subset of $B \cup B' = B$. This is a contradiction to the fact that B is independent in M. Hence w does not use edges from L and thus $p_L(v) = p_L(\sum_{e\in L\cap(\mathrm{supp}(v)\setminus B)} v(e)v_e^B)$.
Let

$$w' = \phi\left(\sum_{e\in L\cap(\mathrm{supp}(v)\setminus B)} v(e)v_e^B\right) = \sum_{e\in L\cap(\mathrm{supp}(v)\setminus B)} v(e)\phi(v_e^B) \in V'.$$

Then $\mathrm{supp}\,(w') \subset B' \cup (\mathrm{supp}\,(v') \cap L) \subset \mathrm{supp}\,(v')$. As B' does not contain elements of $\underline{V'}$ and $\mathrm{supp}\,(v') \cap L$ is finite, the vector space $V' \cap k^{\mathrm{supp}(w')}$ has finite dimension. Thus $\mathrm{supp}\,(w')$ contains an element C of $\mathcal{C}'$ and this necessarily contains an element $z \in L$. Let $v_C \in V'$ such that $\mathrm{supp}\,(v_C) = C$. Then $v' - \frac{v'(z)}{v_C(z)} v_C \in V'$ contains strictly less edges from L in its support than v', so by the induction hypothesis its support is in $\mathcal{S}'$. Then

$$\begin{aligned}
\mathrm{supp}\,(v') &= \mathrm{supp}\left(v' - \frac{v'(z)}{v_C(z)}v_C + \frac{v'(z)}{v_C(z)}v_C\right) \\
&\subset \mathrm{supp}\left(v' - \frac{v'(z)}{v_C(z)}v_C\right) \cup \mathrm{supp}\left(\frac{v'(z)}{v_C(z)}v_C\right) \\
&= \mathrm{supp}\left(v' - \frac{v'(z)}{v_C(z)}v_C\right) \cup C \\
&\subset \mathrm{supp}\,(v') \cup \mathrm{supp}\,(v_C) \cup C,
\end{aligned}$$

so $\mathrm{supp}\,(v') = \mathrm{supp}\left(v' - \frac{v'(z)}{v_C(z)}v_C\right) \cup C$ is a union of elements of $\mathcal{C}'$. □

Claim 2. *$\mathcal{C}'$ satisfies (C1) and (C2).*

Proof. By definition of $\mathcal{C}'$, every element of $\mathcal{C}'$ is non-empty, hence $\emptyset \notin \mathcal{C}'$. Also by the definition of $\mathcal{C}'$, every element of it is minimal non-empty, so no element of $\mathcal{C}'$ can be a proper subset of another one. □

Claim 3. *$\mathcal{C}'$ satisfies (C3).*

Proof. Let $C_1, C_2 \in \mathcal{C}'$, $x \in C_1 \cap C_2$ and $z \in C_1\setminus C_2$. Let $v_1, v_2 \in \underline{V'}$ such that $\mathrm{supp}\,(v_1) = C_1$ and $\mathrm{supp}\,(v_2) = C_2$, then $v_1 - \dfrac{v_1(x)}{v_2(x)} v_2$ has a support which by Claim 1 is a union of elements of $\mathcal{C}'$ and thus contains an element of $\mathcal{C}'$ containing z. So $\mathcal{C}'$

satisfies (C3) if X contains at most one element. By induction it also satisfies (C3) if X is a finite set. Let $(C_x|x \in X)$ be a family of elements of $\mathcal{C}'$, C an element of $\mathcal{C}'$ and $z \in C \backslash \bigcup_{x \in X} C_x$ such that for all x in X, $C_x \cap C = \{x\}$. Then $X_1 = X \cap L$ is finite, so there is an element C_1 of $\mathcal{C}'$ such that $z \in C_1 \subset \left(C \cup \bigcup_{x \in X_1} C_x\right) \backslash X_1$. For each $x \in X \backslash L$ there is a vector $v_x \in V'$ such that $\operatorname{supp}(v_x) = C_x$ and $v_x(x) = 1$. Let $X_2 \subset X \backslash L$ be a maximal set such that $(p_L(v_y))_{y \in X_2}$ is independent in the vector space k^L. The size of X_2 is bounded by the size of L which is finite, so X_2 exists and is a finite set. Then for each x in $X_3 := X \backslash (X_1 \cup X_2)$ there is a unique linear combination $\sum_{y \in X_2} \alpha_{xy} v_y$ such that $v'_x := v_x - \sum_{y \in X_2} \alpha_{xy} v_y$ does not contain edges of L in its support. For all $x \in X_3$, $v'_x(x) = 1$, so by Claim 1 there is an element C'_x of $\mathcal{C}'$ such that $x \in C'_x \subset \operatorname{supp}(v'_x)$. None of the C'_x meets L, so each is a circuit of M. Let $v \in V'$ be a vector such that $\operatorname{supp}(v) = C_1$. Then the family $(C'_x|x \in X_3)$, the vector $\phi^{-1}(v)$ and the set $\operatorname{supp}\left(\phi^{-1}(v)\right) \cap L + z$ meet the requirements of Lemma 2.29 and thus there is a vector $w \in V$ such that $p_{L+z}(w) = p_{L+z}(\phi^{-1}(v))$ (hence $p_{L+z}(\phi(w)) = p_{L+z}(v)$), $\operatorname{supp}(w) \subset \left(\operatorname{supp}\left(\phi^{-1}(v)\right) \cup \bigcup_{x \in X_3} C'_x\right) \backslash X_3$ (hence $\operatorname{supp}(\phi(w))$ is a subset of $\left(\operatorname{supp}(v) \cup \bigcup_{x \in X_3} C'_x\right) \backslash X_3$) and $\operatorname{supp}(w) \backslash (L + z)$ is independent in M (so it is contained in $\mathcal{I}'$). As $\phi(w) \in V'$, by Claim 1 there is $C_2 \in \mathcal{C}'$ such that

$$
\begin{aligned}
z \in C_2 \subset \operatorname{supp}(\phi(w)) &\subset \left(\operatorname{supp}(v) \cup \bigcup_{x \in X_3} C'_x\right) \backslash X_3 \\
&= \left(C_1 \cup \bigcup_{x \in X_3} C'_x\right) \backslash X_3 \\
&\subset \left(\left(C \cup \bigcup_{x \in X_1} C_x\right) \cup \bigcup_{x \in X_3} \operatorname{supp}(v'_x)\right) \backslash (X_1 \cup X_3) \\
&\subset \left(\left(C \cup \bigcup_{x \in X_1} C_x\right) \cup \bigcup_{x \in X_3} \operatorname{supp}(v_x) \cup \bigcup_{y \in X_2} \operatorname{supp}(v_y)\right) \backslash (X_1 \cup X_3) \\
&= \left(C \cup \bigcup_{x \in X_1} C_x \cup \bigcup_{x \in X_3} C_x \cup \bigcup_{y \in X_2} C_y\right) \backslash (X_1 \cup X_3) \\
&= \left(C \cup \bigcup_{x \in X} C_x\right) \backslash (X_1 \cup X_3).
\end{aligned}
$$

As X_2 is finite, there is an element C_3 of $\mathcal{C}'$ such that

$$
\begin{aligned}
z \in C_3 &\subset \left(C_2 \cup \bigcup_{y \in X_2} C_y\right) \backslash X_2 \\
&\subset \left(\left(C \cup \bigcup_{x \in X} C_x\right) \backslash (X_1 \cup X_3) \cup \bigcup_{y \in X_2} C_y\right) \backslash X_2 \\
&= \left(C \cup \bigcup_{x \in X} C_x\right) \backslash X.
\end{aligned}
$$

$\square$

Claim 4. *$\mathcal{I}'$ satisfies (IM), thus $\mathcal{C}'$ satisfies (CM).*

Proof. Let $I \in \mathcal{I}'$ and $X \subset E$ such that $I \subset X$. The proof is by induction on $|I \cap L|$. If $|I \cap L| = 0$, then I is independent in M and by (IM) of M there is a maximal independent set $B \subset X \backslash L$ containing I which then also is a maximal element of $\mathcal{I}'$ contained in $X \backslash L$. As L is finite, there is a maximal $B' \in \mathcal{I}'$ such that $B \subset B' \subset X$. Then B' contains I.
So let $|I \cap L| > 0$ and pick $l \in I \cap L$. As $|(I - l) \cap L| < |I \cap L|$, there is a maximal element B of $\mathcal{I}'$ satisfying $I - l \subset B \subset X$. If $B + l \in \mathcal{I}'$, then $B + l$ is a maximal element of $\mathcal{I}'$ such that $I \subset B + l \subset X$. Otherwise there is $v \in V' \backslash \{0\}$ such that $\operatorname{supp}(v) \subset B + l$. As $B \in \mathcal{I}'$, $v(l) \neq 0$. Similarly as $I \in \mathcal{I}'$, there is $b \in B \backslash I$ such that $v(b) \neq 0$. Then $B + l - b \in \mathcal{I}'$ is a subset of X, contains I and is maximal with these properties. $\square$

The Claims 1 - 4 show that there is a matroid on ground set E such that its set of circuits is $\mathcal{C}'$. Denote this matroid by M'.

Claim 5. *Every cocircuit of M' meets every element of $\underline{V'}$ only in finitely many edges.*

Proof. Let D' be a cocircuit of M. Assume first that D' does not meet L. Then D' is also a cocircuit of M. Let $v' \in V'$ and $v = \phi^{-1} \in V$, then $D' \cap \operatorname{supp}(v') = D' \cap \operatorname{supp}(v)$ is finite. So assume next that D' meets L.
Then $D' \backslash L$ is a scrawl of $M'^*/L = (M' \backslash L)^* = (M \backslash L)^* = M^*/L$ so there is a scrawl D of M^* such that $D' \backslash L \subset D \subset D' \cup L$. Furthermore $D' \backslash L = D \backslash L$ is independent in $M' \restriction_{E \backslash L}$ and thus independent in M. Let B be a base of $M \restriction_D$ containing $D \backslash L$, then D equals $\bigcup_{e \in D \backslash B} C_e^B$ where C_e^B is the fundamental circuit of e in B with respect to M^*. Let $v' \in V'$ and $v = \phi^{-1}(v) \in V$. Then $\operatorname{supp}(v) \backslash L = \operatorname{supp}(v') \backslash L$ and hence

$$\begin{aligned} D' \cap \operatorname{supp}(v') &\subset ((D' \backslash L) \cap (\operatorname{supp}(v') \backslash L)) \cup L \\ &\subset (D \cap \operatorname{supp}(v)) \cup L \\ &\subset \bigcup_{e \in D \backslash B} (C_e^B \cap \operatorname{supp}(v)) \cup L. \end{aligned}$$

As L, $D \backslash B$ and each $C_e^B \cap \operatorname{supp}(v)$ are finite sets, $D' \cap \operatorname{supp}(v')$ is also finite. $\square$

By the definition of $\mathcal{C}'$, $\mathcal{C}' \subset \underline{V'}$, and Claim 1 shows that $\underline{V'} \subset \mathcal{S}'$. Thus by Claim 5 M' is represented by V'. $\square$

Lemma 5.11. *Let M be a tame matroid represented by a vector space $V \subset k^{E(M)}$. Let $E(M) = P_1 \mathbin{\dot\cup} P_2$ be an exact $l+1$-separation of M. Then there are a finite set K of size l and matroids M_i represented by $V_i \subset k^{P_i \cup K}$ s.th. $E(M_i) = P_i \cup K$ and $M = M_1 \oplus_K M_2$.*

Proof. Let $B_i \in \mathcal{B}(M \restriction_{P_i})$ and $B_i' \in \mathcal{B}(M.P_i)$ s. th. $B_i' \subset B_i$. Then $|B_i \backslash B_i'| = l$. Let K be a set which has size l. The maps $p_{B_i \backslash B_i'} : k^{B_1 \cup B_2} \cap V \to k^{B_i \backslash B_i'}$ are isomorphisms: Let $v \in k^{B_1 \cup B_2} \cap V$ such that $p_{B_1 \backslash B_1'}(v) = 0$, then $\operatorname{supp}(v) \cap P_1$ is a subset of B_1'. As $\operatorname{supp}(v) \cap P_1$ is a scrawl of $M.P_1$ and B_1' is a base of $M.P_1$, $\operatorname{supp}(v) \cap P_1$ is the empty set and thus $\operatorname{supp}(v) \subset B_2$. But B_2 is a base of $M \restriction_{P_2}$, so $\operatorname{supp}(v) = \emptyset$ and thus $v = 0$. So $p_{B_1 \backslash B_1'} : k^{B_1 \cup B_2} \cap V \to k^{B_1 \backslash B_1'}$ is injective and

$$\begin{tikzcd}
 & k^{B_1\backslash B_1'} \arrow[dl, "h_1", "\sim"'] \arrow[dr, "\phi_1"', "\sim"] \arrow[rr, hook, "t_1"] & & k^{P_1\cup K} \\
V \cap k^{B_1\cup B_2} & & k^K & \\
 & k^{B_2\backslash B_2'} \arrow[ul, "h_2"', "\sim"] \arrow[ur, "\phi_2", "\sim"'] \arrow[rr, hook, "t_2"] & & k^{P_2\cup K}
\end{tikzcd}$$

FIGURE 2. A commutative diagram illustrating some of the linear functions defined in the proof of Lemma 5.11.

similarly $p_{B_2\backslash B_2'} : k^{B_1\cup B_2} \cap V \to k^{B_2\backslash B_2'}$ is injective. By Remark 2.12 $B_1' \cup B_2$ is a base of M, so for every edge $e \in B_1\backslash B_1'$ there is a fundamental vector, denote it short by v_e. As $(\chi_e)_{e\in B_1\backslash B_1'}$ spans $k^{B_1\backslash B_1'}$ and $p_{B_1\cap B_1'}(v_e) = \chi_e$, this implies that $p_{B_1\backslash B_1'} : k^{B_1\cup B_2}\cap V \to k^{B_1\backslash B_1'}$ is surjective. Similarly $p_{B_2\backslash B_2'} : k^{B_1\cup B_2}\cap V \to k^{B_2\backslash B_2'}$ is surjective. Hence these two maps are isomorphisms. Denote their inverses with h_1 and h_2. Pick an isomorphism $\phi_1 : k^{B_1\backslash B_1'} \to k^K$. Let $\phi_2 = \phi_1 \circ h_1^{-1} \circ h_2$ and define $t_i : k^{B_i\backslash B_i'} \to k^{P_i\cup K}$ by $t_i(x) = p_{P_i} \circ h_i(x) + \phi_i(x)$. Some of these maps are shown in Figure 2. Let $W_i = t_i(k^{B_i\backslash B_i'})$ and $V_i = (V \cap k^{P_i}) + W_i$. By picking an arbitrary bijection from $B_1\backslash B_1'$ to K, ϕ_1 can be seen as an automorphism on $k^{B_1\backslash B_1'}$. Let $U := p_{(P_1\cup B_2)\backslash B_2'}(V \cap k^{P_1\cap B_2})$. Then $(M\restriction_{P_1\cup B_2})/B_2'$ is represented by U and V_1 represents a matroid M_1 by Lemma 5.10 (renaming some edges of a matroid does not change its being a matroid). Similarly V_2 represents a matroid M_2.

In order to show that $M = M_1 \oplus_K M_2$, let first $v \in V(M_1 \oplus_K M_2)$. Then there are $v_i \in V_i$ such that $p_{P_i}(v) = p_{P_i}(v_i)$ and $p_K(v_1) = p_K(v_2)$ and thus $v_i' \in V \cap k^{P_i}, w_i \in W_i$ such that $v_i = v_i' + w_i$. Let $x_i \in V \cap k^{B_1\cap B_2}$ such that $t_i \circ h_1^{-1}(x_i) = w_1$. Then

$$\begin{aligned}\phi_1 \circ h_1^{-1}(x_1) &= p_K \circ t_1 \circ h_1^{-1}(x_1) = p_K(w_1) = p_K(v_1)\\ &= p_K(v_2) = p_K(w_2) = p_K \circ t_2 \circ h_2^{-1}(x_2) = \phi_2 \circ h_2^{-1}(x_2).\end{aligned}$$

As the ϕ_i and h_i are isomorphisms, this implies that $x_1 = x_2$. So $(p_{P_1}(w_1), p_{P_2}(w_2)) = x_1 = x_2$ when identifying k^E with $k^{P_1} \times k^{P_2}$. So $v = v_1' + v_2' + x_1$ is an element of V. This implies that $V_1 \oplus_K V_2 \subset V$ and thus that the support of any vector in $V_1 \oplus_K V_2$ is a scrawl of M.

Let $v \in V$ be a vector such that the support of $p_{P_2}(v)$ is a subset of B_2. Define $z = h_2 \circ p_{B_2\backslash B_2'}(v)$ and $v' = v - w \in V$, then the support of v' is a subset of $P_1 \cup B_2'$. As B_2' is a base of $M.P_2$, $\mathrm{supp}\,(v') \subset P_1$ and thus $v = (p_{P_1}(v' + t_1 \circ h_1^{-1}(w)), p_K \circ t_2 \circ h_2(w))$ is a vector in $V_1 \oplus_K V_2$. This shows that all $v \in V$ only using edges of $P_1 \cup B_2$ have a support which is a scrawl of $M_1 \oplus_K M_2$. Now let C be a circuit of M. If C is a subset of some P_i, then there is a vector $v \in V \cap k^{P_i}$ such that $\mathrm{supp}\,(v) = C$ implying that $v \in V_i$. So in this case there is a vector of V_i such that its support equals C. Assume that C meets both P_1 and P_2. Hence $C \cap P_1$ is independent in M and there is a base B of $M\restriction_{P_1}$ containing $C \cap P_1$. By Remark 2.12, $B \cup B_2'$ is a base of M. Let B' be a base of M such that $B_2 \subset B' \subset B_2 \cup B$. By Remark 2.5, $|(B \cup B_2')\backslash B'| = |B'\backslash(B \cup B_2')| = |B_2\backslash B_2'|$. So there are finitely many edges $e \in B\backslash B'$. For each of these edges there is a fundamental circuit in B' and thus a vector $v_e \in V$ such that its support equals the fundamental circuit. Let $v \in V$ be a vector such that $\operatorname{supp}(v) = C$, then $w = v - \sum_{e \in B \setminus B'} \frac{v(e)}{v_e(e)} v_e$ is an element of V. The support of w is a subset of B'. By Remark 2.5 $B' \cap P_1$ is a base of $M\restriction_{P_1}$ and thus $\operatorname{supp}(w) \cap P_1$, which is a scrawl of $.P_1$, has to be the empty set. Hence the support of w is a subset of P_2 and thus $w \in V \cap k^{P_2} \subset V_1 \oplus_K V_2$. Since $\operatorname{supp}(v_e) \subset P_1 \cup B_2$, for all $e \in B \setminus B'$ it is true that $v_e \in V_1 \oplus_K V_2$. Hence $v = w + \sum_{e \in B \setminus B'} \frac{v(e)}{v_e(e)} v_e$ is an element of $V_1 \oplus_K V_2$. So for every circuit C of M there is a vector in $V_1 \oplus_K V_2$ such that its support equals C. Hence $\mathcal{C}(M) \subset \underline{V_1 \oplus_K V_2} \subset \mathcal{S}(M)$. As $V_1 \oplus_K V_2 \subset V$, the support of every vector of $V_1 \oplus_K V_2$ meets every cocircuit of M in only finitely many edges, so M is represented by $V_1 \oplus_K V_2$. $\square$

Example 5.12. Even if the vector spaces in Lemma 5.11 are constructed to ensure that $M_1 \oplus_K M_2 = M$, they do not necessarily satisfy $V(M) = V_1 \oplus_K V_2$.

Let E be an infinite set and M the matroid with ground set E in which every subset of E is dependent. Let $E = P_1 \mathbin{\dot\cup} P_2$ be a partition of $E(M)$ into two infinite sets. Then the empty set is a base of $M\restriction_{P_1} = M.P_1$ as well as of $M\restriction_{P_2} = M.P_2$, so $E = P_1 \mathbin{\dot\cup} P_2$ is an exact 1-separation of M. Let V be the vector space which is spanned (as a vector space) by the vectors $\{\chi_e | e \in E\} \cup \{\chi_E\}$. As M does not have any cocircuits, it is represented by V. So the construction of Lemma 5.11 can be applied to M, V and $E = P_1 \mathbin{\dot\cup} P_2$. It yields the two vector spaces $V_i \leq k^{P_i}$ (K has size 0 and is thus empty) which contain all those functions $f : P_i \to k$ whose support is finite. Then also every vector in $V_1 \oplus_K V_2$ has finite support, so χ_E is not an element of $V_1 \oplus_K V_2$ and thus $V \neq V_1 \oplus_K V_2$. Of course still $M = M_1 \oplus_K M_2$; V and $V_1 \oplus_K V_2$ are different representations of M.

It is not possible to change the construction in Lemma 5.11 such that $V(M) = V_1 \oplus_K V_2$ but M is still properly glued along K: Let M_1, M_2 be matroids represented by vectorspaces V_1, V_2 such that for $K = E(M_1) \cap E(M_2)$ it is true that $M = M_1 \oplus_K M_2$, $E(M_i) = P_i \mathbin{\dot\cup} K$ and M is properly glued along K. Then by the definition of properly glued and the fact that $E = P_1 \mathbin{\dot\cup} P_2$ is an exact 1-separation of M, K has to be the empty set. Thus $V_1 \oplus_K V_2 \cong V_1 \times V_2$. Assume for a contradiction that $V = V_1 \oplus_K V_2$. As V contains χ_E, χ_{P_1} has to be contained in V_1 and χ_{P_2} has to be contained in V_2. Both of these vectors have infinite support. Let e be an edge which is contained in P_2. Then $\{e\}$ is a circuit of M and thus a circuit of M_2, so $\chi_e \in V_2$. So the vector (χ_{P_1}, χ_e) is a vector in V. Hence the support of (χ_{P_1}, χ_e) contains infinitely many edges (namely $E + e$) and its complement also contains infinitely many edges (namely $P_2 - e$). But the support of every vector in V either contains only finitely many edges or contains all but finitely many edges of E.

But as in Example 2.26, again the only problem is that V contains too many edges: Let M be a matroid which is represented by a vectorspace V which only consists of vectors whose support is a circuit of M and linear combinations of these vectors. Assume that P_1, P_2, K, M_1, M_2, V_1 and V_2 are as in Lemma 5.11. Then it is already shown in the proof of Lemma 5.11 that $V_1 \oplus_K V_2$ is a subset of V. In order to show that $V_1 \oplus_K V_2 = V$, let $v \in V$. Then v is a linear combination $v = \sum_{i \in I} \lambda_i v_i$ of vectors whose support $\operatorname{supp}(v_i) = C_i$ is a circuit of M. As $M = M_1 \oplus_K M_2$, each C_i is a circuit of $M_1 \oplus_K V_2$ and thus there is a vector $w_i \in V_1 \oplus_K V_2$ whose support equals C_i. As $V_1 \oplus_K V_2 \subset V$, w_i is an element of V. As C_i is a minimal non-empty element of $\underline{V}$, w_i has to be a multiple of v_i and thus v_i is an element of $V_1 \oplus_K V_2$. Hence v is also an element of $V_1 \oplus_K V_2$ and thus $V_1 \oplus_K V_2 = V$.

6. Some results about hindrances

In [7] it was conjectured that for any family $(M_k)_{k\in K}$ of matroids on the same ground set for which there is no hindrance there is a covering. This was shown to be equivalent to the packing/covering conjecture [7, Proposition 4.11] and other conjectures (for example intersection [7, Propositon 3.6]). In the same paper it was shown that in some special cases there is no hindrance of $(M_k)_{k\in K}$, implying some special cases of the packing/covering conjecture and other conjectures [7, Theorem 4.16]. Let M_1 and M_2 be two matroids on the same ground set $E(M_1) = E(M_2) = E$.

Definition 6.1. [7, Definiton 4.1] A set (P_1, P_2, e) is called a *hindrance* if

- Each P_j is a subset of $E(M_1) = E(M_2)$ and spans $P_1 \cup P_2 + e$ in M_j
- $e \in E(M)$ and the sets $P_1, P_2, \{e\}$ are pairwise disjoint.

In [7], the definition allows several edges in the place of e by naming the union $P_1 \cup P_2 + e$ (or more edges than e) instead of naming e explicitly, but the existence of one such edge is enough for the purpose of a hindrance.

Remark 6.2. Let (P_1, P_2, e) be a hindrance and define $S = P_1 \cup P_2 + e$.

(1) Let $P'_j \subset P_j$ be a base of $M_j \restriction S$ for $j \in \{1, 2\}$. Then (P'_1, P'_2, e) is a hindrance, too.
(2) Let Q_j be the set of coloops of P_j in $M_j{\restriction_S}$ ($j \in \{1, 2\}$) and $Q = Q_1 \cup Q_2$. Then $(P_1\backslash Q, P_2\backslash Q, e) = (P_1\backslash Q_1, P_2\backslash Q_2, e)$ is a hindrance and neither $M_i{\restriction_{P_i \backslash Q_i}}$ contains a coloop.
(3) As each P_j is spanning in $M_j{\restriction_S}$, there are circuits $C_j, j \in \{1, 2\}$ of M_j such that $e \in C_j \subset P_j + e$. Pick edges $e_j \in C_j$. Then the sets $(P_1 + e - e_1, P_2, e_1)$ and $(P_1, P_2 + e - e_2, e_2)$ are hindrances, too. If P_1 is a base of M_1, then so is $P_1 + e - e_1$, and if P_2 is a base of M_2, then $P_2 + e - e_1$ is a base of M_2, too.

Definition 6.3. [1] Define an order on hindrances with the same edge e as third entry via

$$(P'_1, P'_2, e) \leq (P_1, P_2, e) :\Leftrightarrow P'_1 \subset P_1 \text{ and } P'_2 \subset P_2$$

Lemma 6.4. *Let (P_1, P_2, e) be a hindrance such that each P_j is a base of $P_1 \cup P_2 + e$ in M_j. Then there is a minimal hindrance $(P'_1, P'_2, e) \leq (P_1, P_2, e)$.*

Proof. Let $Q_0 = \{e\}$. For $i \in \mathbb{N}$ and $j \in \{1, 2\}$ define recursively $Q_{2i+j} = \bigcup\{C_f^{P_j} | f \in Q_{2i+j-1}\}$. This is well-defined as $C_f^{P_j}$ is empty if $f \in P_j$. Then $Q_{2i+j}\backslash Q_{2i+j-1}$ is a subset of P_j. Define $P'_j = \bigcup_{i\in\mathbb{N}} Q_{2i+j}\backslash Q_{2i+j-1}$.
For $f \in P'_2 + e$ there is a smallest index $i \in \mathbb{N}$ such that $f \in Q_{2i+1}$ and hence $C_f^{P_1} \subset Q_{2i+2} \subset P'_1 \cup P'_2 + e$. As $C_f^{P_1} - f \subset P_1$ and $(P'_2 + e) \cap P_1 \subset (P_2 + e) \cap P_1 = \emptyset$, $C_f^{P_1} - f$ is a subset of P'_1. So f is spanned in M_1 by P'_1. Similarly, every edge $f \in P'_1 + e$ is spanned in M_2 by P'_2. So (P'_1, P'_2, e) is a hindrance with $(P'_1, P'_2, e) \leq (P_1, P_2, e)$.
For each $i \in \mathbb{N}$, the edges in $Q_{2i+1}\backslash Q_{2i}$ are necessary to span the edges of Q_{2i} in M_1 and the edges in $Q_{2i+2}\backslash Q_{2i+1}$ are necessary to span the edges of Q_{2i+1} in M_2, so any sets $P''_1 \subset P'_1$, $P''_2 \subset P'_2$ either satisfy $P''_1 = P'_1$ and $P''_2 = P'_2$ or that (P''_1, P''_2, e) is not a hindrance. ∎

[1]This is very similar to the definition of an order on waves (and thus on hindrances which are special cases of waves) except of course that the definition of a hindrance here is slightly different.

Corollary 6.5. (i) *For every hindrance (P_1, P_2, e) (not necessarily such that the P_j are bases) there is a minimal hindrance (P'_1, P'_2, e) such that*

$$(P'_1, P'_2, e) \leq (P_1, P_2, e).$$

(ii) *If there exists a hindrance (P_1, P_2, e), then there also exists a hindrance (P'_1, P'_2, e) such that $S' = P'_1 \cup P'_2 + e$ is a scrawl both in M_1 and in M_2.*

Proof. (i) Let (P_1, P_2, e) be a hindrance. By Part 1 of Remark 6.2 there is a hindrance (P''_1, P''_2, e) such that each P''_j is a base of $P''_1 \cup P''_2 + e$ in the corresponding M_j. By Lemma 6.4 there is a minimal hindrance $(P'_1, P'_2, e) \leq (P''_1, P''_2, e)$, so $(P'_1, P'_2, e) \leq (P_1, P_2, e)$ is minimal.

(ii) Let (P_1, P_2, e) be a hindrance. Then by (i) there is a minimal hindrance $(P'_1, P'_2, e) \leq (P_1, P_2, e)$, and by Part 2 of Remark 6.2 there are no coloops in $M_1\restriction_{P'_1 \cup P'_2 + e}$ or in $M_1\restriction_{P'_1 \cup P'_2 + e}$.

$\square$

Lemma 6.6. *Let B_1, B_2 be two bases of the same matroid M and let $x \in E(M)$ be an edge of that matroid. Then*

$$C_x^{B_1} \triangle C_x^{B_2} := (C_x^{B_1} \backslash C_x^{B_2}) \cup (C_x^{B_2} \backslash C_x^{B_1}) \subset \bigcup_{z \in C_x^{B_1} \backslash (B_2 + x)} C_z^{B_2}.$$

Proof. Define $Z = C_x^{B_1} \backslash (B_2 + x)$ and let y be an element of $C_x^{B_1}$. If $y \notin \bigcup_{z \in Z} C_z^{B_2}$, then by (C3) there is a circuit C_y such that

$$y \in C_y \subset (C_x^{B_1} \cup \bigcup_{z \in Z} C_z^{B_2}) \backslash Z \subset B_2 + x$$

which implies that $C_y = C_x^{B_2}$ and $y \in C_x^{B_2}$. Thus $C_x^{B_1} \backslash C_x^{B_2} \subset \bigcup_{z \in Z} C_z^{B_2}$. In particular every $a \in C_x^{B_2}$ is an element of C_x, so contained in $C_x^{B_1} \cup \bigcup_{z \in Z} C_z^{B_2}$. Thus either $a \in C_x^{B_1}$ (which implies that $a \notin C_x^{B_2} \backslash C_x^{B_1}$) or $a \in \bigcup_{z \in Z} C_z^{B_2}$. This proves $C_x^{B_2} \backslash C_x^{B_1} \subset \bigcup_{z \in Z} C_z^{B_2}$. $\square$

Lemma 6.7. *Let (P_1, P_2, e) be a minimal hindrance and $S = P_1 \cup P_2 + e$. Then for each $f \in S$ there is a minimal hindrance (N_1, N_2, f) such that $N_1 \cup N_2 + f = S$.*

Proof. By Part 3 of Remark 6.2 it is true for $e_j \in C_e^{P_j}$ that $(P_1 + e - e_1, P_2, e_1)$ and $(P_1, P_2 + e - e_2, e_2)$ are hindrances.
Assume for a contradiction that $(P_1 + e - e_1, P_2, e_1)$ is not minimal. Then there is a minimal hindrance $(P'_1, P'_2, e) \leq (P_1 + e - e_1, P_2, e_1)$. By the minimality of (P_1, P_2, e) and of (P'_1, P'_2, e) and by Part 1 of Remark 6.2, P_1 and P'_1 are bases of M_1, so the fundamental circuit $C_{e_1}^{P_1 + e - e_1} = C_e^{P_1}$ is a subset of $P'_1 + e_1$ and $C_{e_1}^{P'_1} = C_e^{P_1}$. Thus $(P'_1 + e_1 - e, P'_2, e)$ is a hindrance which is strictly smaller than (P_1, P_2, e). This is a contradiction to the fact that (P_1, P_2, e) is minimal. Similarly, $(P_1, P_2 + e - e_2, e_2)$ is minimal. So if a hindrance is minimal, then the result of any operation of swapping e with an edge in its fundamental circuit in one of the bases is minimal, too.
Define $T_0 = \{(P_1, P_2, e)\}$ and for $i \in \mathbb{N}$, define recursively

$$T_{2i+1} = \{(P'_1, P'_2, f') | P'_1 = P''_1 + f - f' \text{ and } P'_2 = P''_2 \text{ for some } (P''_1, P''_2, f) \in T_{2i} \text{ with } f' \in C_f^{P''_1}\}$$

$$T_{2i+2} = \{(P_1', P_2', f') | P_1' = P_1'' \text{ and } P_2' = P_2'' + f - f' \text{ for some } (P_1'', P_2'', f) \in T_{2i+1} \text{ with } f' \in C_f^{P_2''}\}$$

Also define $R_i = \{f | \exists (P_1', P_2', f') \in T_i : f' = f\}$.
Now the goal is to show by induction that the Q_i as defined in the proof of Lemma 6.4 above coincide with the just defined R_i. $R_0 = \{e\} = Q_0$ and $R_1 = C_e^{P_1} = Q_1$, so let $n \geq 2$ and $r \in R_n$. Then there is a hindrance $(P_1', P_2', f) \in T_{n-1}$ such that $r \in C_f^{P_j'}$ for some $j \in \{1, 2\}$. By Lemma 6.6 it is true that

$$C_f^{P_j'} \triangle C_f^{P_j} \subset \bigcup_{z \in P_j' \setminus P_j} C_z^{P_j}.$$

For an edge $z' \in P_j' \setminus P_j$ there is a smallest index $i \in \mathbb{N}$ such that there is a hindrance $(N_1', N_2', g) \in T_i$ where $z' \notin N_j'$. This index i is smaller than n. But then $z' = g$ and thus $z' \in R_i \subset R_{n-1}$. As $R_{n-1} = Q_{n-1}$ by the induction hypothesis, it is true that $C_{z'}^{P_j} \subset Q_n$. This implies that

$$C_f^{P_j'} \triangle C_f^{P_j} \subset \bigcup_{z \in P_j' \setminus P_j} C_z^{P_j} \subset Q_n.$$

As $(P_1', P_2', f) \in T_{n-1}$, it is true that $f \in R_{n-1} = Q_{n-1}$, hence $C_f^{P_j} \subset Q_n$ and thus

$$\begin{aligned} C_f^{P_j'} &= (C_f^{P_j'} \setminus C_f^{P_j}) \cup (C_f^{P_j'} \cap C_f^{P_j}) \\ &\subset (C_f^{P_j'} \triangle C_f^{P_j}) \cup C_f^{P_j} \subset Q_n \cup Q_n = Q_n. \end{aligned}$$

So $r \in Q_n$.
Let on the other hand $q \in Q_n$, then there is $q' \in Q_{n-1}$ such that $q \in C_{q'}^{P_j}$ for some $j \in \{1, 2\}$. By the induction hypothesis $R_{n-1} = Q_{n-1}$, so there is a hindrance $(P_1', P_2', f) \in T_{n-1}$ such that $f = q'$. Then

$$C_{q'}^{P_j} \setminus C_{q'}^{P_j'} \subset C_{q'}^{P_j} \triangle C_{q'}^{P_j'} = C_{q'}^{P_j'} \triangle C_{q'}^{P_j} \subset \bigcup_{z \in P_j' \setminus P_j} C_z^{P_j}.$$

For $z \in P_j' \setminus P_j$, z is an element of R_i for some smallest index $i \in \mathbb{N}$ where $i < n-1$. This implies that $C_z^{P_j} \subset Q_{i+1} = R_{i+1} \subset R_{n-1}$ and hence $C_{q'}^{P_j} \setminus C_{q'}^{P_j'} \subset R_{n-1}$. So if $q \in C_{q'}^{P_j} \setminus C_{q'}^{P_j'}$, then $q \in R_n$, and otherwise $q \in C_{q'}^{P_j'}$, so there is $(N_1', N_2', g) \in T_n$ such that $g = q$.
So the union of all R_i equals the union of all Q_i and, as (P_1, P_2, e) was chosen minimal, the union of all R_i equals S. This implies that for all $f \in S$ there are an index $n \in \mathbb{N}$ and a hindrance $(P_1', P_2', f') \in T_n$ such that $f' = f$. As all hindrances in the T_i are minimal, so is (P_1', P_2', f'). $\square$

7. Lattices of cyclic flats of finitary matroids

Julie A. Sims proved in [15] that if V is a lattice in which the heights of the elements are bounded, then there is a matroid M such that V is isomorphic to the lattice of cyclic flats of M, using the fact that on every lattice of finite height there is a rank-function. She actually wrote about an "independence space" but that just means a finitary matroid. This result cannot hold for a lattice in which every element has finite height but there is no bound on the height of the elements, as every lattice of cyclic flats of a matroid has a biggest element (Example 7.5). But it is possible to work around this problem by considering lattices of cyclic flats of finite rank. The lattice of all cyclic flats of a finitary matroid can be reconstructed from its lattice of cyclic flats of finite rank.
Our notation for lattices follows [13].

Definition 7.1. A *lattice* is a set V together with two operations $\wedge, \vee : V \times V \to V$ such that

- $\forall v, w \in V : v \wedge (v \vee w) = v = v \vee (v \wedge w)$ (absorption)
- $\forall v, w \in V : v \vee w = w \vee v$ and $v \wedge w = w \wedge v$ (commutativity)
- $\forall v, w, x \in V : v \vee (w \vee x) = (v \vee w) \vee x$ and $v \wedge (w \wedge x) = (v \wedge w) \wedge x$ (associativity).

If V has a least element $0 \in V$, then the height of $v \in V$ is the maximal length of a chain from v to 0 if it exists and infinite otherwise.

Remark 7.2. [13, Vierter Abschnitt, 1.2] Let V be a lattice.

(1) Let v be an element of V, then it is true that $v = v \vee v$ and $v = v \wedge v$ (idempotency).
(2) There is a partial ordering defined on the elements of V by
$$\forall v, w \in V : v \leq w :\Leftrightarrow v \vee w = w.$$
For all $v, w \in V$, $v \vee w = w$ and $v \wedge w = v$ are equivalent statements and it is equivalent to the definition above to define
$$\forall v, w \in V : v \leq w :\Leftrightarrow v \wedge w = v.$$
For all $v, w \in V$, the element $v \vee w$ of V is the least element bigger than v and bigger than w with respect to the partial order and $v \wedge w$ is the greatest element of V smaller than v and smaller than w.
(3) Let $v, w, x \in V$.Then $v \leq w$ and $v \leq x$ is equivalent to the statement that $v \leq w \wedge x$. Similarly, the statement that $v \vee x \leq w$ is equivalent to the statement that $v \leq w$ and $x \leq w$.

Remark 7.3. Let M be a matroid, let V be the set of its cyclic flats and define $\vee, \wedge : V \times V \to V$ via
$$v \vee w = \mathrm{span}_M(v \cup w)$$
$$v \wedge w = \bigcup \{C \in \mathcal{C}(M) | C \subset v \cap w\}.$$
Then $(V, \vee, \wedge)$ is a lattice.

Definition 7.4. Let V be a lattice. A function $r : V \to \mathbb{N}$ is called a *rank-function for V* if it satisfies

- $\forall v, w \in V : (v \neq w$ and $v \vee w = w) \Rightarrow r(v) < r(w)$ (strict monotonicity)
- $\forall v, w \in V : r(v) + r(w) \geq r(v \vee w) + r(v \wedge w)$ (submodularity).

Example 7.5.

The figure on the left shows the Hasse diagram of a lattice V which is the partially ordered set $\mathbb{N}$ with an additional greatest element. There cannot be a rank-function on it, simply because it has an element of infinite height. Nevertheless it is isomorphic to the lattice of cyclic flats of some matroid: Let $V^f = \mathbb{N}$ with the usual order on $\mathbb{N}$. In Proposition 7.9 a matroid will be constructed for V^f such that its lattice of cyclic flats is isomorphic to V.
This problem is a problem for all lattices V with infinite length: Assume there is a matroid M such that V is isomorphic to the lattice of cyclic flats of M. Then V has a greatest element $\mathbb{1}$, namely the element which is mapped to the cyclic flat $E(M)$ of M under the isomorphism. The height of $\mathbb{1}$ is an upper bound on the height of the elements of V and as V has infinite height, the height of $\mathbb{1}$ is necessarily infinite. So the way a rank-function is defined here, no lattice of infinite height on which there is a rank-function can be isomorphic to the lattice of cyclic flats of some matroid. One possible solution of the problem is to not consider the lattice of all cyclic flats of a matroid, but the lattice of cyclic flats of finite rank of a matroid.

Lemma 7.7 will justify the focus on the lattice V' of cyclic flats of finite rank instead of the lattice V of all cyclic flats: In a finitary matroid, the lattice V is the ideal completion of V' and can thus be reconstructed from V'.

Definition 7.6. [13, Vierter Abschnitt, 1.9]
Let V be a lattice. An *ideal* of V is a subset $I \subset V$ which satisfies

- $\forall v, w \in V : v \leq w \in I \Rightarrow v \in I$
- $\forall v, w \in I : v \vee w \in I$.

The *ideal completion* of V is the lattice of all ideals of V ordered by inclusion.

Lemma 7.7. *Let M be a finitary matroid and V^f the set of its cyclic flats of finite rank. Define $\vee, \wedge$ as in Remark 7.3, then $(V^f, \vee, \wedge)$ is a lattice and the lattice $(V, \vee, \wedge)$ of cyclic flats of M is isomorphic to the ideal completion of $(V^f, \vee, \wedge)$.*

Proof. Denote the ideal completion of $(V^f, \vee, \wedge)$ by $(V', \vee', \wedge')$. Define a function $\varphi : (V, \vee, \wedge) \to (V', \vee, \wedge)$ by $F \mapsto \{G \in V^f | G \subset F\}$. Then

- $\varphi(F)$ is well-defined, as it is an ideal of V^f for all $F \in F$.
- φ is injective, because $\bigcup \varphi(F) = F$ for all $F \in V$.
- φ is surjective: Let $J \in V'$ be an ideal of V^f and e an edge of M spanned by $\bigcup J$. Then there is a circuit $C \subset \bigcup J \cup e$ containing e. As M is finitary, C is finite, thus there are finitely many cyclic flats $F_1, ..., F_r \in \mathcal{J}$ of finite rank such that $C \subset F_1 \cup ... \cup F_r + e$. Let $F = F_1 \vee ... \vee F_r$, then F is a flat, thus $e \in F$, and $F \in J$, as J is an ideal. Thus $\bigcup J$ is a cyclic flat and $\varphi(\bigcup J) = J$.
- φ is an isomorphism of lattices, as for all $F, G \subset V$ such that $F \subset G$, $\varphi(F) \subset \varphi(G)$.

So $(V, \vee, \wedge)$ and $(V', \vee', \wedge')$ are isomorphic. $\square$

If a lattice V^f is the lattice of cyclic flats of finite rank of a finitary matroid, then each $v \in V^f$ has a rank $r(v) \in \mathbb{N}$ in M and this rank-function necessarily satisfies the conditions of Definition 7.4. On the other hand, the existence of such a rank-function on a lattice V^f is also sufficient to make V^f arise from the cyclic flats of some matroid, as will soon be shown.

Let V be a lattice and r a rank-function for V. Let $(X_v)_{v\in V}$ be a family of pairwise disjoint infinite sets and $X = \bigcup_{v\in V} X_v$. We want to construct a matroid M on X via r. The following set shall be the set of independent sets of that matroid:

$$\mathcal{J} = \left\{ J \subset X | \forall w \in V : \left| J \cap \bigcup_{v \leq w} X_v \right| \leq r(w) \right\}.$$

Then the set $\mathcal{C}$ containing all subsets of X which are not contained in $\mathcal{J}$ and are minimal subject to that condition will be the set of circuits of M.

Lemma 7.8. *For each element C of $\mathcal{C}$ there is exactly one $w \in V$ such that $|C \cap (\bigcup_{v\leq w} X_v)| > r(w)$. This w satisfies $C \subset \bigcup_{v\leq w} X_v$ and $r(w) + 1 = |C|$.*

Proof. Let $C \in \mathcal{C}$ and define $V_C = \{w \in V| \left|C \cap (\bigcup_{v\leq w} X_v)\right| > r(w)\}$. By the definition of $\mathcal{C}$, V_C is non-empty. Assume for a contradiction that there is an element $w \in V_C$ such that $\mathcal{C} \nsubseteq \bigcup_{v\leq w} X_v$. Then there is an element $c \in C \setminus \bigcup_{v\leq w} X_v$ and thus

$$\left|(C - c) \cap \bigcup_{v \leq w} X_v \right| = \left| C \cap \bigcup_{v \leq w} X_v \right| > r(w)$$

so $C - c$ is not an element of $\mathcal{J}$. This is a contradiction to the minimality of C. Thus for all $w \in V_C$, $C \subset \bigcup_{v\leq w} X_v$. If there were an element $w \in V_C$ such that $|C| \geq r(w) + 2$, then for any $c \in C$ we would have

$$\left|(C - c) \cap \bigcup_{v \leq w} X_v \right| \geq \left| C \cap \bigcup_{v \leq w} X_v \right| - 1 = |C| - 1 \geq r(w) + 2 - 1 > r(w),$$

but then $C - c$ would not be an element of $\mathcal{J}$ which would again contradict the minimality of C. So for all $w \in V_C$, it is true that $|C| = r(w) + 1$.
Let w, x be two elements of V_C, $c \in C$ and $y \in V$ the element of V such that $c \in X_y$. As $C \subset \bigcup_{v\leq w} X_v$, it is true that $y \leq w$, and similarly it is true that $y \leq x$. By Part 3 of Remark 7.2 this implies that $y \leq w \wedge x$. This is true for all $c \in C$, so in particular $C \subset \bigcup_{v\leq w\wedge y} X_v$. Hence

$$\left| C \cap \bigcup_{v \leq w \wedge x} X_v \right| = |C| = r(w) + 1 > r(w \wedge x).$$

This implies that for all $w, x \in V_C$ $w \wedge x$ is an element of V_C, too. As all elements of V_C have rank $|C| - 1$, the rank of $w \wedge x$ equals the rank of w and the rank of x. By the monotonicity of a rank-function of a lattice, this implies that neither w nor x is strictly bigger than $w \wedge x$. Since $w \wedge x \leq w$ and $w \wedge x \leq x$, this implies that $w = w \wedge x = x$, so there are no two different elements of V_C, which thus contains at most one element and hence exactly one element. □

Proposition 7.9. *There is a matroid M such that $\mathcal{J}$ equals the set of independent sets of M.*

Proof. By Lemma 7.8 each $C \in \mathcal{C}$ has finite size. So it is enough to show that the circuit axioms (C1), (C2) and (C3) hold and that for all $S \subset X$ it is true that $S \in \mathcal{J}$ iff it does not contain an element of $\mathcal{C}$.
(C1): By the definition of $\mathcal{J}$, the empty set is an element of $\mathcal{J}$ and thus not an

element of $\mathcal{C}$.
(C2): As $\mathcal{C}$ contains only minimal sets not in $\mathcal{J}$, no element of $\mathcal{C}$ can contain another element of $\mathcal{C}$ as a proper subset.
(C3): As (C1) and (C2) are true, it is enough to show (C3)$'$ which is[2]

$$\forall C_1 \neq C_2 \in \mathcal{C} \forall x \in C_1 \cap C_2 \exists C_3 \in \mathcal{C} : C_3 \subset C_1 \cup C_2 - x.$$

So let C_1, C_2 be two different elements of $\mathcal{C}$ and let x be an element of $C_1 \cap C_2$. For each $c \in C_1 \cup C_2$ let $w(c)$ be the element of V such that $c \in X_{w(c)}$ and for $i \in \{1, 2\}$ let w_i be the unique element of V such that $|C_i \cap \bigcup_{v \le w} X_v| > r(w_i)$ (which exists by Lemma 7.8). By $c \in C_1 \cup C_2$ there is an $i \in \{1, 2\}$ such that $c \in C_i$. This implies by Lemma 7.8 that $c \in \bigcup_{v \le w_i} X_v$, so $w(c) \le w_i$ and thus by associativity $w(c) \le w_1 \vee w_2$. As this is true for all $c \in C_1 \cup C_2$, $C_1 \cup C_2 \subset \bigcup_{v \le w_1 \vee w_2} X_v$. Furthermore, if c is also an element of $C_1 \cap C_2$, then $w(c) \le w_1$ and $w(c) \le w_2$. This implies by Part 3 of Remark 7.2 that $w(c) \le w_1 \wedge w_2$. Thus $C_1 \cap C_2 \subset \bigcup_{v \le w_1 \wedge w_2} X_v$. As C_1 and C_2 are different elements of $\mathcal{C}$, and by (C2) it is impossible that one is a subset of the other, $C_1 \cap C_2$ is a proper subset both of C_1 and of C_2. Because C_1 is an element of $\mathcal{C}$, i.e. a minimal element not in $\mathcal{J}$, every proper subset of it is an element of $\mathcal{J}$, so in particular $C_1 \cap C_2 \in \mathcal{J}$, hence

$$\left| C_1 \cap C_2 \cap \bigcup_{v \le w_1 \wedge w_2} X_v \right| \le r(w_1 \wedge w_2).$$

Because $C_1 \cap C_2 \subset \bigcup_{v \le w_1 \wedge w_2} X_v$, this implies that $|C_1 \cap C_2| \le r(w_1 \wedge w_2)$. So the following equations are true:

$$\begin{aligned}
\left| (C_1 \cup C_2 - x) \cap \bigcup_{v \le w_1 \vee w_2} X_v \right| &= |(C_1 \cup C_2) - x| \\
&= |C_1| + |C_2| - |C_1 \cap C_2| - 1 \\
&= |C_1| - 1 + |C_2| - 1 - |C_1 \cap C_2| + 1 \\
&= r(w_1) + r(w_2) - |C_1 \cap C_2| + 1 \\
&\ge r(w_1) + r(w_2) - r(w_1 \wedge w_2) + 1 \\
&\ge r(w_1 \vee w_2) + 1
\end{aligned}$$

The last inequality is true by the submodularity of the rank-function. So $C_1 \cup C_2 - x$ is not an element of $\mathcal{J}$. Because it is finite, it contains a minimal element C_3 not in $\mathcal{J}$. This proves (C3)$'$.
$S \in \mathcal{J} \Leftrightarrow (\forall C \in \mathcal{C} : C \nsubseteq S)$: Let S be a subset of X which is not contained in $\mathcal{J}$. Then there is an element $w \in V$ such that $|S \cap (\bigcup_{v \le w} X_v)| > r(w)$. Define $S' \subset S \cap (\bigcup_{v \le w} X_v)$ such that $|S'| = r(w) + 1$, then S' is a finite subset of X and is not an element of $\mathcal{J}$, so it contains a minimal element $C \notin \mathcal{J}$. Then C is an element of $\mathcal{C}$ and a subset of S' so in particular a subset of S. Hence S is a superset of an element of $\mathcal{C}$.
Let S be an element of $\mathcal{J}$. By definition of $\mathcal{J}$, all subsets of S are in $\mathcal{J}$, too, so S

[2] this is called (C3) or weak circuits elimination in [14], as this book does not include infinite matroids. Proposition 1.4.12 in the same book shows that (C3)$'$ implies strong circuits elimination which by induction implies (C3) as in this thesis for finitary matroids.

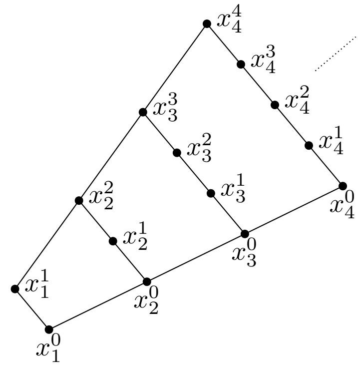

FIGURE 3. A lattice in which every element has finite height on which there is no rank-function.

contains no elements of $\mathcal{C}$.
(CM): This is a straightforward application of Zorn's Lemma. $\square$

Corollary 7.10. *Let $(V^f, \vee, \wedge)$ be a lattice. It arises from the cyclic flats of finite rank of a finitary matroid iff there is a rank-function r on V^f.*

Remark 7.11. Let V be a lattice and V^f be the set of its elements of finite height. If there is a rank-function on V^f and V is isomorphic to the ideal completion of V^f, then V is isomorphic to the lattice of cyclic flats of some matroid.

In contrast to lattices of finite height, there are lattices of infinite height on which there is no rank-function. One class of such lattices are those with at least one element of infinite height (see Example 7.5). But there are also lattices in which each element has finite height and on which there is nevertheless no rank-function:

Example 7.12. Figure 3 shows the Hasse diagram of a lattice L which does not arise as the lattice of cyclic flats of finite rank of a matroid. Assume for a contradiction that L is isomorphic to the lattice of flats of finite rank of some matroid M. Then for each $x_i^j \in L$ there is a cyclic flat $E_i^j \subset E(M)$ which is mapped to x_i^j by this isomorphism. These cyclic flats satisfy for all $n \geq 1$ that

$$r_M(E_1^1) + r_M(E_n^0) \geq r_M(E_1^0) + r_M(E_n^n) \geq r_M(E_1^0) + r_M(E_n^0) + n.$$

The first inequality holds because $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ for all $A, B \subset E(N)$ in all matroids N and the second inequality holds because for all cyclic flats of finite rank F_1, F_2 of a matroid it is true that $(F_1 \subsetneq F_2) \Rightarrow r(F_1) < r(F_2)$. Thus $r_M(E_1^1) \geq r_M(E_1^0) + n$ for all $n \in \mathbb{N}$, which is a contradiction.

Definition 7.13. Let V be a lattice with a least element $0 \in V$. An element $v \in V$ is called *atomic* if the only element of V smaller than v is 0. A lattice is called atomic if for every element $w \in V$ there is an atomic element $v \in V$ such that $w \geq v$.

Corollary 7.14. *There is a matroid such that its lattice of cyclic flats is not atomic.*

Proof. Let $V^f = \mathbb{N}$ be the lattice with the usual ordering on the natural numbers and let $r : V^f \to \mathbb{N}$ be the rank-function $v \mapsto v$ on V^f. Let M be the matroid as constructed in Proposition 7.9 which has V^f as its lattice of cyclic flats of finite rank.

The lattice V of all cyclic flats of M contains only one further element, namely the cyclic flat consisting of all edges of M, called $\mathbb{1}$. This is the lattice from Example 7.5. Consider the dual M^* of M. Then each cyclic flat of M^* is the complement of a cyclic flat of M, hence the lattice of cyclic flats of M^* is isomorphic to the lattice on the same ground set as V but with the ordering inversed. Each $v \in V$ which is smaller than $\mathbb{1}$ is an element of V^f and thus $v + 1$ is also an element of V^f. It is bigger than v and smaller than $\mathbb{1}$, so in V^* there is no atomic element. $\square$

8. Outlook

Of course the most obvious question for further work is whether it is possible to solve Question 1.1 for arbitrary matroids. Let M be a matroid which is not l-nearly finitary for any natural number l. The proof of Theorem 4.4 consists of two steps: The first is the part where the S_i are constructed (using the assumption that M is not l-nearly finitary) and the second step is the part where B and B_{fin} are constructed from the existence of the S_i (using the assumption that M is cofinitary). As already mentioned in Section 4, it should possible by some extra effort to define the S_i such that each $S_{i+1} \backslash S_i$ contains only one edge and thereby to obtain an infinite set F such that for no finite subset $F' \subset F$ there is a $|F'|$-separation $E(M) = P_1 \cup P_2$ such that P_1 is finite and contains F'. The existence of this set then results from M not being l-nearly finitary for any natural number l and thus can be used for arbitrary matroids and is thus true for arbitrary matroids. The second step can then be rewritten to the following Lemma:

Lemma 8.1. *Let M be a cofinitary matroid. Let $F \subset E$ be a set such that for no finite subset $F' \subset F$ there is a $|F'|$-separation $E(M) = P_1 \cup P_2$ such that P_1 is finite and contains F'. Then there are bases B of M and B_{fin} of M_{fin} such that $B \subset B_{fin}$ and $F \subset B_{fin} \backslash B$.*

The proof of this lemma is very similar to the second step of the proof of Theorem 4.4, even if F is finite. This is the converse of Lemma 4.1. In order to confirm Question 1.1 for arbitrary matroids, one possible approach is to try to show Lemma 8.1 for arbitrary matroids. Even if this approach does not work, it can probably be proved for Ψ-matroids (introduced in [5]) and possibly for representable matroids. Lemma 5.9 can be extended without difficulties to a finite sum of matroids. But Ψ-matroids define a possibility to take infinitely many sums simultaneously and maybe it is possible to split a base of such a Ψ-matroid into bases of the glued matroids. Similarly it is possible by Lemma 5.11 to replace a gluing set in a Ψ-matroid M by a gluing set along which M is glued properly. But maybe it is possible to replace all gluing sets simultaneously by ones along which M is glued properly?

Of course the characterisation of lattices of cyclic flats of finite rank of finitary matroids in Corollary 7.10 is not as good a characterisation as one might wish: to show the existence or non-existence of a rank-function is less work than to show that there is or is not a finitary matroid such that the given lattice is the lattice of cyclic flats of finite rank of that matroid. But there is currently no simple characterisation of which lattices have such a rank-function. So further work has to be done on the topic of when such a rank-function exists.

References

[1] Hadi Afzali and Nathan Bowler. thin sums matroids and duality. *Adv. Math*, 271, 2015.

[2] Ron Aharoni and Eli Berger. Menger's theorem for infinite graphs. *Inventiones mathematicae*, 176(1), 2009.

[3] Elad Aigner-Horev, Johannes Carmesin, and Jan-Oliver Fröhlich. infinite matroid union. 2012.

[4] Nathan Bowler and Johannes Carmesin. an excluded minors method for infinite matroids. 2013.

[5] Nathan Bowler and Johannes Carmesin. infinite matroids and determinacy of games. 2014.

[6] Nathan Bowler and Johannes Carmesin. infinite trees of matroids. 2014.

[7] Nathan Bowler and Johannes Carmesin. matroid intersection, base packing and base covering for infinite matroids. 2015.

[8] Henning Bruhn and Reinhard Diestel. infinite matroids in graphs. *in the Infinite Graph Theory special volume of Discrete Math*, 311, 2011.

[9] Henning Bruhn, Reinhard Diestel, Matthias Kriesell, Rudi Pendavingh, and Paul Wollan. axioms for infinite matroids. *Adv. Math*, 239, 2013.

[10] Henning Bruhn and Paul Wollan. finite connectivity in infinite matroids. *Europ. J. Comb.*, 33, 2012.

[11] Reinhard Diestel. *graph theory*. Springer-Verlag, 3 edition, 2006.

[12] D.A. Higgs. infinite graphs and matroids. *Recent Progress in Combinatorics, Proceedings Third Waterloo Conference on Combinatorics*, 1969.

[13] A. G. Kuroš. *Vorlesungen über allgemeine Algebra*. 1964.

[14] James Oxley. *Matroid Theory*. Oxford university press, 2. edition, 2011.

[15] Julie A. Sims. an extension of dilworth's theorem. *Journal of the London Mathematical Society*, s2-16 (3).

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