Highly connected infinite digraphs without edge-disjoint back and forth paths between a certain vertex pair

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Abstract

We construct for all $k \in \mathbb{N}$ a k-edge-connected digraph D with $s, t \in V(D)$ such that there are no edge-disjoint $s \to t$ and $t \to s$ paths. We use in our construction "self-similar" graphs which technique could be useful in other problems as well.

1 Introduction

1.1 Basic notions

In this paper by "path" we mean a finite, simple, directed path. Sometimes we define a path of a digraph $D = (V, A)$ by a finite sequence $v_0, \dots, v_n$ of vertices of D. If there are more than one edges from v_i to v_{i+1} for some $i < n$, then it is not specified which edge is used by the path, so we use this kind of definition only if it does not matter. An $u \to v$ path is a path with initial vertex u and terminal vertex v. Its length is the number of its edges. We call a digraph D connected if for all $u, v \in V(D)$ there is a $u \to v$ path in D. For $U \subseteq V$ let $\mathsf{span}_D(U)$ be the set of those edges of D whose heads and tails are contained in U and let $D[U] = (U, \mathsf{span}_D(U))$. If it is clear what digraph we talk about, then we omit the subscripts.

1.2 Background and Motivation

R. Aharoni and C. Thomassen proved by a construction the following theorem that shows that several theorems about edge-connectivity properties of finite graphs and digraphs become "very" false in the infinite case.

Theorem 1 (R. Aharoni, C. Thomassen [1]). *For all $k \in \mathbb{N}$ there is an infinite graph $G = (V, E)$ and $s, t \in V$ such that E has a k-edge-connected orientation but for each path P between s and t the graph $G = (V, E \setminus E(P))$ is not connected.*

In this article we would like to introduce a similar result. If D is a k-edge-connected finite digraph, then for all $s_1, t_1, \dots, s_k, t_k \in V(D)$ there are pairwise edge-disjoint paths $P_1, \dots, P_k$

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such that P_i is an $s_i \to t_i$ path. This fact is implied by the following Theorem of W. Mader as well as the (strong form of) Edmonds' Branching theorem (see [2] p. 349 Theorem 10.2.1).

Theorem 2 (W. Mader [4])**.** *Let $D = (V, A)$ be a $k+1$-edge-connected, finite digraph and $s, t \in V$. Then there is an $s \to t$ path P such that $(V, A \setminus A(P))$ is k-edge-connected.*

We will show that in the infinite case there is no $k \in \mathbb{N}$ such that k-edge-connectivity guarantees even the existence of edge-disjoint $s_1 \to t_1$ and $s_2 \to t_2$ paths for all s_1, t_1, s_2, t_2 vertices. Not even in the special case where the two ordered vertex pair is the reverse of each other.

2 Main result

Theorem 3. *For all $k \in \mathbb{N}$ there exists a k-edge-connected digraph without back and forth edge-disjoint paths between a certain vertex pair.*

Proof. Let $k \geq 2$ be fixed, $I = \{0, \dots, 2k-1\}$, $I_e = \{i \in I : i \text{ is even }\}$, $I_o = I \setminus I_e$. Denote by I^* the set of finite sequences from I. Let the vertex set V of the digraph is the union of the disjoint sets $\{s_\mu : \mu \in I^*\}$ (we mean $s_\mu = s_\nu$ iff $\mu = \nu$) and $\{t_\mu : \mu \in I^*\}$ ($t_\mu = t_\nu$ iff $\mu = \nu$). If μ is the empty sequence we write simply s, t and we denote the concatenation of sequences by writing them successively. For $\nu \in I^*$ let denote the set $\{r_{\nu\mu} : r \in \{s, t\},\ \mu \in I^*\} \subseteq V$ by V_ν. The edge-set A of the digraph consists of the following edges. For all $\mu \in I^*$ there are k edges in both directions between the two elements of the following pairs: $\{s_\mu, t_{\mu 1}\}$, $\{s_{\mu i}, t_{\mu(i+2)}\}$ $(i = 0, \dots, 2k-3)$, $\{s_{\mu(2k-2)}, t_\mu\}$. Simple directed edges are $(s_\mu, t_{\mu 0}), (t_{\mu i}, s_{\mu(i+1)})_{i \in I_e}, (s_{\mu i}, t_{\mu(i+1)})_{i \in I_o \setminus \{2k-1\}}, (s_{\mu(2k-1)}, t_\mu)$ for all $\mu \in I^*$. Finally $D \overset{\text{def}}{=} (V, A)$ (see figure 1).

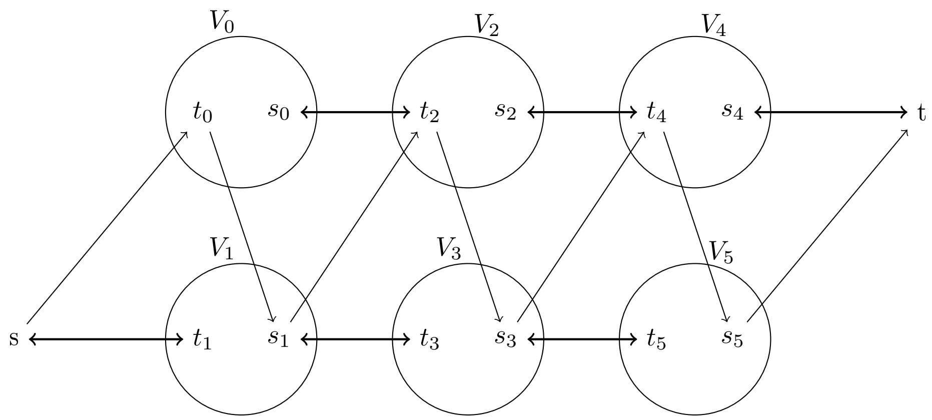

Figure 1: The digraph D in the case $k = 3$. Thick, two-headed arrows stand for k parallel edges in both directions. The (just partially drawn) $D[V_i]$'s are isomorphic to the whole D by Proposition 5.

Remark 4. One can avoid using parallel edges (without losing the desired properties of the digraph) by dividing each of these edges with one-one new vertex and drawing between them $k(k-1)$-many new directed edges, one-one for each ordered pair. One can also achieve k-connectivity instead of k-edge-connectivity by using some similarly easy modification.

Proposition 5. *For $\nu \in I^*$ the function $f_\nu : V \to V_\nu,\ f_\nu(r_\mu) \stackrel{def}{=} r_{\nu\mu}$ ($r \in \{s,t\}$) is an isomorphism between D and $D[V_\nu]$.*

Proof: It is a direct consequence of the definition of the edges since the number of edges from r_μ to $r'_{\mu'}$ are the same as from $r_{\nu\mu}$ to $r'_{\nu\mu'}$ for all $r, r' \in \{s,t\},\ \nu, \mu, \mu' \in I^*$. ●

Proposition 6. *Denote by D_v the digraph that we obtain from D by contracting for all $i \in I$ the set V_i to a vertex v_i. Then D_v is k-edge-connected.*

Proof: In the vertex-sequence $s, v_1, v_3, \ldots, v_{2k-1}$ there are k edges in both directions between the neighboring vertices such as in the sequence $v_0, v_2, \ldots, v_{2k-2}, t$. Finally there are in both directions at least k edges between the vertex sets of the sequences above. ●

For $u \neq v$ we denote by $\lambda(u,v)$ the local edge-connectivity from u to v in D (i.e. $\lambda(u,v) = \min\{|A'| : A' \subseteq A,$ there is no path from u to v in $(V, A \backslash A')\})$ and let $\lambda\{u,v\} \stackrel{\text{def}}{=} \min\{\lambda(u,v), \lambda(v,u)\}$.

Proposition 7. *D is connected.*

Proof: We will show that $\lambda\{s, r_\mu\} \geq 1$ for all $r \in \{s,t\},\ \mu \in I^*$. We will use induction on length of μ (which is denoted by $|\mu|$). Consider first the $|\mu| = 0, 1$ cases directly.

The path $s, t_0, s_1, t_2, s_3, \ldots, t_{2k-2}, s_{2k-1}, t$ shows that $\lambda(s,t) \geq 1$. Using the isomorphism f_i (see Proposition 5) we may fix an $s_i \to t_i$ path P_{s_i,t_i} in $D[V_i]$ for all $i \in I$. The path

$$t, P_{s_{2k-2},t_{2k-2}}, \ldots, P_{s_{2k-2j},t_{2k-2j}}, \ldots, P_{s_0,t_0}, P_{s_1,t_1}, s$$

justifies that $\lambda(t,s) \geq 1$ (thus $\lambda\{s,t\} \geq 1$). Then we may fix a $t_i \to s_i$ path P_{t_i,s_i} in $D[V_i]$ ($i \in I$). The paths

$$s, P_{t_1,s_1}, P_{t_3,s_3}, \ldots, P_{t_{2j+1},s_{2j+1}}, \ldots, P_{t_{2k-1},s_{2k-1}}$$
$$P_{s_{2k-1},t_{2k-1}}, P_{s_{2k-3},t_{2k-3}}, \ldots, P_{s_{2k-1-2j},t_{2k-1-2j}}, \ldots, P_{s_1,t_1}, s$$

certify that $\lambda\{s, r_i\} \geq 1$ if $r \in \{s,t\},\ i \in I_o$. The paths

$$t, P_{s_{2k-2},t_{2k-2}}, P_{s_{2k-4},t_{2k-4}}, \ldots, P_{s_{2k-2-2j},t_{2k-2-2j}} \ldots, P_{s_0,t_0}$$
$$P_{t_0,s_0}, P_{t_2,s_2}, \ldots, P_{t_{2j},s_{2j}}, \ldots, P_{t_{2k-2},s_{2k-2}}, t$$

certify that $\lambda\{t, r_i\} \geq 1$ if $r \in \{s,t\} \geq 1,\ i \in I_e$ and thus (by $\lambda\{s,t\} \geq 1$ and by transitivity) $\lambda\{s, r_i\} \geq 1$ if $r \in \{s,t\},\ i \in I_e$. Hence the cases $\mu \in I*$ with $|\mu| \leq 1$ are settled.

Let be $l \geq 1$ and suppose $\lambda\{s, r_\mu\} \geq 1$ if $r \in \{s,t\},\ \mu \in I^*,\ |\mu| \leq l$. Let $\nu = \mu i$, where $i \in I$ and $|\mu| = l$. By the induction hypothesis we have $\lambda\{s, s_\mu\} \geq 1$. By the induction hypothesis for $l = 1$ we have $\lambda\{s, r_i\} \geq 1$ and so $\lambda\{s_\mu, r_{\mu i}\} \geq 1$ by the isomorphism f_μ. Combining these, we get $\lambda\{s, r_{\mu i}\} \geq 1$. ●

Lemma 8. *D is k-edge-connected.*

Proof: Let $k > l \geq 1$.

Proposition 9. *Let $\mu \in I^*$ arbitrary. If we delete at most l edges of the digraph $D[V_\mu]$ in such a way that its subgraphs $D[V_{\mu i}]$ ($i \in I$) remain connected after the deletion, then $D[V_\mu]$ also remains connected after the deletion.*

Proof: Because the isomorphism f_μ it is enough to deal with the case where μ is the empty sequence. Denote by D' the digraph that we have after the deletion. Let D'_v be the digraph that we get from D' by contracting the sets V_i to a vertex v_i for all $i \in I$. The digraphs $D'[V_i]$ $(i \in I)$ are connected by assumption, thus D' is connected iff D'_v is connected. The digraph D'_v arises by deleting at most $l < k$ edges of the k-edge-connected digraph D_v (see Proposition 6) hence it is connected. ●

We will prove that if D is l-edge-connected, then it is also $l+1$ edge-connected. This is enough since we have already proved 1-connectivity of D in Proposition 7. Assume that D is l-edge-connected. Let $C \subseteq A$, $|C| = l$ arbitrary and $D' \overset{\text{def}}{=} (V, A \setminus C)$. By the definition of $l+1$-edge connectivity we need to show that D' is connected. Suppose for contradiction that it is not. Since the connectivity of the subgraphs $D'[V_i]$ $(i \in I)$ implies the connectivity of D' (by Proposition 9) there is an $i_0 \in I$ such that $D'[V_{i_0}]$ is not connected. Since the connectivity of the subgraphs $D'[V_{i_0 i}]$ $(i \in I)$ implies the connectivity of $D'[V_{i_0}]$ there is an $i_1 \in I$ such that $D'[V_{i_0 i_1}]$ is not connected... By recursion we obtain an infinite sequence $(i_n)_{n\in\mathbb{N}}$ such that the digraphs $D'[V_{i_0 \dots i_n}]$ $(n \in \mathbb{N})$ are all disconnected. Note that the digraphs $D[V_{i_0 \dots i_n}]$ $(n \in \mathbb{N})$ are l-connected because D is l-connected by assumption and they are isomorphic to it, hence necessarily $C \subseteq \mathsf{span}(V_{i_0 \dots i_n})$ for all $n \in \mathbb{N}$. But then

$$C \subseteq \bigcap_{n=0}^{\infty} \mathsf{span}(V_{i_0 \dots i_n}) = \mathsf{span}\left(\bigcap_{n=0}^{\infty} V_{i_0 \dots i_n}\right) = \mathsf{span}(\varnothing) = \varnothing$$

which is a contradiction since $|C| = l \geq 1$. ■

Lemma 10. *There are no edge-disjoint back and forth paths between s and t in D.*

Proof: Suppose, seeking a contradiction, that there are. Let $P_{s,t}$ be an $s \to t$ path and $P_{t,s}$ be a $t \to s$ path such that they are edge-disjoint and have a minimal sum of lengths among these path pairs. For $u, v \in V$ call a set $U \subseteq V$ an uv-cut iff $u \in U$ and $v \notin U$. The set $\{t\} \cup \bigcup\{V_i : i \in I_e\}$ is a ts-cut and its outgoing edges are $\{(t_i, s_{i+1})\}_{i \in I_e}$. Let $i_0 \in I_e$ be the maximal index such that $P_{t,s}$ uses the edge $(t_{i_0} s_{i_0+1})$. Then an initial segment of $P_{t,s}$ is necessarily of the form $t, P_{s_{2k-2},t_{2k-2}}, P_{s_{2k-4},t_{2k-4}}, \dots, P_{s_{i_0},t_{i_0}}, s_{i_0+1}$ where P_{s_i,t_i} is an $s_i \to t_i$ path in $D[V_i]$. The set $T \overset{\text{def}}{=} \{t\} \cup \bigcup\{V_i : i_0 \leq i \in I\}$ is also a ts-cut and all the tails of its outgoing edges are in $\{t_{i_0}, t_{i_0+1}\}$. $P_{t,s}$ has already used the edge (t_{i_0}, s_{i_0+1}) so it may not use another edge with tail t_{i_0} hence $P_{t,s}$ leave T using an edge with tail t_{i_0+1}. But then $P_{t,s}$ contains an $s_{i_0+1} \to t_{i_0+1}$ subpath $P_{s_{i_0+1},t_{i_0+1}}$ in $D[V_{i_0+1}]$.
$S \overset{\text{def}}{=} \{s\} \cup \bigcup\{V_i : i_0 + 1 \geq i \in I\}$ is an st-cut and all the tails of its outgoing edges are in $\{s_{i_0}, s_{i_0+1}\}$. Therefore $P_{s,t}$ has an initial segment in $D[S]$ that terminates in this set. We know that $P_{s,t}$ does not use the edge (t_{i_0}, s_{i_0+1}) because $P_{t,s}$ has already used it. Therefore there is an $m \in \{i_0, i_0+1\}$ such that $P_{s,t}$ has a $t_m \to s_m$ subpath P_{t_m,s_m} in $D[V_m]$. But then the paths P_{t_m,s_m} and P_{s_m,t_m} are proper subpaths of $P_{s,t}$ and $P_{t,s}$ respectively. By Proposition 5 f_m is an isomorphism between D and $D[V_m]$ and thus the inverse-images of the paths P_{t_m,s_m} and P_{s_m,t_m} are edge-disjoint back and forth paths between s and t with strictly less sum of lengths than the added length of paths $P_{s,t}$ and $P_{t,s}$, which contradicts with the choice of $P_{s,t}$ and $P_{t,s}$.■

□

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