Every planar graph with the Liouville property is amenable

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Abstract

We introduce a strengthening of the notion of transience for planar maps in order to relax the standard condition of bounded degree appearing in various results, in particular, the existence of Dirichlet harmonic functions proved by Benjamini & Schramm. As a corollary we obtain that every planar non-amenable graph admits Dirichlet harmonic functions.

1 Introduction

A well-known result of Benjamini & Schramm states that every transient planar graph with bounded vertex degrees admits non-constant harmonic functions with finite Dirichlet energy; we will call such a function a *Dirichlet harmonic function* from now on. In particular, such a graph does not have the Liouville property. Two independent proofs of this theorem were given in [5, 6], one using circle packings and one using square tilings.

The bounded degree condition was essential in both these proofs, and is in fact necessary: consider for example a ray where the nth edge has been duplicated by 2^n parallel edges. Still, there are natural classes of unbounded degree graphs where such obstructions do not occur, and it is interesting to ask whether the above result remains true in them. Recently, planar graphs with unbounded degrees have been attracting a lot of interest, in particular

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due to research on coarse geometry [9], random walks [4, 13] and random planar graphs related to Liouville quantum gravity [1, 21, 2, 3, 4, 8, 12, 14, 15, 18]. Motivated by this, our main result extends the aforementioned result of Benjamini & Schramm to unbounded degree graphs by replacing the transience condition with a stronger one, which we call *UK-transience* and explain below

Theorem 1.1. *Let G be a locally finite UK-transient planar map. Then G admits a Dirichlet harmonic function.*

A *planar map* G, also called a *plane graph*, is a graph endowed with an embedding in the plane. The *roundabout graph* G° is obtained from G by replacing each vertex v with a cycle v° in such a way that the edges incident with v are incident with distinct vertices of v° (of degree 3), preserving their cyclic ordering; see also Section 4. We say that G is *UK-transient* if G° is transient[1]. In Section 4 we relate G° with circle packings of G.

Another way how one might try to strengthen the transience condition is to require that there is a flow f witnessing the transience which does not only have finite Dirichlet energy but finite norm in a different Hilbert space, where we give weights to the edges depending on the degrees of their endvertices. Following up, these ideas, we could show that Theorem 1.1 implies the following

Corollary 1.2. *Let G be a locally finite planar graph G such that there is a flow f of intensity 1 out of some vertex v such that $\sum_{vw\in E(G)}[deg(v)^2 + deg(w)^2]f(vw)^2$ is finite. Then G has a non-constant Dirichlet harmonic function.*

As shown in Section 8, the order of magnitude of the weights here is best-possible. Hence Corollary 1.2 is best-possible, which indicates a way in which Theorem 1.1 is tight.

Our work was partly motivated by a problem from [13], asking whether every simple planar graph with the Liouville property is (vertex-)*amenable*, by which we mean that for every $\epsilon > 0$ there is a finite set S of vertices of G such that less than $\epsilon|S|$ vertices outside S have a neighbour in S. As we show in Section 8,

Theorem 1.3. *Every locally finite non-amenable planar map is UK-transient.*

Combining this with Theorem 1.1 yields a positive answer to the aforementioned problem, and much more. This strengthens a result of Northschield [19], stating that every bounded degree non-amenable planar graph

[1] The authors coined this term in Warwick, UK, where there are many roundabouts.

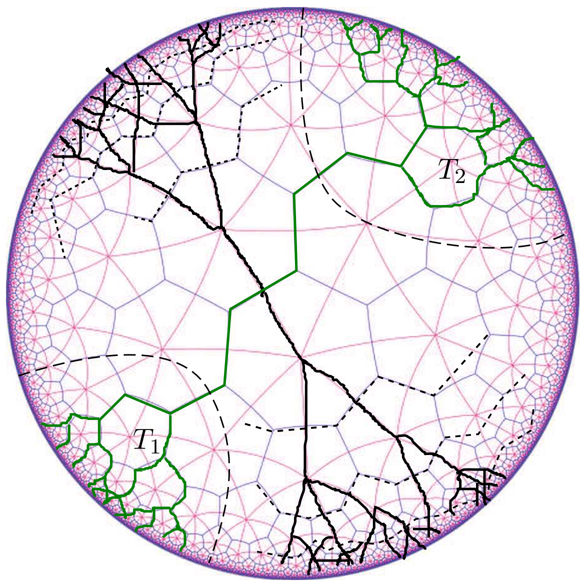

Figure 1: The two subgraphs T_1, T_2 delimited by the dashed curves are transient because of the green flow. The dual of the black flow (dashed) witnesses the fact that the effective conductance between T_1 and T_2 is finite because it has finite energy.

admits non-constant bounded harmonic functions, in two ways: it relaxes the bounded degree condition, and provides Dirichlet rather than bounded harmonic functions.

We think of Theorems 1.1 and 1.3 as indications that the notion of UK-transience is satisfied in many cases, and has strong implications. We expect it to find further applications, and propose some problems in Section 9.

We now give an overview of the proof of Theorem 1.1. As shown in [11], a graph admits Dirichlet harmonic functions if and only if it has two disjoint transient subgraphs T_1, T_2 such that the effective conductance between T_1 and T_2 is finite; see Theorem 3.1. To show that our graphs satisfy this condition, we start with a flow provided by T. Lyons' transience criterion (Theorem 2.1)—this flow lives in an auxiliary graph which for the purposes of this illustration can be thought of as a superimposition of G with its dual— we split that flow into four sub-flows using the square tiling techniques of [13], we use two subflows to obtain T_1, T_2, and we apply a duality argument to the other two subflows to show that the effective conductance between T_1 and T_2 is finite; see Figure 1.

The latter step can be thought of as an occurence of the idea that the effective resistance from the top to the bottom of a rectangle equals the effective conductance (or extremal length [23]) from left to right, with the aforementioned subflows showing finiteness of the top-to- bottom effective resistance. This idea was triggered by another result of Benjamini & Schramm [7], stating that every non-amenable graph contains a non-amenable tree.

2 Preliminaries

A *graph*, or *network*, G is a pair (V, E) where V is a set, called the set of *vertices* (or *nodes*) of G, and E is a set of pairs of elements of V, called the *edges*. In this paper all graphs are simple.

Given a vertex set X, by $E(X)$ we denote those edges with both end-vertices in X. A locally finite graph G is *1-ended* if for every finite vertex S, the graph $G - S$ has only one infinite component.

2.1 Electrical network basics

All graphs in this paper are undirected. However, as we will want to describe flows of electrical current in our networks, we will need to be able to distinguish between the two possible orientations of an edge in order to be able to say in which direction current flows along that edge. A convenient solution is to introduce the set $\overrightarrow{E(G)}$ (or just $\vec{E}$) of *directed edges* of G to be the set of ordered pairs (x, y) such that $xy \in E$. Thus any edge $xy = yx \in E$ corresponds to two elements of $\vec{E}$, which we will denote by $\overrightarrow{xy}$ and $\overrightarrow{yx}$.

An *antisymmetric function* $i : \vec{E} \to \mathbb{R}$ satisfies $i(\overrightarrow{xy}) = -i(\overrightarrow{yx})$ for every edge $xy \in E$. All functions on $\vec{E}$ we will consider will have this property.

Given two dual plane graphs G and G^* and an orientation of the plane, there is a unique bijection * between the directed edges of G and G^* respecting this orientation. If we use * below we shall always assume that we picked some orientation - even if we do not say this explicitly. If F is an edge set, then F^* denotes image of F under *. The function * induces an operator on antisymmetric functions f on the directed edges of G. Given $f : \overrightarrow{E(G)} \to \mathbb{R}$, we denote the induced function from $\overrightarrow{E(G^*)}$ to $\mathbb{R}$ by f^*.

Given a function $i : \vec{E} \to \mathbb{R}$, we say that i satisfies *Kirchhoff's node law* at a vertex x if

$$\partial i(x) := \sum_{y \in N(x)} i(\overrightarrow{xy}) = 0 \qquad (1)$$

holds, where $N(x)$ denotes the set of vertices sharing an edge with x (called the *neighbours* of x).

If i satisfies Kirchhoff's node law at everywhere except at one vertex o, then i is called a *flow from* o. By the *intensity* of i we will mean $\partial i(o)$. Usually we will assume that $\partial i(o) > 0$ when we use this term. Similarly, we define a Kirchhoff's node law at finite vertex sets and flow from a finite set $A \subset V(G)$.

Given $u : V \to \mathbb{R}$, the *induced antisymmetric function* ∂u is given by

$$\partial u(\overrightarrow{xy}) = u(x) - u(y) \tag{2}$$

If $i = \partial u$, we say that the pair i, u satisfies *Ohm's law.*

Suppose that a pair i, u as above satisfies Ohm's law, and i satisfies Kirchhoff's node law. Then, combining (1) with (2) we obtain $\sum_{y\in E(x)}(u(x) - u(y)) = 0$, and solving for $u(x)$ this can be rewritten as

$$u(x) = \frac{\sum_{y\in E(x)} u(y)}{d(x)}, \tag{3}$$

where the *degree* $d(x)$ of x is the number of edges incident with x.

If a function u satisfies the formula (3), then we say that u is *harmonic* at x. Note that the above implication can be reversed to yield that if u is harmonic at a vertex then it satisfies Kirchhoff's node law there. In other words, if the pair i, u satisfies Ohm's law, then u is harmonic at a vertex x if and only if i satisfies Kirchhoff's node law at x.

A function $u : V \to \mathbb{R}$ is *harmonic* if it is harmonic at every $x \in V$.

The (Dirichlet) *energy* of $i : \vec{E} \to \mathbb{R}$ is defined by

$$E(i) := \sum_{e\in\vec{E}} i^2(e).$$

Similarly, we define the energy of a function $u : V \to \mathbb{R}$ by $E(u) := \sum_{xy\in E} (v(x) - v(y))^2$. We call u a *Dirichlet harmonic function* if u is harmonic and $E(u) < \infty$. We write $\mathcal{O}_{HD}$ for the class of graphs on which all Dirichlet harmonic functions are constant.

A *potential* on the network N is a function $u : V \to \mathbb{R}$. The *boundary* of the potential u is the set of vertices at which u is not harmonic.

A *walk* in G is a sequence of incident vertices and edges $x_0e_{01}x_1e_{12}x_2 \dots x_k$ (where the x_j are vertices and the e_{jl} edges). A walk as above is *closed* if $x_k = x_0$. Kirchhoff's cycle law postulates that for every closed walk as above we have

$$\sum_{0\leq n<k} i(\overrightarrow{x_n x_{n+1}}) = 0. \tag{4}$$

It is not hard to check that i satisfies Kirchhoff's cycle law if it does so for every *injective* closed walk, i.e. one for which the x_j are distinct for $0 \leq j < k$. Moreover, this is the case if and only if there is a potential u with $i = \partial u$.

2.2 Random walk basics

All random walks in this paper are simple and take place in discrete time, that is, if our the random walker is at a vertex x of our graph G at time n, then it is at each of the $d(x)$ neighbours of x with equal probability $1/d(x)$ at time $n+1$. The starting vertex of our random walk will always be deterministic, and usually denoted by o.

G is called *transient*, if the probability to visit any fixed vertex is strictly less than 1. We will make heavy use of T. Lyons classical characterisation of transience in terms of flows:

Theorem 2.1 ([17, 16])**.** *A locally finite graph G is transient if and only if for some (and hence for every) vertex $o \in V(G)$, G admits a flow from o with finite energy.*

If G is transient, then we can define a flow i out of any vertex o as follows. For every vertex $v \in V$, let $h(v)$ be the probability $p_v(o)$ that random walk from v will ever reach o. Thus $h(o) = 1$. Note that h is harmonic at every $v \neq o$. Let $i(\overrightarrow{xy}) := h(x) - h(y)$. By our discussion in Subsection 2.1, i is a flow out of o, and we call it the *random walk flow* out of o.

3 Known facts

3.1 HD facts

We shall use following characterisation of the locally finite graphs admitting Dirichlet harmonic functions:

Theorem 3.1 ([11])**.** *A locally finite graph G is not in $\mathcal{O}_{HD}$ if and only if there are transient vertex-disjoint subgraphs A and B such that there is a potential ρ of finite energy which is constant on A and B but takes different values on them.*

Corollary 3.2. *A locally finite graph G is not in $\mathcal{O}_{HD}$ if and only if there is a flow f and a potential ρ both of finite energy such that the supports of f and $\partial(\rho)$ intersect in precisely one edge.*

Proof. We may without loss of generality assume that the two graphs A and B of Theorem 3.1 are joined by an edge xy. Given two vertex-disjoint subgraphs A and B, there is a flow f of finite energy with $f(\overrightarrow{xy})$ nonzero and whose support is included in $(A \cup B) + xy$ if and only if A and B are transient. Thus Corollary 3.2 follows from Theorem 3.1. $\square$

Next, we give a new independent functional analytic proof of the 'if'-implication of Corollary 3.2. For that we need the following:

Lemma 3.3. *Let H be a Hilbert space space and V and W two orthogonal subspaces such that the orthogonal complement $V^\perp$ of V is not orthogonal to $W^\perp$. Then $V^\perp \cap W^\perp$ is nontrivial.*

Proof. Then $V^\perp + W^\perp = V \oplus W \oplus (V^\perp \cap W^\perp)$. By assumption, there are $v \in V^\perp$ and $w \in W^\perp$ with $< v|w > \neq 0$. Thus $v \in W \oplus (V^\perp \cap W^\perp)$ and $w \in V \oplus (V^\perp \cap W^\perp)$. Since V and W are orthogonal, the projection of v to $V \oplus (V^\perp \cap W^\perp)$ is contained in $(V^\perp \cap W^\perp)$. This projection is a nontrivial by assumption, completing the proof. $\square$

Proof of the 'if'-implication of Corollary 3.2. We consider the Hilbert space of antisymmetric functions on the edges with finite Dirichlet energy. Its scalar product is given by $\langle f \mid g \rangle = \sum_{e \in E(G)} f(e)g(e)$. Let C be its subspace generated by the characteristic functions of the finite cycles, and D be its subspace generated by the atomic bonds $b(v)$ given by the characteristic functions of the set of edges incident with a vertex v. Note that $f \in D^\perp$ and $\partial(\rho) \in C^\perp$. Thus by Lemma 3.3, there is some nontrivial $h \in C^\perp \cap D^\perp$, which is an antisymmetric function induced by a non-constant Dirichlet harmonic function. $\square$

A *cut* of a graph G is the set of edges between a set of vertices $U \subset V(G)$ and its complement $V(G) \setminus U$.

Corollary 3.4 ([22])**.** *Let G be a locally finite graph with a finite cut b such that $G - b$ has two transient components. Then G is not in $\mathcal{O}_{HD}$.*

Proof. Just apply Theorem 3.1 with any potential ρ which is constant on any component of $G - b$ and assigns different values on two transient components of $G - b$. $\square$

Theorem 3.5 ([11])**.** *Let H be a connected locally finite graph. Let G be a locally finite graph obtained from H by adding for each $n \in \mathbb{N}$ a path P_n of length 2^n such that P_n meets H and the other P_n only in its starting vertices. Then $G \in \mathcal{O}_{HD}$ if and only if $H \in \mathcal{O}_{HD}$.*

Given a locally finite graph G, an antisymmetric function f on $\overrightarrow{E(G)}$ *witnesses that a subgraph H of G is transient* if the restriction $\bar{f}$ of f to $\overrightarrow{E(H)}$ is a flow from some finite vertex set of finite energy. We can change $\bar{f}$ at finitely many edges to get a flow from a single vertex of finite energy. Thus $\bar{f}$ implies that H is transient by Theorem 2.1.

Recall that a *bond* of a graph is a minimal separating edge-set (i.e. a minimal nonempty cut).

Remark 3.6. *Let G and G^* be locally finite dual plane graphs. Let f be a flow of G of finite energy. Then one of the following is true.*

A) The function f^* satisfies Kirchhoff's cycle law;

B) there is a finite bond b of G such that f witnesses that the two components of $G - b$ are transient.

Proof. If f^* violates Kirchhoff's cycle law at a finite cycle C of G^*, then C considered as an edge set of G is a bond b and f witnesses that the two components of $G - b$ are transient. $\square$

3.2 Electrical network facts

The following 'Monotone-Voltage Paths' lemma can be found in [16, Corollary 3.3]

Lemma 3.7. *Let G be a transient connected network and v the voltage function from the unit current flow i from a vertex o to ∞ with $v(\infty) = 0$. For every vertex x, there is a path from o to x along which v is monotone.*

4 UK-transience

Given a locally finite plane graph G, informally the *roundabout graph* G° is obtained from G by replacing each vertex v by a roundabout of length equal to the degree of v so that every vertex gets degree 3. Formally, the vertex set of G° is the set of pairs (v, e) where e is an edge and v is an endvertex of e. The embedding of G gives us a cyclic order C_v of the set of edges incident with the vertex v. The edges of G° are of two types, for each edge $e = vw$ we have an edge joining (v, e) and (w, e). For any two edges e and f adjacent in the cyclic order C_v, we have an edge between (v, e) and (v, f).

Note that the roundabout graph G° is like G a plane graph.

In a slight abuse of notation, we shall suppress the inclusion map which maps the edge $e = vw$ to $\{(v, e), (w, e)\}$ in our notation, and we will just write things like $E(G) \subseteq E(G^\circ)$. The edges *going out off* a roundabout are those with precisely one endvertex in the roundabout. We say that a graph G is *UK-transient* if its roundabout graph G° is transient.

Remark 4.1. *Every cut of G is a cut of G°. Conversely, every cut b of G° with $b \subseteq E(G)$ is also a cut of G.* □

Remark 4.2. *We remark that the roundabout graph depends on the embedding of G. Thus UK-transience is a property of plane graphs and not of planar graphs. Indeed, let G be the graph obtained from T_2 by attaching 2^n leaves at each vertex at level n. It is straightforward to check that there is a non-UK-transient embedding of G in the plane as well as a UK-transient one. Still UK-transience implies transience, in the sense that if G admits a UK-transient embedding, then G is transient:*

Lemma 4.3. *If G° is transient, then so is G.*

Proof. Since G° is transient, it admits a flow f of finite energy from some vertex $o \in V(G^\circ)$ by Lyons' criterion, Theorem 2.1. We will show that f induces a flow of finite energy in G.

For a vertex $v \in V(G^\circ)$, let us denote by v° the set of vertices lying in the same roundabout as v. Note that f satisfies Kirchhoff's node law at every v° except o°. Therefore, the restriction f' of f to $\overrightarrow{E(G)}$ satisfies Kirchhoff's node law at every vertex of G except the vertex o°. In other words, f' is a flow from o°. Its energy is bounded from above by that of f, and so G is transient by Theorem 2.1. □

In the following we will often use the notation $G^{*\circ}$, by which we mean that we apply first * and then $^\circ$. Thus $G^{*\circ}$ is the roundabout graph of the dual of G.

The *plane line graph* $G^\diamond$ of a plane graph G is the plane graph obtained from the roundabout graph G° by contracting all non-roundabout edges. Another way to define $G^\diamond$, explaining the name we chose, is by letting the vertex set of $G^\diamond$ be the set of midpoints of edges of G and joining two such points with an arc whenever the corresponding edges are incident with a common vertex v of G and lie in the boundary of a common face of v. It is clear from this definition that

$$G^\diamond = G^{*\diamond}. \tag{5}$$

A third equivalent definition of $G^\diamond$ can be given by considering a circle packing P of G, letting $V(G^\diamond)$ be the set of intersection points of circles of

P, and letting the arcs in P between these points be the edges of $G^\diamond$. A fourth definition of $G^\diamond$ is as the dual of the bipartite graph G', with $V(G')$ consisting of the vertices and faces of G, and $E(G')$ joining each vertex of G to each of its incident faces. see this

Lemma 4.4. *Let G be a locally finite plane graph. Then G° is transient if and only if $G^\diamond$ is.*

Proof. This follows easily from Theorem 2.1: if G° has a flow f of finite energy from $o \in V(G^\circ)$, then f induces such a flow f' in $G^\diamond$ from the vertex corresponding to o by just restricting f to $E(G^\diamond) \subset E(G^\circ)$.

Conversely, given a flow f' in $G^\diamond$ as above, we can construct a flow f on G° by letting $f(e) = f'(e)$ for every $e \in E(G^\diamond)$ and letting $f(e)$ be the unique value that makes both endvertices of e satisfy Kirchhoff's node law, unless those vertices correspond to o in which case we let $f(e)$ be the unique value that makes exactly one endvertex of e satisfy Kirchhoff's node law. That such values always exist is an easy fact about Kirchhoff's node law. The energy $E(f)$ of f is finite because the contribution of each vertex to $E(f)$ is bounded above by a constant times the contribution of its corresponding vertex in $G^\diamond$ to $E(f')$. $\square$

Lemma 4.4, combined with the fact that $G^\diamond = G^{*\diamond}$ (5), immediately yields

Corollary 4.5. *If G° is transient, then so is $G^{*\circ}$.*

Another way to state Corollary 4.5 is to say that G is UK-transient if and only if G^* is UK-transient.

5 Square tilings and the two crossing flows

In this section we use the theory of square tilings of transient planar graphs in order to find the special flows in our UK-transient G mentioned in the introduction. Square tilings in our sense were introduced in [5], and generalise a classical construction of Brooks et. al. [10] from finite plane graphs to infinite transient ones.

Let $\mathcal{C}$ denote the cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0, 1]$, or more generally, a cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0, a]$ for some real $a > 0$ (which turns out to coincide with the effective resistance from a vertex o to infinity). A *square tiling* of a plane graph G is a mapping τ assigning to each edge e of G a square $\tau(e)$ contained in $\mathcal{C}$, where we allow $\tau(e)$ to be a 'trivial square' consisting of just a point (see Figure 2 for an example). A nice property of square tilings is that every vertex $x \in V$ can be associated with a horizontal line segment $\tau(x) \subset \mathcal{C}$ such that for every edge e incident with x, $\tau(e)$ is tangent to $\tau(x)$.

The construction of this τ is based on the random walk flow i out of a root vertex o (as defined in Subsection 2.2): the side length of the square $\tau(e)$ is chosen to be $|i(e)|$, and the placement of that square incide $\mathcal{C}$ is decided by a coordinate system where potentials of vertices induced by the flow i are used as coordinates. For example, the top circle of the cylinder $\mathcal{C}$ is the 'line segment' corresponding to o, because o has the highest potential. All other vertices and edges accumulate towards the base of $\mathcal{C}$, because their potentials (which equal the probability for random walk to return to o, normalised by the height of $\mathcal{C}$) converge to 0; see [13] for details.

We let $w(\tau(e))$ denote the width of the square $\tau(e)$. Our square tilings always have the following properties which we will use below:

1. Two of the sides of $\tau(e)$ are always parallel to the boundary circles of $\mathcal{C}$;
2. $w(\tau(e)) = |i(e)|$ for every $e \in \vec{E}$, where i denotes the random walk flow out of o;
3. the interiors of any two such squares $\tau(e), \tau(f)$ are disjoint;
4. every point of $\mathcal{C}$ lies in $\tau(e)$ for some $e \in E$;
5. every vertex x can be associated with a horizontal line segment $\tau(x) \subset \mathcal{C}$ so that for every edge e incident with x, $\tau(e)$ is tangent to $\tau(x)$, and every point of $\tau(x)$ is in $\tau(f)$ for some edge f incident with x, and
6. every face F can be associated with a vertical line segment $\tau(F) \subset \mathcal{C}$ so that for every edge e in the boundary of F, $\tau(e)$ is tangent to $\tau(F)$.

It was shown in [5] that a plane graph G admits a square tiling exactly when G is *uniquely absorbing*. We say that G is *uniquely absorbing*, if for every finite subgraph G_0 there is exactly one connected component D of $\mathbb{R}^2 \setminus G_0$ which is *absorbing*, that is, random walk on G visits $G \setminus D$ only finitely many times with positive probability (in particular, G is transient).

A *meridian* of $\mathcal{C}$ is a vertical line of the form $\{x\} \times \{0, 1\} \subset \mathcal{C}$ for some $x \in \mathbb{R}/\mathbb{Z}$. An important property of meridians that we will use below is that the net flow i crossing any meridian is zero; see [13, Lemma 6.6] for a more precise statement.

Lemma 5.1. *Let G and G^* be locally finite dual plane graphs. If G° is transient, then there are flows f and h of finite energy in the roundabout graphs G° and $G^{*\circ}$ respectively whose supports intersect in a single edge (of $E(G) = E(G^*)$).*

Here the graphs G° and $G^{*\circ}$ have precisely the edge set $E(G) = E(G^*)$ in common. In the proof below we think of $G^\diamond$ as being constructed from G° and $G^{*\circ}$ by contracting $E(G) = E(G^*)$. This way we can consider the roundabout of $v \in V(G)$ as a cycle of $G^\diamond$.

Proof. We will first find appropriate auxiliary flows f', h' in $G^\diamond$ and use them to induce the desired flows f on G° and h on $G^{*\circ}$ by sending some flow along $E(G)$.

We distinguish two cases, according to whether $G^\diamond$ is uniquely absorbing.

If $G^\diamond$ is uniquely absorbing, then [5] provides a square tiling of $G^\diamond$ on a cylinder $\mathcal{C}$ as described above, with o being an arbitrary vertex of $G^\diamond$.

Given a vertex $x \in V(G^\diamond)$, we let $|x|$ denote the 'strip' of the cylinder $\mathcal{C}$ whose horizontal span coincides with that of the line segment $\tau(x)$ (as described in item 5). Then $\tau(x)$ separates $|x|$ into two rectangles, and we denote the bottom one (that is, the one not meeting $\tau(o)$) by $\lceil x \rceil$.

Next, we associate to this x a flow $\check{x}$ out of x that 'lives in $\lceil x \rceil$'. To define the flow $\check{x}$, for every $e \in \vec{E}(G^\diamond)$ with $i(e) \geq 0$, where i is the random walk flow out of o, let $\check{x}(e) := w(\tau(e) \cap \lceil x \rceil)$ be the width of the rectangle $\tau(e) \cap \lceil x \rceil \subset \mathcal{C}$ corresponding to e. (Thus if $\tau(e)$ is contained in $\lceil x \rceil$, then $\check{x}(e) = i(e)$ by (2), and if $\lceil x \rceil$ dissects $\tau(e)$ then $\check{x}(e) < i(e)$.) Naturally, we extend $\check{x}$ to the remaining directed edges in the unique way that makes $\check{x}$ antisymmetric. By the aforementioned property of meridians proved in [13, Lemma 6.6], $\check{x}$ is indeed a flow out of x.

More generally, if M, M' are two meridians intersecting $\tau(x)$, we let $\lceil MxM' \rceil$ denote the rectangle of $\mathcal{C}$ bounded by M, x, M' and the bottom circle of $\mathcal{C}$, and define the flow out of x that *lives in* $\lceil MxM' \rceil$ similarly to $\check{x}$, except that we replace the rectangle $\lceil x \rceil$ with $\lceil MxM' \rceil$ in that definition.

Our plan is to find four vertices $x_1, \ldots, x_4$ far enough from each other on $\mathcal{C}$ and flows f_i out of those vertices that live in appropriate disjoint rectangles, and combine these flows pairwise to obtain f', h'.

Now more precisely, we claim that we can choose four vertices $x_i, 1 \leq i \leq 4$ in $G^\diamond$, a flow f_i out of each x_i, and a path P_i from x_i to o, so that these objects satisfy the following properties

1. $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ for $i \neq j$; even stronger, no roundabout of G° meets both $\text{supp}(f_i)$ and $\text{supp}(f_j)$;

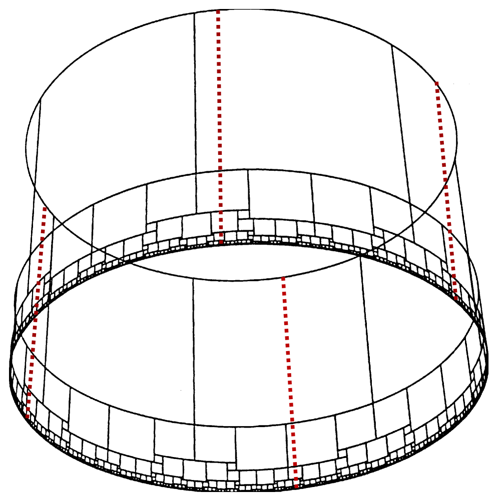

Figure 2: An example of a square tiling, with the four meridians M_i of Lemma 5.1 in dotted lines.

2. for every i and every edge e of P_i, no edge of the roundabout of G° containing e is in the support of any $f_j, 1 \leq j \leq 4$, and

3. the roundabout of G° containing the first edge of P_i does not contain x_j and does not contain any edge of P_j for $j \neq i$.

Before proving that such a choice is possible, let us first see how it helps us construct the desired flows f, h.

We claim that there is a tree T contained in G (we really mean G and not $G^\diamond$) such that the set of leaves of T is $\{r_1, r_2, r_3, r_4\}$, where r_i denotes the roundabout of G° containing x_i, and such that no edge of $G^\diamond$ lying in a roundabout corresponding to a vertex in T is in the support of any f_i. Indeed, consider the subgraph H of G induced by the vertices of G whose roundabouts meet $\bigcup_{1 \leq i \leq 4} P_i$; that subgraph is connected since all P_i meet o, and its roundabouts avoid the supports of the f_i by (2). Letting T be a spanning tree of H, we can now use (3) to deduce that each r_i is a distinct leaf of T. If T has any further leaves, we can recursively prune them untill its set of leaves is $\{r_1, r_2, r_3, r_4\}$. Let $T^\diamond$ denote the subgraph of $G^\diamond$ spanned by the roundabouts in T.

There are two possible shapes for this $T^\diamond$ depending on whether T has a vertex of degree 4, as depicted in Figure 3. Assume without loss of generality that x_1, x_2, x_3, x_4 appear in that cyclic order along the outer face of $T^\diamond$. Consider first the case where T has no vertex of degree 4. Easily, we can

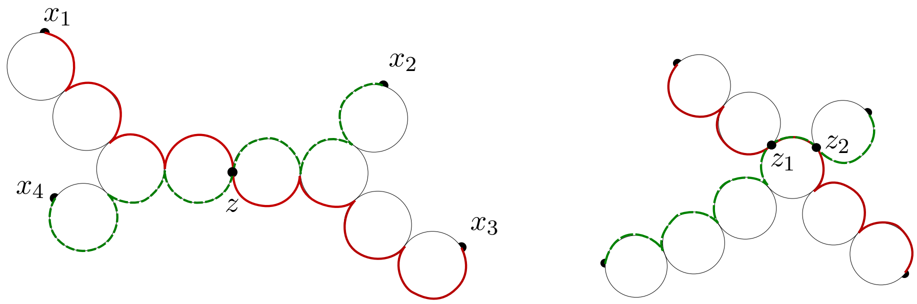

Figure 3: The 'tree' $T^\diamond$ and the paths P, Q.

find an x_3-x_1 path P and an x_4-x_2 path Q such that $E(P) \cap E(Q) = \emptyset$, and there is a unique vertex $z \in T^\diamond$ at which P, Q *cross*, that is, P contains two opposite edges of z and Q contains the other two (and so for every other vertex v in $P \cap Q$, we have no crossing at v). Figure 3 shows how to choose these paths P, Q.

In the other case, where T has a vertex of degree 4, we choose P, Q so that we have exactly two vertices z_1, z_2 meeting $P \cup Q$ in three edges, and all other vertices meet $P \cup Q$ in at most two edges; see the right side of Figure 3.

We can now construct the desired flow f' from a finite flow along P and an appropriate linear combination of f_1, f_3, where the coefficients, one positive and one negative, are tuned in such a way that Kirchhoff's node law (1) is satisfied at x_1 and x_3. Similarly, the flow h' can be constructed using a linear combination of f_2, f_4, and a finite flow along Q.

Note that f' induces a flow f on G° and h' induces a flow h on $G^{*\circ}$ by sending appropriate amounts of flow along the edges of G or G^* (as explained in the proof of Lemma 4.4). We claim that, in the case where T has no vertex of degree 4, the only edge in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$ is the edge e_z of G corresponding to the vertex z of $T^\diamond$, while in the case where T does have a vertex of degree 4, the only edge in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$ is one of the two edges e_{z_1}, e_{z_2}. Indeed, the supports of the f_i meet no common roundabouts by (1), and as P, Q lie in T, the choice of T combined with (2) implies that no edge in $\mathrm{supp}(f_i)$ contributes to $\mathrm{supp}(f) \cap \mathrm{supp}(h)$. Thus the only possible intersections come from vertices of $G^\diamond$ in $P \cap Q$.

Now in the case where T has no vertex of degree 4, note that every vertex in $P \cap Q$ has all its 4 edges in $P \cup Q$. It is now straightforward to check using the definitions of the graphs $G^\circ, G^\diamond, G^{*\circ}$ that z is the only vertex whose edge is in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$, as z was the only vertex at which P and Q cross.

In the case where T does have a vertex of degree 4, similar arguments apply, and it is again straightforward to check that exactly one of e_{z_1}, e_{z_2} is in $\text{supp}(f) \cap \text{supp}(h)$ (which of the two depends on which of f', h' we use to induce a flow in G° and which in $G^{*\circ}$).

Thus, in the uniquely absorbing case, it only remains to prove that we can indeed choose vertices x_i, flows f_i, and paths P_i with properties (1), (2) and (3) above.

For this, recall that the length of the circumference of $\mathcal{C}$ is 1, and let $M_i, 1 \leq 4$ denote the meridian of $\mathcal{C}$ whose width coordinate is $i/4 \pmod 1$. For each i, let $h_i \in (0, \frac{1}{16})$ be small enough that every roundabout of G° meeting M_i at a point whose height coordinate is less than h_i has width less than $1/8$, where the width of a roundabout O is defined to be the maximum width of a line segment contained in $\tau[O]$; such a choice is possible because $\tau[O]$ is two squares wide at each horizontal level by (6) (where we use the fact that O bounds a face of $G^\diamond$), and a square that starts close to the bottom of $\mathcal{C}$ cannot be very wide. In addition, we choose h_i even smaller, if needed, to ensure that if x is a vertex such that $\tau(x)$ meets M_i below height h_i, then $w(\tau(x)) < 1/8$; this is possible because there are are only finitely many edges e with $w(\tau(e))$ greater than any fixed constant since $\mathcal{C}$ has finite area, and $\tau(x)$ is at most three squares $\tau(e)$ wide by (5) and the fact that $G^\diamond$ is 4-regular.

Let $\lceil h_i M_i \rceil$ denote the subset of M_i with height coordinates ranging between zero and h_i, and $\lfloor h_i M_i \rfloor$ the subset of M_i with height coordinates ranging between h_i and 1.

For every $i \leq 4$, there is a lowermost edge e_i meeting $\lceil h_i M_i \rceil$ such that the roundabout O_i of G° containing e_i also contains an edge g_i meeting $\lfloor h_i M_i \rfloor$ (Figure 4); this is true because $\lfloor h_i M_i \rfloor$, being closed, only meets finitely many squares of positive area, and so there are finitely many roundabouts to choose from. There is at least one to choose from: a roundabout whose image contains the point of M_i at height h_i.

Let x_i denote the endvertex of e_i whose height coordinate is lower, and note that $\tau(x_i)$ meets M_i. Let M'_i be a meridian meeting $\tau(e_i)$ (and in particular $\tau(x_i)$) close enough to M_i, but distinct from M_i, that the rectangle $\lceil M_i x_i M'_i \rceil$ bounded by M_i, x_i, M'_i and the bottom circle of $\mathcal{C}$, meets the τ image of no roundabout meeting $\lfloor h_i M_i \rfloor$; such a M'_i exists because, by the choice of e_i, O_i, no roundabout meeting $\lfloor h_i M_i \rfloor$ has an edge e meeting $\lceil M_i x_i M'_i \rceil$, or we would have chosen e instead of e_i. As we can choose M'_i as close to M_i as we wish, we may assume that $d(M_i, M'_i) < 1/16$, which will be useful later.

Let f_i be the flow out of x_i that lives in $\lceil M_i x_i M'_i \rceil$, as defined above. We

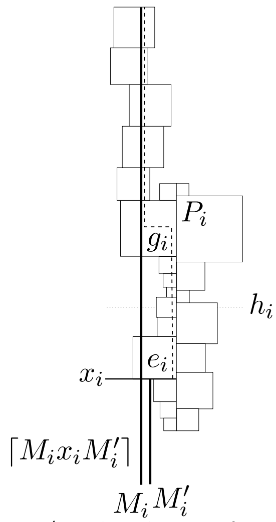

Figure 4: The choice of x_i, f_i and P_i.

claim that

If $e \in \text{supp}(f_i)$, then $\tau(e)$ is contained in the open vertical strip of radius 1/8 centered at M_i. (6)

Indeed, by the definition of f_i, if $e \in \text{supp}(f_i)$, then $\tau(e)$ intersects the interior of $\lceil M_i x_i M_i' \rceil$. Then $\tau(e)$ cannot have a point at height higher that h_i, which we recall is less than 1/16, because it would have to intersect the interior of $\tau(e_i)$ in that case, contradicting (3). Thus the height of $\tau(e)$ is at most 1/16, and being a square, so is its width. Together with our assumption that $d(M_i, M_i') < 1/16$, this proves our claim.

Note that (6), combined with the choice of the M_i, immediately implies that $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ for $i \neq j$; in fact, it even implies the stronger statement of (1), because by (6) if edges e, f lie in a common roundabout then $\tau(e), \tau(f)$ must meet a common meridian.

It remains to construct the paths P_i: we let P_i start with the x_i-g_i path in O_i containing e_i, and continue with the g_i-o path consisting of all the edges whose τ-image meets M_i above $\tau(f_i)$. To make the later path well-defined, we would like M_i to meet no trivial squares $\tau(e)$ of zero width. This can easily be achieved: since $G^\diamond$ has only countably many edges, and every trivial square meets just one meridian, we can arrange for our 4 M_i to be among the uncountably many remaining ones by rotating $\mathcal{C}$ appropriately. The fact that the edges whose τ-image meets M_i above $\tau(g_i)$ form a g_i-o path now follows from (5) and the fact that $\tau(o)$ is the top circle of $\mathcal{C}$. In fact, by the above argument, we can even assume that M_i does not meet the boundary of any square $\tau(e)$, and so M_i uniquely determines that g_i-o path. Note that by construction,

$$\text{every edge of } P_i \text{ is in a roundabout } O \text{ such that } \tau[O] \text{ meets } M_i. \tag{7}$$

To see that (2) is satisfied, recall that we chose h_i small enough that every roundabout of G° meeting M_i at a point whose height coordinate is less than h_i has width less than $1/8$, and P_i only uses roundabouts meeting M_i. Thus for $e \in E(P_i)$, $\tau(e)$ is contained in the vertical strip of radius $1/8$ centered at M_i. On the other hand, (6) says that the support of f_j is contained in the strip of radius $1/8$ centered at M_j, and so (2) follows from the fact that $d(M_i, M_j) \geq 1/4$.

Finally, we can prove (3) by a similar argument, now using the fact that $w(\tau(x_j)) < 1/8$ by the second part of our definition of h_j, and the fact that the roundabout containing the first edge e_i of P_i is contained in the strip of radius $1/8$ centered at M_i and every roundabout containing an edge of P_j meets M_j by (7).

Suppose now $G^\diamond$ is not uniquely absorbing. Then for some finite subgraph G_0 we have at least two absorbing components D_1, D_2 in $\mathbb{R}^2 \setminus G_0$. By elementary topological arguments, G_0 contains a cycle C such that both the interior I and the exterior O of C contain transient subgraphs of $G^\diamond$.

If any of these subgraphs I, O is uniquely absorbing, then we can repeat the above arguments to that subgraph to obtain the two desired flows.

Hence it remains to consider the case where there is a cycle C_I in I and a cycle C_O in O that further separate each of I, O into two transient sides. In fact, we can iterate this argument as often as we like, to obtain many distinct transient subgraphs separated from any given cycle. Let us iterate it often enough to obtain four disjoint cycles $C_i, 1 \leq 4$, and inside each C_i a cycle D_i such that the interior of D_i is transient and no roundabout ot G° meets any two of these eight cycles.

We now apply Theorem 2.1 to each of the four interior sides of the D_i to obtain four transience currents f_i out of vertices x_i, such that the support of f_i is contained in D_i. We can then combine those flows pairwise in a way similar to the uniquely absorbing case to obtain the two desired flows f', h', and from them f, h: we can let o be an arbitrary vertex outside all C_i, and define T and the paths P, Q similarly. The fact that $|\operatorname{supp}(f) \cap \operatorname{supp}(h)| = 1$ follows from the same graph-theoretic arguments about the structure of $G^\diamond$, for which we did not need the square tiling. $\square$

6 Harmonic functions on plane graphs

In this section, we use Theorem 3.1 to prove a new existence criterion for Dirichlet harmonic functions, Theorem 6.3 below, which is used in the proof of Theorem 1.1. Before proving Theorem 6.3, we prove the following which we think is interesting in its own right, and which motivated the main result of this section.

Theorem 6.1. *Let G and G^* be locally finite 1-ended dual plane graphs. Then the following are equivalent:*

1. *$G \notin \mathcal{O}_{HD}$;*
2. *$G^* \notin \mathcal{O}_{HD}$;*
3. *there are flows f and h of finite energy of G and G^* respectively whose supports intersect in a single edge.*

Proof. By symmetry, it suffices to show that 1 is equivalent to 3. If $G \notin \mathcal{O}_{HD}$, then let f and ρ be as in Corollary 3.2. Then f and $\partial\rho$ witness 3.

For the converse suppose there are flows f and h as in 3. Then h^* satisfies Kirchhoff's cycle law by Remark 3.6 because 2 in that remark cannot be fulfilled as G is 1-ended and transient graphs are infinite. Thus h^* is induced by a potential ρ, which together with f witnesses that $G \notin \mathcal{O}_{HD}$ by Corollary 3.2. □

Example 6.2. We give a simple example that neither 2 nor 3 imply 1 in Theorem 6.1 if we leave out the assumption that G and G^* are 1-ended. Let H be the graph obtained from disjoint cycles C_n of length 2^n by gluing C_n and C_{n+1} together at a single edge for each n that are distinct for different n. We obtain the graph G from a triangle by gluing two copies of H at distinct edges of the triangle.

In the next theorem, we propose a strengthening of 3 which implies that $G \notin \mathcal{O}_{HD}$ - even if G has more than one end.

Theorem 6.3. *Let G and G^* be locally finite dual plane graphs such that there are flows f and h of finite energy in the roundabout graphs G° and $G^{*\circ}$ respectively whose supports intersect in a single edge. Then $G \notin \mathcal{O}_{HD}$.*

Proof. Let $h[G^*]$ be the restriction of h to $E(G^*)$, which is a flow of G^* as h satisfies Kirchhoff's node law at the set of vertices of each roundabout.

Case 1: $h[G^*]^*$ satisfies Kirchhoff's cycle lawin G. Then let ρ be a potential induced by $h[G^*]^*$, and let $f[G]$ be restriction of f to $E(G)$, which is a flow of G as f satisfies Kirchhoff's node law at the set of vertices of each roundabout. Then f and ρ witnesses that $G \notin \mathcal{O}_{HD}$ by Corollary 3.2.

Having dealt which case 1, the remaining case is by Remark 3.6:

Case 2: There is a finite bond b of G^* such that $h[G^*]$ witnesses that the two components D_1 and D_2 of $G^* - b$ are transient. The bond b considered as an edge set of G is the set of edges of a cycle C, see Figure 5.

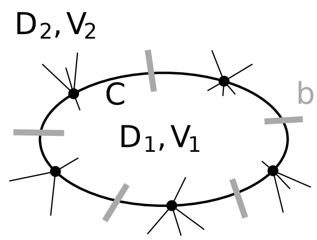

Figure 5: The cycle C, drawn thick, separates V_1 from V_2. In the dual, the bond b, drawn grey, separates D_1 from D_2.

Without loss of generality all edges of D_1 are contained in the interior of C, and the edges of D_2 in the exterior of C. Let V_1 be the set of vertices of G contained in the interior of C, and V_2 those vertices in the exterior. Since the set X of edges incident with a vertex of C contains a cut X' separating V_1 from V_2, by Corollary 3.4 it remains to show that $G[V_1]$ and $G[V_2]$ are both transient.

To see that $G[V_1]$ is transient, it suffices to show that $G[V_1]^\circ$ is transient by Lemma 4.3. Note that $G[V_1]$ and $G^*[D_1]$ are both locally finite and have locally finite duals. Moreover, the dual of $G[V_1]$ can be obtained from G^* by contracting all edges not in $G[V_1]$ (considered as edges of G^*). Thus the dual of $G[V_1]$ can be obtained from $G^*[D_1]$ by identifying finitely many vertices (and deleting finitely many loops). Since transience is invariant under changing finitely many edges or vertices, it remains to show that $G^*[D_1]^\circ$ is transient by Corollary 4.5. However, this is witnessed by $h[G^*]$. Summing up, the transience of $G[V_1]$ is inherited from $G^*[D_1]^\circ$ via $G[V_1]^\circ$.

Similarly, $G[V_2]$ is transient. Thus $G \notin \mathcal{O}_{HD}$ by Lemma 4.3 applied to X' and the $G[V_i]$.

□

7 Proof of the main result

Before proving Theorem 1.1, we need the following.

Lemma 7.1. *Let G be a locally finite plane UK-transient graph. Then there is a locally finite plane UK-transient supergraph H of G such that its dual H^* is locally finite, and $H \in \mathcal{O}_{HD}$ if and only if $G \in \mathcal{O}_{HD}$.*

Proof. To make sure that H^* is locally finite, we let G' be a supergraph of G obtained by 'triangulating' every infinite face of G in such a way that each vertex of G receives at most 2 new edges per incident face (any finite number would do in place of 2); this is easy to do. As $V(G)$ is countable, the set of newly added edges is countable. Take an enumeration of the set of newly added edges and subdivide the n-th edge 2^n-times. Call the resulting graph H. Note that H is locally finite and all its faces are finite. The roundabout graph of H has a subgraph which can be obtained from the roundabout graph of G by subdividing each edge at most twice.

Thus H is UK-transient. By Theorem 3.5 H is in $\mathcal{O}_{HD}$ if and only if G is in $\mathcal{O}_{HD}$, thus H has the desired properties. □

Proof of Theorem 1.1. By Lemma 7.1, we may assume without loss of generality that G^* is locally finite. Thus the theorem follows from combining Lemma 5.1 with Theorem 6.3. □

8 Applications

A vertex is in the *boundary* ∂X of some vertex set X if it is not in X but adjacent to a vertex in X. An infinite graph G is *non-amenable* if there is a constant $\gamma > 0$ such that the boundary ∂S has size at least $\gamma \cdot |S|$ for every finite vertex set S of G. The supremum of such values for γ is the *Cheeger-constant $Ch(G)$ of G.*

Lemma 8.1. *If a locally finite plane graph G is non-amenable, then so is its roundabout graph.*

Proof. Let X be a finite vertex set of the roundabout graph of G. Let $\overline{X}$ be the set of those vertices of G whose roundabouts meet X.

Sublemma 8.2. *Less than $6 \cdot |\overline{X}|$ vertices of X have all their neighbours in X.*

Proof. Let Y be the set of those vertices of X with all their neighbours in X. If $(v,e) \in Y$, then $(w,e) \in X$ where w is the other endvertex of e. Thus $|Y| \leq 2 \cdot |E(\overline{X})|$. As $(\overline{X}, E(\overline{X}))$ is plane, it has average degree less than 6. Thus $|E(\overline{X})| < 3 \cdot |\overline{X}|$, and thus $|Y| < 6 \cdot |\overline{X}|$. $\square$

If $|X| \geq 12 \cdot |\overline{X}|$, then at least $|X|/2$ vertices of X have a neighbour outside X. As the roundabout graph has maximal degree 3, the neighbourhood of X has then size at least $|X|/6$. Thus we may assume that $|X| < 12 \cdot |\overline{X}|$. Let $\overline{\overline{X}}$ be the set of those vertices of $\overline{X}$ whose whole roundabout is in X. Let $\epsilon = (|\overline{X}| - |\overline{\overline{X}}|)/|\overline{X}|$.

Sublemma 8.3. $|\partial X| > \frac{\epsilon}{12}|X|$

Proof. The roundabout of some $x \in \overline{X} \setminus \overline{\overline{X}}$ contains a vertex of ∂X, in formulas: $|\partial X| \geq |\overline{X} \setminus \overline{\overline{X}}| = \epsilon \cdot |\overline{X}|$. Thus the lemma follows from the assumption that $|X| < 12 \cdot |\overline{X}|$. $\square$

Sublemma 8.4. $|\partial X| \geq K(\epsilon) \cdot |X|$*, where* $K(\epsilon) = \frac{Ch(G) \cdot (1-\epsilon) - \epsilon}{12}$.

Proof. Each vertex in $\partial \overline{\overline{X}}$ is in $\overline{X} \setminus \overline{\overline{X}}$ or its roundabout contains a vertex of ∂X. Thus we estimate:

$$|\partial X| \geq |\partial \overline{\overline{X}}| - |\overline{X} \setminus \overline{\overline{X}}| \geq Ch(G) \cdot |\overline{\overline{X}}| - \epsilon |\overline{X}|$$

Note that $|\overline{\overline{X}}| = (1-\epsilon) \cdot |\overline{X}|$. Thus $|\partial X| \geq K(\epsilon) \cdot |X|$, where $K(\epsilon) = \frac{Ch(G)(1-\epsilon)-\epsilon}{12}$. $\square$

There is a positive constant δ - only depending on $Ch(G)$ - such that $K(\delta') \geq Ch(G)/24$ for all $\delta' \leq \delta$. Let γ be the minimum of $\frac{\delta}{12}$ and $\frac{Ch(G)}{24}$. Then $|\partial X| \geq \gamma \cdot |X|$ by Sublemma 8.3 and Sublemma 8.4. Hence the roundabout graph of G is non-amenable. $\square$

Proof of Theorem 1.3 (already mentioned in the Introduction). If G is non-amenable, then so is G° by Lemma 8.1. Every non-amenable locally finite graph is transient as it contains a subtree with positive Cheeger-constant by a result of Benjamini and Schramm [7]. $\square$

Corollary 8.5. *Every locally finite planar non-amenable graph G admits a non-constant Dirichlet harmonic function.*

Proof. Just combine Theorem 1.3 and Theorem 1.1. $\square$

Corollary 8.6. *Let G be a locally finite planar graph G such that there is a flow f of intensity 1 out of some vertex v such that $\sum_{v\in V(G)} deg(v)\left(\sum_{e|v\in e}|f(e)|\right)^2$ is finite. Then G has a non-constant Dirichlet harmonic function.*

Proof. For a vertex z of G°, we denote by $\vec{e_z}$ the unique directed edge not in any roundabout and pointing towards z.

By Theorem 1.1, it remains to extend f to a flow of G° from some vertex v' in the roundabout of v of finite energy by assigning values to the edges of the roundabout. At each roundabout C for a vertex $w\neq v$ of G, this is a finite Dirichlet-Problem: We want to find a function g_w assigning values to the directed edges of C such that at the vertex z it accumulates $-f(\vec{e_z})$. As f satisfies Kirchhoff's node law at w, the sum of the $f(\vec{e_z})$ is 0.

It is well-known that there is such a g_w and it is unique up to adding a multiple of the constant flow around C. Choosing g_w of minimal energy ensures for every $k\in C$ that $|g_w(k)|\leq\sum_{e|w\in e}|f(e)|$ since otherwise we could add a constant flow to g_w decreasing the energy. Pick a vertex v' in the roundabout for v. As above, there is a function g_v at the roundabout for v which at the vertex $z\neq v'$ accumulates $-f(\vec{e_z})$, and accumulates $1-f(\vec{e_{v'}})$ at v'.

Then f together with the g_x defines a flow of G° from v' of intensity 1, whose energy is bounded by $\sum_{v\in V(G)} deg(v)\left(\sum_{e|v\in e}|f(e)|\right)^2$, and thus finite.

□

Given a locally finite graph G, for $e=vw$ we let $r(e)=deg(v)^2+deg(w)^2$. The graph G is *super transient* if there is a flow from some vertex of intensity 1 such that its r-weighted energy is finite, that is, $\sum_{e\in E(G)} f(e)^2 r(e)$ is finite. Note that super transience implies transience. Note that G is super transient if and only if the graph $G[r]$ is transient, where we obtain $G[r]$ from G by subdividing each edge e $r(e)$-many times.

Corollary 8.7. *Every super transient planar locally finite graph G has a non-constant Dirichlet harmonic function.*

Proof. By Cauchy-Schwarz, $\left(\sum_{e|v\in e}|f(e)|\right)^2\leq deg(v)\sum_{e|v\in e} f(e)^2$. Thus this follows from Corollary 8.6 □

We can now re-prove the result of [5] that motivated our work:

Corollary 8.8 ([5])**.** *Every transient planar graph of bounded degree has a non-constant Dirichlet harmonic function.*

Proof. A transient bounded degree graph is super transient, so this follows from Corollary 8.7. □

We remark that if we omit the assumption of planarity, then Corollary 8.7 and Corollary 8.6 become false as the example of the 3-dimensional grid $\mathbb{Z}^3$ shows. Indeed, it is in $\mathcal{O}_{HD}$ but transient and thus super transient as its degrees are uniformly bounded. The next example shows that Corollary 8.6 is best-possible.

Example 8.9. In this example, we show that the order of magnitude in Corollary 8.6 is best possible. More precisely, we construct a locally finite planar graph G without non-constant Dirichlet-harmonic functions but still with a flow f out of some vertex such that for every $\epsilon > 0$ the term $E_\epsilon(f) = \sum_{v\in V(G)} deg(v)^{(1-\epsilon)} \left(\sum_{e|v\in e} |f(e)|\right)^2$ is finite.

In this construction, we rely on the fact that the 2-dimensional grid $\mathbb{Z}^2$ has a subdivision T of the infinite binary tree T_2 such that edges at level n are subdivided at most 2^n-times. It is straightforward to construct this subdivision T recursively and we leave the details to the reader. We obtain G from $\mathbb{Z}^2$ by contracting for each edge e of T_2 all but one of its subdivision edges.

As the branch set of each vertex of G is finite, G and its dual are 1-ended. Moreover, the dual of G is obtained from $\mathbb{Z}^2$ by deleting edges. Thus by Theorem 6.1, $G \in \mathcal{O}_{HD}$.

Next we construct f. The subtree S of G consisting of those edges of T that are not contracted is isomorphic to T_2. Let f be the flow on T_2 which assigns edges at level n the value 2^{-n}. Thus f induces a flow on G with support S.

Next we estimate $E_\epsilon(f)$. A vertex v at level n of S has degree at most $8 \cdot 2^n$. Thus

$$E_\epsilon(f) \leq 1000 \cdot \sum_{n\in\mathbb{N}} 2^n \cdot 2^{n(1-\epsilon)} \cdot 2^{-2n} = 1000 \cdot \sum_{n\in\mathbb{N}} 2^{-\epsilon n}$$

Hence $E_\epsilon(f)$ is finite, completing this example.

9 Further remarks

As mentioned in the introduction, we expect our notion of UK-transience to find further applications. For example, we expect that the results of [20, Section 2] generalise from bounded-degree non-amenable planar maps to UK-transient ones.

A lot of this paper is motivated by [13], the main result of which states that the Poisson boundary of every bounded degree, uniquely absorbing, plane graph coincides with the boundary of the square tiling; this had been asked by Benjamini & Schramm [6]. We can now ask whether this generalises to graphs of unbounded degree using UK-transience:

Problem 1. *Does the Poisson boundary of every uniquely absorbing, UK-transient plane graph coincide with the boundary of its square tiling?*

A closely related result of [1] states that the Poisson boundary of every 1-ended triangulation of the plane coincides with the boundary of its circle packing. Again, we ask for a similar generalisation:

Problem 2. *Does the Poisson boundary of every 1-ended, UK-transient, triangulation of the plane coincide with the boundary of its circle packing?*

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