A Minor Characterisation of Normally Spanned Sets of Vertices

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1 Introduction

Let T be a rooted tree in a graph G. Then T is called *normal* in G if the endvertices of every T-path in G are comparable in the tree-order of T. Furthermore, we call a set of vertices U of G *normally spanned* if there exists a normal tree T in G such that $U \subseteq V(T)$. For example, every countable set of vertices of a graph is normally spanned. Moreover, a minor H of G is supported by $U \subseteq V(G)$ if, and only if for every branch set X_v of H exists a vertex $u \in U$ such that $u \in X_v$. For more definitions, see Section 2.

The following theorem was conjectured by Halin in [12, Conjecture 7.6] and proven by Pitz in [17] in 2020:

Theorem 1.1 (Pitz)**.** *A connected graph has a normal spanning tree if, and only if every minor of it has countable colouring number.*

The main result of this thesis will be the following generalisation:

Theorem 1.2. *A set of vertices U of a connected graph is normally spanned in G if, and only if every minor of G supported by U has countable colouring number (Definition 2.14).*

For an extended version of Theorem 1.2 containing more equivalences, see Theorem 3.2. In Section 3.3, we will explain more precisely how we will prove this.

To both statements in Theorem 1.2 it is also equivalent that every minor of G supported by U has a normal spanning tree. This follows with Theorem 1.1.

Additionally, in [17], Pitz has proven a forbidden minor characterisation for normal spanning trees. With this characterisation it follows that there is a forbidden minor characterisation of normally spanned sets of vertices, i.e. we find a set of graphs $\mathcal{X}$ such that U is normally spanned if, and only if there is no minor supported by U (Definition 2.12) lying in $\mathcal{X}$. We will study this later.

Moreover, Jung has already given a characterisation for normally spanned sets of vertices, that is:

Theorem 1.3 (Jung)**.** *A set of vertices U of a connected graph is normally spanned in G if, and only if U is a countable union of dispersed sets (Definition 2.6).*

In practice, however, it is easier to check whether a set of vertices is fat $TK^{\aleph_0}$-dispersed (Definition 2.23). There is a very efficient criterion of Halin from [11] to have a normal spanning tree:

Theorem 1.4 (Halin)**.** *Every connected graph without a $TK^{\aleph_0}$ has a normal spanning tree.*

This theorem was again strengthened by Diestel in [6] to the following:

Theorem 1.5 (Diestel)**.** *Every connected graph without a fat $TK^{\aleph_0}$ (Definition 2.22) has a normal spanning tree.*

The proof of this theorem was updated again in [17, §6]. Following Diestel's Normal Spanning Tree Criterion 1.5, Pitz proved the following generalisation in [18]:

Theorem 1.6 (Pitz)**.** *A set of vertices U of a connected graph G is normally spanned in G if U is fat $TK^{\aleph_0}$-dispersed in G.*

We give a new proof for this theorem by applying our forbidden minor characterisation. On top of that, we give another characterisation to be a normally spanned set of vertices, namely:

Theorem 1.7. *A set of vertices U of a connected graph G is normally spanned in G if, and only if U is a countable union of fat $TK^{\aleph_0}$-dispersed sets.*

This theorem is adapted from both Jung's and Diestel's criterion. For the case $U = V(G)$ Pitz has already shown the theorem in [18].

Now, we present two applications which show why the existence of normal spanning trees is useful:

Normal spanning trees are useful for understanding the end-spaces of graphs. For example, if T is a normal spanning tree of G, then every end of G contains exactly one normal ray of T; see [7, Lemma 8.2.3] for more details.

Let G be a graph and $\omega \in \Omega(G)$. We say that a ray $R \in \omega$ *devours* the end ω if for every ray $R' \in \omega$ we have $R \cap R' \neq \emptyset$. It is easy to see that every ray in a normal spanning tree devours its corresponding end of G. End-devouring rays are for example useful in [10] and [13].

Also, normal trees can serve as complementary structures for stars, combs, dominating stars and dominated combs in an arbitrary graph. Here, a comb is the union of a ray R (the comb's spine) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the teeth of this comb. Given a set of vertices U, a comb attached to U is a comb with all its teeth in U. Furthermore a star attached to U is a subdivided infinite star with all leaves in U.

For example, we can characterise the graphs that do not contain an infinite comb or an infinite star, respectively, attached to a given set of vertices:

Theorem 1.8 (Bürger and Kurkofka)**.** *Let G be a connected graph and let $U \subseteq V(G)$ be a set of vertices. Then G contains a comb attached to U if, and only if there is no rayless normal tree T in G such that $U \subseteq V(T)$. Furthermore G contains a star attached to U if, and only if there is no locally finite normal tree T in G such that $U \subseteq V(T)$ and all whose rays are undominated in G.*

For more details, see [2–5].

2 Preliminaries

2.1 Basic Definitions

We follow the notation in [7]. Below, we list some definitions that we will use particularly frequently. Some of them are also from [7]. Furthermore, throughout the thesis we take many definitions verbatim from [17].

Definition 2.1. Let G be a graph and $H \subseteq G$ be a subgraph of G. We write $N(H)$ for the set of vertices in $G - H$ with a neighbour in H.

Definition 2.2 (U-Component)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let H be a subgraph of G. Let D be a component of $G - H$. We call D a U-component if, and only if there is a vertex $u \in U$ such that $u \in V(D)$.

Definition 2.3 (Finite Adhesion)**.** Let G be a graph. Let H be a subgraph of a graph G. We say that H has finite adhesion in G if, and only if for all components D of $G - H$ holds that $N(D)$ is finite.

2.2 Normality

Definition 2.4 (Normal Tree)**.** Let G be a graph. Let T be a rooted tree in G. Then T is called normal in G if, and only if the endvertices of every T-path in G are comparable in the tree-order of T.

Definition 2.5 (Normally Spanned)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. We call U normally spanned if, and only if there exists a normal tree T in G such that $U \subseteq V(T)$.

Definition 2.6 (Dispersed Set)**.** Let G be a graph. Let $X \subseteq V(G)$ be a set of vertices of G. Then, X is a dispersed set if, and only if for every ray $R \subseteq G$ there is a finite set of vertices $S \subseteq V(G)$ that separates X and R.

Definition 2.7 (Cofinal Set)**.** Let $(A, \leq)$ be a preordered set and let $B \subseteq A$ be a subset of A. Then, B is cofinal in A if, and only if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Theorem 2.8 (Jung)**.** *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then, the following are equivalent:*

(1) *U is normally spanned in G,*

(2) *U is a countable union of dispersed sets in G,*

(3) *There is a normal tree of G with root r that cofinally contains U for any vertex $r \in V(G)$.*

Definition 2.9 (Normal Spanning Tree)**.** Let G be a graph. We call T a normal spanning tree of G if T is a normal tree of G and $V(G) \subseteq V(T)$.

2.3 Minors

Definition 2.10 (Minor)**.** Let G be a graph. We call a graph H a minor of G, written $H \preccurlyeq G$, if, and only if to every vertex $v \in H$ we can assign a connected set $X_v \subseteq V(G)$ such that for all $w \neq v \in H$ it is true that $X_v \cap X_w = \emptyset$ and such that G contains a X_v–X_w edge if there is a v–w edge in H. We call the X_v branch sets.

Definition 2.11 (Countable Branch Set)**.** Let G be a graph. Let $H \preccurlyeq G$. We say that H has countable branch sets if, and only if for every $v \in H$ we have that the branch set X_v in G is countable.

Definition 2.12 (Minor Supported by U)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let H be a minor of G. We call H a minor of G supported by U if, and only if for every branch set X_v of H exists a vertex $u \in U$ such that $u \in X_v$. We also say that U supports a minor H of G.

Remark 2.13. Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then a subgraph or a minor of a minor of G supported by U is a minor of G supported by U itself.

2.4 Countable Colouring Number

Definition 2.14 (Countable Colouring Number)**.** Let G be a graph. Then G has countable colouring number if, and only if there is a well-order $\leq$ on $V(G)$ such that every vertex of G has only finitely many neighbours preceding it in $\leq$.

The idea of this definition is that a graph with countable colouring number is colourable with countably many colours:

Definition 2.15 (Chromatic Number)**.** Let G be a graph. A vertex colouring of a $V(G)$ is a map $\varphi : V(G) \to \mathcal{C}$ such that $c(x) \neq c(y)$ whenever x and y are adjacent in G. The elements of the set $\mathcal{C}$ are called the available colours. The chromatic number of G is the smallest cardinal $\chi(G)$ such that there is a function $\varphi : V(G) \to \mathcal{C}$ with $|\mathcal{C}| = \chi(G)$.

Proposition 2.1. *Let G be a graph with countable colouring number. Then the chromatic number of G is at most countable.*

Proof. Let G be a graph with countable colouring number. Enumerate the vertices $\{v_i \in V(G) : i \in I\}$ of G so that the order of the numbering witnesses the property of G having countable colouring number. Recursively we define a function

$$\varphi : V(G) \to \mathbb{N}$$

such that for all adjacent $x \neq y \in V(G)$ it is satisfied that $\varphi(x) \neq \varphi(y)$. Suppose that we have already defined $\varphi(v_j)$ for all $j < i$. Next, define the set $\mathcal{A}_i$ that contains all adjacent vertices v_j of v_i for $j < i$. Then $\mathcal{A}_i$ is finite, since G has countable colouring number and we have chosen the enumeration to witness this. Let n be the unique minimal element of $\mathbb{N} \setminus \varphi(\mathcal{A}_i)$ and define $\varphi(v_i) := n$. Then for all adjacent $v_j < v_i$ with $j < i$ it is true that $\varphi(v_j) \neq \varphi(v_i)$. Hence in the end, indeed φ is the desired function that proves that the chromatic number of G is at most countable. $\square$

2.5 Forbidden Minors

Definition 2.16 ((λ, λ^+)-Graph)**.** A (λ, λ^+)-graph for some infinite cardinal λ is a bipartite graph (A, B) such that $|A| = \lambda$, $|B| = \lambda^+$, and every vertex in B has infinite degree.

Definition 2.17 (Cofinality)**.** Let A be a partially ordered set. The cofinality $\mathrm{cf}(A)$ is the least of the cardinalities of the cofinal subsets of A.

Definition 2.18 (Regular and Singular)**.** Let κ be an ordinal. We say that κ is regular, if $\mathrm{cf}(\kappa) = \kappa$. Else, we say that κ is singular.

Definition 2.19 (Stationary Set)**.** Let ι, κ and λ be ordinals such that it is $\iota = \{\kappa : \kappa < \iota\}$ and λ is any limit ordinal. Let λ be any limit ordinal. A subset $A \subseteq \lambda$ is a club-set in λ if it is

(a) closed, i.e. for all limits $\mu < \lambda$ we have that $\sup(A \cap \mu) = \mu$ implies $\mu \in A$ and

(b) unbounded, i.e. $\sup(A) = \lambda$.

A subset $S \subseteq \lambda$ is stationary in λ if S meets every club-set of λ.

For more details, see [15, §III.6]

Definition 2.20 ((κ, S)-Graph)**.** A (κ, S)-graph for some regular uncountable cardinal κ and some stationary set $S \subseteq \kappa$ of cofinality ω ordinals is a graph with vertex set $V(G) = \kappa$ such that $N(s) \cap \{v \in \kappa : v < s\}$ is countable with supremum s for all $s \in S$.

Theorem 2.21 (Komjáth, Bowler, Carmesin and Reiher)**.** *Let G be a graph. Then G has countable colouring number if, and only if G contains neither a (λ, λ^+)-graph nor a (κ, S)-graph as a subgraph.*

For details see [16].

2.6 Fat $TK^{\aleph_0}$s

Definition 2.22 (Fat $TK^{\aleph_0}$)**.** A $TK^{\aleph_0}$ is any subdivision of the countable clique $K^{\aleph_0}$. A fat $TK^{\aleph_0}$ is any subdivision of the multigraph obtained from a $K^{\aleph_0}$ by replacing every edge with $\aleph_1$ parallel edges.

Definition 2.23 (Fat $TK^{\aleph_0}$-Dispersed)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices. We say that U is fat $TK^{\aleph_0}$-dispersed in G if, and only if for every fat $TK^{\aleph_0}$ in G the branch vertices of it can be separated from U by a finite set of vertices.

3 Motivation

3.1 Pitz's Theorem

Summarized from two different publications, Pitz has shown the following theorem:

Theorem 3.1. *Let G be a connected graph. Then, the following are equivalent:*

(A) G has a normal spanning tree,

(B) every minor of G has a normal spanning tree,[1]

(C) every minor of G has countable colouring number,

(D) every minor of G with countable branch sets has countable colouring number,

(E) every minor of G is neither a (λ, λ^+)-graph nor a (κ, S)-graph,

(F) every minor of G is neither a (λ, λ^+)-graph nor a (κ, S)-graph with countable branch sets,

(G) $V(G)$ is a countable union of fat $TK^{\aleph_0}$-dispersed sets in G.

The equivalences of (A) and (C) to (F) are contained in [17]. The equivalence between (A) and (G) is contained in [18]. The equivalence between (A) and (B) was observed by Halin in [12]. In this thesis, we generalise this theorem as follows:

[1] I.e. the property of having a normal spanning tree is closed under minors. This was not proven by Pitz, but by Halin in [12].

3.2 Theorem

Theorem 3.2. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then, the following are equivalent:*

(1) U is normally spanned in G,

(2) every minor of G supported by U has a normal spanning tree,

(3) every minor of G supported by U has countable colouring number,

(4) every minor of G supported by U with countable branch sets has countable colouring number,

(5) every minor of G supported by U is neither a (λ, λ^+)-graph nor a (κ, S)-graph,

(6) every minor of G supported by U is neither a (λ, λ^+)-graph nor a (κ, S)-graph with countable branch sets,

(7) U is a countable union of fat $TK^{\aleph_0}$-dispersed sets in G.

This theorem is the local version of Theorem 3.1. This means that for $U = V(G)$ the statements of both theorems are equivalent. In the following we prove Theorem 3.2.

3.3 Organisation and Proof Sketch

The work is divided into three parts. In the first part we show the implications from (1) to (2), from (2) to (3), from (3) to (5), from (5) to (6) and from (6) to (4). These proofs are fairly straightforward. We show them in Section 4.

$$\begin{tikzcd}
 & & (2) \arrow[r, Rightarrow, "\checkmark"] & (3) \arrow[dr, Rightarrow, "\textit{Section 4}"] & \\
(7) \arrow[r, Leftrightarrow, "\textit{Section 13}"] & (1) \arrow[ur, Rightarrow, "\textit{Section 4}"] & & & (5) \arrow[dl, Rightarrow, "\checkmark"] \\
 & & (4) \arrow[ul, Rightarrow, "\textit{Section 5--12}"] & (6) \arrow[l, Rightarrow, "\textit{Section 4}"'] &
\end{tikzcd}$$

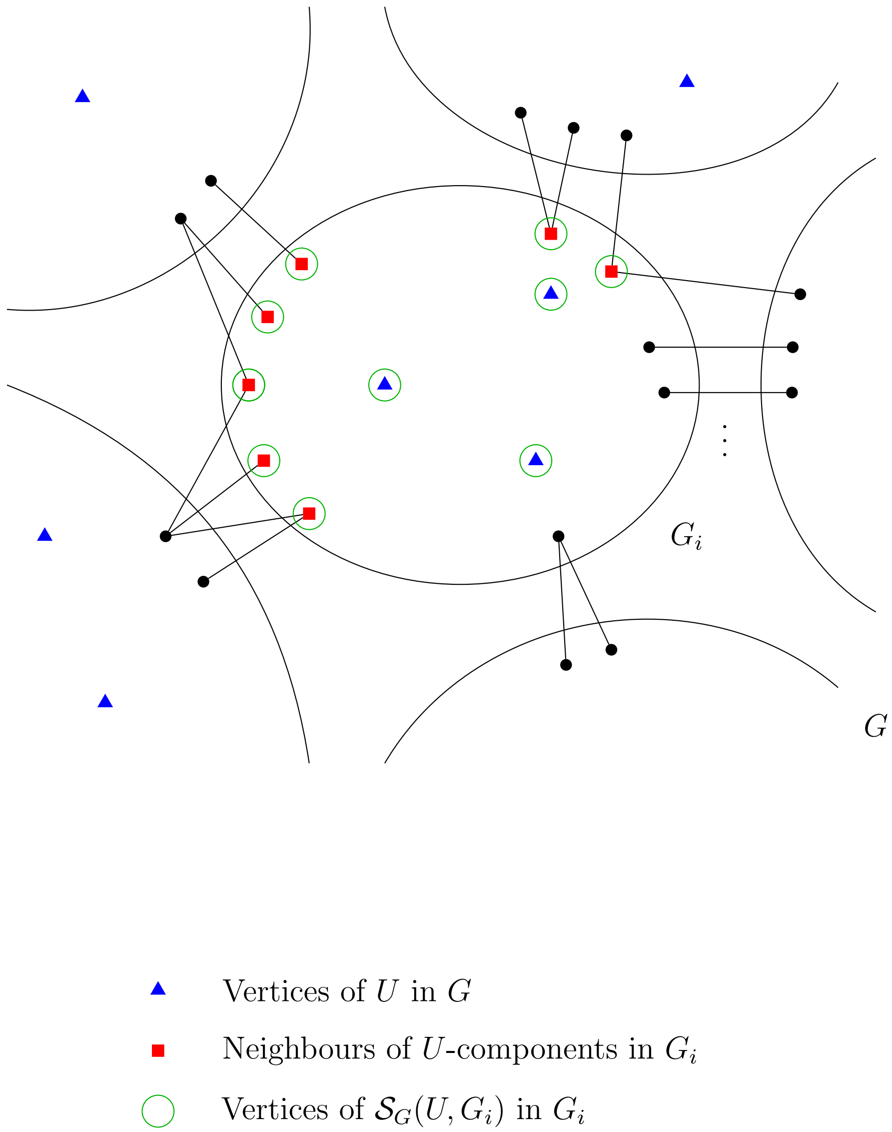
The second part of the thesis is the main part. Here we show the implication from (4) to (1). This proves to be the most difficult part and needs a lot of preparation. This goes from Section 5 to 12. Here, we give a rough plan of how the proof works. Since there are many analogies to [17], we will also outline the similarities and differences.

We will use the proof from [17] as a basis. There, the author proves the global version of the statement, that is (D) to (A).

We prove (4) to (1) by induction on the cardinality of U while Pitz proves (D) to (A) by induction on the cardinality of $V(G)$. These are similar approaches because in Pitz's proof, $U = V(G)$ and thus $|U| = |V(G)|$. Then we decompose a certain subgraph of G which contains all of U. Pitz therefore decomposes the whole graph. We obtain a chain of smaller graphs G_i than U (or G, respectively) for which the induction assumption holds. Recursively we build an increasing chain of normal trees T_i in G such that T_i covers $V(G_i) \cap U$. The union of these normal trees is then the required normal tree containing U. For Pitz this is then a normal spanning tree. For the construction of the normal trees we need that every G_i has *finite adhesion in G towards U*. That means that every neighbourhood of a U-component of $G - G_i$ is finite. (See also Section 8.) In fact, in Pitz's case it is necessary to check this property for all components. This property is essential for extending the normal tree T_i to T_{i+1}. In the end, we combine the normal trees T_i to a normal tree in G covering U. We obtain these G_i's by using the Decomposition Lemma 10.4. In this lemma, the countable colouring number of the minors supported by U contributes.

Even though we prove a stronger statement than Pitz, a part our proof becomes easier. This is because Pitz obtains from his induction hypothesis spanning trees of each G_i which are only normal in G_i. Before he can combine them to a normal spanning tree of G, he has to show that all these trees are also normal in G, which is not trivial ([17, Claim 4.2]). The advantage of our proof is that we do not need this step, because our stronger induction hypothesis gives us trees covering $U \cap V(G_i)$ which are normal in G and not just in G_i.

To be precise, we do not apply the induction hypothesis to $U \cap V(G_i)$, but to a superset of it called $\mathcal{S}_G(U, G_i)$. In addition to $U \cap V(G_i)$, this set of vertices contains the vertices adjacent to any U-component of $G - G_i$.


G_i
G
Vertices of U in G
Neighbours of U-components in G_i
Vertices of $\mathcal{S}_G(U, G_i)$ in G_i

So $\mathcal{S}_G(U, G_i)$ can be thought of as a canonical separator of U and G_i in G, which lies in G_i. The special thing about $\mathcal{S}_G(U, G_i)$ is that it carries the information of the location of the set of vertices U in G and passes it in a compressed way to the subgraph G_i. Now, G_i knows not only its "own" vertices from U, but also knows in which direction the other vertices from U are located. This is a new concept which Pitz does not use. For us it is important, because we only have to extend the normal tree in the direction of U as already mentioned before. Pitz, on the other hand, must extend the normal tree in all directions, otherwise he cannot cover all of $V(G)$. Since every minor of G supported by U with countable branch sets has countable colouring number and by the fact that every G_i has finite adhesion towards U, we follow that every minor of G supported by $\mathcal{S}_G(U, G_i)$ has countable colouring number. By the induction assumption, we will then know that $\mathcal{S}_G(U, G_i)$ is normally spanned in G. We use this to extend the normal tree. The new normal tree will then contain not only $G_i \cap U$, but it will contain all of $\mathcal{S}_G(U, G_i)$ cofinally. This is important because it is thus satisfied that the normal trees also have finite adhesion into the components in G into which we want to build further. To look at all this in detail, see Section 11 and 12.

As mentioned before, for the main result we need the Decomposition Lemma, which decomposes a subgraph of G containing U into smaller graphs G_i such that each G_i has finite adhesion in G towards U. Also, we want that for every U-component D of $G - G_i$ that there are infinitely many U-components of $G - G_i$ with the same neighbourhood as D. Pitz has already proven in his paper a Decomposition Lemma in which each of these G_i has finite adhesion[2] in G. However, he may also more strongly assume that each minor of G has countable colouring number. Since we must restrict ourselves to minors supported by U, we cannot simply adopt Pitz's lemma. In Section 10.1 we see that in general it is not possible to build the G_i's with finite adhesion if only the minors supported by U have countable colouring number. So our aim is to modify the lemma so that it matches our case, i.e. that the G_i will have finite adhesion but only towards U. For the proof of this, we follow the proof of Pitz.

Moreover, Pitz uses and proves several auxiliary statements which he uses for the proof. We also have to adapt all of these to our case. Most proofs are

[2]finite adhesion "in all directions", see Definition 2.3

analogous or at least use the same proof idea. So I would like to emphasise here only what has changed.

In the proof of Lemma 9.4 we will build a barricade as minor supported by U. Barricades can be thought of as (λ, λ^+)-graphs that have a more general shape. While it was sufficient in Pitz's proof to build a barricade as an ordinary minor of G, we have to work a bit more. We do the preliminary work for this in Section 5.

Furthermore, we will also introduce normal semi-partition trees. These can be thought of as a generalisation of normal trees of a graph, where the normal trees can be embedded in the graph not as subgraphs, but as minors. Such a generalisation already exists for normal spanning trees and is called normal partition tree. Pitz investigates and uses these for his proof, but we will investigate and use normal semi-partition trees. To be precise, we will define, study and use normal semi-partition trees supported by U. All this happens in the sections from 5 to 7 and 9.

In Section 10.2 we finally prove the Decomposition Lemma for T-graphs. From this follows directly the needed Decomposition Lemma for graphs (in Section 10.3) as in [17].

Note that in order to show the theorem throughout the proof, we must ensure that all minors have countable branch sets.

The third part of this thesis is the application of the previous theorem, i.e. the equivalences from (1) to (6). There, we show the equivalence from (1) to (7). We will use a statement of Pitz from [18] for this, but will prove it differently. This part is written down in Section 13.

4 Minor Characterisation for Normally Spanned Sets of Vertices

4.1 Theorem

We first consider the first six equivalences of our main result:

Theorem 4.1. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then, the following are equivalent:*

(1) *U is normally spanned in G,*

(2) *every minor of G supported by U has a normal spanning tree,*

(3) *every minor of G supported by U has countable colouring number,*

(4) *every minor of G supported by U with countable branch sets has countable colouring number,*

(5) *every minor of G supported by U is neither a (λ, λ^+)-graph nor a (κ, S)-graph,*

(6) *every minor of G supported by U is neither a (λ, λ^+)-graph nor a (κ, S)-graph with countable branch sets.*

Here, we prove the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (4).

(1) ⇒ (2)*:* Let G be a connected graph. Let $U \subseteq V$ be a set of vertices of G. Further suppose that U is normally spanned. Let T be the normal tree in G such that $U \subseteq V(T)$. By Jung [Theorem 2.8] U is a countable union

$$U = \bigcup_{n \in \mathbb{N}} U_n$$

of dispersed sets. Let G' be a minor of G supported by U. Let $n \in \mathbb{N}$. Define

$$U'_n := \{v \in V(G') : X_v \cap U_n \neq \emptyset\}.$$

Claim: U'_n is a dispersed set in G'.

Let R' be a ray in G'. Let $R^* := \{v \in V(G) : \exists w \in V(R') : v \in X_w\}$. Then R^* is a connected induced subgraph in G. Let $R \subseteq R^*$ be a ray in G such that R meets every branch set X_w for $w \in R'$. Let $S \subseteq V(G)$ be the finite set of vertices that separates U_n and R in G. Define $S' = \{v \in V(G') : X_v \cap S \neq \emptyset\}$. Since S is finite, S' is a finite set of vertices in G'. We show that S' separates U'_n and R' in G'. Assume not. Then there is an U'_n–R' path P' that does not meet S'. Consider $P^* := \{v \in V(G) : \exists w \in V(P') : v \in X_w\}$ similarly as before. In P^* we find an U_n–R path P that does not meet S by construction. A contradiction.

Since G' is a minor of G supported by U and $U \subseteq V(T)$,

$$V(G') = \bigcup_{n \in \mathbb{N}} U'_n.$$

Hence there exists a normal spanning tree T' of G' by Jung [Theorem 2.8].

(2) ⇒ (3)*:* This implication follows directly with [17]. Here, we show it in detail:

Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let H be a minor of G supported by U. Suppose that H has a normal spanning tree T. We define a well-order on $V(H)$ witnessing that H has countable colouring number. For this, let L_i be the ith level of T and consider a well-order $\leq_i$ of L_i for all $i \in \mathbb{N}$. Let $v, v' \in V(H)$ and let $i, i' \in \mathbb{N}$ such that $v \in L_i$ and $v' \in L_{i'}$. We define $v \leq v'$ if, and only if $i < i'$ or $i = i'$ and $v \leq_i v'$. This defines a well-order $(V(H), \leq)$ of $V(T) = V(H)$.

Next, consider a vertex $v \in V(H)$. We show that there are only finitely many neighbours $w \in N(v)$ with $w \leq v$. First, find $i \in \mathbb{N}$ such that $v \in L_i$. Then, the smaller neighbours of v must be contained in $\bigcup_{h \leq i} L_h$ by definition of $(V(H), \leq)$. Further, observe that all smaller neighbours of v must than lie in $\lceil v \rceil_T$. Since $\lceil v \rceil_T$ is finite for normal spanning trees, there are at most finitely many smaller neighbours of v.

(3) $\Rightarrow$ (5)*:* Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Suppose that every minor of G supported by U has countable colouring number. Let H be a minor of G supported by U. By Theorem 2.21 it follows that H does not contain a (λ, λ^+)-graph or a (κ, S)-graph as a subgraph. In particular, H is neither a (λ, λ^+)-graph nor a (κ, S)-graph.

(5) $\Rightarrow$ (6)*:* Follows directly.

(6) $\Rightarrow$ (4)*:* Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Suppose that every minor of G supported by U with countable branch sets is neither a (λ, λ^+)-graph nor a (κ, S)-graph. Let H be a minor of G supported by U with countable branch sets. Let H' be a subgraph of H. Then $H' \preccurlyeq H \preccurlyeq G$ is a minor of G supported by U with countable branch sets. By assumption, H' is neither a (λ, λ^+)-graph nor a (κ, S)-graph. With Theorem 2.21 it follows that H has countable colouring number.

To close the ring closure, the only implication missing now is from (4) to (1). As already announced, we need some preparation for this. We will now start with this in the following section:

5 Barricades

5.1 Definition

Definition 5.1 (Barricade)**.** A barricade is a bipartite graph with bipartition (A, B) such that $|A| < |B|$ and every vertex of B has infinitely many neighbours in A.

Example 5.2. A (λ, λ^+)-graph is a barricade for some infinite cardinal λ.

5.2 Preparation

Lemma 5.3. *A barricade H with bipartition (A, B) has a subgraph that is also a barricade H' with bipartition (A', B') such that $A' \subseteq A$, $B' \subseteq B$ and every vertex $a \in A'$ has more than $|A|$ many neighbours in B'.*

A stronger theorem was already stated and observed in [16, Lemma 2.4]. We show it here, but the proof is nearly the same.

Proof. Let H be a barricade with bipartition (A, B). Define a subgraph H' of H by deleting all $a \in A$ and $b \in N(a)$ if, and only if

$$|N(a)| \leq |A|$$

for $a \in A$. Further, define

$$A' := A \cap V(H')$$

and

$$B' := B \cap V(H').$$

Then, (A', B') is a bipartition with $A' \subseteq A$ and $B' \subseteq B$. Note that we delete at most $|A|$ many $|N(a)|$ and every set of vertices $N(a)$ in B that we delete satisfies $|N(a)| \leq |A|$, i.e. we delete at most $|A| \cdot |A|$ many vertices of B. Then, with $|A| < |B|$ it follows that $|B'| = |B|$.

We show that every $b \in B' \subseteq V(H')$ has infinitely many neighbours in $A' \subseteq V(H')$. For this, let $b \in B'$. In H we know that b has infinitely many neighbours in A. Since we do not delete b for H', we know that there is no $a \in A \subseteq V(H)$ such that $b \in N(a)$. Thus we do not delete any $a \in N(b)$. Hence b also has infinitely many neighbours in H'.

Note that $|A|$ must be infinite, since every $b \in B'$ has infinitely many neighbours in A'. However, it is **not necessary** that $|A'| = |A|$. But indeed, $|A'| \leq |A| < |B| = |B'|$. By construction, we only keep $a \in A$ which had more than $|A|$ many neighbours in B and we deleted at most $|A| \cdot |A|$ many of them. Thus for every $a \in A'$ there are more than $|A|$ many neighbours in B'. $\square$

5.3 Barricades Supported by U

Definition 5.4 (Barricade Supported by U)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. A barricade of G supported by U is a minor of G which is a barricade that is supported by U.

With the previous Lemma 5.3, we can now prove the following lemma, which will be relevant and helpful later in Lemma 9.4:

Lemma 5.5. *Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let G have a minor with countable branch sets that is a barricade with bipartition (A, B) such that the B-side is supported by U, i.e. for every vertex $b \in B$ the corresponding branch set in G contains a vertex of U. Then there is a barricade of G supported by U with countable branch sets.*

Proof. Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let G have a minor H with countable branch sets that is a barricade with bipartition (A, B) such that the B-side is supported by U. With Lemma 5.3 find a barricade H' with bipartition (A', B') such that H' is a subgraph of H with $A' \subseteq A$, $B' \subseteq B$ and every vertex $a \in A'$ has more than $|A|$ many neighbours in B'. Then, $H' \preccurlyeq H \preccurlyeq G$. Also, H' has countable branch sets in G. Note that the B'-side is supported by U since $B' \subseteq B$.

Shortly, the idea of the proof is to find a matching M of A' in (A', B') and then to contract the edges in M. After that, this defines a minor H'' of H' with countable branch sets that is a barricade such that every branch set of H'' contains a vertex $b \in B'$. Since the B'-side of $H' \preccurlyeq G$ is supported by U, for every $b \in B'$ there is a vertex $u \in U$ that is contained in the branch set of b in G. Because of that, H'' is a minor of G supported by U and hence the desired barricade of G supported by U with countable branch sets.

For this, enumerate $A' = \{a_i : i \in |A'|\}$. For every $i \in |A'|$ we build a branch set Y_i in H' that contains a_i and a vertex $b \in B'$. Start with V_0 and consider a_0. Since every $a \in A'$ has neighbours in B', the set $N(a_0)$ is not empty. Choose a vertex $b_0 \in N(a_0) \subseteq B'$ in H'. Define

$$V_0 := \{a_0, b_0\}.$$

Now we consider a_i and suppose that we have already defined Y_j such that a_j and a vertex of B is in Y_j for every $j < i$. Also we suppose that $V_k \cap Y_j = \emptyset$ for two distinct indices $k \neq j < i$. For V_i consider a_i. Define

$$B_i := \{b \in B' : \text{there exists } j < i \text{ such that } b \in Y_j\}$$

and $N_i := N(a_i) \setminus B_i$. Since every $a \in A'$ has more than $|A|$ many neighbours in B' we have that $|A'| < |N(a_i)|$. Thus in step i is $|B_i| < |A'| < |N(a_i)|$. Therefore $N_i \neq \emptyset$. Choose a vertex $b_i \in N_i$ and define

$$V_i := \{a_i, b_i\}.$$

In the end, contracting all V_i for every $i < |A'|$ to a vertex defines the minor $H'' \preccurlyeq H' \preccurlyeq G$ with countable branch sets that is supported by U.

It remains to show that H'' is a barricade. First, define A'' as the set containing all contracted V_i's for every $i < |A'|$. Define

$$B'' = B' \setminus \bigcup_{i<|A'|} B_i.$$

By definition, $A'' \cap B'' = \emptyset$. Thus we get a bipartition (A'', B'') of H''. By construction, $|A''| = |A'|$ and $|B''| = |B'| - |A'|$. Since $|A'| < |B'|$ we have that

$$|A''| = |A'| < |B'| - |A'| = |B''|.$$

Since every $b \in B' \subseteq V(H')$ has infinitely many neighbours in $A' \subseteq V(H')$ and by the fact that we do not change the neighbourhood of the $b \in B'' \subseteq V(H'')$ by contracting the V_i's, we still have that every $b \in B'' \subseteq V(H'')$ has infinitely many neighbours in $A'' \subseteq V(H'')$. □

Now that we have examined properties of barricades, we will not encounter them again until Section 10. Next, we define and examine normal semi-partition trees supported by U:

6 Normal Semi-Partition Trees Supported by U

6.1 T-Graphs

Before we get to know normal semi-partition trees, we briefly review definitions and terminologies of T-graphs:

Definition 6.1 (Order Tree)**.** A partially ordered set $(T, \leq)$ is called an order tree if, and only if

(a) it has a unique minimal element (called the root) and

(b) all subsets of the form $\lceil t \rceil = \lceil t \rceil_T := \{t' \in T : t' \leq t\}$ are well-ordered.

Definition 6.2 (T-Graph)**.** An order tree T is normal in a graph G if, and only if

(a) $V(G) = T$ and

(b) the two vertices v and w of any edge $\{v, w\}$ of G are comparable in T.

We call G a T-graph if, and only if

(a) T is normal in G and

(b) the set of lower neighbours in G of any $t \in T$ is cofinal in $\lceil \mathring{t} \rceil$.

Example 6.3 (ω_1-Graph)**.** An ω_1-graph is a T-graph for the well-order $T = (\omega_1, \leq)$.

Example 6.4 (Aronszajn Tree-Graph)**.** Let T be an order tree. Then T is an Aronszajn tree if, and only if

(a) T has height ω_1,

(b) all levels of T are countable and

(c) T has no uncountable branches.

An Aronszajn tree-graph is a T-graph for an Aronszajn tree T.

Definition 6.5 (Terminology of T-Graphs)**.** Let T be an order tree. A maximal chain in T is called a branch of T. The height of T is the supremum of the order types of its branches. The height of a $t \in T$ is the order type of $\lceil \mathring{t} \rceil := \lceil t \rceil \setminus \{t\}$. The set of all $t \in T$ at height i is the ith level of T. We use the intuitive interpretation of a tree-order as expressing height also informally. For example, we say that $t \in T$ is above $t' \in T$ if $t > t'$.

Let $X \subseteq T$ be a set. The down-closure of X is defined as

$$\lceil X \rceil = \lceil X \rceil_T := \bigcup \{\lceil x \rceil : x \in X\}.$$

We say that X is down-closed, or X is a rooted subtree, if $X = \lceil X \rceil$. A subset of T that is an order tree under the order induced by T is a subtree of T if along with any two comparable points t and t' it contains the interval $\{x \in T : t \leq x \leq t'\}$ in T between them.

If $t < t'$ but there is no point between t and t', we call t' a successor of t. If t is not a successor of any point it is called a limit.

6.2 Normal Semi-Partition Trees Supported by U

We now define the normal semi-partition trees. Notice that the definition is very similar to the definition of normal partition trees introduced in [1].

Definition 6.6 (Semi-Partition Trees)**.** Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let $\{V_t : t \in T\}$ be a set of pairwise disjoint, non-empty sets of vertices V_t of $V(G)$. We call this a semi-partition. If this semi-partition is such that for every $t \in T$ there is a $u \in U$ with $u \in V_t$, we say that the semi-partition is supported by U. If the index set T of a semi-partition is an order tree $(T, \leq)$, we call $(T, \leq)$ a semi-partition tree for G. If the semi-partition is supported by U, we say that the semi-partition tree for G is supported by U. Whenever we speak of a semi-partition tree T for G, we shall assume that it comes with a fixed semi-partition $\{V_t : t \in T\}$. Similarly when we speak of a semi-partition tree T for G supported by U, we shall assume that it comes with a fixed semi-partition $\{V_t : t \in T\}$ supported by U.

Definition 6.7. Let G be a connected graph. For a set of points $X \subseteq T$ of a semi-partition tree, we write

$$G(X) := G\big[\bigcup\{V_t : t \in X\}\big]$$

for the corresponding induced subgraph of G. For vertices $v \in V(G(T))$, we write $t(v)$ for the vertex $t \in T$ such that $v \in V_t$.

Definition 6.8 ($\dot{G}$)**.** Let G be a connected graph. Let T be a semi-partition tree for G. We denote by $\dot{G} = G/T$ the graph obtained from G by deleting all $v \in V(G)$ if, and only if there is no $t \in T$ such that $v \in V_t$; and after that by contracting the sets V_t for $t \in T$. We may then identify T with the set of vertices of $\dot{G}$. Thus, t and $t' \in T$ become adjacent vertices of $\dot{G}$ if, and only if G contains a V_t–$V_{t'}$ edge.

Let $U \subseteq V(G)$ be a set of vertices of G. Note that $\dot{G}$ is a minor of G supported by U if, and only if T is a semi-partition tree for G supported by U.

Definition 6.9 (Normal Semi-Partition Trees)**.** Let G be a connected graph. Let T be a semi-partition tree for G. We call T a normal semi-partition tree for G if the following properties hold:

(a) $\dot{G}$ is a T-graph,

(b) for every $G(T)$-path P with endvertices u and v in $G(T)$, the points $t(u)$ and $t(v)$ are comparable in the tree order of T,

(c) For every $t \in T$, the set V_t is connected in G (so $\dot{G}$ is a minor of G),

(d) for every $t \in T$, we have either $|V_t|$ is finite or $|V_t| = \mathrm{cf}(\mathrm{height}(t))$.

Note that in this definition, we allow that $|V_t|$ is finite in place of $|V_t| = 1$.

Remark 6.10. Rooted subtrees of normal semi-partition trees supported by U are normal semi-partition trees supported by U, too.

Remark 6.11. For a (normal semi-partition) subtree $T' \subseteq T$ of a normal semi-partition tree, note that $G(T')$ is connected: For every $t \in T'$ we have by definition of a normal semi-partition tree that V_t is connected. It is enough to show that $\dot{G}'$ is connected. We show that for every $t \in \dot{G}'$ there is a path to the root $r \in \dot{G}'$ of T'. Since $\dot{G}'$ is a T'-graph, the set of lower neighbours in $\dot{G}'$ of t are cofinal in $\lceil \mathring{t} \rceil$. Thus we find a lower neighbour t_0 of t in $\lceil t \rceil$. If $t_0 = r$, we are done. Otherwise recursively find a sequence $t_0, t_1, t_2, \dots \in \lceil t \rceil$ such that t_{i+1} is a lower neighbour of t_i for $i \geq 0$. This works as before by choosing t_{i+1} in the set of lower neighbours in $\dot{G}'$ of t_i. It remains to show that this sequence is finite and its last element is the root. Consider the sequence as a subset A of $\lceil t \rceil$. Since $t_0 \in A$ we know that A is a non-empty set. By the fact that $\lceil t \rceil$ is well-ordered, A has a minimal element. By construction of the sequence, this is the last element of the sequence, which must be the root r. Hence, the sequence is finite. Now, $P := tt_0 \dots r$ is the required path.

6.3 Existence of Normal Semi-Partition Trees Supported by U

We already know by [1] that for every graph there is a normal partition tree. Pitz references this in his paper in [17, Lemma 3.4] and uses this statement for his proofs. We also need this statement for our normal semi-partition trees. We show:

Lemma 6.12. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then there is a normal semi-partition tree T of G supported by U with $U \subseteq G(T)$.*

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of size κ. Fix an enumeration $\{u_i : i < \kappa\}$ of U. We recursively build a $\subseteq$-increasing sequence $\{T_i : i \leq \kappa\}$ of normal semi-partition trees of G supported by U such that for all $j < i$ every $u_j \in G(T_i)$. For $i = 1$, define an order tree $T_1 := \{t\}$. Further, define $V_t := \{u_0\}$. This defines a normal semi-partition tree T_1 supported by U and $u_0 \in G(T_1)$.

Now, suppose that we have already defined the normal semi-partition tree T_i of G supported by U for $i < \kappa$ and that for all $j < i$ every $u_j \in G(T_i)$. To define T_{i+1}, consider $G - G(T_i)$. If there is no component D with $u_i \in V(D)$, define $T_{i+1} := T_i$. Otherwise let D be the component with $u_i \in V(D)$. Define

$$C := \{t \in T_i : N(D) \cap V_t \neq \emptyset\}.$$

Since T_i is normal, C is a chain in the tree-order of T_i.

In the case that $|C|$ is finite, choose the unique maximal element $t' \in C$. Let $d' \in D$ be a neighbour of a vertex $v' \in V_{t'}$ in G. By the fact that D is connected, we find a u_i–d' path P in D. Let t be a point that is not in T_i already. Define $V_t := V(P)$. Now, let T_{i+1} be obtained through T_i by placing t directly above t' as a new successor.

For the other case, suppose that $|C| := \zeta$ is infinite. Let

$$C' := \{t_n \in C : n < \mathrm{cf}(\zeta)\}$$

be a cofinal subchain in C. For every $n < \mathrm{cf}(\zeta)$ find a neighbour $d_n \in D$ of a vertex $v_n \in V_{t_n}$ in G. Note that some neighbours can be the same vertex.

First, suppose that

$$\big|\{d_n \in D : n < \mathrm{cf}(\zeta)\}\big| < \mathrm{cf}(\zeta).$$

Then there is an $n' < \mathrm{cf}(\zeta)$ such that

$$\big|\{t \in C' : d_{n'} \in N(V_t)\}\big| = \mathrm{cf}(\zeta).$$

This follows by the fact that $\mathrm{cf}(\zeta)$ is a regular[3] cardinal. Furthermore, observe that $\{t_n \in C' : N(d_{n'}) \cap V_{t_n}\}$ is a cofinal subchain of C' and hence of C. Additionally, since D is connected, we find a d_n–u_i path P in D. Define $V_t := V(P)$. Now, let T_{i+1} be obtained through T_i by placing t above $\lceil C \rceil$ as a limit point.

Suppose now that

$$\big|\{d_n \in D : n < \mathrm{cf}(\zeta)\}\big| = \mathrm{cf}(\zeta).$$

Since D is connected, find for every $n < \mathrm{cf}(\zeta)$ a u_i–d_n path P_n in D. Define

$$P := \bigcup_{n < \mathrm{cf}(\zeta)} P_n.$$

Since every P_n is finite, $|V(P)| = \mathrm{cf}(\zeta)$. Since $u_i \in P_n$ for all $n < \mathrm{cf}(\zeta)$, indeed P is connected. Again, define $V_t := V(P)$. As before, let T_{i+1} be obtained through T_i by placing t above $\lceil C \rceil$ as a limit point.

[3]This is the reason why we defined the subchain C' of C.

By construction, T_{i+1} is a semi-partition tree of G supported by U and for all $j < i+1$ every $u_j \in G(T_{i+1})$. It remains to check that T_{i+1} is a normal semi-partition tree of G. For property (a), consider $\dot{G}_{i+1}$ of T_{i+1}. By induction hypothesis, $\dot{G}_i$ is a T_i-graph. We have to show that $\dot{G}_{i+1}$ is a T_{i+1}-graph. Since T_i is an order tree, it has a unique minimal element and thus, T_{i+1} has the corresponding partial order and minimal element as well. Since we add a point t above a chain of T_i, it follows that T_{i+1} is again a partial order. Also all subsets $\lceil s \rceil_{T_i}$ are well-ordered in T_i for $s \in T_i$, so they are well-ordered in T_{i+1} as well. As before, $\lceil \mathring{t} \rceil_{T_{i+1}}$ is a well-order since we add t above a chain of T_i. We show that the set of lower neighbours of $t \in T_{i+1}$ is cofinal in $\lceil \mathring{t} \rceil$. In the first case, indeed t' is the lower neighbour of t by construction and since t is a successor, t' is cofinal in $\lceil \mathring{t} \rceil$. In the other cases, we have the vertex $d_{n'} \in V_t$ or the vertices $d_n \in V_t$ for $n < \mathrm{cf}(\zeta)$; which gives us the existence of the set of lower neighbours of $t \in T_{i+1}$ that is cofinal in $\lceil \mathring{t} \rceil$ by construction. To see that the set of lower neighbours of any $s \in T_{i+1}$ in $\dot{G}_{i+1}$ is cofinal in $\lceil \mathring{s} \rceil$, recall that this is already true for T_i by induction hypothesis. It remains to show that any two adjacent vertices u, u' of $V(\dot{G}_{i+1}) = T_{i+1}$ are comparable in T_{i+1}. Stronger, we show property (b) that any two adjacent vertices u, u' of $\dot{G}_{i+1}$ for which there is a $G(T_{i+1})$-path in G are comparable in T_{i+1}. If $u, u' \in T_i$, this follows from the induction hypothesis. Now suppose that $u = t$. But t and u' are comparable by the fact that we placed t above every element of C in T_i. For property (c), note that V_t is connected in G. For property (d) define $\zeta' := \big|\lceil C \rceil\big|$. Observe that $\zeta' = \mathrm{cf}(\zeta)$. In the end, $|V_t|$ is finite or

$$|V_t| = \mathrm{cf}(\zeta) = \mathrm{cf}(\mathrm{cf}((\zeta)) = \mathrm{cf}(\zeta') = \mathrm{cf}(\mathrm{height}(t)).$$

Thus, by induction hypothesis, both properties are satisfied.

For all limits $\ell \leq \kappa$ define

$$T_\ell := \bigcup_{i<\ell} T_i.$$

We have to show that for all limits $\ell \leq \kappa$ every T_ℓ is a normal semi-partition tree of G supported by U such that for all $j < \ell$ every $u_j \in G(T_\ell)$. So let $\ell \leq \kappa$ be a limit. By construction, T_ℓ is a semi-partition tree of G. To see that T_ℓ is supported by U, let $i < \ell$. Then there is a $t \in T_{i+1}$ such that $u_i \in V_t$.

Hence this t must be in T_ℓ as well. It remains to check that T_ℓ is a normal semi-partition tree of G. For property (a), we have to show that $\dot{G}_\ell$ is a T_ℓ-graph. First, T_ℓ is a partially ordered set, since for all $i < \ell$ we have that T_i is a partially ordered set. Since T_i has a unique minimal element for all $i < \ell$, T_ℓ has the same one as well. Also all subsets of the form $\lceil t' \rceil_{T_i}$ are well-ordered in T_i for every $i < \ell$, so they are well-ordered in T_ℓ as well. To show that T_ℓ is normal, note that $V(\dot{G}_i) = T_i$ for all $i < \ell$. Hence $V(\dot{G}_\ell) = T_\ell$. To see that the set of lower neighbours of any point $s \in T_\ell$ is cofinal in $\lceil \mathring{s} \rceil_{T_\ell}$, find an $i < \ell$ such that $s \in T_i$. In T_i the set of lower neighbours of any $s \in T_i$ is cofinal in $\lceil \mathring{s} \rceil_{T_i}$. Since the lower neighbours do not expand of s in T_ℓ, we have that s is cofinal in $\lceil \mathring{s} \rceil_{T_\ell}$, too. It remains to show that any two adjacent vertices u, u' of $V(\dot{G}_\ell) = T_\ell$ are comparable in T_ℓ. Stronger, we show property (b) that any two adjacent vertices u, u' of $\dot{G}_\ell$ for which there is a $G(T_\ell)$-path in G are comparable in T_ℓ. Since u and $u' \in T_\ell$, there must exist an $i < \ell$ such that s and $s' \in T_i$. Here, both of them are comparable. Thus they are comparable in T_ℓ as well. For property (c), consider a V_t in G for $t \in T_\ell$. Find an $i < \ell$ such that $t \in T_i$. Here, V_t is connected in G and so it is for T_ℓ. For property (d), consider a V_t in G for $t \in T_\ell$. Find an $i < \ell$ such that $t \in T_i$. Here, $|V_t|$ is finite or $|V_t| = \mathrm{cf}(\mathrm{height}(t))$ and so it is for T_ℓ.

In the end, $T := T_\kappa$ is the desired normal semi-partition tree. $\square$

We now know the definition of normal semi-partition trees supported by U and that they exist for every graph such that they cover all of U. Next, we want to find out more properties about normal semi-partition trees:

7 Normal Semi-Partition Trees Supported by U Have Countable Branches and Branch Sets

Our main result in this section is that in our case the normal semi-partition trees supported by U have countable branches and branch sets. "In our case" here means that we consider only graphs with a fixed set of vertices U, so that each minor supported by U with countable branch sets has countable colouring number.

7.1 Definitions

Definition 7.1 (TK^μ)**.** Let μ be a cardinal. Analogously to a $TK^{\aleph_0}$ (Definition 2.22) is a TK^μ any subdivision of the clique K^μ.

Definition 7.2 (IK^μ)**.** Let μ be a cardinal. A graph G is an IK^μ if its vertex set admits a partition $\{X_v : x \in V(K^\mu)\}$ into connected subsets X_v such that for distinct vertices $v, w \in K^\mu$ there is a X_v–X_w edge in G. The sets X_v are the branch sets of the IK^μ.

Definition 7.3 (Compatible)**.** Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Further let μ be a cardinal. Let K be an IK^μ of G supported by U. We call a TK^μ in G compatible with K if, and only if for every branch vertex v of the TK^μ there is a branch set X of K such that $v \in X$ and for two distinct branch vertices $v \neq v'$ holds that they are contained in two different branch sets of K.

Definition 7.4 (Direct Path)**.** For a cardinal μ, let $\mathcal{B}$ be the set of the branch vertices of a TK^μ. We call the unique path between two vertices $v \in \mathcal{B}$ and $w \in \mathcal{B}$ that does not meet any other vertex of $\mathcal{B}$ the direct path between v and w.

Definition 7.5 (Spread)**.** Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Further let μ be a cardinal. Let K be an IK^μ of G supported by U. Consider a TK^μ in G that is compatible with K. We call a TK^μ in G spread in K if, and only if for all branch vertices $v \in \mathcal{B}$ and $w \in \mathcal{B}$, every branch set of K which is traversed by the direct path between v and w does not contain any vertex of $\mathcal{B} \setminus \{v, w\}$.

7.2 Preparation

Lemma 7.6. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let K be an $IK^{\aleph_1}$ of G supported by U. Then there is a $TK^{\aleph_1}$ in G that is compatible with K and spread in K.*

A global version of this lemma similar to this statement has already been proven by Jung in paper [14]. The statement of the theorem is that a graph which contains an uncountable clique minor also contains a topological uncountable clique minor. The following proof uses the same proof ideas as Jung used.

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let K be an $IK^{\aleph_1}$ of G supported by U. Let K' be a subgraph-minimal $IK^{\aleph_1}$ in G such that every branch set of K' is contained in a branch set of K and such that every branch set of K contains a branch set of K'. Such a graph K' exists by Jung [14]. We show that there is a $TK^{\aleph_1}$ in G that is compatible with K' and spread in K'. Then this $TK^{\aleph_1}$ is also compatible with K and spread in K.

Claim: Every branch set of K' contains a vertex of uncountable degree in K'.

Let X be a branch set of K'. Consider $Y := X \cup N_{K'}(X)$. Then Y is uncountable, since $N_{K'}(X)$ is uncountable. Let $y \in Y$. Consider the distance classes D_n of vertices in Y with distance n to y. Since Y is uncountable there exists an uncountable distance class. Consider the minimal n such that the distance class D_n is uncountable. Notice that n is at least 1. By minimality, D_{n-1} is countable. It follows that there is an $x \in D_{n-1}$ that has uncountably many neighbours in D_n and hence in G'. Since every vertex of $N_{G'}(X)$ has degree 1 in Y by minimality of K', it follows that $x \in X$.

Let $\mathcal{V}$ be a set of vertices of K' with uncountable degree and such that two distinct vertices in $\mathcal{V}$ are contained in two disjoint branch sets of K'. In the following we construct a $TK^{\aleph_1}$ in K' with branch vertices $\{v_i : i < \omega_1\} \subseteq \mathcal{V}$. Write K_i for the branch set of K' that contains v_i for all $i < \omega_1$. The direct path of the $TK^{\aleph_1}$ between v_i and v_j for $i < j < \omega_1$ will be called $P_{i,j}$ and this path will not traverse any branch set of $\{K_k : k < \omega_1, k \neq i, k \neq j\}$. Indeed, the $TK^{\aleph_1}$ is compatible with K' and spread in K' then. To do so, we use transfinite recursion over $i < \omega_1$.

First, let $i = 0$. Let $v_0 \in \mathcal{V}$ be an arbitrary vertex. Next, suppose that $i > 0$ and that we have defined branch vertices v_j and direct paths $P_{j,j'}$ for all $j < j' < i$ with the properties from above. Note that $\bigcup_{j<j'<i} P_{j,j'}$ is countable because $i < \omega_1$. Thus we can choose a vertex v_i in $\mathcal{V}$ such that the branch set K_i which contains v_i is disjoint to $\{P_{j,j'} : j < j' < i\}$.

Recursively define the path system $\{P_{j,i} : j < i\}$. In step $\ell < i$, suppose that we have already defined $P_{j,i}$ for all $j < \ell$. Let $\mathcal{W}$ be the countable set of all branch sets of K' which contain vertices of the set

$$\bigcup\{P_{j,j'} : j < j' < i \vee (j < \ell \wedge j' = i)\}.$$

Our aim is to find a direct path $P_{\ell,i}$ between v_ℓ and v_i such that $P_{\ell,i}$ does not traverse any branch set of $\mathcal{W} \setminus \{K_\ell, K_i\}$.

For every neighbour $z \in N_{K'}(v_i)$, find a path P_z in K' which starts with v_i and contains z such that all vertices except for the last vertex of P_z are contained in K_i. Such a path P_z exists for every neighbour z by minimality of K'. Let $K_{i(z)}$ be the branch set of K' in which contains the last vertex of P_z. Note that the branch sets $K_{i(z)}$ for $z \in N_{K'}(v_i)$ are pairwise distinct, again by minimality of K'. Since $N_{K'}(v_i)$ is uncountable by the choice of v_i but $\mathcal{W}$ is only countable, there is a neighbour z of v_i, such that $K_{i(z)} \notin \mathcal{W}$. Similarly, find a path $Q_{z'}$ in K' which starts with v_ℓ such that all vertices except for the last vertex of $Q_{z'}$ are contained in K_ℓ and the last vertex is contained in a branch set $K_{i(z')}$ such that $K_{i(z')} \notin \mathcal{W}$. Let w be the last vertex of P_z and let w' be the last vertex of $Q_{z'}$. Since $K_{i(z)}$ and $K_{i(z')}$ are connected and adjacent in K', there is a path P between w and w' in $K_{i(z)} \cup K_{i(z')}$. Now we obtain the path $P_{\ell,i}$ between v_ℓ and v_i by connecting the three paths P_z, P and $Q_{z'}$. □

Lemma 7.7. *Let G be a connected graph and let $U \subseteq V(G)$ be a set of vertices such that G contains a $K^{\aleph_1}$ minor supported by U. Then G has a $K^{\aleph_1}$ minor supported by U with countable branch sets.*

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let K be an $IK^{\aleph_1}$ in G supported by U. By Lemma 7.6, there is a $TK^{\aleph_1}$ in G that is compatible with K and spread in K. Enumerate the branch vertices $\{v_i : i < \omega_1\}$ of the $TK^{\aleph_1}$. Consider the branch set K_i of K such that $v_i \in K_i$. Let $u_i \in U \cap K_i$. Find a u_i–v_i path Q_i in K_i. Additionally, find the direct paths $P_{j,i}$ between v_j and v_i in the $TK^{\aleph_1}$ for all $j < i < \aleph_1$. Our aim is to construct a $K^{\aleph_1}$ minor of G supported by U with countable branch sets. For this, we recursively construct sets of vertices V_i of G such that

(i) V_i is countable for every $i < \omega_1$,

(ii) V_i is connected for every $i < \omega_1$,

(iii) $v_i \in V_i$ for every $i < \omega_1$,

(iv) there is a $u \in U$ with $u \in V_i$,

(v) $V_i \cap V_j = \emptyset$ for $i \neq j < \omega_1$,

(vi) V_i and V_j are pairwise adjacent for $i \neq j < \omega_1$,

(vii) V_i only contains vertices of the direct paths $P_{j,i}$ for $j < i$ and vertices of the path Q_i.

Then the minor of G with branch sets V_i is the desired $K^{\aleph_1}$ minor.

For $i = 0$, simply define $V_0 := Q_0$. Indeed, V_0 satisfies the properties (i) – (vii). For $i > 0$, suppose that we have already defined the branch sets V_j with $j < i$ such that the properties (i) – (vii) from above hold. Let $\tilde{P}_{j,i}$ be the maximal subpath of $P_{j,i}$ with the property that it contains v_i and such that it is disjoint to V_j. Then, define

$$V_i := Q_i \cup \bigcup_{j<i} \tilde{P}_{j,i}.$$

We show that V_i satisfies the properties (i) – (vii): For property (i) note that V_i is countable, since we only consider countable unions of finite sets. Furthermore it is easy to see that the properties (ii) – (iv) and (vii) are true.

For property (v) we show that V_i is disjoint to V_j for all $j < i$. Thus let $j < i$. By (vii), V_j only contains vertices of the direct paths $P_{k,j}$ for $k < j$ and of the path Q_j. The paths $\tilde{P}_{\ell,i}$ for $\ell < i$ are disjoint to all vertices of V_j which are contained in paths $P_{k,j}$ for $k < j$ because they can only possibly intersect in the path $\tilde{P}_{j,i}$. However, $\tilde{P}_{j,i}$ is disjoint to V_j by definition. Next, it is clear that Q_i is disjoint to Q_j because Q_i is contained in K_i and Q_j in K_j. Furthermore, Q_i is also disjoint to all vertices of V_j which are contained in paths $P_{k,j}$ for $k < j$ because these paths do not intersect K_i by the definition of spread. By the same arguments, it follows that Q_j is disjoint to the paths $\tilde{P}_{\ell,i}$ for $\ell < i$. Hence V_i is indeed disjoint to V_j.

Finally, for property (vi) note that V_i is adjacent to V_j by definition of $\tilde{P}_{j,i}$ and because $v_j \in V_j$. □

7.3 T has Countable Branches and Branch Sets

We now prove the main result of the section:

Lemma 7.8. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let T be a normal semi-partition tree for G supported by U. Let every minor of G supported by U with countable branch sets not contain an uncountable clique minor supported by U. Then*

(i) all branches of T are at most countable.

(ii) for all $t \in T$ the branch sets V_t in G are at most countable.

Pitz has already shown an analogous result in [17, Lemma 3.5] for normal partition trees. We use the same proof idea as he did.

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let T be a normal semi-partition tree for G supported by U. Let every minor of G supported by U with countable branch sets not contain an uncountable clique minor supported by U.

(i): Suppose for a contradiction that there is an uncountable branch B in T. Now, ω_1 is isomorphic to an initial segment of B, because B cannot be isomorphic to a proper initial segment of ω_1, since B is uncountable. Thus it exists an initial segment B' of B such that $B' \cong \omega_1$. Now the vertices V_t with $t \in B'$ form an ω_1-graph as a subgraph of $\dot{G}$. But then, by [8, Proposition 3.5] we find a $K^{\aleph_1}$ minor. Since $\dot{G}$ is a minor of G supported by U, the ω_1-graph as a subgraph (and thus minor) and the $K^{\aleph_1}$ minor are minors of G supported by U as well. But then by Lemma 7.7 there is a $K^{\aleph_1}$ minor supported by U with countable branch sets in G. A contradiction.

(ii): By the definition of a normal semi-partition tree it is true that either $|V_t|$ is finite or $|V_t| = \mathrm{cf}(\mathrm{height}(t))$ for all $t \in T$. By the first part of this Lemma, $\mathrm{height}(t)$ is countable for all $t \in T$. Since the cofinality of a countable ordinal is countable, indeed $|V_t|$ is countable, too.

$\square$

Corollary 7.9. *If the properties of Lemma 7.8 are true, $\dot{G}$ has countable branch sets.*

We have now learned a lot about normal semi-partition trees and will return to them later. In the next section, however, we will not encounter them for the moment. Instead, we will now deal with finite adhesion towards U:

8 Finite Adhesion and Finite Adhesion Towards U

Pitz uses the concept of finite adhesion in his paper. (See also [17]). We will also make use of this, but mainly towards U. We have already mentioned the term in Section 3.3. We will now define both terms again formally and give a useful lemma.

8.1 Definitions

Definition 8.1 (Adhesion Set)**.** Let G be a graph. Let H be a subgraph of G. We call a set of vertices $A \subseteq V(H)$ an adhesion set of H in G is there is a component D of $G - H$ such that $A = N(D)$.

To repeat Definition 2.3:

Definition 8.2 (Finite Adhesion)**.** Let G be a graph. Let H be a subgraph of G. We say that H has finite adhesion in G if, and only if for all components D of $G - H$ holds that $N(D)$ is finite.

In other words, H has finite adhesion in G if, and only if all adhesion sets of H in G are finite.

Definition 8.3 (U-Adhesion Set)**.** Let G be a graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let H be a subgraph of G. We call a set of vertices $A \subseteq V(H)$ an U-adhesion set of H in G is there is a U-component D of $G - H$ such that $A = N(D)$.

Definition 8.4 (Finite Adhesion towards U)**.** Let G be a graph. Further, let $U \subseteq V(G)$ be a set of vertices of G. Let H be a subgraph of G. We say that H has finite adhesion in G towards U if, and only if for all U-components D of $G - H$ holds that $N(D)$ is finite.

In other words, H has finite adhesion in G towards U if, and only if all U-adhesion sets of H in G are finite.

8.2 Finite Adhesion Towards U is Closed under Uncountable Unions

Lemma 8.5. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let $\{G_i \subseteq G : i < \omega_1\}$ be an increasing sequence of subgraphs with finite adhesion in G towards U. Then*

$$G' := \bigcup_{i<\omega_1} G_i$$

has finite adhesion in G towards U.

A similar statement was already remarked in [17, Remark 3.2].

Proof. We prove the statement by contraposition. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let $\{G_i \subseteq G : i < \omega_1\}$ be an increasing sequence of subgraphs. Let $G' := \bigcup_{i<\omega_1} G_i$. Let D be a U-component of $G - G'$ with

$$|N(D) \cup V(G')| = \infty.$$

We show that there is an $i_0 < \omega_1$ such that

$$|N(D) \cap V(G_{i_0})| = \infty.$$

Recursively define a sequence i_n for $n \in \mathbb{N}$: First, let $i_1 := 0$. In the case that $|N(D) \cap V(G_{i_1})| = \infty$, we stop the recursion and define $i_0 := i_1$. Otherwise, we have to continue. Next, suppose that we still have to continue after step n. Our aim is to find $i_{n+1} < \omega_1$ such that

$$|N(D) \cap V(G_{i_n})| < |N(D) \cap V(G_{i_{n+1}})|.$$

Hence, suppose for a contradiction that i_{n+1} does not exist. Thus it follows that $|N(D) \cap V(G_i)|$ is bounded by $|N(D) \cap V(G_{i_n})|$. Furthermore

$$N(D) \cap V(G_{i_n}) \subseteq N(D) \cap V(G_j)$$

for all $j \leq i_n$. Together this implies

$$N(D) \cap V(G_{i_n}) = N(D) \cap V(G_j)$$

for all $j \leq i_n$. We conclude that

$$\begin{aligned}
\infty = |N(D) \cap V(G')| &= \left| N(D) \cap V\left(\bigcup_{i<\omega_1} G_i\right)\right| \\
&= \left| N(D) \cap \bigcup_{i<\omega_1} V(G_i)\right| = \left|\bigcup_{i<\omega_1} (N(D) \cap V(G_i))\right| \\
&= |N(D) \cap V(G_{i_n})| < \infty.
\end{aligned}$$

Assuming we never run into the trivial case, we now have a sequence $(i_n)_{n\in\mathbb{N}}$ with $i_n < \omega_1$ such that $|N(D) \cap V(G_{i_n})| < \infty$ is strictly increasing. Since $\mathrm{cf}(\omega_1) = \omega_1$[4] and by the fact that the sequence $(i_n)_{n\in\mathbb{N}}$ has length ω_0, we find an ordinal $i_0 < \omega_1$ which bounds the sequence from above. Thus, it remains to show that $|N(D) \cap V(G_{i_0})| = \infty$. We know that $i_n < i_0$ implies $G_{i_n} \subseteq G_{i_0}$ for all $n \in \mathbb{N}$, so

$$\begin{aligned}
|N(D) \cap V(G_{i_0})| \geq & \left| N(D) \cap V\left(\bigcup_{n\in\mathbb{N}} G_{i_n}\right)\right| \\
&= \left| N(D) \cap \bigcup_{n\in\mathbb{N}} V(G_{i_n})\right| = \left|\bigcup_{n\in\mathbb{N}} (N(D) \cap V(G_{i_n}))\right| \\
&= sup_{n\in\mathbb{N}} |N(D) \cap V(G_{i_n})| = \infty.
\end{aligned}$$

The penultimate equation holds because the sequence of sets $N(D) \cap V(G_{i_n})$ is monotone increasing. $\square$

The aim of the next chapter is to show the Closure Lemma. Here the new concepts of finite adhesion towards U and normal semi-partition trees supported by U will enter.

[4]Note that for a singular cardinal as ω the statement would fail: For example consider the double ladder with every vertex in U.

9 Normal Semi-Partition Trees Supported by U with Finite Adhesion Towards U

9.1 Preparation

Theorem 9.1. *The following do not have countable colouring number:*

(*i*) *Barricades*

(*ii*) *Aronszajn tree-graphs*

For more details, see [16, Lemma 2.4] and [8, Theorem 7.1], respectively.

Lemma 9.2. *Let G be a connected graph. Let T be a normal semi-partition tree for G. Let $T' \subseteq T$ be down-closed. Let D be a component of $G - G(T')$ with $D \cap G(T) \neq \emptyset$. Then there exists a unique T-minimal element t_D of*

$$t(D) := \{t(v) : v \in V(D) \cap V(G(T))\}.$$

Proof. Let G be a connected graph. Let T be a normal semi-partition tree for G. Let $T' \subseteq T$ be down-closed. Let D be a component of $G - G(T')$ with $D \cap G(T) \neq \emptyset$. Suppose for a contradiction that there are two minimal elements t_D, t'_D of $t(D)$. Then $\lfloor t_D \rfloor$ and $\lfloor t'_D \rfloor$ are disjoint. By connectedness of D there is a $G(\lfloor t_D \rfloor)$–$G(\lfloor t'_D \rfloor)$ path P in D. By following P, we obtain a sequence of points $t_1, t_2, \dots t_n$ in T such that $t_1 \in \lfloor t_D \rfloor$ and $t_n \in \lfloor t_{D'} \rfloor$, we have that t_i and t_{i+1} are comparable for all $i < n$. This sequence must contain a point of $\lceil t_D \rceil \cap \lceil t'_D \rceil$. However, by minimality of t_D and t'_D, it follows that P avoids $G(\lceil t_D \rceil) \cap G(\lceil t'_D \rceil)$. This is a contradiction. □

Lemma 9.3. *Let G be a connected graph. Let T be a normal semi-partition tree for G. Let $T' \subseteq T$ be down-closed. Let D be a component of $G - G(T')$ with $D \cap G(T) \neq \emptyset$. Let t_D be the unique minimal element of*

$$\{t(v) : v \in V(D) \cap V(G(T))\}$$

given by Lemma 9.2. Then

$$T'' := T' \cup \{t_D\}$$

is down-closed.

Proof. Let G be a connected graph. Let T be a normal semi-partition tree for G. Let $T' \subseteq T$ be down-closed. Moreover, let D be a component of $G - G(T')$ with $D \cap G(T) \neq \emptyset$. Let t_D be the unique T-minimal element of $\{t(v) : v \in V(D) \cap V(G(T))\}$ given by Lemma 9.2. Let

$$T'' := T' \cup \{t_D\}.$$

We show that between T' and t_D there is no other element of T. In other words, we show that for all $t \in T$, if $t < t_D$ then $t \in T'$. For a contradiction we assume that between t_D and T' there is a $t \in T$, i.e. it exists a $t \in T$ such that $t < t_D$ and $t \notin T'$. Since the lower neighbours of t_D are cofinal in $\lceil \mathring{t}_D \rceil$ and $t \in \lceil \mathring{t}_D \rceil$ we find a lower neighbour s of t_D with $t \leq s$. Now, since T' is down-closed, $t \leq s$ and $t \notin T'$, also $s \notin T'$. Next, our aim is to show that the choice for t_D was not minimal and that s would have been the correct choice. For this we have to show that D has a vertex $v \in V_s$. We show stronger that $V_s \subseteq D$. First notice that V_s is disjoint to $G(T')$. By connectedness, indeed V_s is contained in a component $D' \subseteq G - G(T')$. In contrast to t we know that for s there exists an V_{t_D}–V_s edge in G. Observe by the same argument as before that $V_{t_D} \subseteq D$ since it is disjoint to $G - G(T')$ as well. Hence, $D = D'$. A contradiction to the minimal choice of t_D. □

Now we come to the Closure Lemma. It is called like this because we close a set under desirable properties.

9.2 Closure Lemma

Lemma 9.4. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Let T be a normal semi-partition tree for G supported by U with $U \subseteq G(T)$. Let $X \subseteq T$ be an infinite set. Then X is included in a rooted (normal semi-partition) subtree $T' \subseteq T$ with $|X| = |T'|$ such that $G(T')$ has finite adhesion in G towards U.*

This lemma is also an analogue of [17, Lemma 3.7]. In the proof we will benefit for the first time from all the preliminary work. This is because we construct barricades supported by U and Aronszajn trees supported by U to obtain a contradiction to Theorem 9.1. After all, we have already understood these constructions sufficiently well in the previous sections. Additionally, the plan is to build T' in a normal semi-partition tree T, about which we also studied in detail. The lemma itself will then be of great help to us in proving the Decomposition Lemma 10.2. We follow the proof idea of Pitz:

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Let T be a normal semi-partition tree for G supported by U with $U \subseteq G(T)$. Let $X \subseteq T$ be infinite.

We recursively build a $\subseteq$-increasing sequence $\{T_i : i < \omega_1\}$ of rooted (normal semi-partition) subtrees of T by letting $T_0 = \lceil X \rceil_T$. For successor steps, suppose that T_i is already defined. Let D be a U-component of $G - G(T_i)$. Since T is supported by U, we have $D \cap G(T) \neq \emptyset$. Let t_D be the unique T-minimal element of $\{t(v) : v \in V(D) \cap V(G(T))\}$ given by Lemma 9.2. We define

$$T_{i+1} := T_i \cup \{t_D : D \text{ is a } U\text{-component of } G - G(T_i) \text{ with } |N(D)| = \infty\},$$

which then is down-closed by Lemma 9.3. Further, let

$$T_\ell := \bigcup_{i<\ell} T_i$$

for limit ordinals $\ell < \omega_1$. In the end, we set

$$T' := \bigcup_{i<\omega_1} T_i.$$

By construction, T' is a rooted (normal semi-partition) subtree of T including X.

To see that $G(T_i)$ has finite adhesion in G towards U, suppose for a contradiction that there is a U-component D of $G - G(T')$ with $|N(D)| = \infty$. Since $\mathrm{cf}(\omega_1) = \omega_1 > \omega_0$, there is some $i_0 < \omega_1$ such that $|N(D) \cap G(T_{i_0})| = \infty$. Hence for all $i_0 \leq i < \omega_1$, the unique U-component[5] D_i of $G - G(T_i)$ containing D also satisfies $|N(D_i) \cap G(T_i)| = \infty$, because $N(D) \cap G(T_{i_0}) \subseteq N(D_i) \cap G(T_i)$. Then $\{t_{D_i} : i_0 \leq i < \omega_1\}$ forms an uncountable chain in T, which gives rise to the existence of an uncountable branch in T. A contradiction to Lemma 7.8.

It remains to show that $|T'| = |X|$. Observe that since T contains no uncountable chains by Lemma 7.8, we have $|T_0| = |X|$. We now prove by transfinite induction on $i < \omega_1$ that $|T_i| = |X|$. The cases where i is a limit are clear, because they are countable. Thus, suppose $i = j + 1$. By induction hypothesis, $|T_j| = |X|$. We show that $|T_i| = |T_j| = |X|$. Suppose for a contradiction that $|T_i| > |T_j|$. We construct a minor of G with countable branch sets that is a barricade (A, B) such that the B-side is supported by U[6]. Define

$$A := V(G(T_j)).$$

For B, consider all U-components D of $G - G(T_j)$ with $t_D \in T_i - T_j$. By definition of t_D it is true that $|N(D)| = \infty$. Let $N \subseteq N(D)$ be a countable subset of $N(D)$. Find for every $n \in N$ a neighbour $d_n \in N(n) \cap D$. Also, let $u_D \in U \cap D$. Then, u_D and all d_n are at most countable many vertices in D. Find a tree T_D in D of countable size that contains these vertices. Finally, define B by contracting the trees T_D. Since every tree T_D in $G - G(T_j)$ contains at least one vertex of U, the B-side is supported by U. Remember

[5] Indeed, there is a $u \in U$ such that $u \in D$ because of the choice of t_D.

[6] i.e. for every vertex $b \in B$ the corresponding branch set in G contains a vertex of U

by Lemma 7.8 that $|G(T_j)| = |T_j|$. Then by assumption,

$$\begin{aligned}|A| = |G(T_j)| = |T_j| &< |T_i - T_j| \\ &= \big|\{t_D \in T_i : t_D \text{ was added in step } i\}\big| \\ &\leq \big|\{t_D \in T_i - T_j\}\big| = |B|.\end{aligned}$$

For the minor we only keep the edges between both sides. Thus, the minor is bipartite. By only adding t_D to T_j, that are in U-components D of $G - G(T_j)$ with $|N(D)| = \infty$, and by keeping the infinite degree with the construction of the trees T_D, every vertex $b \in B$ has infinitely many neighbours. Thereby we find a barricade (A, B) as a minor in G with countable branch sets such that the B-side is supported by U. Using Lemma 5.5, we also find a barricade as a minor of G with countable branch sets such that the minor is supported by U. This minor has countable colouring number by assumption. A contradiction to Theorem 9.1(i).

In the case that X is uncountable, observe that

$$|T'| = \left|\bigcup_{i<\omega_1} T_i\right| \leq \aleph_1 \cdot |X| = |X|.$$

Also $|T'| \geq |X|$, since $X \subseteq T'$. Thus we have $|T'| = |X|$. For the other case, i.e. if X is countable, we also have

$$|T'| = \left|\bigcup_{i<\omega_1} T_i\right| \leq \aleph_1 \cdot |X|.$$

Suppose for a contradiction that $|T'| = \aleph_1$. Since $T_0 \subseteq T'$ is a rooted (normal semi-partition) subtree we have that $\dot{G}[T_0]$ is connected as shown in Remark 6.11. Then, construct T'' with root r by contracting the rooted (normal semi-partition) subtree T_0 to a vertex r in T'. This contraction results in a minor G'' of $\dot{G}$. Since X is countable by assumption and by Lemma 7.8, also T_0 is countable. Hence G'' has countable branch sets in $\dot{G}$ and because of Lemma 7.8 also in G. Call T'' the order tree of G''. Since T' is normal in G, also T'' is normal in G''. By the fact that for a $s \in T'$ the set of lower neighbours in G is cofinal in $\lceil \mathring{s} \rceil$, we have that for any $s \in T''$ the set of lower neighbours in G'' is cofinal in $\lceil \mathring{s} \rceil$. Thus G'' is a T''-graph. Since $\dot{G}$ is a minor supported by U, also G'' is a minor supported by U. In the 0th level of T'' is only the root. Additionally, by construction and by Lemma 9.3, points in $T_i - \bigcup_{j<i} T_j$ for $i \geq 1$ belong to the ith level of T''. Since

$$\left| T_i - \bigcup_{j<i} T_j \right| \leq |T_i| = |X|$$

and X is countable (in this case), all levels of T'' are countable. In the end, since T'' like T' and T contains no uncountable chains, if follows that T'' is an Aronszajn tree such that the branch sets of G'' are countable. Since $G'' \preccurlyeq \dot{G} \preccurlyeq G$, there is an Aronszajn tree minor of G with countable branch sets. This minor is supported by U, because it is a minor of $\dot{G}$, which is a minor supported by U. Thus it has countable colouring number by assumption. A contradiction to Theorem 9.1(ii). $\square$

Finally, after all the preliminary work, we turn to the section about the Decomposition Lemma:

10 Decomposition Lemma

10.1 Motivation

In [17], Pitz decomposes the graph into induced subgraphs, so that these have finite adhesion in G. We also want to do this, but Pitz's precondition is stronger: He assumes that all minors of G with countable branch sets have countable colouring number. In the following example we see that our precondition, i.e. that only the minors of G supported by U with countable branch sets have countable colouring number for a fixed set of vertices U of G, is too weak. In fact, we stronger show that the precondition is also too weak if all minors of G supported by U have countable colouring number:

Example 10.1. There is a graph G and a set of vertices $U \subseteq V(G)$ with the property that all minors of G supported by U have countable colouring number and such that there is no normal semi-partition tree T supported by U such that $U \subseteq G(T)$ and $G(T)$ has finite adhesion.

Proof. We consider the cases where U is countable or uncountable. Note that there is no finite U with the properties from above.

U countable: Consider $G := K^{\aleph_1}$ and let U be an arbitrary countable set of vertices of G. Then, a minor of G supported by U has at most countable many vertices and hence has countable colouring number. Let T be a normal semi-partition tree that is supported by U such that $U \subseteq G(T)$. Then, T has at most countable many branch sets, since U is countable. Additionally, every branch set is at most countable by definition of normal semi-partition trees. Hence, T cannot have finite adhesion.

U uncountable: Now, consider a vertex v and uncountably many disjoint copies $(K_i : i < \omega_1)$ of a $K^{\aleph_1}$. From each $K_i^{\aleph_1}$, choose a vertex v_i and add a v–v_i edge. Call the constructed graph G. Now, let U be an uncountable set of vertices of $G - v$ such that for all $i < \omega_1$ it is true that $K_i^{\aleph_1}$ contains countably many vertices of U.

Let H be a minor of G supported by U. We define a well-order on $V(H)$ that shows that H has countable colouring number. If H has a branch set X_t such that $v \in X_t$, let t be the first element in the well-order. Next, let $i < \omega_1$. Define

$$S_i := \{s \in V(H) : V_s \subseteq V(K_i^{\aleph_1})\}.$$

Find a well-order on S_i. As in the upper case, the well-order on S_i has the countable colouring number property. Now concatenate all the well-orders the other according to the order of their indices, leaving t as the first element of the well-order. Indeed, this is a well-order of $V(H)$ showing that H has countable colouring number.

Additionally, note that H does not contain an uncountable clique minor supported by U. Let T be a normal semi-partition tree that is supported by U such that $U \subseteq G(T)$. By Lemma 7.8, every branch set of T is countable. Since $U \subseteq G(T)$, for every $i < \omega_1$ there is a branch set V_t of T such that $V_t \cap V(K_i^{\aleph_1}) \neq \emptyset$. In other words, T goes in every $K_i^{\aleph_1}$. Hence T does not have finite adhesion as before. □

Consequently, we will only make a weaker conclusion. In fact, we will show that we can find a decomposition in graphs with finite adhesion towards U. In other words, we can only say something about neighbourhoods of U-components. We will see later, however, that we are only interested in these. Additionally, we will also use another property, which is listed below. We start analogously to [17] with the more general and expanded Decomposition Lemma:

10.2 Decomposition Lemma for T-Graphs

Lemma 10.2. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Let T be a normal semi-partition tree for G supported by U with $U \subseteq G(T)$. Then T can be written as a continuous increasing union*

$$T = \bigcup_{i<\mathrm{cf}(\kappa)} T_i$$

of infinite, $< \kappa$-sized rooted (normal semi-partition) subtrees T_i such that

(i) all graphs $G(T_i)$ have finite adhesion in G towards U,

(ii) for every U-adhesion set S of $G(T_i)$ in G there are infinitely many U-components D of $G - G(T_i)$ with $N(D) = S$.

We again use the same proof idea as Pitz in [17, Lemma 3.6], but adapted to our situation. Furthermore, we extend the proof so that the property (ii) holds. Lemma 9.4 is used here frequently:

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Let T be a normal semi-partition tree for G supported by U with $U \subseteq G(T)$.

By the second part of Lemma 7.8, we have $|G(T)| = |\dot{G}|$. Since $U \subseteq G(T)$, it follows that

$$\kappa = |U| \leq |G(T)| = |\dot{G}| = |T|.$$

Since $\dot{G}$ is supported by U, there is for every $t \in T$ a vertex $u \in U$ such that u is contained in the branch set V_t of $G(T)$. Thus there are at most κ many branch sets in $G(T)$. Hence

$$\kappa = |U| \geq |\dot{G}| = |T|.^{7}$$

Summarised we have that $\kappa = |T|$. Fix an enumeration $\{t_i : i < \kappa\}$ of the points of T.

We distinguish between the two cases whether κ is a regular or a singular cardinal:

[7] By the second part of Lemma 7.8 we even get $|U| \geq |G(T)|$.

κ is a regular uncountable cardinal: For the first case, suppose that κ is a regular uncountable cardinal. Consider $\dot{G}$. Since T is supported by U, it follows that $\dot{G}$ is a minor of G supported by U. With Lemma 7.8 it follows that $\dot{G}$ has countable branch sets. Hence by assumption, $\dot{G}$ has countable colouring number. Find a well-order $\dot{\leq}$ of $V(\dot{G})$ of order type $|\dot{G}|$[8] witnessing that $\dot{G}$ has countable colouring number, i.e. such that for every vertex $v \in V(\dot{G})$ there are at most finitely many neighbours $u \dot{\leq} v$.

We recursively define a continuous increasing sequence $\{T_i : 0 \leq i < \kappa\}$ such that

(a) T_i is an infinite rooted normal semi-partition subtree of T for all $0 \leq i < \kappa$[9],

(b) $t_i \in T_{i+1}$ for all $0 \leq i < \kappa$,

(c) the points of T_i form a proper initial segment of $(V(\dot{G}), \dot{\leq})$ for all $0 \leq i < \kappa$,[10]

(d) for all $0 \leq i < \kappa$ and for every finite subset $S \subseteq V(G(T_i))$ there are either 0 or κ many U-components D of $G - G(T_i)$ with $N(D) = S$,

(e) $G(T_i)$ has finite adhesion in G towards U for all $0 \leq i < \kappa$.

Then T_i is as desired for all $0 \leq i < \kappa$.

First, define $T_{-1} := \lceil \{t_i : i < \omega\} \rceil_T$. Then $T_{-1} \subseteq T$ is an infinite set of size $\omega < \kappa$. Also, define $t_{-1} := \emptyset$.

Now, suppose that T_i is defined and for all $0 \leq j \leq i$ suppose that T_j satisfies the properties (a) – (e). If $t_i \in T_i$, simply define $T_{i+1} := T_i$. Otherwise, let $T_i^0 := T_i \cup \lceil t_i \rceil$. For $n \in \mathbb{N}$ suppose that we have already defined T_i^{3n}. Use Lemma 9.4 to define a rooted (normal semi-partition) subtree $T_i^{3n+1} \subseteq T$ that contains T_i^{3n} and with $|T_i^{3n}| = |T_i^{3n+1}|$ such that $G(T_i^{3n+1})$ has finite adhesion in G towards U. Further, for $n \in \mathbb{N}$ suppose that we have already defined T_i^{3n+1}. Let T_i^{3n+2} be the smallest (normal semi-partition) subtree of T including the down-closure of T_i^{3n+1}

[8] see e.g. [9, Corollary 2.1]

[9] and thus supported by U

[10] Then $|T_i| < \kappa$ for all $0 \leq i < \kappa$.

in $(V(\dot{G}), \dot{\leq})$.[11] Now suppose that for $n \in \mathbb{N}$ we have already defined T_i^{3n+2}. Let $\mathcal{D}$ be the set of all U-components D of $G - G(T_i^{3n+2})$ for which $N(D)$ is finite and there exist only less than κ many U-components D' of $G - G(T_i^{3n+2})$ with $N(D) = N(D')$. For every $D \in \mathcal{D}$, choose a vertex $u_D \in D \cap U$ and let t_D be the point in T such that V_{t_D} contains u_D. Let T_i^{3n+3} be the smallest (normal semi-partition) subtree of T which contains $T_i^{3n+2} \cup \{t_D : D \in \mathcal{D}\}$.

In the end, define

$$T_{i+1} := \bigcup_{n \in \mathbb{N}} T_i^n.$$

By construction, T_{i+1} is a rooted (normal semi-partition) subtree of T with $t_i \in T_{i+1}$ and hence the properties (a) and (b) are satisfied.

For property (c), we have to show that the points of T_{i+1} form a proper initial segment of $(V(\dot{G}), \dot{\leq})$. First we show that the points of T_{i+1} form an initial segment of $(V(\dot{G}), \dot{\leq})$. Now, let $t \in T_{i+1}$. Then there is an $0 \leq k < \kappa$ such that $t \in T_k$. Choose k minimal with that property. Then $k =: j + 1$ is a successor or zero such that we constructed T_j^n for all $n \in \mathbb{N}_0$. For T_j find an $n \in \mathbb{N}$ such that $t_i \in T_j^{3n+2}$. By construction, all preceding points of t with reference to $\dot{\leq}$ are contained in T_j^{3n+2}. To see that this initial segment is proper, we first show by induction on n that $|T_i^n| < \kappa$. Since $|T_i| < \kappa$ it follows with Lemma 7.8 that $|T_i^0| < \kappa$. Now suppose that $|T_i^{3n}| < \kappa$. Then $|T_i^{3n+1}| < \kappa$ by Lemma 9.4. Next, observe that the points of T_i^{3n+1} cannot be a cofinal chain in $(V(\dot{G}), \dot{\leq})$, since $|T_i^{3n+1}| < \kappa = \mathrm{cf}(\kappa)$. This follows because κ is regular by assumption. Hence the down-closure of T_i^{3n+1} in $(V(\dot{G}), \dot{\leq})$ has size $< \kappa$. Now take the down-closure of these points in T. This defines the rooted (normal semi-partition) subtree T_i^{3n+2} of T of size $< \kappa$.[12] Next, we show that also $|T_i^{3n+3}| < \kappa$. This follows because $|T_i^{3n+2}| < \kappa$ and therefore there are only $< \kappa$ finite subsets of $V(G(T_i^{3n+2}))$. Thus the set $\mathcal{D}$ from the construction of T_i^{3n+3} has size $< \kappa$ and it follows that also $|T_i^{3n+3}| < \kappa$. This proves that $|T_i^n| < \kappa$ for all $n \in \mathbb{N}$.

Since κ is a regular uncountable cardinal, it has uncountable cofinality.

[11]For every $t \in T_i^{3n+1}$ add all $t' \in T$ if $t' \dot{\leq} t$.

[12]Since the down-closure of $< \kappa$ many points is still $< \kappa$, because the branches are countable.

Since T_{i+1} is the countable union of all T_i^n and all of them are of size $< \kappa$, indeed $|T_{i+1}| < \kappa$. Thus the initial segment is proper.

Now we show property (d). Let S be a finite subset of $V(G(T_{i+1}))$ and let $\mathcal{D}_S$ be the set of all U-components D of $G - G(T_{i+1})$ with $N(D) = S$. We have to show that $|\mathcal{D}_S| = 0$ or $|\mathcal{D}_S| = \kappa$. Since S is finite, there exists $n \in \mathbb{N}$ such that $S \subseteq V(G(T_i^{3n+2}))$. Let $\mathcal{D}'_S$ be the set of all U-components D of $G - G(T_i^{3n+2})$ with $N(D) = S$. If $|\mathcal{D}'_S| = \kappa$, then also $|\mathcal{D}_S| = \kappa$ since $|V(G(T_{i+1}))| < \kappa$ by (c). Now suppose that $|\mathcal{D}'_S| < \kappa$. Let $\mathcal{D}''_S$ be the set of all U-components D of $G - G(T_i^{3n+3})$ with $N(D) = S$ and note that $\mathcal{D}_S \subseteq \mathcal{D}''_S$ since $S \subseteq V(G(T_i^{3n+2})) \subseteq V(G(T_i^{3n+3}))$. Then we have that $\mathcal{D}''_S = \emptyset$: Indeed, if D is any component in $\mathcal{D}''_S$, then D is also contained in $\mathcal{D}'_S$ because $S \subseteq V(G(T_i^{3n+2}))$. However, from $D \in \mathcal{D}'_S$ it follows that $D \notin \mathcal{D}''_S$ by construction of T_i^{3n+3}, a contradiction. Hence $|\mathcal{D}''_S| = 0$ and it follows that $|\mathcal{D}_S| = 0$.

It remains to show property (e), i.e. that $G(T_{i+1})$ has finite adhesion in G towards U. Suppose for a contradiction that there is a U-component D of $G - G(T_{i+1})$ with infinitely many neighbours in $G(T_{i+1})$. Consider

$$t(D) := \{t(v) : v \in V(D) \cap G(T)\}.$$

Fix $u \in U$ such that $u \in D$. By the fact that $U \subseteq G(T)$, it follows that $t(u) \in t(D)$. Thus $t(D) \neq \emptyset$. Let t_D be the unique T-minimal element of $t(D)$ (given by Lemma 9.2).

Claim: t_D is a limit of T.

First note that for all $-1 \leq i < \kappa$ it holds that $T_{i+1} \neq \emptyset$, since the trees are nested and $T_{-1} \neq \emptyset$. This means that $t_D \neq 0$. Next, let $x <_T t_D$. Then it follows that $x \in T_{i+1}$ by Lemma 9.3. Hence there exists an $n \in \mathbb{N}$ such that $x \in T_i^{3n+1}$. Since T_i^{3n+1} is down-closed, we have that for $v \in N(D)$ with $t(v) \leq x$ it follows that $t(v) \in T_i^{3n+1}$. Thus $v \in G\left(T_i^{3n+1}\right)$. Since $v \in N(D)$, we have that $v \in G\left(T_i^{3n+1}\right) \cap N(D)$. Now, $\left|G\left(T_i^{3n+1}\right) \cap N(D)\right|$ must be finite, since $G\left(T_i^{3n+1}\right)$ has finite adhesion in G towards U. Since T_i^{3n+1} is down-closed, it holds that

$$\lceil x \rceil_{T_i^{3n+1}} \subseteq T_i^{3n+1}.$$

Thus

$$G\big(\lceil x\rceil_{T_i^{3n+1}}\big)\cap N(D)\subseteq G(T_i^{3n+1})\cap N(D)$$

and hence $\big|G\big(\lceil x\rceil_{T_i^{3n+1}}\big)\cap N(D)\big|$ is finite, too. In other words, only finitely many neighbours $v\in N(D)$ satisfy $t(v)\leq_T x$. Since we suppose that $|N(D)|$ is infinite, now it follows that at least one neighbour $v\in N(D)$ satisfies $x<_T t(v)<_T t_D$. Hence t_D is a limit.

By the definition of a T-graph, t_D has infinitely many neighbours below it in $\dot{G}$ and hence in T_{i+1}. However, since T_{i+1} forms an initial segment in $(V(\dot{G}),\dot{\leq})$ by (c) that does not contain t_D by the choice of t_D, it follows that t_D is preceded by infinitely many of its neighbours in $\dot{\leq}$. A contradiction to the choice of $\dot{\leq}$. Thus $G(T_i)$ has finite adhesion in G towards U.

Now let $0<\ell<\kappa$ be a limit. Define

$$T_\ell:=\bigcup_{i<\ell}T_i.$$

Then, T_ℓ is an infinite rooted normal semi-partition subtree of T. Indeed, the points of T_ℓ form an initial segment of $(V(\dot{G}),\dot{\leq})$. Further, $G(T_\ell)$ has finite adhesion in G towards U, which can be shown analogously as before. Thus we have that the properties (a) and (e) hold. Moreover, for property (b) there is nothing to show. Property (d) can be proven similarly to the successor case. For property (c) it remains to show that the points of T_ℓ form a proper initial segment of $(V(\dot{G}),\dot{\leq})$. By assumption $\mathrm{cf}(\kappa)=\kappa$ is a regular uncountable cardinal. Thus we have that $\ell<\kappa=\mathrm{cf}(\kappa)$. Since T_ℓ is the union of λ many subtrees T_i with size $<\kappa$, indeed $|T_\ell|<\kappa$. Thus the initial segment is proper.

κ is a singular uncountable cardinal: Now suppose that κ is a singular uncountable cardinal. First, we enumerate $V(T) = \{t_i : i < \kappa\}$ and let $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ be a continuous increasing sequence of cardinals with limit κ, where $\kappa_0 > \mathrm{cf}(\kappa)$ is uncountable. We build a family

$$\{T_{i,j} : i < \mathrm{cf}(\kappa), j < \omega_1\}$$

of infinite rooted (normal semi-partition) subtrees of T with $G(T_{i,j})$ of finite adhesion in G towards U, such that $|T_{i,j}| = \kappa_i$ for all i and j. For this, we do a nested recursion on i and j. When we come to choose $T_{i,j}$, we will already have chosen all $T_{i',j'}$ with $j' < j$, or with both $j' = j$ and $i' < i$. Whenever we have just selected such a subtree $T_{i,j}$, we fix immediately an enumeration $\{t^k_{i,j} : k < \kappa_i\}$ of this tree. We impose the following conditions for all $i < \mathrm{cf}(\kappa)$ and $j < \omega_1$ on this construction:

(*a*) $A_i := \{t_k : k < \kappa_i\} \subseteq T_{i,0}$,

(*b*) $B_{i,j} := \bigcup \{T_{i',j'} : i' \leq i, j' \leq j, (i',j') \neq (i,j)\} \subseteq T_{i,j}$,

(*c*) $C_{i,j+1} := \{t^k_{i',j} : k < \kappa_i, i < i' < \mathrm{cf}(\kappa)\} \subseteq T_{i,j+1}$.

(*d*) Let $\mathcal{D}$ be the set of all U-components D of $G - G(T_{i,j})$ for which $N(D)$ is finite and there exist at most κ_i many U-components D' of $G - G(T_{i,j})$ with with $N(D) = N(D')$. For every $D \in \mathcal{D}$, choose a vertex $u_D \in D \cap U$ and let t_D be the point in T such that V_{t_D} contains u_D. Then

$$D_{i,j+1} := \{t_D : D \in \mathcal{D}\} \subseteq T_{i,j+1}.$$

These three conditions specify a vertex subset $X_{i,j} \subseteq T$ which has to be included in $T_{i,j}$. Using Lemma 9.4, we let $T_{i,j}$ be a (normal semi-partition) subtree that contains $X_{i,j}$ and that has the same size as $X_{i,j}$. Then, $G(T_{i,j})$ has finite adhesion in G towards U for all $i < \mathrm{cf}(\kappa)$ and $j < \omega_1$.

Claim: For all $i < \mathrm{cf}(\kappa)$ and $j < \omega_1$ we have $|T_{i,j}| = \kappa_i$.

We prove the claim by a nested transfinite induction on i and j. Suppose that $|T_{i',j'}| = \kappa_{i'}$ for all i', j' with $j' < j$, or with both $j' = j$ and

$i' < i$. We have to show that $|T_{i,j}| = \kappa_i$ and by Lemma Lemma 9.4 it suffices to show that $|X_{i,j}| = \kappa_i$.

If $j = 0$, then $X_{i,0} = A_i \cup B_{i,0}$. Indeed, $|A_i| = |\{t_k : k < \kappa_i\}| = \kappa_i$. We also have $|B_{i,j}| = |\bigcup\{T_{i',0} : i' < i\}| \leq \kappa_i$ because $|T_{i'}| = \kappa_{i'}$ for all $i' < i$ and the $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ sequence is increasing. Thus it holds that $|X_{i,0}| = \kappa_i$.

Now, let $0 < j < \omega_1$ and suppose that $j = h + 1$ is a successor ordinal, then $X_{i,j} = B_{i,j} \cup C_{i,j} \cup D_{i,j}$. Then

$$|B_{i,j}| = \left|\bigcup\{T_{i',j'} : i' \leq i, j' \leq j, (i', j') \neq (i, j)\}\right| \leq \kappa_i$$

because the $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ sequence is increasing and

$$|C_{i,j}| = \left|\{t^k_{i',h} : k < \kappa_i, i < i' < \mathrm{cf}(\kappa)\}\right| = \kappa_i \cdot \mathrm{cf}(\kappa) = \kappa_i.$$

It is clear that $|D_{i,j}| \leq \kappa_i$. Thus, $|X_{i,j}| = \kappa_i$.

Finally, let $0 < j < \omega_1$ and suppose that j is a limit ordinal, then $X_{i,j} = B_{i,j}$. Hence,

$$|B_{i,j}| = \left|\bigcup\{T_{i',j'} : i' \leq i, j' \leq j, (i', j') \neq (i, j)\}\right| \leq \kappa_i$$

as before and therefore $|X_{i,j}| = \kappa_i$. This completes the proof of the claim.

Now, we define

$$T_i := \bigcup_{j<\omega_1} T_{i,j}$$

for all $i < \mathrm{cf}(\kappa)$. We show that for all $i < \mathrm{cf}(\kappa)$ the T_i are infinite, $< \kappa$-sized (normal semi-partition) subtrees of T such that all graphs $G(T_i)$ have finite adhesion in G towards U.

Let $i < \mathrm{cf}(\kappa)$. Since $T_{0,0} \subseteq T_i$ by property (b), indeed $\kappa_0 = |T_{0,0}| \leq |T_i|$. Since κ_0 is infinite, $|T_i|$ is infinite, too. By the fact that for all $j < \omega_1$ it holds that $G(T_{i,j})$ has finite adhesion in G towards U, it follows by Lemma 8.5 that also $G(T_i)$ has finite adhesion in G towards U. To see that the sequence $\{T_i : i < \mathrm{cf}(\kappa)\}$ is continuous, we show in the following that $T_\ell = \bigcup_{i<\ell} T_i$ for all $\ell < \mathrm{cf}(\kappa)$. Let $\ell < \mathrm{cf}(\kappa)$ be a limit and $j < \omega_1$.

Since the $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ sequence is continuous, $\kappa_\ell = \bigcup_{i<\ell} \kappa_i$. Thus

$$\begin{aligned} T_{\ell,j} &= \{t^k_{\ell,j} : k < \kappa_\ell\} = \{t^k_{\ell,j} : k < \textstyle\bigcup_{i<\ell} \kappa_i\} \\ &= \textstyle\bigcup_{i<\ell} \{t^k_{\ell,j} : k < \kappa_i\}. \end{aligned}$$

With property (c), $T_{\ell,j}$ is a subset of $\bigcup_{i<\ell} T_{i,j+1} \subseteq \bigcup_{i<\ell} T_i$. Hence is $T_j \subseteq \bigcup_{i<\ell} T_i$.

Furthermore, for the other inclusion, it is enough to show that the sequence $\{T_i : i < \mathrm{cf}(\kappa)\}$ is increasing. But this follows by property (b).

It is left to show that for all $0 \leq i < \mathrm{cf}(\kappa)$ and for every finite subset S of $V(G(T_i))$ there are either 0 or infinitely many U-components D of $G - G(T_i)$ with $N(D) = S$. The proof works similarly as in the regular case. Let S be a finite subset of $V(G(T_i))$ and let $\mathcal{D}_S$ be the set of all U-components D of $G - G(T_i)$ with $N(D) = S$. We show that $|\mathcal{D}_S| = 0$ or $|\mathcal{D}_S| > \kappa_i$. Since S is finite and $T_i = \bigcup_{j<\omega_1} T_{i,j}$, there exists $j^* < \omega_1$ such that $S \subseteq V(G(T_{i,j^*}))$. Let $\mathcal{D}'_S$ be the set of all U-components D of $G - G(T_{i,j^*})$ with $N(D) = S$. If $|\mathcal{D}'_S| > \kappa_i$, then also $|\mathcal{D}_S| > \kappa_i$ since $|V(G(T_i))| = \kappa_i$. Now suppose that $|\mathcal{D}'_S| \leq \kappa_i$. Let $\mathcal{D}''_S$ be the set of all U-components D of $G - G(T_{i,j^*+1})$ with $N(D) = S$ and note that $\mathcal{D}_S \subseteq \mathcal{D}''_S$ since $S \subseteq V(G(T_{i,j^*})) \subseteq V(G(T_{i,j^*+1}))$. Then we have that $\mathcal{D}''_S = \emptyset$ by (d). It follows that $|\mathcal{D}_S| = 0$.

□

Remark 10.3. Note that in the setting of Lemma 10.2 also the T_i are normal semi-partition subtrees supported by U for all $i < \mathrm{cf}(\kappa)$.

Analogous to [17], the more general and expanded Decomposition Lemma 10.2, which we have just proved, now directly leads to the Decomposition Lemma 10.4, which we will use later in the main proof in Section 12:

10.3 Decomposition Lemma

Lemma 10.4. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Then U can be covered by a continuous increasing union*

$$U \subseteq \bigcup_{i<\sigma} G_i$$

of infinite, $< \kappa$-sized connected induced subgraphs G_i such that

(I) all graphs G_i have finite adhesion in G towards U,

(II) for every U-adhesion set S of G_i in G there are infinitely many U-components D of $G - G_i$ with $N(D) = S$.

The analogous Decomposition Lemma of Pitz can be found in [17, Lemma 3.3].

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of uncountable size κ. Let every minor of G supported by U with countable branch sets have countable colouring number. Let T be a normal semi-partition tree for G supported by U with $U \subseteq G(T)$. Indeed, T exists because of Lemma 6.12. By Lemma 10.2, T can be written as a continuous increasing union

$$T = \bigcup_{i<\mathrm{cf}(\kappa)} T_i$$

of infinite, $< \kappa$-sized rooted (normal semi-partition) subtrees T_i satisfying Lemma 10.2 (i) and Lemma 10.2 (ii).

Define $\sigma := \mathrm{cf}(\kappa)$ and $G_i := G(T_i)$. Then

$$\bigcup_{i<\sigma} G_i = \bigcup_{i<\mathrm{cf}(\kappa)} G(T_i)$$

is a continuous increasing union that covers $G(T)$ and thus also covers U, since $U \subseteq G(T)$. Since for all $i < \sigma$ it is satisfied that T_i is infinite, also G_i is infinite for all $i < \sigma$. By definition of $G(T_i)$, for all $i < \sigma$ we have that G_i is an induced subgraph of G. By Lemma 7.8 and since every $|T_i| < \kappa$ it follows that $|G_i| < \kappa$ for all $i < \sigma$. Finally, property (I) and (II) follow directly from Lemma 10.2 (i) and Lemma 10.2 (ii), respectively. □

Remark 10.5. Consider the same setup as in Lemma 10.4. Remember that every T_i in Lemma 10.2 is supported by U. By the choice of G_i in the proof of Lemma 10.4 it follows that every $\dot{G}_i$ is supported by U, too.

We now come to the definition of $\mathcal{S}_G(U, H)$ for a subgraph H and a set of vertices U of a graph G:

11 $\mathcal{S}_G(U, H)$

11.1 Definition

Definition 11.1. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let $H \subseteq G$ be a subgraph of G. Let

$$N := \bigcup_{D \in \mathcal{D}} N(D),$$

where $\mathcal{D}$ is the set of all U-components of $G - H$. We define the set of vertices

$$\mathcal{S}_G(U, H) := (H \cap U) \cup N.$$

So far we have understood this set as a canonical separator of U and H in G. In the following, we investigate further properties, which we will then use for the main proof in Section 12.

11.2 Direction Lemma

Lemma 11.2. *Let G be a connected graph. Let U be a subset of $V(G)$. Let H and H' be subgraphs of G with $H \subseteq H'$. Let D be a connected subgraph of $G - \mathcal{S}_G(U, H)$ that meets $\mathcal{S}_G(U, H')$. Then D is contained in a U-component of $G - H$.*

Proof. Let G be a connected graph. Let U be a subset of $V(G)$. Further let H and H' be subgraphs of G with $H \subseteq H'$. Let D be a connected subgraph of $G - \mathcal{S}_G(U, H)$ that meets $\mathcal{S}_G(U, H')$. If D contains a vertex of U, it clearly holds that D is contained in a U-component of $G - H$.

If D does not contain a vertex of U, then D must contain a neighbour v of a U-component C of $G - H'$ by definition of $\mathcal{S}_G(U, H')$. Then define $B := G[V(D) \cup V(C)]$. It is true that B is connected because D and C are connected and there is an edge from $v \in D$ to C. It is left to show that B is disjoint to H, then it follows that $D \subseteq B$ is contained in a U-component of $G - H$. Suppose for a contradiction that B meets H. Note that C contains a vertex $u \in G - H' \subseteq G - H$. Since B is connected and contains u, it follows that B meets H in a neighbour of a U-component of

$G - H$. Therefore, B meets $\mathcal{S}_G(U, H)$. However, both D and C are disjoint to $\mathcal{S}_G(U, H)$, a contradiction. □

11.3 Trees Cofinally Containing a Superset of $\mathcal{S}_G(U, H)$ Have Finite Adhesion Towards U

Lemma 11.3. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Let $H \subseteq G$ be a subgraph of finite adhesion in G towards U. Let $X \subseteq V(H)$ be a superset of $\mathcal{S}_G(U, H)$. Let T be a rooted tree in G that cofinally contains X. Then any U-component D of $G - H$ satisfies $|D \cap T| < \infty$.*

Pitz proves a similar auxiliary Lemma in [17, Lemma 4.3]. We are orienting ourselves to his proof idea.

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Moreover, let $H \subseteq G$ be a subgraph of finite adhesion in G towards U. Let $X \subseteq V(H)$ be a superset of $\mathcal{S}_G(U, H)$. Let T be a rooted tree in G that cofinally contains X. Let D be a U-component of $G - H$. Suppose for a contradiction that $|D \cap T| = \infty$. We recursively construct disjoint paths $P_n \subseteq \lfloor d_n \rfloor_T$ with endvertices $d_n \in D \cap T$ and $h_n \in X$. Suppose that the paths $P_1, \ldots, P_n \subseteq T$ have already been constructed. Define

$$Q := \left\lceil \bigcup_{m \leq n} V(P_m) \right\rceil_T .$$

Since for all n we have that P_n is finite and we have finitely many paths, the union of them is finite. Since the down-closure of finitely many vertices in a graph theoretical tree is finite, the down-closure of the union of all P_n is finite as well. Thus Q is finite. By assumption, $D \cap T$ is infinite and hence there is a $d_{n+1} \in (D \cap T) \setminus Q$. Since X is cofinal in T, there is a vertex $h_{n+1} \in X \subseteq V(H)$ above d_{n+1}. Let P_{n+1} be the unique path in T from d_{n+1} to h_{n+1}. Since Q is down-closed, we have $\lfloor d_{n+1} \rfloor \cap Q = \emptyset$. Since $P_{n+1} \subseteq \lfloor d_{n+1} \rfloor$, it follows that P_{n+1} is disjoint from $P_1, \ldots, P_n$.

However, the existence of infinitely many pairwise disjoint paths from D to $\mathcal{S}_G(U, H)$ in G give rise to the existence of infinitely many pairwise disjoint paths from D to H in G. However, this contradicts that H has finite adhesion in G towards U. □

Lemma 11.4. *Let G be a connected graph. Let U be a subset of $V(G)$. Let H be a subgraph of G with finite adhesion in G towards U. Further let $X \subseteq V(H)$ be a superset of $\mathcal{S}_G(U, H)$. Let T be a rooted tree in G that cofinally contains X. Further let C be a component of $G - T$ that is contained in a U-component of $G - H$. Then $|N(C)|$ is finite.*

Again, Pitz proves a similar Lemma in [17, Claim 4.1]. We follow his proof idea.

Proof. Let G be a connected graph. Let U be a subset of $V(G)$. Let H be a subgraph of G with finite adhesion in G towards U. Let $X \subseteq V(H)$ be a superset of $\mathcal{S}_G(U, H)$. Let T be a rooted tree in G that cofinally contains X. Further let C be a component of $G - T$ that is contained in a U-component D of $G - H$. Consider the neighbours of C in T. Observe that

$$N(C) = (N(C) \cap V(H)) \cup (N(C) \setminus V(H)).$$

First note that $N(C) \cap V(H) \subseteq N(D) \cap V(H)$. Since H has finite adhesion in G towards U, we have that $|N(C) \cap V(H)| \leq |N(D) \cap V(H)|$ is finite. In addition, observe that $N(C) \setminus V(H) \subseteq T \cap D$. Use Lemma 11.3 to see that $|T \cap D|$ is finite. Then, $|N(C) \setminus V(H)| \leq |T \cap D|$ is finite as well. All together,

$$|N(C)| \leq |N(C) \cap V(H)| + |N(C) \setminus V(H)|$$

is finite. Hence, T has finite adhesion in G towards U. $\square$

11.4 Minors Supported by $\mathcal{S}_G(U, H)$ Have Countable Colouring Number

Lemma 11.5. *Let G be a graph. Let H be a minor of G with finite branch sets. If H has countable colouring number, then G has countable colouring number.*

Proof. Let G be a graph. Let H be a minor of G with finite branch sets $(X_v : v \in V(H))$ and countable colouring number. Let $\dot{\leq}$ be a well-order of $V(H)$ witnessing that H has countable colouring number. For every $v \in V(H)$, fix a linear order $\leq_v$ of the finite set X_v. We define a well-order $\leq$ of $V(G)$. Let $a, b \in V(G)$ and let v and w be the vertices of H such that $a \in X_v$ and

$b \in X_w$. We define $a \leq b$ if and only if either $v \neq w$ and $v \dot{\leq} w$ or if $v = w$ and $a \leq_v b$.

We show that $\leq$ witnesses that G has countable colouring number. So let b be any vertex of G and let w be the vertex of H such that $b \in X_w$. Suppose for a contradiction that there are infinitely many vertices $\{a_i : i < \omega\}$ of G with $a_i \leq b$. For every a_i, let v_i be the vertex of H such that $a_i \in X_{v_i}$. Since the branch sets of H are finite, the set $\{v_i : i < \omega\}$ is infinite. But every v_i with $v_i \neq w$ is adjacent to w in H because a_i is adjacent to b in G. Further, for every v_i it holds that $v_i \dot{\leq} w$, since $a_i \leq b$. It follows that there are infinitely many neighbours v_i of w in H with $v_i \dot{\leq} w$, a contradiction to the choice of $\dot{\leq}$. $\square$

Lemma 11.6. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let every minor of G supported by U with countable branch sets have countable colouring number. Let $H \subseteq G$ be an induced subgraph of G such that H has finite adhesion in G towards U and such that for all U-adhesion sets S of H there are infinitely many U-components D with $S = N(D)$. Then every minor of G supported by $\mathcal{S}_G(U, H)$ with countable branch sets has countable colouring number.*

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Let every minor of G supported by U with countable branch sets have countable colouring number. Let $H \subseteq G$ be an induced subgraph of G such that H has finite adhesion in G towards U and such that for all U-adhesion sets S of H there are infinitely many U-components D with $S = N(D)$. Write $\tilde{H}$ for the subgraph of G induced by the union of H with all components D of $G - H$ with $D \cap U = \emptyset$. Note that the U-components of $G - H$ are the same as the components of $G - \tilde{H}$ and the U-adhesion sets of H are the same as the adhesion sets of $\tilde{H}$. We write $\mathcal{N}$ for the set of all adhesion sets of $\tilde{H}$. Now let $G^{\mathcal{S}}$ be a minor of G supported by $\mathcal{S}_G(U, H)$ with countable branch sets $(X_a : a \in V(G^{\mathcal{S}}))$. We write A for the subset of $V(G^{\mathcal{S}})$ containing all vertices $a \in V(G^{\mathcal{S}})$ such that X_a contains a neighbour of a component of $G - \tilde{H}$. Choose a function $\varphi : A \to \mathcal{N}$ such that $X_a \cap \varphi(a) \neq \emptyset$ for all $a \in A$. Our aim is to find countable pairwise disjoint connected subsets $(X'_a : a \in V(G^{\mathcal{S}})$ of $V(G)$ such that

(i) $G^{\mathcal{S}}$ is a minor of G with branch sets $(X'_a : a \in V(G^{\mathcal{S}}))$,

(ii) for every N in the image of φ there is a vertex $a \in \varphi^{-1}(N)$ such that X'_a contains a vertex from U.

First, we show why finding branch sets $(X'_a : a \in V(G^{\mathcal{S}}))$ as above completes the proof. Consider the minor G^U of $G^{\mathcal{S}}$ where we contract the set $\varphi^{-1}(N)$ for all $N \in \mathcal{N}$ to a single vertex. Note that the set $\varphi^{-1}(N)$ is finite because N is finite as $\tilde{H}$ has finite adhesion. Then G^U is a minor of G with countable branch sets supported by U by (ii). By assumption, G^U has countable colouring number. Since G^U is also a minor of $G^{\mathcal{S}}$ with finite branch sets, Lemma 11.5 implies that $G^{\mathcal{S}}$ has countable colouring number as desired.

Now it remains to show that there exist countable pairwise disjoint connected subsets $(X'_a : a \in V(G^{\mathcal{S}}))$ of $V(G)$ satisfying (i) and (ii). For all $a \in V(G^{\mathcal{S}})$, we define X'_a as a superset of $X_a \cap V(\tilde{H})$ in three steps.

In the first step, we define $\bar{X}_a$ for all $a \in V(G^{\mathcal{S}})$ with the properties that $X_a \cap V(\tilde{H}) = \bar{X}_a \cap V(\tilde{H})$ and that for every $N \in \mathcal{N}$ there are only finitely many components D of $G - \tilde{H}$ with $\bar{X}_a \cap D \neq \emptyset$ and $N(D) = N$. For that, let $\mathcal{D}_a$ be a minimal set of components of $G - \tilde{H}$ with the property that $\bar{X}_a := X_a \cap (\tilde{H} \cup \bigcup \mathcal{D}_a)$ is connected. Note that $\mathcal{D}_a$ exists because X_a is connected in G. Now let $N \in \mathcal{N}$. Since N is finite, there are only finitely many pairs of vertices v, w in N for which there exists a v–w path with inner vertices in $X_a \cap D$ for a component D of $G - \tilde{H}$ with $N(D) = N$. For every pair of vertices, it is enough for $\mathcal{D}_a$ to contain only one such component D. Thus, by minimality of $\mathcal{D}_a$, there are only finitely many components D of $G - \tilde{H}$ with $\bar{X}_a \cap D \neq \emptyset$ and $N(D) = N$, as required.

Furthermore, we claim that for every $N \in \mathcal{N}$, there are only finitely many components D of $G - \tilde{H}$ with $\bigcup \{\bar{X}_a : a \in V(G^{\mathcal{S}})\} \cap D \neq \emptyset$ and $N(D) = N$: We already know that for all $a \in V(G^{\mathcal{S}})$, there are only finitely many components D of $G - \tilde{H}$ with $\bar{X}_a \cap D \neq \emptyset$ and $N(D) = N$. But since N is finite, there are only finitely many $a \in V(G^{\mathcal{S}})$ for which $\bar{X}_a$ meets N. Thus there are only finitely many $a \in V(G^{\mathcal{S}})$ for which $\bar{X}_a$ meets a component D with $N(D) = N$, and each $\bar{X}_a$ meets only finitely components D with $N(D) = N$. This proves the claim above.

In the second step, for every N in the image of φ we choose a vertex $a^* \in \varphi^{-1}(N)$ and a component D of $G - \tilde{H}$ with $\bigcup \{\bar{X}_a : a \in V(G^{\mathcal{S}})\} \cap D \neq \emptyset$

and $N(D) = N$. Let P be a $\bar{X}_{a^*}$–U path in G with inner vertices in D. Note that P exists because it is true that $\bar{X}_{a^*} \cap N = X_{a^*} \cap N = X_{a^*} \cap \varphi(a^*) \neq \emptyset$ and because D is connected. We define the set $\hat{X}_{a^*}$ by adding all vertices of P to $\bar{X}_{a^*}$. For all other $a \in V(G^{\mathcal{S}})$, define $\hat{X}_a := \bar{X}_a$. This ensures property (ii). Also, for every $N \in \mathbb{N}$, there are only finitely many components D of $G - \tilde{H}$ with $\bigcup\{\hat{X}_a : a \in V(G^{\mathcal{S}})\} \cap D \neq \emptyset$ and $N(D) = N$. This is true because the same is true for the sets $\bar{X}_a$ and for every $N \in \mathcal{N}$ we added at most one path P to a component D of $G - \tilde{H}$ with $N(D) = N$.

In the third step, we further expand $\hat{X}_a$ for every $a \in V(G^{\mathcal{S}})$ to a set X'_a with the property that there is an X'_a–X'_b edge in G if there is an X_a–X_b edge in G.[13] We also make sure that the sets X'_a are pairwise disjoint and connected. Then property (i) holds. Enumerate all edges ab of $G^{\mathcal{S}}$ such that there is an X_a–X_b edge in G but no $\hat{X}_a$–$\hat{X}_b$ edge by $\{a_ib_i : i < \mu\}$ for a cardinal μ. For every $i < \mu$, there must exist a X_{a_i}–X_{b_i} edge with at least one endvertex inside some component D of $G - \tilde{H}$. Hence either X_{a_i} or X_{b_i} contains a vertex $d_i \in D$. Let us assume without loss of generality that $d_i \in X_{a_i}$. (Otherwise, we rename a_i and b_i.) Further, X_{a_i} and X_{b_i} meet $N_i := N(D) \in \mathcal{N}$ and thus also $\hat{X}_{a_i}$ and $\hat{X}_{b_i}$ meet N_i. Note that every $N \in \mathcal{N}$ only occurs as N_i for finitely many $i < \mu$ since N is finite. In each step i, we will expand $\hat{X}_{a_i}$ by adding a finite connected set of vertices Y_i such that there is a Y_i–$\hat{X}_{a_i}$ edge and a Y_i–$\hat{X}_{b_i}$ edge in G and such that Y_i is contained in a component D of $G - \tilde{H}$ with $N(D) = N_i$. Suppose that we have already done this for all $j < i$ for an $i < \mu$. Let $\mathcal{D}_i$ be the set of components of $G - \tilde{H}$ with $N(D) = N_i$. By assumption, $\mathcal{D}_i$ is infinite. Note that there are only finitely many $j < \mu$ with $N_j = N_i$ and therefore only finitely many $j < \mu$ for which Y_j is contained in a component from $\mathcal{D}_i$. Further, there are only finitely many components $D \in \mathcal{D}_i$ with $\bigcup\{\hat{X}_a : a \in V(G^{\mathcal{S}})\} \cap D_i \neq \emptyset$. Hence there exists a component $D_i \in \mathcal{D}_i$ such that D_i is disjoint to $\hat{X}_a$ for all $a \in V(G^{\mathcal{S}})$ and disjoint to Y_j for all $j < i$. Next, since $\hat{X}_{a_i}$ and $\hat{X}_{b_i}$ meet $N_i = N(D_i)$, there is an $\hat{X}_{a_i}$–$\hat{X}_{b_i}$ path P in G with inner vertices in D_i. Let Y_i be the set of inner vertices of P.

In the end, for all $a \in V(G^{\mathcal{S}})$ define X'_a as the union of $\hat{X}_a$ with all sets Y_i for $i < \mu$ such that $a = a_i$. By construction, it follows that indeed there is an X'_a–X'_b edge in G if there is an X_a–X_b edge in G.

[13]It can also happen that there is a X'_a–X'_b but no X_a–X_b. Then $G^{\mathcal{S}}$ is still a minor of G with branch sets $(X'_a : a \in V(G^{\mathcal{S}}))$ by Definition 2.10.

By construction and by the fact that $\bar{X}_a$ is connected, also X'_a is connected for every $a \in V(G^{\mathcal{S}})$. Further, we have $X'_a \cap X'_b = \emptyset$ for $a \neq b \in V(G^{\mathcal{S}})$. It remains to show that X'_a is countable for all $a \in V(G^{\mathcal{S}})$. Note that $\hat{X}_a$ is countable since X_a is countable. Now suppose for a contradiction that X'_a is uncountable. Since also $\hat{X}_a$ is countable, this means that we added uncountable many sets Y_i to $\hat{X}_a$. Thus there is an uncountable set $I \subseteq \mu$ with $a = a_i$ for all $i \in I$. Since every $N \in \mathcal{N}$ only occurs finitely often as N_i for $i < \mu$, it follows that there is an uncountable subset $J \subseteq I$ such that $N_j \neq N_{j'}$ for all $j \neq j' \in J$. For every $j \in J$, we know that there is a vertex $d_j \in X_a$ such that d_j is contained in a component D of $G - \tilde{H}$ with $N(D) = D_j$. Since J is uncountable, we have that also X_a is uncountable, a contradiction. This proves that X'_a is countable. $\square$

12 Minor Characterisation with Countable Colouring Number

12.1 Theorem

Theorem 12.1. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then, the following are equivalent:*

(1) *U is normally spanned in G,*

(4) *every minor of G supported by U with countable branch sets has countable colouring number.*

Here, too, we roughly follow Pitz's proof structure.

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of size κ.

(1) $\Rightarrow$ (4)*:* ✓ (see Section 12)

(4) $\Rightarrow$ (1)*:* Suppose that every minor of G supported by U with countable branch sets has countable colouring number. We prove the implication by induction on κ. For the start of the induction suppose that κ is countable. By Jung [Theorem 2.8], U is normally spanned in G. For the induction step, suppose that κ is uncountable. Suppose that all $X \subseteq V(G)$ of size $< \kappa$ are normally spanned in G, if every minor of G supported by X with countable branch sets has countable colouring number. With Lemma 10.4, find a continuous increasing chain $\{G_i : i < \sigma\}$ of infinite, $< \kappa$-sized connected induced subgraphs G_i that cover U and such that all G_i have finite adhesion in G towards U. Further, we may assume that for every U-adhesion set S of $G - G(T_i)$, there are infinitely many U-components D of $G - G(T_i)$ with $N(D) = S$.

Define

$$\mathbb{S}_i := \bigcup_{j \leq i} \mathcal{S}_G(U, G_j).$$

We construct by recursion on $i < \sigma$ a sequence of normal trees $\{T_i : i < \sigma\}$ in G extending each other all with the same root, such that each T_i contains $\mathbb{S}_i$ cofinally. In the end, define

$$T := \bigcup_{i<\sigma} T_i.$$

Then, T is the desired normal tree in G that covers $U \subseteq \mathbb{S}_\sigma$.

It remains to describe the recursive construction. Let $i < \sigma$. First, let $i = 0$ and consider $\mathcal{S}_G(U, G_0) = \mathbb{S}_0$. By Lemma 11.6, every minor of G supported by $\mathbb{S}_0$ with countable branch sets has countable colouring number. Since $|\mathbb{S}_0| \leq |G_0| < \kappa$, we find a normal tree in G that contains $\mathbb{S}_0$ by induction hypothesis. In particular, by Jung [Theorem 2.8], we find a normal tree T_0 in G that contains $\mathbb{S}_0$ cofinally.

Next, suppose that $i = \ell$ is a limit. Define

$$T_\ell := \bigcup_{j<\ell} T_j.$$

Then T_ℓ is a normal tree in G with the same root as T_j and also extending T_j for all $j < \ell$. Also, T_ℓ contains $\bigcup_{j<\ell} \mathbb{S}_j$ cofinally, since $\mathbb{S}_j \subseteq \mathbb{S}_\ell$ for all $j < \ell$. We show that $\mathbb{S}_\ell = \bigcup_{j<\ell} \mathbb{S}_j$, which shows that T_j contains $\mathbb{S}_\ell$ cofinally. It is clear that $\bigcup_{j<\ell} \mathbb{S}_j \subseteq \mathbb{S}_\ell$. For the other inclusion, we have to show that $S_G(U, G_\ell) \subseteq \bigcup_{j<\ell} \mathbb{S}_j$. Since the G_i sequence is continuous, it follows that every vertex of $V(G_\ell) \cap U$ is contained in $V(G_j) \cap U \subseteq \mathbb{S}_j$ for some $j < \ell$. Now let S be a U-adhesion set of G_ℓ. Since the G_i sequence is continuous and S is finite, we have that $S \subseteq G_j$ for some $j < \ell$. Then S is also a U-adhesion set of G_j and therefore $S \subseteq S_G(U, G_j) \subseteq \mathbb{S}_j$. This completes the proof that $S_G(U, G_\ell) \subseteq \bigcup_{j<\ell} \mathbb{S}_j$.

Now suppose that $i = j + 1$ is a successor. Assume that we have already defined a normal tree T_j in G that cofinally contains $\mathbb{S}_j$. Next, we construct T_i. For an index set A, enumerate all components $\{D_\alpha : \alpha < A\}$ of $G - T_j$ that meet $\mathbb{S}_i$. Note that those components must meet $\mathcal{S}_G(U, G_i)$. Let $\alpha \in A$. Since D_α is a connected subgraph of G such that it avoids $S_G(U, G_j) \subseteq T_j$ and meets $\mathcal{S}_G(U, G_i)$, by Lemma 11.2 [14] note that D_α is contained in a U-component of $G - G_j$. Since T_j is normal in G, the

[14] for $H = G_j$ and $H' = G_i$

neighbourhood of D_α forms a chain in T_j. Because of Lemma 11.4[15] this chain is finite. Thus there exists a maximal element $t_{D_\alpha} \in N(D_\alpha)$ in the tree-order of T_j. Choose a neighbour r_{D_α} of t_{D_α} in D_α.

Claim: $\mathcal{S}_G(U, G_i)$ is a countable union of dispersed sets in G.

> Consider a minor of G supported by $\mathcal{S}_G(U, G_i)$ with countable branch sets. By Lemma 11.6, this minor has countable colouring number. Now, since $|\mathcal{S}_G(U, G_i)| \leq |G_i| < \kappa$, we know that $\mathcal{S}_G(U, G_i)$ is normally spanned in G by induction hypothesis. Hence by Jung [Theorem 2.8], $\mathcal{S}_G(U, G_i)$ is a countable union of dispersed sets in G.

Hence, also $\mathcal{S}_G(U, G_i) \cap V(D_\alpha) = \mathbb{S}_i \cap V(D_\alpha)$ is a countable union of dispersed sets in D_α. By Jung [Theorem 2.8], there is a normal tree $T_{D_\alpha} \subseteq D_\alpha$ with root r_{D_α} cofinally containing $\mathbb{S}_i \cap V(D_\alpha)$. Call e_{D_α} the edge between t_{D_α} and r_{D_α}. Define

$$T_i := T_j \cup \bigcup_{\alpha < A} T_{D_\alpha} \cup \bigcup_{\alpha < A} \{e_{D_\alpha}\}.$$

By construction, T_i is a normal tree in G with the same root as T_j containing $\mathbb{S}_i$ cofinally and extending T_j.

□

We now prove the last remaining equivalence of Theorem 3.2, namely between (1) ⇔ (7):

[15] applied with $X = \mathbb{S}_j$

13 Minor Characterisation with Fat $TK^{\aleph_0}$-Dispersed Sets

13.1 Theorems

Lemma 13.1. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices that is fat $TK^{\aleph_0}$-dispersed (Definition 2.23) in G. Then U is normally spanned in G.*

The theorem has already been proven by Pitz in [18]. We prove the theorem here again in a different way. Here we use the results of the previous sections by applying Theorem 4.1. Notice that we improve the definition of being "fat $TK^{\aleph_0}$-dispersed" here. In the paper it should have been chosen the same as ours.

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices. Suppose that U is fat $TK^{\aleph_0}$-dispersed in G. Assume that U is not normally spanned in G. By Theorem 4.1, there is a minor H of G supported by U that is a (λ, λ^+)-graph or a (κ, S)-graph with countable branch sets.

For the following, call a fat $TK^{\aleph_0}$ in G compatible with H if, and only if for every branch vertex v of the fat $TK^{\aleph_0}$ there is a branch set X of H such that $v \in X$ and for two distinct branch vertices $v \neq v'$ holds that they are contained in two different branch sets of H.

H is a (λ, λ^+)-graph: As in [17, Theorem 6.1], find a fat $TK^{\aleph_0}$ in G that is compatible with H. Enumerate all branch vertices of the fat $TK^{\aleph_0}$, say $\{v_n : n \in \mathbb{N}\}$. Let $n \in \mathbb{N}$ and consider the branch vertex v_n. Let X be the branch set of H such that $v_n \in X$. Since H is supported by U, find a vertex $u \in U$ with $u \in X$. Since the branch set is connected, find a v_n–u path P_n in X. Then, $\{P_n : n \in \mathbb{N}\}$ is a set of pairwise vertex-disjoint paths between U and the branch vertices of the fat $TK^{\aleph_0}$. A contradiction to the fact that U is fat $TK^{\aleph_0}$-dispersed in G.

H is a (κ, S)-graph: Again, as in [17, Theorem 6.1], find a fat $TK^{\aleph_0}$ in G that is compatible with H. As before, we find infinitely many pairwise vertex-disjoint paths between U and the branch vertices of the fat $TK^{\aleph_0}$. A contradiction. $\square$

Theorem 13.2. *Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G. Then, the following are equivalent:*

(1) *U is normally spanned in G,*

(7) *U is a countable union of fat $TK^{\aleph_0}$-dispersed sets in G.*

Proof. Let G be a connected graph. Let $U \subseteq V(G)$ be a set of vertices of G.

(1) $\Rightarrow$ (7)*:* Let U be normally spanned in G. Then, U is a countable union

$$U = \bigcup_{n \in \mathbb{N}} U_n$$

of dispersed sets in G, by Jung [Theorem 2.8]. We show that for all $n \in \mathbb{N}$ it is true that U_n is fat $TK^{\aleph_0}$-dispersed in G. Suppose for a contradiction that there is an $n \in \mathbb{N}$ such that U_n is not fat $TK^{\aleph_0}$-dispersed in G. This means that there is a fat $TK^{\aleph_0}$ in G such that U cannot be separated from the branch vertices of this fat $TK^{\aleph_0}$ by a finite set of vertices in G. Hence there exist infinitely many pairwise vertex-disjoint path $\{P_i : i \in \mathbb{N}\}$ from U to the branch vertices of the fat $TK^{\aleph_0}$ in G. Let R be a ray in the fat $TK^{\aleph_0}$ such that every branch vertex of the fat $TK^{\aleph_0}$ is contained in R. However, for every $i \in \mathbb{N}$ we have that an initial segment of P_i is an R–U_n path, which means that there is no finite set in G that can separate U_n and R. A contradiction to the fact that U_n is dispersed in G.

(7) $\Rightarrow$ (1)*:* Let

$$U = \bigcup_{n \in \mathbb{N}} U_n$$

be a countable union such that for all $n \in \mathbb{N}$ we have that U_n is a fat $TK^{\aleph_0}$-dispersed set in G. Let $n \in \mathbb{N}$. By Lemma 13.1, U_n is normally spanned in G. By Jung [Theorem 2.8],

$$U_n = \bigcup_{i \in \mathbb{N}} U_n^i$$

is a countable union of dispersed sets in G. Together,

$$U = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} U_n^i$$

is a countable union of dispersed sets in G. Hence, by Jung [Theorem 2.8], U is normally spanned in G.

$\square$

References

[1] J.-M. Brochet and R. Diestel. Normal tree orders for infinite graphs. *Transactions of the American Mathematical Society*, 345:871–895, 1994.

[2] C. Bürger and J. Kurkofka. Duality theorems for stars and combs I: Arbitrary stars and combs. *Journal of Graph Theory*, 99:525–554, 2021.

[3] C. Bürger and J. Kurkofka. Duality theorems for stars and combs II: Dominating stars and dominated combs. *Journal of Graph Theory*, 99:555–572, 2021.

[4] C. Bürger and J. Kurkofka. Duality theorems for stars and combs III: Undominated combs. *Journal of Graph Theory*, 100:127–139, 2021.

[5] C. Bürger and J. Kurkofka. Duality theorems for stars and combs IV: Undominating stars. *Journal of Graph Theory*, 100:140–162, 2022.

[6] R. Diestel. A simple existence criterion for normal spanning trees. *The Electronic Journal of Combinatorics*, 23, 2016. P2.33.

[7] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[8] R. Diestel and I. Leader. Normal spanning trees, Aronszajn trees and excluded minors. *Journal of the London Mathematical Society*, 63:16–32, 2001.

[9] J. Erde, P. Gollin, A. Joó, P. Knappe, and M. Pitz. A Cantor-Bernstein-type theorem for spanning trees in infinite graphs. *Journal of Combinatorial Theory Series B*, 149:16–22, 2021.

[10] A. Georgakopoulos. Infinite Hamilton cycles in squares of locally finite graphs. *Advances in Mathematics*, 220:670–705, 2009.

[11] R. Halin. Simplicial decompositions of infinite graphs. *Annals of Discrete Mathematics*, 3:93–109, 1978. Elsevier.

[12] R. Halin. Miscellaneous problems on infinite graphs. *Journal of Graph Theory*, 35:128–151, 2000.

[13] Ruben Melcher Jan Kurkofka and Max Pitz. A strengthening of Halin's grid theorem. *Journal of the London Mathematical Society*, 68(4):1009–1013, 2022.

[14] H. A. Jung. Zusammenzüge und Unterteilungen von Graphen. *Mathematische Nachrichten*, 35:241–267, 1967.

[15] K. Kunen. *Set theory*, volume 34 of *College Publications*. Studies in Logic, London, 2011.

[16] P. Komjáth N. Bowler, J. Carmesin and C. Reiher. The colouring number of infinite graphs. *Combinatorica*, 39(6):1225–1235, 2019.

[17] M. Pitz. Proof of Halin's Normal Spanning Tree Conjecture. *Israel Journal of Mathematics*, 246:353–370, 2021.

[18] M. Pitz. Quickly proving Diestel's normal spanning tree criterion. *The Electronic Journal of Combinatorics*, 28(3), 2021. P3.59.