Blocks and 2-blocks of graph-like spaces

von

Hendrik Heine

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Gutachter:
Prof. Dr. Nathan Bowler

1 Introduction

The ideas spawning from the method of tree decompositions have led to many fruitful avenues of research. There is now a theory of abstract separations, first introduced by Reinhard Diestel and Sang-il Oum in the first, 2014 version of [6], which generalizes the well known concepts of separating sets in graphs and the connectivity of edge partitions in matroids. This has led to the simplification, unification and generalization of numerous theorems and proofs about decompositions.

One type of theorem that was affected are the results about splitting objects along their separations of some low order, where the order corresponds to the size of the separator in the graph case and the value of the connectivity function for matroids. For separations of order two in particular, it is now known for both infinite graphs ([11]) and infinite matroids ([1]) that the torsos of their parts are 3-connected, cycles/circuits or bonds/cocircuits.

Given these results, it seems natural to ask a question of the same type for objects that share some of the same features enabling the proofs for graphs and matroids. One such object, generalizing graphs by adding a topological structure, are the graph-like spaces defined in [3] as a way to represent graphic matroids, which are based on the graph-like continua of Thomassen and Vella ([12]).

In this thesis we first give the definitions required and examine some properties of pseudo-arcs in graph-like spaces, as defined in [3], which for many purposes can be treated similarly to paths in graphs. An example of this is the proof of a version of Menger's theorem for pseudo-arcs which we give. Afterwards we apply the framework of separations to graph-like spaces and define both a decomposition into blocks as well as one along the separations of order two of a graph-like space, prove that these blocks are always 2-connected and give, using methods of [11], a description of the torsos of the latter decomposition analogous to those for graphs and matroids given certain niceness conditions. We also explore some of their further properties.

In particular, we will show that for those graph-like spaces which induce matroids in the sense of [3] the parts of these decompositions of graph-like spaces correspond in a natural way to those of their familiar equivalents in the associated matroid. We will also consider the issue of reconstructing a graph-like space from the parts of our decomposition along 2-separations and find that, while it is not possible in general or even for all graph-like spaces inducing matroids, it can be done for compact graph-like spaces.

2 Preliminaries

2.1 Separation systems

We will use the concept of separations systems as a framework for our decompositions, the reference used for the definitions in this subsection is [6].

A *separation system* is a triple $(S, \leq, ^*)$, where $(S, \leq)$ is a partial order and * is an involution on S such that $s \leq t$ if and only if $t^* \leq s^*$ for all $s, t \in S$. We call the elements of S *(oriented) separations* and sets of the forms $\{s, s^*\}$ for $s \in S$ unoriented separations.

A separation s is called *small* if $s \leq s^*$, *trivial* if there exists a separation t such that $s < t$ and $s < t^*$ and *degenerate* if $s = s^*$. A pair of unoriented separations is *nested* if they contain comparable elements and a pair of ordered separations (s, t) is *nested* if the unoriented separation containing s is nested with the one containing t. A separation in some separation system is *good* if it is nested with any other separation of that system and neither it nor its inverse is small.

A *partial orientation* of a separation system S is a subset of S meeting each unoriented separation at most once. It is an *orientation* of S if it meets each unoriented separation exactly once. A partial orientation is *consistent* if it contains no separations $s \neq t$ such that $s^* < t$.

We call a set of separations $T \subseteq S$ *regular* if it contains no small separations and nested if all pairs of separations contained in it are nested. A *tree set* is a nested separation system containing no trivial or degenerate elements.

During the construction of a decomposition from a tree set we shall need the following special case of [6, Lemma 4.1].

Lemma 2.1. *Let S be a regular tree set and P a consistent partial orientation of S. Then P extends to a consistent orientation and for any maximal Element of P there is a unique consistent orientation in which that element is maximal.*

An important example of a tree set given in [8] is the set of oriented edges of a tree. Formally, for some tree T let $\overrightarrow{E}(T)$ be the set of all pairs (x, y) of vertices of T such that x and y are adjacent, let * be the involution exchanging the components of each pair and let $(a, b) \leq (c, d)$ if $(a, b) = (c, d)$ or the unique $\{a, b\}$-$\{c, d\}$-path meets b and c. Since the path must have some pair of endvertices, $\overrightarrow{E}(T)$ clearly is a regular tree set. An orientation o of $\overrightarrow{E}(T)$ *points towards* some vertex v of T, if no element of o has v in its first component. An orientation o of $\overrightarrow{E}(T)$ points towards an end ω of T if for every $(a, b) \in o$ the end ω lives in the component of $T - ab$ containing b. Indeed, the following lemma shows that these are the only two possibilites.

Lemma 2.2. *Every consistent orientation of $\overrightarrow{E}(T)$ for some tree T points to a vertex or an end of T.*

Proof. Let D be the directed graph induced by the edges of some consistent orientation o. Since two edges starting at the same vertex would contradict the consistency of o, each vertex has at most one edge directed outwards. If D has a vertex with no such edge, then o points to that vertex by definition. Otherwise we can find a directed ray R in D simply by starting at an arbitrary vertex and following its unique edge directed outwards. We claim that o points towards the end of this ray. Let $(a, b) \in o$ be arbitrary. Then by the consistency of o the unique $\{a, b\}$-R-path in T must meet b, completing the proof. □

2.2 Infinite matroids

While matroids have been researched for some time, a proper notion of infinite matroids was only established recently in [2], which the following definitions and statements are based on. For our purposes a *matroid* M on some ground set E will be defined as a set of subsets of E satisfying the following conditions:

1. $\emptyset \in M$
2. M is down-closed with regard to inclusion
3. For any $I \in M$ that is not maximal with regard to inclusion and any $J \in M$ that is there exists an $x \in J \setminus I$ such that $I \cup \{x\} \in M$.
4. For any $I \in M$ and $Z \subseteq E$ there exists an inclusion-maximal element of M that contains I and is contained in Z.

For the rest of this subsection we fix a matroid M on a ground set E.

Any element of E will be called an *edge*. The elements of M are called *independent* and subsets of E that are not elements of M are called *dependent*. An inclusion-maximal independent set is called a *base* and an inclusion-minimal dependent set is called a *circuit*.

A key property of circuits, *circuit elimination*, asserts that for any circuit C, any family $(e_i)_{i \in I}$ of different edges of C, any family $(C_i)_{i \in I}$ such that C_i contains e_i, but no e_j for $j \neq i$ and any z contained in C, but not any C_i, there exists a circuit D contained in the union of C and all the C_i such that

$z \in D$, but $e_i \notin D$ for all $i \in I$. Furthermore, the set of circuits of a matroid defines it uniquely.

The *dual* M^* of M is the matroid on E containing all the subsets of the complements of the maximal elements of M. The *restriction* of M to some $X \subseteq E$ is the matroid on the ground set X containing all those elements of M contained in X. The *contraction* onto X is the dual of the restriction of the dual of M to X.

A circuit of the dual of M is called a *cocircuit* of M. An edge e is called a *loop* if $\{e\}$ is a circuit and a *coloop* if $\{e\}$ is a cocircuit. If an edge is not a loop, it is contained in some cocircuit and dually, if an edge is not a coloop, it is contained in some circuit.

The notion of connectivity for infinite matroids was introduced in [5], where the connectivity function κ_M for a matroid M was defined as the function mapping each subset $X \subseteq E$ to the number of elements that need to be deleted from the union of a base of M restricted to X and a base of M restricted to the complement of X to obtain a base of M. They proved that this is well-defined and that this is in fact the same number of edges that need to be deleted from any base of M restricted to X to obtain a base of M contracted onto X. Furthermore, they showed that $\kappa_M = \kappa_{M^*}$ and that $\kappa_M(X) = 0$ if and only if there is no circuit meeting both X and X^c.

Writing $e \sim f$ for two edges $e.f$ if $e = f$ or if e and f are contained in a common circuit, a *component* of M is an equivalence class under $\sim$. M is called *connected* if all its edges are equivalent.

A *separation* of M is a pair (F, F^c) for any $F \subseteq E$ such that both F and F^c contain at least $\kappa_M(F) + 1$ edges. Its *order* $|(F, F^c)|$ is $\kappa_M(F) + 1$. A separation of order k is called a k-separation. Let $S_k(M)$ be the set of all k-separations of M. The *inverse* $(F, F^c)^*$ of a separation (F, F^c) is (F^c, F) and we write $(F_1, F_1^c) \leq (F_2, F_2^c)$ if $F_1 \subseteq F_2$. Then $(S_k, \leq, ^*)$ is a separation system. M is *k-connected* for some natural number k if all separations of order less than k are small. As shown in [5], a matroid is connected if and only if it is 1-connected. We write $g_k(M)$ for the set of separations which are good in S_k. As noted in the introduction, the decomposition of a matroid along its 2- separations from [1] will become important. We shall only state part of it here, for which we need some definitions. Later, we will also need a key lemma from the proof.

Proposition 2.3. *If M is connected, then for any infinite chain $(A_i, B_i)_{i \in I}$ of good separations of M, the intersection of the B_i is empty. In particular, there are no $\omega + 1$-chains of good separations.*

A *tree decomposition* of a matroid is a pair of a tree T and a (possibly trivial) partition $(R_v)_{v \in V(T)}$ of subsets E such that for any edge e of T the partition of the edges that has two edges $f \in R_v$ and $g \in R_w$ on the same side if and only if v and w lie in the same component of $T - e$ always corresponds to an unordered separation, the *induced unoriented separation of* e. If these are always k-separations, we say that it has *uniform adhesion* k. We call the R_v the *parts* of the tree decomposition.

For the rest of this section fix a tree decomposition $D = (T, (R_v)_{v \in V(T)})$. Let ϕ_v for some $v \in V(T)$ be the function mapping any subset X of M to $X \cap R_v$ together with vw for every w such that X meets some R_z where z is in the component of $T - vw$ containing w. The authors prove that for any $v \in V(T)$ there is a matroid on the set consisting of R_v together with all the edges of T adjacent with v, which has as its circuits all those sets $\phi_v(C)$ for a circuit C of M that are not singleton edges outside R_v. This is called the *torso* M_v of that R_v. For the rest of this subsection assume M to be connected and E to have at least three elements. We are now able to state the existence of their decomposition.

Theorem 2.4. *There exists a tree decomposition of uniform adhesion* 2 *of* M *such that any torso* M_v *of it is* 3*-connected, a circuit or a cocircuit and such that the union of all the induced unordered separations of the edges of* T *is the set of good separations of* M*.*

We will also need a special case of [1, Lemma 4.11], for which we will assume D to be as in Theorem 2.4.

Lemma 2.5. *Let* (F, F^c) *be a* 2*-separation of* M *and* $v \in V(T)$*. If* $|\phi_v(F)| \geq 2$ *and* $|\phi_v(F)^c| \geq 2$*, then* $(\phi_v(F), \phi_v(F^c))$ *is a* 2*-separation of* M_v*.*

We obtain the following corollary.

Corollary 2.6. *Let* (F, F^c) *be a* 2*-separation of* G *that is not good. Then there exists some* $v \in V(T)$ *such that* $(\phi_v(F), \phi_v(F^c))$ *is a* 2*-separation of* M_v*.*

Proof. Since (F, F^c) is not good, there must exists some $v \in V(T)$ meeting both F and F^c. But since it cannot cross any good separations, such a v must be unique. But then $|\phi_v(F)| \geq 2$ and $|\phi_v(F)^c| \geq 2$, since (F, F^c) is a 2-separation and we are done by Lemma 2.5. □

2.3 Graph-like spaces

A *graph-like space*, as defined in [3], is a topological space G together with a vertex set V, an edge set E and a continuous map $t_e^G : [0,1] \mapsto G$ for every $e \in E$ satisfying the following conditions:

1. V and $(0,1) \times E$ are disjoint.

2. The underlying set of G is $V \cup (0,1) \times E$.

3. $\{t_e^G(0), t_e^G(1)\} \subseteq V$ for every $e \in E$.

4. $\{t_e^G(x); x \in (0,1)\} \subseteq (0,1) \times \{e\}$ for every $e \in E$

5. $t_e(G)$ restricted to $(0,1)$ is an open map for every $e \in E$.

6. For $v, w \in V$ there exist disjoint open subsets U, U' of G such that $v \in U$, $w \in U'$ and $V(G) \subseteq U \cup U'$.

In this subsection G will always be a graph-like space. For an edge e of G we call $(0,1) \times e$ the set of *inner points* of e and write $G - e$ for the subspace obtained by deleting these points. We will write $V(G)$ for the vertex set and $E(G)$ for the edge set of G. We call $t_e^G(0)$ and $t_e^G(1)$ the *endvertices* of an edge e and call e a *loop* if they are the same. Two edges are called *parallel* if they have the same endvertices. For a set of edges E we write $V(E)$ for the set of endvertices of edges in E. A map of graph-like spaces is a continuous map from G to a graph-like space H that arises from a map $\phi_V : V(G) \to V(H)$ and a map $\phi_E : E(G) \to V(G) \cup (E(G) \times \{+, -\}$ (where $+$ and $-$ are chosen such that this is a disjoint union) by mapping any vertex as in ϕ_V and any inner edge point (x, e) to $\phi_E(e)$ if that is a vertex, to (x, f) if $\phi_E((x, e))$ has the form $(f, +)$ and to $(1 - x, f)$ if it has the form $(f, -)$.

A *graph-like subspace* H of G is a graph-like space such that its topological space is a subspace of that of G, its vertex and edge sets are subsets of those of G and the map for any edge of H is the same as the map for the same edge in G. An important example of a graph-like subspace is $G[X]$, the *graph-like subspace of* G *induced by* X, for some $X \subseteq V(G)$, which is defined as the graph-like space obtained from G by deleting all vertices in $V(G) \setminus X$ and all edges that do not have both endvertices in X. Another example is the *graph-like subspace of* G *induced by* F for some edge set F, which is the graph-like space obtained from G by restricting to the closure of F.

We call a compact, topologically connected graph-like subspace A of G a *pseudo-line (with endvertices $x, y \in V(A)$)* if for every edge e the vertices x and y are contained in different topological components of $A - e$ and for every $v, w \in V(A)$ there exists some edge f such that v and w are in different topological components of $G - f$. A map of graph-like spaces from a pseudo-line L to G is called a *pseudo-path* between the images of the endpoints of L, an injective such map is a *pseudo-arc.* We will call the images of the endvertices of L *endvertices* of the pseudo-path. Given two pseudo-lines and a pair of one endvertex of each their *concatenation* is the space obtained by taking their direct sum and identifying the two given vertices. This is again a pseudo-line by [3]. Given two pseudo-paths $f : L \to G$ and $g : M \to G$ such that $f(x) = g(y)$, where x and y are endvertices of their respective pseudo-lines, their concatenation is given by the function from the concatenation of L and M at x and y that agrees with f and g on their domains. Clearly this is a pseudo-path.

The space G is *pseudo-arc connected* if there exists a pseudo-arc between any two vertices of G. A *pseudo-arc component* with respect to pseudo-arcs is a maximal vertex set X such that $G[X]$ is connected.

Since pseudo-arc connectedness will play a much more important role in this thesis than topological connectedness for reasons that will become clearer in the section about separations, we will depart from standard usage in that whenever we speak of connectedness or components in graph-like spaces it is meant with respect to pseudo-arcs if not stated otherwise. Furthermore, we will say that some set of vertices X *separates* two others A and B if there is no pseudo-arc between any $a \in A$ and $b \in B$ not meeting X and similarly a set of edges E separates them if this hold for its set of inner points. A *pseudo-cycle* is the union of two pseudo-lines with the same endvertices which are otherwise disjoint.

Given sets X and Y in a topological space H with some designated set V an (X, Y)*-witness* in H (with respect to V) is a pair (U, W) of disjoint open sets in H such that $X \subseteq U$, $Y \subseteq W$ and $V \subseteq U \cup W$. A *witness* in H (with respect to V) is just any (X, Y)-witness for nonempty subsets X, Y of V. Clearly the sixth condition for graph-like spaces is equivalent to the existence of an (X, Y)-witness (with respect to the vertex set) for all finite vertex sets (X, Y). For graph-like spaces it is always assumed that witnesses and (X, Y)-witnesses are with respect to their vertex set. In a graph-like space a *cross-edge* of a witness (U, W) is an edge with an endvertex in both U and W.

A *topological cut* in G is a set of edges F such that there exists a witness (U, W) in G with F as its set of cross-edges. A graph-like space G *induces a matroid* M if the edge sets of pseudo-cycles of G are exactly the circuits of M and the minimal topological cuts of G are exactly the cocircuits of M.

To define a contraction in G let first $\equiv_C$ for some edge set C be the equivalence relation on $V(G)$ such that $u \equiv_C v$ for $u, v \in V(G)$ if and only if the set of crossing edges of every $(\{u\}, \{v\})$-witness meets C. Then the function f_C mapping each vertex of G to its equivalence class, all inner edge points of edges within an equivalence class to that class and every other point to itself defines a quotient topology, which is a graph-like space with the set of equivalence classes as vertex set and the edges not in C as edge set. We say that this graph-like space is obtained from G by *contracting* C.

We will need the following three results from [3], [4] and [9] respectively.

Lemma 2.7. *Any pseudo-line is the closure of its inner points of edges.*

Theorem 2.8. *Every compact graph-like space induces a matroid.*

Theorem 2.9. *If G is compact and topologically connected G is connected.*

A *tree-like space*, as defined in [9], is a compact graph-like space, such that between any two vertices there is a unique pseudo-line. We will write $L_T(s, t)$ for the unique pseudo-line between s and t in a tree-like space T.

The fact that for purposes of connectivity pseudo-arcs and pseudo-paths are equivalent concepts will often prove useful. To prove this, we will add an edge to our graph-like space. Thus for any set X of pairs of different elements of $V(G)$ we will write $G \oplus X$ for some fixed graph-like space constructed by choosing any family $(e_x)_{x\in X}$ not meeting $E(G)$, forming the sum of G and $[0, 1] \times \{e_x; x \in X\}$ and identifying $(0, e_x)$ and $(1, e_x)$ arbitrarily with the two elements of x.

Proposition 2.10. *If there is a pseudo-path f from $x \in V(G)$ to $y \in V(G)$ in G, there is a pseudo-arc from x to y in G, whose image is contained in the image of f.*

Proof. The image of f is a compact graph-like space. Therefore $X = G \oplus \{x, y\}$ is compact. Then X induces a matroid by Theorem 2.8. Assume first that e is a coloop in M. Then there must be some topological cut (A, B) such that e is the only edge with an endvertex in A and B. Adding the inner points of any other edge to the side its endvertices lie on and restricting to $X - e$ gives open sets A' and B' partitioning $X - e$, but this space is connected. Therefore, there exists some cycle C in M containing e. Let g be an isomorphism of graph-like spaces between some pseudo-circle P and a subgraph-like space containing exactly the edges of C and let f be the edge of P mapped to e. Then the restriction of g to $P - f$ is a pseudo-arc from x to y contained in the image of f. □

This also implies that no two components can share a vertex since then their union would easily be seen to be connected by concatenating pseudo-arcs. By Zorn's Lemma every vertex is contained in some component, so the components partition the vertices, as we would expect. We can also use this to obtain a sort of decomposition into pseudo-cycles of two pseudo-arcs with the same endvertices, similar to the one of a closed walk into circles for a finite graph.

Corollary 2.11. *If there are two different pseudo-lines L_1 and L_2 from $x \in V(G)$ to $y \in V(G)$, then for any $e \in E(L_1) \triangle E(L_2)$ there exists a pseudo-cycle in G containing e and using only edges of L_1 and L_2.*

Proof. Let e be any edge of L_1 not contained in L_2. Let a be its endpoint closer to x in L_1 and b the other. Concatenating the segment of L_1 from a to x, the whole of L_2 and the segment of L_2 from y to b gives a pseudo-path from a to b not using e. By Proposition 2.10 the result follows. □

Often times in this thesis, when the compactness of pseudo-lines is used, it is only the following key consequence of it that is necessary.

Lemma 2.12. *Any linearly ordered ascending net in a pseudo-line converges to its supremum.*

Proof. Since pseudo-lines are compact, the net A has a convergent subnet B. Then B can clearly only converge to the supremum of A. Since A is linearly ordered and ascending, A must do the same. □

We can use this to prove statements treating pseudo-arcs basically as if they were paths in graphs, as in our proof of the finite Menger's theorem for pseudo-arcs in the next section and the following lemma, which we shall need later to show that a certain contraction minor of a matroid is connected. For concatenations of segments of pseudo-lines we will use the same notation as we would for paths in graphs.

Lemma 2.13. *Let $v, w \in V(G)$ such that $G - v - w$ is connected and let L_1 and L_2 be two pseudo-lines with endvertices v and w. Then for any $e_1 \in E(L_1)$ and $e_2 \in E(L_2)$ with $V(e_i) \neq \{v, w\}$ for $1 \leq i \leq 2$ there exists a pseudo-line with endvertices v and w containing e_1 and e_2 or a pseudo-cycle containing e_1 and e_2, but not both v and w*

Proof. If either e_1 or e_2 is contained in both pseudo-lines, we are done.

First, we will consider the case that L_1 and L_2 are disjoint except for v and w. Choose a pseudo-line L between L_1 and L_2 avoiding v and w. By

Lemma 2.12 L has a last vertex x in L_1 and a first vertex after x in L_2, which we call y. If e_1 and e_2 are both closer to v than x and y, then we obtain a cycle vL_1xLyL_2v avoiding w containing both e_1 and e_2. If e_1 is closer to v than x and e_2 is closer to w than y we obtain a pseudo-line vL_1xLyL_2w from v to w. The other two possibilities are analogous.

Now let us consider the case in which L_1 and L_2 are not disjoint. By Lemma 2.12 there is a vertex x of L_1 closest to w on L_2 before e_2 and a vertex y of L_1 on L_2 closest to v that is closer to w than e_2. We have $x \neq v$ or $y \neq w$ by assumption. If e_1 is in the segment of L_1 between x and y, then xL_2yL_1x is a pseudo-cycle as required. Otherwise w.l.o.g. e_1 is closer to v than both x and y. If x is closer to v on L_1 than y then $VL_1xL_2yL_1$ is a pseudo-line as required. The other case is symmetrical. □

Since we want to work with separations based on vertices, which cannot completely capture the conncectivity of multigraphs, it is useful to restrict ourselves to some analogue of simple graphs. Thus we call a graph-like space *simple* if it has no multiple edges or loops. A simple subgraph-like space H of G is a *simplification* of G if their vertex-sets are the same and any edge missing from H is a loop or parallel to an edge of H. Unlike for graphs, however, this operation does not necessarily leave the components unchanged.

Example 2.14. Consider the curve $f(t) = (t, 0)$ for $t \in [0, 1]$ and the curve g defined piecewise as $g(t) = (t, 1 - 2^{n+2}|t - (1 - 3 \cdot (1/2)^{n+2})|)$ for $t \in [1 - (1/2)^n, 1 - (1/2)^{n+1}]$ where $n \geq 0$ and $g(t) = (1, 0)$ for $t = 1$. Let A and B be their respective images. Then $[0, 1]^2$ restricted to $A \cup B$ is a graph-like space G with vertex set $V = (A \cap B) \cup \{(1, 0)\}$ and edge set $E = E_f \cup E_g$, where E_f and E_g correspond to the sections between successive vertices on f and g, respectively. Let H be the simplification of G obtained by deleting E_f. Clearly, the edges in E_f define a pseudo-line from $(0, 0)$ to $(0, 1)$. However, any such pseudo-line in H would need to contain all the edges in E_g, but then it would not be closed in $[0, 1]^2$ and thus not compact.

If a graph-like space is well-behaved enough to allow us to subdivide edges, we can however use this to obtain a simple graph-like space preserving the components. Because we view graph-like spaces through the lens of their pseudo-arcs, it will be useful to have a notion of closure based only on pseudo-arcs. We call a graph-like subspace H of G *pseudo-arc closed* if every pseudo-arc in G starting in H has a last vertex mapped to H. For this notion to be helpful to us, some pseudo-arc closed sets should naturally occur in our decompositions. Indeed, since our decompositions will be based on

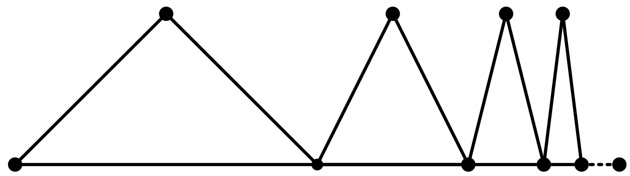

Figure 1: Example 2.14

separations whose separators are finite sets of vertices, the following lemma shows this.

Lemma 2.15. *Any graph-like subspace H of G such that there exists some closed set X of vertices such that any pseudo-arc from H to its complement meets X is pseudo-arc closed.*

Proof. Let $f : P \to G$ be any pseudo-arc starting in H. If it ends in H then its last vertex is its last vertex in H. Otherwise let Z be the set of vertices of P mapped to X. Let z be the supremum of Z. Then by Lemma 2.12 $f(z) \in X$. Since the segment of f starting at z does not meet X again, it cannot meet H either, so z is the last vertex of f mapped to H. □

It would be very useful if pseudo-arc closed spaces automatically turned out to be closed. This clearly cannot be the case, since we can consider any topologically totally disconnected space as an edgeless graph-like space, but for our purposes the following will suffice.

Lemma 2.16. *If G is compact and H is a pseudo-arc closed graph-like subspace with finitely many topological components, then H is closed.*

Proof. Assume that there is some $x \in \bar{H} \setminus H$. This must clearly be a vertex. Since H has finitely many components, there exists some component C of H such that $x \in \bar{C}$. Since C is topologically connected, so is $\bar{C}$. Then $\bar{C}$ is a topologically connected, compact graph-like space, so by Theorem 2.9 it is pseudo-arc connected. Therefore there is some pseudo-line L with x as one endvertex and one endvertex in C that is contained in $\bar{C}$. Since inner points of edges are never in the closure of graph-like subspaces that don't already contain them, all edges of L are contained in H. Since H is pseudo-arc closed, there is a vertex h of H closest to x. Clearly, $c \neq x$. But then the part of L between h and x is a nontrivial pseudo-line containing no edges, contradicting Lemma 2.7. □

2.4 Ultrafilter limits

To aid the reconstruction efforts mentioned before, some standard topological tools are needed, in particular a method of taking limits that chooses among different limit points.

For this section let D be a directed set, $\mathcal{U}$ an ultrafilter on D and X a topological space. We call $\mathcal{U}$ *cofinal* if every $U \in \mathcal{U}$ is cofinal in D.

For a family $(X_d)_{d \in D}$ of subsets of X we define its *(ultrafilter) limit* (with respect to $\mathcal{U}$) $\lim_{\mathcal{U}}(X_d)_{d \in D}$ as the set of all those $x \in X$ such that for every open neighborhood V of x we have $\{d \in D; V \cap X_d \neq \emptyset\} \in \mathcal{U}$. This is a known generalization of the limits of ultrafilters discussed in sources like [10].

For this to be a useful concept, it should relate in some way to the usual limits. Indeed, ultrafilter limits encompass all the possible limits of subnets.

Lemma 2.17. *A point $x \in X$ is a limit point of a net $(x_d)_{d \in D}$ if and only if there exists some cofinal ultrafilter $\mathcal{V}$ such that $x \in \lim_{\mathcal{V}}(\{x_d\})_{d \in D}$. A point $x \in X$ is the limit of a net $(x_d)_{d \in D}$ if and only for all cofinal ultrafilters $\mathcal{V}$ we have $x \in \lim_{\mathcal{V}}(\{x_d\})_{d \in D}$.*

Proof. For the forward direction of the first statement let $\mathcal{F}$ be the filter generated by $\{d \in D; x_D \in V\}$ for open neighborhoods V of x and all complements of noncofinal subsets. Clearly, all the sets in $\mathcal{F}$ are cofinal, so in particular they are nonempty. Then let $\mathcal{V}$ be an extension of $\mathcal{F}$ to an ultrafilter. This is a cofinal ultrafilter and $x \in \lim_{\mathcal{V}}(\{x_d\})_{d \in D}$ by definition. For its backward direction, if $x \in \lim_{\mathcal{V}}(\{x_d\})_{d \in D}$ for some cofinal ultrafilter $\mathcal{V}$, then $\{d \in D; x_d \in V\}$ is cofinal in D for every open neighborhood V of x, so x is a limit point of $(x_d)_{d \in D}$.

The forward direction of the second statement is immediate from the fact that the complement of any set eventually containing D is not cofinal. To prove its backward direction, assume that x is not the limit of $(x_d)_{d \in D}$. Then there exists some open neighborhood V of x such that $Z = \{d \in D; x_d \notin V\}$ is cofinal. The set Z together with the complement of any set that is not cofinal generates a cofinal filter, which can be extended to a cofinal ultrafilter $\mathcal{V}$. By construction $x \notin \lim_{\mathcal{V}}(\{x_d\})_{d \in D}$ □

The proof of the following lemma is based on [10, Theorem 3.48].

Lemma 2.18. *Let $(X_d)_{d \in D}$ be a family of subsets of X. If X is Hausdorff and $\{d \in D; |X_d| \leq 1\} \in \mathcal{U}$, then $\lim_{\mathcal{U}}(X_d)_{d \in D}$ has at most one element. If X is compact and $\{d \in D; X_d \neq \emptyset\} \in \mathcal{U}$ then $\lim_{\mathcal{U}}(X_d)_{d \in D}$ has at least one element.*

Proof. For the first statement let $U = \{d \in D; |X_d| \leq 1\}$ and let $x \neq y \in \lim_{\mathcal{U}}(X_d)_{d\in D}$. Since X is Hausdorff, there are open neighborhoods V_1 of x and V_2 of y that are disjoint. Let $U_i = \{d \in D; X_d \cap V_i \neq \emptyset\}$ for $i \in \{1, 2\}$. By definition these are contained in $\mathcal{U}$ and thus so is $U_1 \cap U_2$. By disjointness we have $U_1 \cap U_2 \subseteq U^c$, so $U \notin \mathcal{U}$.

For the second statement let $U = \{d \in D; X_d \neq \emptyset\}$ and let $(x_d)_{d\in U}$ be a choice of one element of each X_d. If $\lim_{\mathcal{U}}(X_d)_{d\in D}$ is empty, then by definition we may choose for each $x \in X$ an open neighborhood V_x such that $U_x := \{d \in D; V_x \cap X_d = \emptyset\} \in \mathcal{U}$. Since X is compact this open cover has a finite subcover $V_{y_1}, \ldots, V_{y_n}$. Then $U^c = U_{y_1} \cap \cdots \cap U_{y_n} \in \mathcal{U}$, so $U \notin \mathcal{U}$. □

In compact Haussdorf spaces being a continuum (closed and connected) is preserved under taking ultrafilter limits.

Proposition 2.19. *If X is compact and Hausdorff and $(X_d)_{d\in D}$ is a family of closed, connected subsets of X, then $\lim_{\mathcal{U}}(X_d)_{d\in D}$ is closed and connected.*

Proof. We will write Y for $\lim_{\mathcal{U}}(X_d)_{d\in D}$. For any point x in the closure of Y any of its open neighborhoods V contains a point $y \in Y$. Thus V is also a neighborhood of y. We then have $\{d \in D; V \cap X_d \neq \emptyset\} \in \mathcal{U}$ and so $x \in Y$. This proves that Y is closed. Now assume that Y is disconnected. Then there are nonempty closed sets Y_1 and Y_2 partitioning Y. Since X is compact and Hausdorff, X is normal. Thus there are disjoint open sets V_1 and V_2 containing Y_1 and Y_2 respectively. Because V_1 and V_2 are open neighborhood of points in Y we have $U_i := \{d \in D; V_i \cap X_d\} \in \mathcal{U}$ for $i \in \{1, 2\}$. Let $Z_d = X_d \setminus (V_1 \cup V_2)$. We have $Z_d \neq \emptyset$ for all $d \in U_1 \cap U_2$ by the connectivity of X_d. Thus by Lemma 2.18 $\lim_{\mathcal{U}}(Z_d)_{d\in D}$ is nonempty, but it must be a subset of Y by definition of the ultrafilter limit. W.l.o.g. there exists a $z \in Y_1 \cap \lim_{\mathcal{U}}(Z_d)_{d\in D}$. But V_1 is a neighborhood of z not meeting any Z_d, a contradiction. □

This means that the limit of edge-disjoint continua in a compact graph-like space can only be a single vertex, where a family of subsets of a graph-like space is *edge-disjoint* if no two of them contain inner points of the same edge.

Corollary 2.20. *If G is a compact graph-like space, D is unbounded, $\mathcal{U}$ is cofinal and $(X_d)_{d\in D}$ is a family of closed, connected edge-disjoint subsets of G, then $\lim_{\mathcal{U}}(X_d)_{d\in D}$ is a single vertex.*

Proof. By Proposition 2.19 $\lim_{\mathcal{U}}(X_d)_{d\in D}$ is connected. If it is not a single vertex, it contains an inner point p of an edge e, since $V(G)$ is totally disconnected. But then by Lemma 2.17 it is the limit of a net of points of which at most one is an inner point of e, contradicting the fact that p has an open neighborhood consisting only of inner points of e. □

3 Menger's theorem for pseudo-arcs

As promised, we will give a proof of a finite version of Menger's theorem for pseudo-arcs in graph-like spaces by treating pseudo-arcs as though they were paths. The details of the proof closely follow the third (augmenting paths) proof of Menger's theorem in [7]. The augmenting arcs method in graph-like spaces was first used in [12].

Since the proofs this is based on often refer to the degree of vertices, we need an equivalent. This will be number of directions at a vertex, where a *direction* at a vertex v in a graph-like space G is an equivalence class of nontrivial pseudo-lines with v as an endvertex, two such pseudo-lines being equivalent if there exists another such pseudo-line that is contained in both. A *leaf* of a tree-like space is a vertex at which there at most one direction. The following lemma is a translation of a trivial fact for graph-theoretical trees.

Lemma 3.1. *Any tree-like space with an edge has at least two leaves.*

Proof. Given a tree-like space T, let P be the set of all pairs (v, d), where $v \in V(T)$ and d is a direction at v. We define a relation on P by setting $(v, d) \leq (w, e)$ for (v, d) and (w, e) in P if any element of d can be obtained from one of e by deleting its segment from w to v. Clearly $\leq$ is reflexive and transitive.

To prove that it is antisymmetric, let us assume that $(v, d) \leq (w, e)$ and $(w, e) \leq (v, d)$. If $v = w$, then also $d = e$. So we may assume that $v \neq w$. Since $(w, e) \leq (v, d)$ there must be some $L \in d$ meeting w. But then this pseudo-line cannot be obtained from one with w as an endvertex, a contradiction. Thus $\leq$ is a partial order.

Let us first prove that T has a leaf using Zorn's Lemma. So let C be any chain in P. If C is empty, then any direction is an upper bound and one exists because T has an edge, so we may assume that C is nonempty. Since T is compact, the net $(v)_{(v,d)} \in C$ has a limit point x. Since T is a tree-like space, the unique pseudo-lines $L_{(v,d)}$ between x and v where $(v, d) \in C$ are all contained in the same direction at x. So if x is not a leaf already, there is some other direction e at x. Since the elements of e meet L_v only at x the concatenation of $L_{(v,d)}$ and any of them is contained in d and so $(v, d) \leq (x, e)$. By Zorn's Lemma, P has a maximal element (y, f). Now if there was another direction g at y then for any $G \in g$, the other endvertex of G together with the direction of G at that vertex would contradict maximality. Thus y is the desired leaf.

Now we know that T has some leaf y, which gives us some maximal element (y, f) of P. Let P' be the subset of P that is not below (y, f). The other endvertex of any element F of f, together with the direction of F at that vertex is an element of P', so the empty chain still has an upper bound. For any other chain the construction before still gives an upper bound, since it cannot lie below (y, f) if the elements of C do not. Thus P' has a maximal element (z, h) and it is a leaf as before. Since y is a leaf, $y \neq z$ and so T has two leaves. □

Let us now fix a graph-like space G, subsets $A, B \subseteq V(G)$ and a finite set of disjoint pseudo-arcs $\mathcal{P}$ from A to B. For these we can now formulate an equivalent to alternating paths in graphs. We call a pseudo-path $f : L \to G$ *alternating* if it satisfies the following conditions:

1. It starts in $A \setminus V(\mathcal{P})$.
2. No two edges of L are mapped to the same edge.
3. For any edge e contained in both $f(L)$ and an element p of $\mathcal{P}$ the endvertices of e are ordered differently in f and p.
4. We have $|f^{-1}(v)| \leq 1$ for any vertex $v \notin V(\mathcal{P})$.
5. If we have $f(l) = v$ for some $v \in V(\mathcal{P})$ and $l \in V(L)$ such that v is not the final vertex of f then there exists a one-sided interval at l such that f is injective on it and in its image there is a connected part of some $p \in \mathcal{P}$.

Let us first prove that these pseudo-paths are actually augmenting.

Lemma 3.2. *If there is an alternating pseudo-path f ending in $B \setminus V(\mathcal{P})$, there exists a set of disjoint pseudo-arcs $\mathcal{Q}$ from A to B with $|\mathcal{Q}| > |\mathcal{P}|$.*

Proof. Let T be the union of the images of f and the elements of $\mathcal{P}$. Since this is a finite union, T is compact. Let T' be the subspace of T induced by $E(\mathcal{P}) \triangle E(f(L))$. Then T' is closed and thus also compact and so are the topological components of T'. By Theorem 2.9 they are then pseudo-arc connected.

Let A' be the set of starting vertices of f and the $p \in \mathcal{P}$ and B' the set of their ending vertices. Now consider the set $\mathcal{Q}$ of those components of T'

meeting $A' \cup B'$. Since there is only one direction in T' at any vertex of $A' \cup B'$, these vertices cannot be contained in a pseudo-cycle. But since there are no more than two directions at any vertex of T', there indeed cannot be a pseudo-cycle contained in any element of $\mathcal{Q}$, since there would need to be a pseudo-arc from that pseudo-cycle to any vertex not contained in it. Thus $\mathcal{Q}$ is a set of tree-like spaces.

Let $Q \in \mathcal{Q}$ be given. Since there is at least one direction at any vertex, Q has an edge. By Lemma 3.1 Q then has two leaves q_1 and q_2. Any vertex of Q must be contained in $L_T(q_1, q_2)$ since otherwise there would be a pseudo-line between them, contradicting the fact that there are no more than two directions at any vertex of T'. Thus $\mathcal{Q}$ is indeed a set of pseudo-lines and since there is just one direction at its endvertices, they must be contained in $A' \cup B'$. In particular $|\mathcal{Q}| > |\mathcal{P}|$.

Now it suffices to show that no element of $\mathcal{Q}$ has both endvertices in A' or B'. Assume for a contradiction that $q \in \mathcal{Q}$ is a pseudo-arc from x to y and let p_1 and p_2 be the elements of $\mathcal{P}$ containing x and y respectively. By Lemma 2.12 q has a last vertex a on p_1 and a first vertex after a on p_2. Then there exists an interval $I \subseteq L$ such that $f(I)$ is a pseudo-path between a and b. But no matter in which order f traverses a or b, by the fifth condition it would have to traverse an edge of p_1 or p_2 in the same order as in its pseudo-arc, contradicting the third condition. □

Now it suffices to show that if augmentation fails, we can find an appropriate separator.

Lemma 3.3. *If there is no alternating pseudo-path ending in $B \setminus V(\mathcal{P})$, there exists a choice of one vertex from each element of $\mathcal{P}$ separating A and B.*

Proof. For every $p \in \mathcal{P}$ let x_p be the supremum in p of the set of all vertices v such that there exists an alternating pseudo-path ending in v and let X be the set of these vertices. We claim that X separates A and B.

Let us assume for contradiction that there exists a pseudo-arc f from A to B avoiding X. From Lemma 2.12 and the fact that f does not alternate we know that f has a first vertex in $V(P)$, which cannot appear after x_p in the ordering of its pseudo-arc. The subspace S induced by the parts of all the $p \in \mathcal{P}$ until x_p is closed and f meets it by the above argument, so by Lemma 2.12 f has a last vertex y in S, which lies on some $r \in \mathcal{P}$. Since f avoids X, $y \neq x_r$, so there exists a z on r after y such that there is an alternating pseudo-path g that ends in z.

Let z' be the first vertex of g on the segment of r between z and y and let g' be the segment of g until z' followed by the reversed segment of r until y. Then g' is alternating pseudo-path ending in y. Since g' meets $V(\mathcal{P})$ only in S and y is the last vertex of f in S, the part of f after y can meet g' only outside of $V(\mathcal{P})$. If it does not, then let g'' be the concatenation of these two pseudo-paths. If it does, then let c be the infimum in g' of those l in its preimage such that $g'(l)$ lies on the part of f after y. Then $g'(c)$ is contained in the part of f starting from y. Let g'' be the concatenation of g' until c and f starting from its last occurence of $g'(c)$.

In both cases, g'' satisfies the conditions of an alternating pseudo-path in so far as the subspace S is concerned. But it cannot meet the parts of each $r \in \mathcal{P}$ starting from x_r because if it did meet this closed space, it would have a first vertex s in it by Lemma 2.12 and then s would contradict the choice of X, so indeed g'' is alternating with respect to $\mathcal{P}$. □

Corollary 3.4. *For any natural number k the sets A and B can be separated by at most k vertices if and only if there is no set of $k+1$ disjoint pseudo-arcs from A to B.*

Proof. The forward direction is trivial. For the other direction, assume that there is no set of $k+1$ disjoint pseudo-arcs from A to B. Let $\mathcal{P}$ be a set of disjoint pseudo-arcs from A to B of maximal size. Then Lemma 3.2 implies that there can be no alternating pseudo-path ending in $B \setminus V(\mathcal{P})$, so by Lemma 3.3 there is a set of vertices of size $|\mathcal{P}| \leq k$ separating A and B. □

Motivated by the fact that for graph-like spaces inducing matroids only the vertices contained in a pseudo-cycle matter for the matroid, we call a graph-like space G *irredundant* if every vertex of G is contained in some pseudo-cycle.

For these spaces, we can deduce from Menger's theorem that if no vertex disconnects them, then every two vertices are contained in a common cycle. Indeed, first we obtain a fan version of Menger for $k = 2$ by applying Menger with the single vertex replaced by a pseudo-cycle.

Corollary 3.5. *Let G be an irredundant graph-like space and let v be a vertex such that $G - v$ is connected. Then v can be separated from a set of vertices B not containing v by a vertex $w \neq v$ if and only if there is no set of two pseudo-arcs from v to B that are disjoint except for v.*

Proof. Let C be some pseudo-cycle including v. By Corollary 3.4 there exists either a vertex $w \neq v$ separating B and C or two disjoint pseudo-arcs from B to C. In the first case we are done, so let f and g be two such pseudo-arcs.

Since C is compact, f and g have last vertices on C. Since these are different, we can add segments of the pseudo-cycle disjoint except for v to obtain the required pseudo-arcs. $\square$

Afterwards we apply this version to our two vertices by again replacing one of them with a pseudo-cycle.

Corollary 3.6. *Let G be an irredundant graph-like space and let $v \neq w$ be vertices such that $G - v$ and $G - w$ are connected. Then v can be separated from w by a vertex $z \notin \{v, w\}$ if and only if there is no set of two pseudo-arcs from v to w that are disjoint except for v and w.*

Proof. Let C be some pseudo-cycle including w. By Corollary 3.5 there exists either a vertex $z \neq v, w$ separating v and C or two disjoint pseudo-arcs from v to C. In the first case we are done, so let f and g be two such pseudo-arcs. Since C is compact, f and g have last vertices on C. Since these are different, we can add segments of the pseudo-cycle disjoint except for w to obtain the required pseudo-arcs. $\square$

4 Separations

In order to apply the theory of separation systems explained above, we need a notion of separation for graph-like spaces. In analogy to the vertex separations of graphs, which can be defined as pair of sets of vertices, which together cover all of them, such that there is no finite path from one of them to the other avoiding their intersection, given a graph-like space G for us a *separation* is a pair (A, B) of subsets of $V(G)$ with $A \cup B = V(G)$ such that every pseudo-arc from A to B meets $A \cap B$. Its *order* $|(A, B)|$ is $|A \cap B|$. We will call a separation of order k a *k-separation*.

As stated before, we will only consider simple graph-like spaces, so we can define a simple graph-like space to be *k-connected* if and only if every l-separation for any $l < k$ is small and avoid making exceptions for loops and double edges. Then clearly G is 1-connected if and only if any two vertices can be connected by a pseudo-arc.

As usual for separations of this form, we define the *inverse* $(A, B)^*$ of a separation (A, B) as (B, A) and a relation $\leq$ between separations as follows: $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$. Clearly, if (A, B) is a k-separation, then so is (B, A). Thus * is an involution on the set of k-separations. By definition of $\leq$, * is also order-reversing, i.e. $(A, B) \leq (C, D)$ if and only if $(C, D)^* \leq (A, B)^*$. Define $S_k(G)$ for each finite k as the set of all k-separations of G. Then $(S_k, \leq, ^*)$ is a separation system.

A separation is small if and only if one side contains every vertex of G. We call such separations *improper* and the other separations *proper*. We will write $g_k(G)$ for the set of good separations of S_k. By definition g_k forms a regular tree set. A basic criterion for a sensible definition is the existence of meets and joins. Indeed, for finite order separations there are always meets and joins of finite order.

Lemma 4.1. *Let (A, B) and (C, D) be finite order separations. Then $(A \cup C, B \cap D)$ is a finite order separation.*

Proof. We have $(A \cup C) \cap (B \cap D) = (A \cap B \cap D) \cup (C \cap B \cap D) \subseteq V(G)$ and since any point that is not contained in A or C must be contained in both B and D we also have $(A \cup C) \cup (B \cap D) = G$. Thus, if $(A \cup C, B \cap D)$ is not a separation, then there must be some pseudo-arc f from some $y \in A \cup C$ to some $z \in B \cap D$ not meeting $(A \cup C) \cap (B \cap D)$.

Since $(A \cap B) \cup (C \cap D)$ is finite and f crosses at least one of (A, B) or (C, D) there is some last x on f. This gives a pseudo-arc g from x to z avoiding one of $A \cap B$ and $C \cap D$, which therefore must lie completely in D or B, respectively. Therefore, $z \in (A \cap B \cap D) \cup (C \cap B \cap D)$, a contradiction.

Furthermore, since $(A \cup C) \cap (B \cap D) \subseteq (A \cap B) \cup (C \cap D)$, $(A \cup C, B \cap D)$ has finite order. □

As for tree decompositions in graphs, the parts of our decompositions will be obtained as an intersection of the second components of all separations belonging to some orientation and it will be useful to have a short notation for this. Thus for any set o of separations of a graph-like space we will write P_o for the set $\bigcap_{(A,B) \in o} B$.

Since we want the decomposition to display the connectivity of the graph-like space, it would be problematic for a part that lies between nonempty parts to be empty. Fortunately, this does not happen.

Lemma 4.2. *Let C be a chain of finite order separations in some 1-connected graph-like space G, such that P_C and P_{C^*} are nonempty. Then for any partition (C_1, C_2) of C with $c_1 < c_2$ for all $(c_1, c_2) \in C_1 \times C_2$ the set $P_{C_1 \cup C_2^*}$ is nonempty.*

Proof. Let $a \in P_C^*$ and $b \in P_C$ be given and let f be a pseudo-arc from a to b. Clearly, the image of f must meet $A \cap B$ for every $(A, B) \in C_1$. Let $x_{(A,B)}$ be the last vertex on f in $A \cap B$ and let x be their supremum. By choice of the $x_{(A,B)}$ we have $f(x) \in B$ for all $(A, B) \in C_1$. Now it suffices to show that $f(x) \in E$ for all $(E, F) \in C_2$.

Assume for contradiction that $f(x) \in F - E$. Then f must meet $E \cap F$ somewhere before x, let y be the last such point. Since $y < x$, there must be some $x_{(A,B)}$ between them. But then $x_{(A,B)} \in E - F$, contradicting $(A.B) < (E, F)$. □

Now we will investigate the connectivity of the sides of separations that have the lowest order possible in G. For these, making the separator complete suffices to give its sides the same connectivity as the original space.

Proposition 4.3. *Let (A, B) be a proper k-separation in a k-connected graph-like space space G for some finite k. Then $G[A] \oplus \{\{v, w\}; v \neq w \in A \cap B\}$ and $G[B] \oplus \{\{v, w\}; v \neq w \in A \cap B\}$ are k-connected.*

Proof. By symmmetry it suffices to prove this for A. Let (C, D) be an proper l-separation for $l < k$ of $G[A] \oplus \{\{v, w\}; v \neq w \in A \cap B\}$. Since G is k-connected there is some pseudo-arc f from C to D avoiding $C \cap D$ in G. Replacing each segment of f between two elements v and w of $A \cap B$ such that there is no $x \in A \cap B$ between v and w on f with the edge added between them makes this into a pseudo-arc in $G[A] \oplus \{\{v, w\}; v \neq w \in A \cap B\}$ because replacing a segment of a pseudo-line between two vertices with a single edge always leaves a pseudo-line. This contradicts the fact that (C, D) was a separation. □

To apply this to the space we started with we need some relation between the separations of some space and that space with some edges added.

Lemma 4.4. *If (A, B) is a separation of G and X is a finite set of pairs, whose union is contained in A or B, then (A, B) is a separation of $G \oplus X$.*

Proof. By symmetry we may assume w.l.o.g. that $\bigcup X \subseteq A$. It suffices to show that (A, B) is a separation of $G \oplus X$. If not, there is some pseudo-arc f from A to B avoiding $A \cap B$. Let x be the supremum of the points mapped to inner points of new edges. Then $f(x) \in A$ and f restricted to its points from x to its end in B is a pseudo-arc in G from A to B avoiding $A \cap B$, a contradiction. □

Together these place a helpful restriction on the separations occuring in the sides of these maximum order separations.

Corollary 4.5. *Let (A, B) be a proper k-separation in a k-connected graph-like space space G for some finite k. Then any proper l-separation (C, D) of A or B for $l < k$ has a vertex of $A \cap B$ in $C \setminus D$ and $D \setminus C$.*

Proof. By symmetry, it suffices to prove this for A. Let (C, D) be a proper separation not satisfying this. By Lemma 4.4 this can then be extended to a separation of $A \oplus \{\{v, w\}; v \neq w \in A \cap B\}$, contradicting Proposition 4.3. $\square$

It may perhaps seem more straightforward to use as separations *topological separations*, i.e. pairs (A, B) of vertex sets of some graph-like space G such that $A \cup B \supseteq V(G)$ and such that no topological component $G - (A \cap B)$ meets both A and B with order and involution defined as before. The following example, however, shows that chains of topological separations in topologically connected simple graph-like spaces do not necessarily have suprema even when they do have upper bounds, whereas we will see later that this does not occur for our separations in simple graph-like spaces connected in our sense, a feature that will be used heavily in our proofs.

Example 4.6. Let $V = \omega \times 2 \times 2 \cup (\{\omega\} \times 2\}$, $E_1 = \{((a, 0, b), (a+1, 0, b)); a \in \omega, b \in 2\}$, $E_2 = \{((a, 0, b), (a, 1, b)); a \in \omega, b \in 2\}$ and $E = E_1 \cup E_2$.

Let v^ϵ for some $v \in V$ and $0 < \epsilon \leq 1$ consist of v together with $(0, \epsilon) \times \{e\}$ for all $e \in E$ with v as their first component and $(1 - \epsilon, 1) \times \{e\}$ for all $e \in E$ with v as their second component. Let τ^* be the set of open intervals of edges e and let τ_v for $v \in \omega \times 2 \times 2$ be the set of all v^ϵ for $0 < \epsilon \leq 1$. Furthermore, let $\tau_{(\omega,i)} = \{\{(\omega, i)\} \cup \bigcup_{k>n; b\in 2} \tau^\epsilon_{(k,i,b)} \cup \bigcup_{e\in F(n,i)} (0, 1) \times \{e\}; n \in \omega; 0 < \epsilon \leq 1\}$ for $i \in 2$, where $F(n, i)$ for $n \in \omega$ and $i \in 2$ is $E \cap \{(k, i, c); k > n, c \in 2\}^2$. Then we define τ to be the topology on $V \cup ((0, 1) \times E)$ induced by the union of τ^* and the τ_v for $v \in V$.

Let $v_1, v_2 \in V$ be two different vertices. W.l.o.g. $v_i \neq (\omega, i)$ for $i \in 2$. Choose $U_i^* \in \tau_{(\omega,i)}$ avoiding v_i and not containing the midpoint of any edge whose endvertices are not both contained in U_i^* and let U_i' be obtained from U_i^* by taking its union with $v_i^{1/3}$. Now let $U_1 = U_1'$ and let U_2 be obtained from U_2' by adding $v^{1/3}$ for any $v \in V \setminus (U_1' \cup U_2')$. Then (U_1, U_2) is a $(\{v_1\}, \{v_2\})$-witness in $(V \cup ((0, 1) \times E), \tau)$. Thus this space with vertex set V, edge set E and maps chosen as the second projections forms a simple graph-like space G.

Since the graph (V, E) has just three components and two of them contain a sequence converging to the third, G is topologically connected. With a similar argument deleting some vertex of the form $(a, 0, b)$ for $a \in \omega, b \in 2$ always leaves two topological components and the other vertices do not topologically disconnect G. For each vertex v topologically disconnecting G let s_v be the separation with separator v, whose first component is finite. Let C be the chain $s_{(a,0,0)}$ for $a \in \omega$. Then the upper bounds for C are exactly the separations $s_{(b,0,0)}$ for $b \in \omega$, but they do not have a minimum.

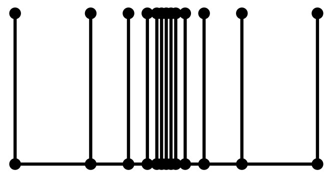

Figure 2: Example 4.6

5 Tree-like decompositions

The goal of this section is to introduce a notion similar to tree decompositions for graph-like spaces. Since chains of separations of any length can occur, ordinary trees can not necessarily capture the structure of a graph-like space. To avoid this problem, we will use a tree-like space instead, as in [9]. Therefore we start by defining a few terms for tree-like spaces in analogy to concepts in ordinary trees to help us work with them.

Clearly, any vertex other than v will occur in exactly one direction at v, which we call its *direction from* v. An *adjacent direction* D of a tree-like space T is a direction at any $v \in V(T)$ that is represented by a pseudo-arc with just one edge. We write v^D for the unique vertex besides v contained in every pseudo-arc representing D. A direction that is not an adjacent direction is a *limit direction*. Clearly, a limit direction can only be a direction at one vertex v, which we call its *end*. Call $t \in V(T)$ a *limit point* of T if there is a limit direction at t and *necessary* if $v \in \overline{V(T) - v}$.

For our analogous decomposition we can keep two of the conditions for tree decompositions of graphs similar, but since the edges of a graph-like space do not completely determine its connectivity, the condition that every edge must be contained in some part is not sufficient. We instead mandate the fact that intersections of neighboring parts form separators, which is a corollary in the case of graphs.

A *tree-like decomposition* of a graph-like space G is a pair $(T, (V_t)_{t\in V(T)})$, where T is a tree-like space and $V_t \subseteq V(G)$, satisfying the following conditions:

1. $\bigcup_{t\in V(T)} V_t = V(G)$
2. $V_s \cap V_t \subseteq V_x$ for all $s, t, x \in V(T)$ with $x \in L_T(s,t)$
3. For every $s, t \in V(T)$ and every edge $e \in E(L_T(s,t))$ between a and b we have that every pseudo-arc in G from V_s to V_t meets $V_a \cap V_b$.

It is *proper* if every empty V_t is a necessary leaf.

For any $t \in V(T)$ and direction d at t we will write $V(t,d)$ for all those vertices which occur in V_s for some s whose direction from t is d and $\mathring{V}(t,d)$ for $V(t,d)\setminus V_t$. By the second condition given any $t \in V(T)$ each $v \in V(G)\setminus V_t$ is contained in $V(t,d)$ for exactly one direction d at t.

A tree-like decomposition $D = (T, (V_d)_{d \in D})$ has *adhesion* $\leq k$ (or $< k$) for some cardinal k if for every $s,t \in V(T)$ we have $|V_s \cap V_t| \leq k$ (or $|V_s \cap V_t| < k$).

For compact graph-like spaces there are a few useful topological properties of tree-like decompositions that we will note here. These will be useful for our project of reconstructing a graph-like space from a decomposition.

Lemma 5.1. *Let $(T, (V_t)_{t \in V(T)})$ be a tree-like decomposition of a compact, connected graph-like space G of adhesion $< \aleph_0$. Then for any $t \in V(T)$ that is not a limit point and any direction d at t we have that $G[V(G) \setminus \mathring{V}(t,d)]$ is closed.*

Proof. Since t is not a limit point, d is an adjacent direction, so every pseudo-arc with an endpoint in $G[V(G) \setminus \mathring{V}(t,d)]$ that meets its complement must meet $V_t \cap V_{t^d}$. Therefore it is pseudo-arc closed by Lemma 2.15 and has finitely many components, which implies that it is closed by Lemma 2.16. □

Lemma 5.2. *Let $D = (T, (V_t)_{t \in V(T)})$ be a tree-like decomposition of a compact, connected graph-like space G of adhesion $< \aleph_0$. Then any graph-like subspace of G with vertex set V_t for some $t \in V(T)$ that is not a limit point is closed.*

Proof. It suffices to prove that $G[V_t]$ is closed since the set of inner points of any edge is open. Let d be any direction at t. Let $X(d) = G[V(G) \setminus \mathring{V}(t,d)]$. Since t is not a limit point, d is an adjacent direction, so every pseudo-arc with an endpoint in $X(d)$ that meets its complement must meet $V_t \cap V_{t^d}$. Therefore $X(d)$ is pseudo-arc closed by Lemma 2.15 and has finitely many components. This implies that it is closed by Lemma 2.16. As the intersection of the $X(d)$, $G[V_t]$ is then also closed. □

Lemma 5.3. *Let $D = (T, (V_t)_{t \in V(T)})$ be a tree-like decomposition of a connected graph-like space G of adhesion $< \aleph_0$. If G is compact, then any graph-like subspace of G with vertex set $V(t,d)$ for some $t \in V(T)$ that is not a limit point and some direction d at t is closed. If G is $|G_t \cap G_{t^d}|$-connected, then $G[V(T,d)] - E(G[V_t \cap V_{t^d}])$ is connected.*

Proof. For the first assertion it suffices to prove that $X = G[V(t,d)]$ is closed since the set of inner points of any edge is open. Since t is not a limit point,

d is an adjacent direction, so every pseudo-arc with an endpoint in X that meets its complement must meet $G_t \cap G_{t^d}$. Therefore $X(d)$ is pseudo-arc closed by Lemma 2.15 and has finitely many components. This implies that it is closed by Lemma 2.16.

For the second assertion, let S consist of all but one vertex of $V_t \cap V_{t^d}$. Let $v, w \in V(t, d)$ be given. Then there exists some pseudo-arc f between them avoiding S. But if this pseudo-arc had a vertex x outside of $V(t, d)$, both the sections toward and from x would need to meet $G_t \cap G_{t^d}$ in different vertices, so it would meet S, a contradiction. □

In the following, our goal is to construct a proper tree-like decomposition of a graph-like space from a nested set S of proper separations of G. The basis for this decomposition will be the tree-like space given by the following result from [9].

Theorem 5.4. *For any tree set τ there exists a tree-like space T such that the vertices of T are the consistent orientations of T, its edges are the unoriented separations of τ, an unoriented separation e is incident with two orientations differing exactly in e and for two $s, t \in \tau$ we have $s \leq t$ if and only if $L_T(v_s, v_{t^*}) \subseteq L_T(v_{s^*}, v_{t^*}) \subseteq L_T(v_{s^*}, v_t)$, where v_x for some separation x is the orientation incident with $\{x, x^*\}$ containing x.*

Let us write $D'(S)$ for the pair $(T(S), (P_o)_{o \in V(T((S)))})$.

Proposition 5.5. *$D'(S)$ is a tree-like decomposition of G.*

Proof. For the first condition, let any $x \in V(G)$ be given. The set $\{(A, B); x \in B - A\}$ is a partial consistent orientation. Thus, by Lemma 2.1 it can be extended to a consistent orientation o_x. Then we have $x \in P_{o_x}$ by definition.

For the second condition, let $o, p, x \in O(S)$ with $x \in L_{T'(S,f)}(o, p)$ and $a \in P_o \cap P_p$ be given. Then $a \in P_{o \cup p}$ and in particular $a \in P_x$ since $x \subseteq o \cup p$.

Now let s, t, e, a, b be given as in the third condition and let f be a pseudo-arc from V_s to V_t. Now by choice of e we have that e is an unoriented separation such that s and t contain different orientations of it. Thus f must meet its separator $V_a \cap V_b$. □

To make $D'(S)$ proper, we can just ignore those vertices which are associated with empty parts and not needed as limits because this leaves a connected space. To make this formal, write $O^*(S)$ for the set of those $o \in O(S)$ with P_o nonempty. Then, let $T(S)$ be the subspace of $T'(S)$ induced by $\overline{O^*(S)}$ and write $D(S)$ for the pair $(T(S)[O^*(S)], (P_o)_{o \in O^*(S)})$.

Lemma 5.6. *If G is 1-connected and nonempty and S consists only of finite order separations, $D(S)$ is a proper tree-like decomposition.*

Proof. If G is nonempty, then so is $O^*(S)$. So to show that $T(S)$ is a tree-like space it suffices to prove that $L_{T'(S,f)}(a,b)$ does not contain any c with P_c empty if P_a and P_b are nonempty. This follows from Lemma 4.2. That $D(S)$ is a tree-like decomposition is then a direct corollary of Proposition 5.5.

To show that it is proper, let V_o be an empty part of $D(S)$. Then o is clearly necessary. If o is not a leaf, then there exist two pseudo-arcs ending in o that are disjoint except for v. Then by Lemma 4.2 one of these must consist only of empty parts, which can then not all be necessary, a contradiction. □

It remains to give a definition of torso analogous to that for graphs. We will only do so for tree-decompositions of adhesion ≤ 2 because this is all we need. Since edges added for the torso for graphs in some sense represent paths through the rest of the graph we will analogously have the added edges represent pseudo-arcs. Unlike in graphs, however it can matter which pseudo-arc between two vertices is chosen. We will see in our chapter about 2-separations that in this case the adjacencies we obtain for the torso are the same as if we would add an edge between any two vertices occuring together in another part. Since we want to keep the torsos simple, we do not add edges between vertices which are already adjacent. To make the correspondence to matroids simpler, we make an exception for parts with just two vertices, however.

Let G be some graph-like space, $V \subseteq V(G)$ and let F be a set of nontrivial pseudo-lines in G with disjoint interiors meeting $G[V]$ exactly in their endvertices. Then we will write $G \boxplus F$ for a graph-like space obtained from $G[V] \cup \bigcup F$ by contracting all but one edge of each $f \in F$.

Given a tree-like decomposition $D = (T, (V_i)_{i \in I})$ of a graph-like space G of adhesion ≤ 2 a torso of a part V_i is a space of the form $G[V_i] \boxplus F$, where if $|V_i| > 2$ then F is a set consisting of a choice of one nontrivial pseudo-line $L_{x,y}$ with x, y as its endvertices, whose interior does not meet $G[V_i]$ for any nonadjacent pair $x \neq y \in V(G)$ for which any such $L_{x,y}$ exists and if $|V_i| \leq 2$ then F is a set containing a choice of one nontrivial pseudo-line $L_{x,y,d}$ with x and y as its endvertices, whose interior does not meet $G[V_i]$ and that is contained in parts associated with d, for any pair $x \neq y$ and any direction d at v for which any such $L_{x,y,d}$ exists. The definition of the torso guarantees its existence for any tree-like decomposition of adhesion ≤ 2, the following lemma shows that for graph-like spaces inducing matroids any two torsos of the same part are similar.

Lemma 5.7. *If G represents a matroid and $D = (T, (V_i)_{i \in I})$ is a tree-like decomposition of G of adhesion ≤ 2, then the set of pseudo-cycles of any two torsos of some part V_i is the same (up to renaming the added edges). Furthermore the set of pseudo-cycles of the torso then does not depend on the set and topology of the edges between pairs of vertices between which an edge of the torso would be added if they were nonadjacent.*

Proof. Let C be a cycle of a torso for some choice of pseudo-lines F and let F' be some other choice of pseudo-lines. Since C is a pseudo-cycle in this torso, the space C' obtained from C by replacing every added edge by its associated pseudo-arc in F is a pseudo-cycle in G. Applying Corollary 2.11 to each pair of pseudo-arcs corresponding to the added edges used by C in F and F' gives a set of pseudo-cycles J_e for each added edge e.

Let E_e be a set with a choice of one edge from each element of J_e. If there exists some $f \in C$ that is not an added edge, then we apply circuit elimination on C with the union of the J_e at the E_e and f.

If there exists no such edge, then in particular there are more than two added edges. Then choose one of them arbitrarily as f and let E' be the set of added edges except for f. We can then apply circuit elimination on C with all the elements J_e for $e \in E'$ at all the edges of E_e for $e \in E'$ and some element of J_f. We may now apply the first case to D, J_f and E_f.

In both cases we obtain a cycle C^* using only pseudo-arcs of F' whenever it leaves V_i. Thus replacing those pseudo-arcs with their corresponding edges gives a pseudo-cycle of the torso, which must be C. The proof of the second assertion is similar. □

From now on we will assume we have chosen some way of deciding on one torso for any part of any tree-like decomposition we consider and use the definite article for that torso.

6 Blocks

Our goal for this section is to define a decomposition of some fixed connected graph-like space G with no loops into blocks. This will be done in three steps: first we characterize the good 1-separations of G, then show that any graph-like space with no good 1-separation is 2-connected and finally prove that any separation of the parts of the decomposition lifts to one of G. Let us start by making an observation about 1-separations.

Lemma 6.1. *Two 1-separations (A, B) and (C, D) can cross only if $A \cap B = C \cap D$.*

Proof. Let (A, B) and (C, D) be two crossing separations and let v be the unique element of $A \cap B$ and w the unique element of $C \cap D$. Assume for a contradiction that $v \neq w$. W.l.o.g. we may assume that $w \in A$ and $v \in C$. By Lemma 4.1 $(A \cup C, B \cap D)$ is a separation, but $(A \cup C) \cap (B \cap D) = (A \cap B \cap D) \cup (C \cap B \cap D) = (\{v\} \cap D) \cup (\{w\} \cap B) = \emptyset$, contradicting the fact that G is 1-connected. □

We can use this to reach our first goal and characterize the good 1-separations of G.

Proposition 6.2. *A proper* 1*-separation* (A, B) *of* G *is good if and only if* A *or* B *contains just one component of* $G - (A \cap B)$ *.*

Proof. For the forward direction let C_1 and C_2 be two components in A and D_1 and D_2 be two in B. Then moving C_2 to B and D_2 to A gives a separation crossing (A, B). For the other direction, let (A, B) be a separation satisfying the condition. W.l.o.g. we can assume A is the side with just one component. We know that $A \cap B$ has exactly one element, call it v. Clearly. (A, B) cannot cross any separation with intersection v since $A + v$ must always be contained in one side. Now Lemma 6.1 gives the result. □

Similar to the known decomposition theorems for graphs, the sets P_o for some $o \in g_1$ are good candidates for our blocks. The first requirement they need to satisfy to fill this role is that they should not be separated by any 1-separations, as our next statement guarantees.

Corollary 6.3. *For any* $o \in O(g_1)$ *and any* 1*-separation* (A, B) *the set* P_o *is completely contained in* A *or* B*.*

Proof. Let v be the unique vertex of $A \cap B$. If P_o met multiple components of $G - v$, let C and D be two. Then $(C + v, G - C)$ is a 1-separation satisfying the conditions of Proposition 6.2. So o must contain it or its inverse. Therefore one of C or D cannot be contained in P_o. □

In particular, this implies that any space with no good 1-separation is 2-connected, which was the second objective.

This, however, is not enough to make this a sensible notion of block, since blocks should also induce 2-connected spaces themselves. To prove that they do, we will show that separations of $G[P_o]$ induce ones of G.

Lemma 6.4. *For any separation* (A, B) *of* $G[P_o]$ *for some set of* 1*-separation* o *there exist some separation* (E, F) *of* G *with* $A \subseteq E$*,* $B \subseteq F$ *and* $A \cap B = E \cap F$*.*

Proof. Let (A, B) be a separation of P_o and assume that f is a pseudo-arc from some $a \in A$ to some $b \in B$ avoiding $A \cap B$. Since (A, B) is a separation in P_o, f contains some vertex $c \notin P_o$. Let $(C, D) \in o$ be some separation with $c \in C$. Then the part of f from a to c and from c to b both meet $C \cap D$, but this only has one element, contradicting the injectivity of f. Therefore $A \cap B$ separates $A - (A \cap B)$ and $B - (A \cap B)$ in G and we may arrange the components of $G - (A \cap B)$ not meeting either set arbitrarily to get a separation with the required properties. □

Having checked off all requirements, the desired statement is now immediate.

Corollary 6.5. *For any $o \in O(g_1)$ the subgraph-like space $G[P_o]$ is 2-connected.*

Proof. P_o has no proper 0-separation because this would contradict the 1-connectivity of G by Lemma 6.4. P_o has no proper 1-separation because this would contradict Corollary 6.3 by Lemma 6.4. □

With this result it now makes sense to call $D(g_1(G))$ the *block decomposition* of G and its parts *blocks*. Since blocks never meet both sides of a 1-separation, they are uniquely determined as the maximal such sets by construction. We can also characterize them using Corollary 6.5.

Proposition 6.6. *The nonempty blocks of G are exactly the maximal vertex sets X such that $G[X]$ is 2-connected.*

Proof. If X is a nonempty block, then $G[X]$ is 2-connected by Corollary 6.5. Any superset Y of X containing some $y \notin X$ can not be 2-connected, since y is separated from any $x \in X$ by some vertex v in G and thus also in $G[Y]$. If $G[X]$ is 2-connected, then there exists no good 1-separation in G separating vertices of X and so X is contained in some block Y. Since $G[Y]$ is 2-connected, X cannot be a strict subset of Y if it is maximal with this property. □

Let us now investigate some other properties of $D(g_1(G))$. For this, we will first prove the existence of suprema for chains of 1-separations.

Lemma 6.7. *Let C be a chain of proper 1-separations of G with no maximal element such that P_C is nonempty and $A \setminus B$ is nonempty for any $(A, B) \in C$. Then there is a 1-separation (E, F) that is a supremum for C, such that $E \setminus F = \bigcup_{(A,B) \in C} A \setminus B$ and $F = P_C$.*

Proof. Let $b \in P_C$ and $a \in V(G) \setminus P_C$ be arbitrary and let f be any pseudo-arc from a to b. For any $(A, B) \in C$ with $a \in A$ the image of f contains the unique vertex of $A \cap B$, say $x_{(A,B)}$ is mapped to it. This forms a linearly ordered, increasing net, which converges to its supremum by Lemma 2.12. Since f is continuous, the net of the $\bigcup A \cap B$ converges to its image x. We have $x \in B$ for all $(A, B) \in C$, so $x \in P_C$. Since the set of (A, B) considered is always a final segment of C, this x is independent of the choice of a, b and f. Therefore x separates P_C from all those vertices which lie in $A \setminus B$ for any $(A, B) \in C$. Thus we can obtain a proper separation (E, F) that is an upper bound for C by putting into E exactly those components of $G - x$ that meet $A \setminus B$ for some $(A, B) \in C$.

Now $E \setminus F$ will contain only vertices that lie in $A \setminus B$ for some $(A, B) \in C$. Indeed, if not then there is some pseudo-arc l avoiding x from an element of P_C to a vertex in its complement, but then the net of $x_{(A,B)}$ as above on it would converge to x, a contradiction. Therefore $F = P_C$.

Finally, assume that (I, J) is any smaller 1-separation. If $I \cap J = \{x\}$ then clearly it cannot be an upper bound. So we may assume that $I \cap J = \{y\}$ with $y \neq x$. Then clearly $y \in E$. If (I, J) was an upper bound, then for any $(A, B) \in C$, $a \in A \setminus B$, and pseudo-arc f from a to x, the image of f would need to meet y. But by choice of x there is some $(X, Y) \in C$ such that the part of f between y and x meets $X \cap Y$, a contradiction. □

In particular for chains of good separations we obtain good suprema.

Corollary 6.8. *Any chain $C \subseteq g_1$ such that there are $o \in O^*(g_1)$, $p \in O(g_1)$ with $C \subseteq o$, $C^* \subseteq p$, $|P_C| > 1$ and $o \triangle p = C \cup C^*$ has a supremum in g_1.*

Proof. Let C be such a chain. If C has a maximal element, we are done. Otherwise let (E, F) be the supremum for this chain given by Lemma 6.7. Since $|P_C| > 1$, this is clearly a proper separation. By Proposition 6.2 it suffices to show that $E \cap F$ does not separate any two vertices a and b such that there exists a separation $(A, B) \in C$ with $a, b \in A \setminus B$. If it does then any pseudo-arc f from a to b meets $E \cap F$ in its unique vertex x. But since $x \in P_o$, the parts of f on either side of x meet $A \cap B$, contradicting the injectivity of f. □

When working with parts of tree-like decompositions, those that are easiest to handle are those which do not correspond to limit points in the tree-like space. This makes the following corollary very helpful, since it tells us that all other parts are of a few types and in particular are all *trivial*, by which we mean that they have at most one vertex.

Corollary 6.9. *For any block V_o exactly one of the following four statements is true:*

1. *o is a leaf, o contains a chain with no supremum in g_1 and V_o is trivial*
2. *o contains a chain and the inverse of its supremum and V_o is trivial and nonempty*
3. *o contains the supremum of every chain it contains and V_o is trivial and nonempty*
4. *o contains the supremum of every chain it contains and no two of its maximal elements have the same separator and V_o is nonempty*

Proof. If o contains a chain with no supremum in g_1, then by Corollary 6.8 o is a leaf and has at most one vertex. Otherwise let us first assume the nonempty block V_o contains a chain C without a supremum. Since by assumption C has a limit $(E, F) \in g_1$, we then have $(F, E) \in o$. But then $V_o \subseteq E \cap F$, as required. Now let us assume that o has two maximal elements (A, B) and (E, F) with the same separator. By Proposition 6.2 one side of each separation contains just one component. But since $(F, E) \leq (A, B)$, this must be B and F respectively. Therefore V_o contains just the separating vertex. □

This tool can help us obtain some results which let us infer some of the connectivity properties from its block decomposition. These will be useful for analyzing 2-separations in the next section. For some block V_o of the block decomposition of G let X_o be the set of vertices contained in V_o and at least one other block.

Corollary 6.10. *For any block V_o such that o is not a leaf $(G - V_o + X_o, V_o)$ is a separation.*

Proof. If o does not contain a supremum for some chain, then by Corollary 6.9 V_o has just one vertex w and there is some chain C such that o contains the inverse of its supremum $(E.F)$. Then $E \cap F = \{w\}$ and flipping (F, E) in o gives an orientation o' with $w \in V_{o'}$, so $w \in X_o$. Otherwise by Zorn's Lemma any element lies under some maximal one and since the separator of any maximal element is clearly contained in X_o, any pseudo-arc f from V_o to its complement must meet X_o. □

Lemma 6.11. *If V_o, V_p are blocks and v is a vertex of $L_{T(g_1)}(o, p)$ besides o and p then X_v separates V_o from V_p.*

Proof. Let us first consider the case that V_v and thus X_v by Corollary 6.10 consists of a single vertex w. Then there is a pseudo-arc f from V_o to w. This pseudo-arc must use the separating vertices of any separations corresponding to the edges of $L_{T(g_1)}(o, v)$ in order. By Lemma 2.12 these must then converge to w. Since pseudo-arcs are closed and any arc from V_o to V_p must also include these vertices, w separates V_o and V_p. If V_v has more than one vertex, then by Corollary 6.9 any chain of separations in o has a supremum in o so any element of o lies below some maximal element in o. Indeed, otherwise applying Zorn's Lemma to the set of all elements not below any maximal element of o would give us a maximal element in this set, which would then also be maximal in o, a contradiction. Since the separator of any maximal separation in o is clearly contained in X_o any pseudo-arc from V_o to V_p must meet X_o. □

Lemma 6.12. *Let f be a pseudo arc from $x \in P_o$ to $y \in P_p$ in G, where V_o and V_p are blocks. Then $L_{T(g_1)}(o, p)$ has as nontrivial vertices exactly o, p (if nontrivial) and those v such that the image of f meets V_v in at least two vertices.*

Proof. Let $v \notin \{o, p\}$ be a nontrivial vertex contained in $L_{T(g_1)}(o, p)$. Then any element of v lies under some maximal one by Corollary 6.9 and Zorn's Lemma. Therefore v has two incident edges in $L_{T(g_1)}(o, p)$. The image of f must contain the separating vertices a and b of the associated separations (A, B) and (C, D). By Corollary 6.9 these cannot be the same.

Conversely $L_{T(g_1)}(o, p)$ contains o and p by definition, so let a nontrivial vertex $v \notin \{o, p\}$ be given and assume it is not contained in $L_{T(g_1)}(o, p)$, but f meets V_v in two vertices a and b. Now there exists some separation $(A, B) \in v \setminus (o \cup p)$. At least one of a and b, w.l.o.g. a, is not contained in $A \cap B$. Then the parts of f from a to o and from o to a must both meet $A \cap B$, contradicting the injectivity of f. □

7 2-Blocks

Using our results about blocks, in this section we will define a decomposition along the 2-separations of some fixed simple 2-connected graph-like space G. For this, we will follow steps similar to the last section.

Thus we start by looking for a characterization of good 2-separations. From our discussion of separations earlier the following is immediate.

Corollary 7.1. *For any proper 2-separation (A, B) the subspaces A and B are 1-connected.*

Proof. By symmetry it suffices to show this for A. Let v and w be the vertices of $A \cap B$. If there was a proper 0-separation (C, D) of A, then $(C+v, D+v)$ or $(C+w, D+w)$ would be a a proper 1-separation with v and w together in A or B, contradicting Corollary 4.5. $\square$

This is useful to prove the characterization we were looking for.

Proposition 7.2. *A 2-separation (A, B) is good if and only if A or B is 2-connected and A or B contains just one component of $G - (A \cap B)$.*

Proof. Let v and w be the vertices of $A \cap B$. If both A and B are not 2-connected, then there are proper 1-separation (C, D) of A and (E, F) of B. By Corollary 4.5 we may assume that $v \in (C \setminus D) \cap (E \setminus F)$ and $w \in (D \setminus C) \cap (F \setminus E)$. Then $(C \cup E, D \cup F)$ is a proper 2-separation crossing (A, B). If both A and B contain at least two components of $G - (A \cap B)$ exchanging one component from A with one from B gives a proper 2-separation crossing (A, B).

For the other direction, assume that (C, D) crosses (A, B). Then the second condition implies that $A \cap B \neq C \cap D$. W.l.o.g. we may assume that A is 2-connected. Then $C \cap D$ must be contained in A, since otherwise A would lie completely in C or D. But $C \cap D$ also has to meet B since otherwise B would lie completely in C or D by Corollary 7.1. So $C \cap D$ must consist of one vertex $v \in A \cap B$ and one vertex $w \in A \setminus B$. This however contradicts Corollary 4.5. $\square$

Note that the condition that one side should be 2-connected is only a restriction if the separator leaves exactly two components, otherwise, as we prove next, it is always satisfied.

Lemma 7.3. *If a proper 2-separation (A, B) has two components in A (or B) then A (or B) is 2-connected.*

Proof. By symmetry it suffices to show this for A. Let $A \cap B = \{v, w\}$. Assume that (C, D) is a proper 1-separation of A with separator $\{x\}$. By Corollary 4.5 v and w lie on different sides and neither is x. So x lies in one of the components of $A - \{v, w\}$. Let F be some other component. Moving all the other components in A to B gives a proper 2-separation, so by Corollary 7.1 the subspace defined by $F \cup \{v, w\}$ is 1-connected. Thus there is some pseudo-arc f from v to w in this space. But then f avoids x, a contradiction. $\square$

This, together with our characterization implies that to prove that G has a good 2-separation, it is sufficient to find any proper 2-separation with a 2-connected side.

Lemma 7.4. *If there is a proper 2-separation (A, B) in G, such that A or B is 2-connected , then there exists a good 2-separation with separator $A \cap B$ in G.*

Proof. W.l.o.g. A is 2-connected. Let v and w be the two elements of $A \cap B$. If (A, B) is not good, then by Lemma 7.2 both A and B must contain at least two components. Let (C, D) be the separation with all but one component from A moved to B. Then by Lemma 7.3 D is 2-connected. This implies that (C, D) is good by Proposition 7.2. □

These tools will help us with our second step of proving that G is 3-connected or a pseudo-cycle if it does not have a good 2-separation. The strategy for this follows that used to verify a similar statement for infinite graphs in [11, Theorem 6].

Just as the block decomposition of each side of a 2-separation of a 2-connected graph is always a path, it turns out to always be a pseudo-line in our case.

Proposition 7.5. *For any proper 2-separation (A, B) of G with $A \cap B = \{v, w\}$ the block decomposition of A and B is a pseudo-line, whose endpoints are the unique blocks containing v and w, respectively.*

Proof. By symmetry it suffices to show this for A. Let us first show that v is contained in just one block V_o. If not, then v lies in the separator of some proper 1-separation of A, contradicting Corollary 4.5. Similarly, w is contained in just one block V_p. If there was some vertex x of the tree-like space not on the pseudo-line between o and p, there must be some pseudo-line from x to o. It must contain at least one edge e not on the pseudo-line from o to p. Then the associated separation does not separate v and w, a contradiction to Corollary 4.5. □

Moreover, in both cases no block has more than two vertices.

Lemma 7.6. *Let (A, B) be any proper 2-separation of G. If any block of the block decomposition of A or B has three vertices, then G has a good 2-separation.*

Proof. By symmetry it suffices to show this for A. If the decomposition has just one block then A is 2-connected by Corollary 6.5, so then there is a good 2-separation by Lemma 7.4. Otherwise the two vertices $v, w \in A \cap B$ lie in different blocks V_o and V_p.

First, let us assume that one of these blocks, w.l.o.g. V_o has at least three vertices. Since by Corollary 6.5 V_o is 2-connected, by Lemma 7.4 it suffices to show that there is no pseudo-arc from V_o to its complement not meeting X_o or v. By Corollary 6.10 any such pseudo-arc f cannot be contained in A. Thus it must meet w. But w is separated from V_o by X_o in A, so this is impossible.

If any other block V_x has at least three vertices, then similarly it suffices to show that X_x separates V_x from its complement and any pseudo-arc contradicting this would need to meet v or w, which X_x separates from V_x. □

Using these two results, we now emulate the proof of [11, Theorem 6].

Proposition 7.7. *If G has no good 2-separation, then it is 3-connected or a pseudo-cycle.*

Proof. If G is 3-connected then we are done. So we may assume that there is a proper 2-separation (A, B) with $A \cap B = \{v, w\}$. We want to show that A and B are pseudo-lines from v to w. By symmetry it suffices to show this for A. To this end, we will first prove that in A there is a unique pseudo-line between any two vertices.

By Corollary 7.1 A is 1-connected, so there is a pseudo-line between any two vertices. Now let L and K be two pseudo-arcs between x and y. We want to prove that they are identical. It suffices to show that the edge set of L is equal to that of K. Indeed, by Lemma 2.7 they are the closure of their interior points of edges. So let e be any edge of L. By the third condition for tree-like decompositions the endvertices of e are contained in some block V_o. Then by Lemma 6.12 K also has two vertices a and b in V_o. By Lemma 7.6, however, every block has only two vertices, so these are the endvertices of e. Since no pseudo-arc between two vertices of a block can leave that block, K must then also contain e. By symmetry the converse holds as well.

Thus it now suffices to prove that A is equal to the unique pseudo-line L between v and w in A. If A has any vertex u not in L, then u lies in some block V_t. By Proposition 7.5 t lies on the pseudo-line between a and b. If V_t is a trivial block, then L must contain u by Lemma 6.11. If V_t is nontrivial, then L must contain u by Lemma 6.12. In both cases this is a contradiction. If A has any edge $f \notin L$, then it connects two vertices of L, but these can also be connected along L, a contradiction. □

What we would like to prove now to satisfy our third goal is a statement similar to Lemma 6.4 which would allow us to characterize the parts of our decomposition. From the similar theorems for graphs and matroids we know that we should expect to lift separations not from the parts themselves, but from their torsos and so obtain a description of the parts in terms of these. Unfortunately, as the following example shows, this will not yield a satisfactory result in the general case, since we may end up with torsos which are not even connected.

Example 7.8. Let $V = \omega \times 2 \times 3 \cup \{\omega\}$, $E_1 = \{((a,b,c),(a,b,d)); a \in \omega, b \in 2, c \in 3, d \in 3 \setminus \{c\}\}$, $E_2 = \{((a,b,c),(a+1,b,0)); a \in \omega, b \in 2, c \in \{1,2\}\}$, $E_3 = \{((a,0,0),(a,1,0)); a \in 3\}\}$ and $E = E_1 \cup E_2 \cup E_3$.

Let v^ϵ for some $v \in V$ and $0 < \epsilon \leq 1$ consist of v together with $(0,\epsilon) \times \{e\}$ for all $e \in E$ with v as their first component and $(1-\epsilon,1) \times \{e\}$ for all $e \in E$ with v as their second component. Let τ^* be the set of open intervals of edges e and let τ_v for $v \in \omega \times 2 \times 3$ be the set of all v^ϵ for $0 < \epsilon \leq 1$. Moreover, let $\tau_\omega = \{\{\omega\} \cup \bigcup_{k>n; b\in 2, c\in 2} \tau^\epsilon_{(k,b,c)} \cup \bigcup_{e \in F(n)} (0,1) \times \{e\}; n \in \omega; 0 < \epsilon \leq 1\}$, where $F(n)$ for $n \in \omega$ is $E \cap \{(a,b,c); a > n, c \in 2\}^2$. Then we define τ to be the topology on $V \cup ((0,1) \times E)$ induced by the union of τ^* and the τ_v for $v \in V$.

Let $v_1, v_2 \in V$ be two different vertices. W.l.o.g. $v_2 \neq \omega$. Choose $U_1' \in \tau_\omega$ avoiding v_2 and not containing the midpoint of any edge whose endvertices are not both in U_1' and let U_1 be obtained from U_1' by taking its union with $v_1^{1/3}$. Let $U_2' = v_2^{1/3}$ and let U_2 be obtained from U_2' by adding $v^{1/3}$ for any $v \in V \setminus (U_1 \cup U_2')$. Then (U_1, U_2) is a $(\{v_1\},\{v_2\})$-witness with regard to V in $(V \cup ((0,1) \times E), \tau)$. Thus this space with vertex set V, edge set E and maps chosen as the second projections forms a simple graph-like space H.

Since the underlying graph (V,E) has just two components and the vertices in $\omega \times 1 \times 2$ induce a pseudo-line connecting them, H is connected. Moreover, deleting any finite set of vertices whose deletion does not disconnect $(V - \{\omega\}, E)$ cannot disconnect H since it leaves a final segment of this pseudo-line. In particular, since $(V - \{\omega\}, E)$ is 2-connected, so is H and since all separators of size 2 in $(V - \{\omega\}, E)$ are of the form $\{(a,b,0),(a+1,b,0)\}$ for $a \in \omega$ and $b \in 2$, so are those of H. Clearly, all these actually are separators in H and they leave exactly two components, one of them finite. Since none of the 2-separations of H cross, the 2-block decomposition of H is a star. Let V_t be the part corresponding to the center vertex and let T be the torso of this part where all the pseudo-arcs chosen are finite paths with only a vertex with a 3 in the last component as inner vertices. Any topologically connected nontrivial subspace of T including ω must include infinitely many

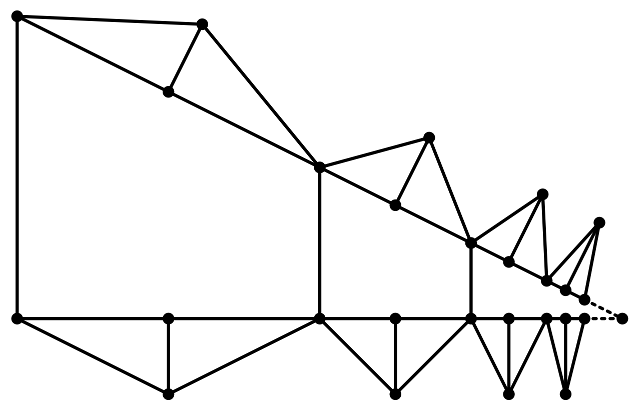

Figure 3: Example 7.8

of the edges added in the torso. But since any vertex of T has an open neighborhood not meeting the midpoint of any of these edges, any such subspace can not be compact. Therefore T contains no nontrivial pseudo-arc ending in ω and is thus not connected.

The problem in this example is that replacing sections of a pseudo-arc with other pseudo-arcs is only guaranteed to result in a pseudo-arc for finitely many such substitutions. Thus we call a graph-like space G *stable*, if for every pseudo-line L with endpoints v, w in G, set I of edge-disjoint connected segments of C, none of which trivial or the whole of L, and family $(L_i)_{i\in I}$ of pseudo-lines in G such that the endvertices of L_i are those of i, L_i and i are different and the interior of L_i and L_j never meets for $i \neq j$ there exists a pseudo-line L' with endvertices v and w contained in $L \setminus \bigcup I \cup \bigcup\{L_i; i \in I\}$. Before we continue toward a lifting result for stable graph-like space, we should make sure that this class of spaces is nontrivial. Indeed, all compact graph-like spaces are stable.

Proposition 7.9. *If G is compact, then G is stable.*

Proof. Since G is compact, G represents a matroid by Theorem 2.8. Given L, v, w, I and $(L_i)_{i\in I}$ as in the definition of stable, define H to be G restricted to the edges of L and of the L_i. Then H is also compact and so is

$H' = H \oplus \{\{v, w\}\}$. Thus H' represents a matroid. Call the added edge e. For each i let J_i be a set of cycles as in Corollary 2.11 applied to L and L_i and choose an edge j_i of L_i from each.. Then we may use circuit elimination on $E(L) + e$, keeping e and eliminating the j_i using J_i. The resulting circuit then contains the desired pseudo-line with endpoints v and w. □

Whether all graph-like spaces inducing matroids are stable is not known. For the rest of this section, we will assume G to be stable. This now allows us to prove the desired lifting lemma.

Lemma 7.10. *For any separation (A, B) of the torso T_o of some part V_o of $D(g_2)$, there exists some separation (E, F) of G with $A \subseteq E$, $B \subseteq F$ and $A \cap B = E \cap F$.*

Proof. Let (A, B) be a separation of T_o and assume that f is a pseudo-arc from some $a \in A$ to some $b \in B$ avoiding $A \cap B$. Since (A, B) is a separation of the torso, f contains vertices outside of V_o. For any such vertex c, let v_c be the supremum of vertices of V_o before c and w_c be the infimum of vertices of V_o after c. We can now use the stability of G to obtain a pseudo-arc from v to w contained in V_o together with the pseudo-lines selected for the torso. Replacing these with their equivalent edges we obtain a pseudo-arc in T_o still avoiding $A \cap B$, a contradiction. Therefore $A \cap B$ separates $A - (A \cap B)$ and $B - (A \cap B)$ in G and we may arrange the other components arbitrarily to get a separation with the required properties. □

For 2-separations we can preserve the property of being good under lifting.

Lemma 7.11. *For any good 2-separation (A, B) of the torso T_o of some part V_o of $D(g_2)$, there exists some good 2-separation (E, F) of G with $A \cap B = E \cap F$ and V_o meeting both $E \setminus F$ and $F \setminus E$.*

Proof. If $G - (A \cap B)$ has at least three components, then by Lemma 7.3 and Proposition 7.2, arranging the components such that one component meeting $A \setminus B$ is alone on on one side and the other components are on the other side gives a good 2-separation fulfilling the requirements. So by Lemma 7.10 we may assume that it has two components, one, say C, containing $A \setminus B$ and one, say D, containing $B \setminus A$. By Lemma 7.2 (A, B) has a 2-connected side, say A. Then it suffices to show that $C \cup (A \cap B)$ is 2-connected, so let us assume for a contradiction that it is not.

By Corollary 7.1 it is 1-connected. Thus we may assume that $C \cup (A \cap B)$ has a proper 1-separation (K, L). By Corollary 4.5 $A \cap B$ has a vertex $c \in K \setminus L$ and a vertex $d \in L \setminus K$. Let v be the unique vertex of $K \cap L$.

If $v \in T_o$, then the pseudo-arcs corresponding to each additional edge of the torso can contain v only if it was one of its endvertices. Therefore, the pseudo-arc from c to d avoiding v in $T_o[A]$, which exists since this set is 2-connected, can be turned into a pseudo-path f from K to L avoiding v by replacing any added edges by the corresponding pseudo-arcs. This, however, contradicts (K, L) being a separation by Proposition 2.10.

So $v \notin T_o$ and v must meet one of the pseudo-arcs corresponding to the added edges since T_o is connected. Since these have disjoint interiors v can in fact only meet one. Let a be one of the endvertices of this added edge. Then the pseudo-arc from c to d avoiding a in $T_o[A]$ can be turned into a pseudo-path f from K to L avoiding v by replacing any added edges by the corresponding pseudo-arcs. This again contradicts (K, L) being a separation by Proposition 2.10.

Thus $(C \cup (A \cap B), D \cup (A \cap B))$ is as required. □

From this we can now deduce the desired characterization.

Theorem 7.12. *The torsos of the parts of the* 2*-block decomposition of* G *are* 3*-connected, pseudo-cycles or bonds.*

Proof. The torso T_o has no proper 0-separations or 1-separations since this would contradict the 2-connectivity of G by Lemma 7.10. The torso has no good 2-separations by Lemma 7.11, since otherwise o would need to orient the lifted separation. If T_o has only two vertices, then it is clearly a bond. Otherwise T_o is simple. Then it is 3-connected or a pseudo-cycle by Proposition 7.7. □

Even if a graph-like space is not stable, its nonempty 2-blocks which are 3-connected are still uniquely determined as the maximal sets of size at least two which do not lie on different sides of any 2-separation by construction.

In the rest of this section, we consider some additional properties of the 2-block decomposition. In particular, we will justify our definition of torsos by showing that edges are added as they would be for graphs. For this we will first note that like chains of 1-separations, chains of 2-separations have suprema if they have upper bounds.

Lemma 7.13. *Let* C *be a chain of proper* 2*-separations of* G *with no maximal element such that* $|P_C| \geq 2$ *and* $(A \setminus B)$ *is nonempty for any* $(A, B) \in C$*. Then there is a* 2*-separation* (E, F) *that is a supremum for* C *such that* $F = P_{X \cup C}$*.*

Proof. By Corollary 3.4 there are two disjoint pseudo-arcs f and g from $(\bigcup_{(A,B)\in C} A)$ to P_C. For any $(A,B) \in C$ we have that the image of f and g contain one of the two vertices in $A \cap B$, say $x^f_{(A,B)}$ and $x^g_{(A,B)}$ are mapped to it. These form linearly ordered, increasing nets, which converge to their suprema x^f and x^g by Lemma 2.12. Since f and g are continuous, the nets of the respective images converge to $x_1 = f(x^f)$ and $x_2 = g(x^g)$, respectively. We have $x_1, x_2 \in B$ for all $(A,B) \in C$, so $x_1, x_2 \in P_{X\cup C}$. Any other pseudo-arc h into P_C from $(\bigcup_{(A,B)\in C} A)$ must then contain a subnet of one these two nets and since the net of the vertices of separators of elements of C met by h must converge, it must converge to x_1 or x_2. Therefore $\{x_1, x_2\}$ separates P_C from all those vertices which lie in $A \setminus B$ for any $(A,B) \in C$. Thus we can obtain a proper separation (E,F) that is an upper bound for C by putting into E exactly those components of $G - \{x_1, x_2\}$ that meet $A \setminus B$ for some $(A,B) \in C$.

Now $E \setminus F$ will contain only vertices that lie in $A \setminus B$ for some $(A,B) \in C$. Indeed, if not then there is some pseudo-arc l avoiding x_1 and x_2 to some such vertex, but then the net of $x_{(A,B)}$ as above on it would converge to one of these vertices, a contradiction.

Finally, assume that (I,J) is any smaller 2-separation. If $I \cap J = \{x_1, x_2\}$ then clearly it cannot be an upper bound. So we may assume that there is some $z \in (G \cap H) \cap (E \setminus F)$. If (I,J) were an upper bound then for any pair f and g of disjoint pseudo-arcs from $(\bigcup_{(A,B)\in C} A) \cap P_X$ to $P_{X\cup C}$ one would need to meet z, w.l.o.g. f. Then the net of the $f(x_{(A,B)})$ as above has x_1 or x_2 as a supremum, so there is some $(A,B) \in C$ such that $x_{(A,B)}$ lies between z and x_1 on f, a contradiction. □

Moreover, a supremum of a chain of good 2-separations is again good.

Corollary 7.14. *Any chain $C \subseteq g_2$ such that there are $o \in O^*(g_2)$ and $p \in O(g_2)$ with $C \subseteq o$, $C^* \subseteq p$, $|P_C| > 2$ and $o \triangle p = C \cup C^*$ has a supremum in g_2.*

Proof. Let C be such a chain. If C has a maximal element we are done. Otherwise by Lemma 7.13 there is a 2-separation (E,F) that is a supremum for C. Since $|P_C| > 2$, it is proper. Let us first show that E contains just one component of $G \setminus (E \cap F)$. For that it suffices to show that $E \cap F$ does not separate any two vertices a and b such that there exists a separation $(A,B) \in C$ with $a, b \in A \setminus B$. If it does then any pseudo-arc f from a to b meets $E \cap F$ in one of its vertices x or y. But since $x, y \in P_o$, the parts of f on either side of x or y meet $A \cap B$. Since f is injective, this happens in different vertices. Therefore either vertex of $A \cap B$ already separates a and b, contradicting the 2-connectivity of G.

Now by Proposition 7.2 it suffices to show that E or F is 2-connected. If not, then by Corollary 7.1 there are proper 1-separations (I, J) of E and (K, L) of F. W.l.o.g. we may assume $x \in (I \setminus J) \cap (K \setminus L)$ and $y \in (J \setminus I) \cap (L \setminus K)$ by Corollary 4.5. Then $(I \cup K, J \cup L)$ is a proper 2-separation. Since (I, J) and (K, L) were proper, there are vertices $z_1 \in I \setminus J$, $z_2 \in J \setminus I$, $z_3 \in K \setminus L$ and $z_4 \in L \setminus K$. We choose them outside of $E \cap F$ if possible. If z_1 and z_2 are not contained in $E \cap F$, there exists some $(A, B) \in C$ such that $z_1, z_2 \in A \setminus B$. This, however, implies that (A, B) will cross $(I \cup K, J \cup L)$, a contradiction. Otherwise w.l.o.g. $z_1 = x$, so the unique vertex v of $I \cap J$ separates x from every other vertex of E. By Corollary 7.1 v and x are then adjacent. Since $v \notin P_C$, there is a final segment C' of C such that x is in the separator of every element of C'. If $z_2 = y$ a similar argument would show that y was in the separator for every element of some final segment of C', contradicting the fact that any chain of good 2-separations with the same separator has at most two elements. Thus there is some $(A, B) \in C'$ with $z_2 \in A \setminus B$. Now the separations (A, B) and $(I \cup K, J \cup L)$ cross, a contradiction. □

Thus all the parts with at least three vertices are again not limit points.

Corollary 7.15. *For any 2-block V_o exactly one of the following four statements is true:*

1. *o is a leaf, o contains a chain with no supremum in g_2 and V_o has at most two vertices*
2. *o contains a chain and the inverse of its supremum and the torso of V_o is a bond*
3. *o contains the supremum of every chain it contains and the torso of V_o is a bond*
4. *o contains the supremum of every chain it contains and no two of its maximal elements have the same separator and V_o is nonempty*

Proof. If o contains a chain with no supremum in g_2, then by Corollary 7.14 o is a leaf and $|V_o| \leq 2$. Otherwise let us first assume the nonempty block V_o contains a chain C, but not its supremum. Since by Corollary 7.14 C has a limit $(E, F) \in g_1$, we then have $(F, E) \in o$. But then $V_o \subseteq E \cap F$, so its torso must be a bond. Now let us assume that o has two distinct maximal elements (A, B) and (E, F) with the same separator. By Proposition 7.2 at least one side of (A, B) and and at least one side of (E, F) contain just one component of $G - (A \cap B)$. But since (A, B) and (E, F) are nested, we have $(F, E) \leq (A, B)$, so in particular $F \subseteq A$ and $B \subseteq E$. If B contained more than one component of $G - (A \cap B)$, then E would also contain all these components and F would contain just the one of A, so actually $(A, B) = (E, F)$, a contradiction. Therefore B contains just one component of $G - (A \cap B)$ and analogously so does F. Since the single components in B and F must be distinct, V_o contains just the separating vertices and its torso is a bond. □

We can use this to prove that the added edges of such parts are determined only by the overlap with neighboring parts.

Lemma 7.16. *If V_o is a nontrivial 2-block, edges are added in the torso exactly between those nonadjacent pairs $\{x, y\}$ such that there exists some 2-block V_p with $x, y \in V_p$ and $o \neq p$.*

Proof. If an edge between x and y is added in the torso, then x and y are nonadjacent and there exists some pseudo-line L from x to y, such that the interior of L does not meet V_o. In particular there exists some $p \in V(T)$ such that L meets $V_p \setminus V_o$. We have that $o \setminus p$ is a chain. But by Corollary 7.15 any chain in o has a supremum (A, B). Let o' be obtained from o by replacing (A, B) with (B, A). Then $V_{o'}$ contains x and y.

For the other direction, if x and y are nonadjacent and there exists some other 2-block V_p such that $x, y \in V_p$ then there is some 2-separation with separator $\{x, y\}$ contained in o. By Corollary 7.1 there then exists some pseudo-arc between x and y, whose interior does not meet V_o. □

Since the set of added edges for 2-blocks with at most two vertices is easy to determine, this is enough to reach our goal.

Lemma 7.17. *If V_o is a 2-block with two vertices x and y then the set of added edges of the torso is in bijection with the set of directions at f.*

Proof. It suffices to show that every direction D at o indeed gives rise to an edge in the torso. Let p be some vertex whose direction from o is D. By Corollary 3.4 there exist two disjoint pseudo-arcs from V_o to V_p. Adding a pseudo-arc between their endvertices gives a pseudo-path between x and y, in whose image we can find a pseudo-arc f between x and y by Proposition 2.10. Then f witnesses that D gives rise to an edge. □

Corollary 7.18. *The underlying graph of the torso of some part V_o is uniquely determined by $T(g_1)$ and $G[V_o]$.*

Proof. This is a direct consequence of Lemma 7.16 and Lemma 7.17 together with the fact that there cannot be edges added to parts with at most one vertex. □

8 Graph-like spaces inducing matroids

In this section we will observe a natural correspondence between parts of the tree decomposition of Theorem 2.4 and 2-blocks of a graph-like space inducing that matroid. First, however, we should note that the similar correspondence for blocks is obvious.

Proposition 8.1. *Let G be a 1-connected, simple graph-like space representing a matroid M. Then there is a bijection g between the components of M and the nontrivial blocks of G such that for any component C the block $g(C)$ contains exactly the edges of C.*

Proof. By the third condition for tree-like decompositions, any edge is contained in some block and since any two blocks meet in at most one vertex, it cannot be contained in multiple. Let us first show that any two edges e, f in the same component C of M are contained in the same block. There exists some pseudo-cycle D containing e and f. Since no pseudo-cycle can meet both sides of a proper 1-separation in more than the separating vertex, there cannot be a good 1-separation separating two endvertices of e and f. Now this defines a function g mapping each component C of M to the block containing all its edges. The surjectivity of g is immediate from the fact that each nontrivial block contains at least one edge.

So it remains to prove that g is injective. Let E and F be components of M and let $e \in E$ and $f \in F$ be any edges. If e and f lie in the same block, then by Corollary 3.4 there are two disjoint pseudo-arcs between their endvertices. This gives a pseudo-cycle containing both e and f, so $E = F$. Therefore g is injective. □

To obtain a similar result for 2-blocks, we will use statements translating separations back and forth between graph-like space and matroids. To make sure that a good separation is translated to a good one, a criterion for 2-separations of matroids being good will be useful.

Proposition 8.2. *Let M be a connected matroid and let (E, E^c) be a 2-separation of M. Then (E, E^c) is good if and only if M restricted to E or E^c is connected and M contracted onto E or E^c is connected.*

Proof. To prove the forward direction, first assume that M restricted to E and E^c are both disconnected and let C and D be a component of each. We want to prove that $(E', E'^c) = (E - C + D, E^c - D + C)$ is a 2-separation of M. Since components are nonempty and M is connected, it suffices to prove that for any bases B_1 and B_2 of M restricted to E' and E'^c respectively there exists an edge e of their union, such that $B_1 \cup B_2 - e$ is independent. For that, let $B_E = (B_1 \setminus D) \cup (B_2 \cap C)$ and $B_{E^c} = (B_2 \setminus C) \cup (B_1 \cap D)$. These are independent sets in M restricted to E and E^c respectively, so there exists some e such that $B_E \cup B_{E^c} - e$ is independent in M. But this is the same set as $B_1 \cup B_2 - e$, so we are done.

For the second assertion, assume that (E, E^c) is good. Since the connectivity function is unaffected by taking duals, (E, E^c) is a good 2-separation of M^*. As proven before, M^* restricted onto E or E^c is connected. Thus M contracted onto E or E^c is connected.

To prove the other direction, we will assume that (E, E^c) is not good. W.l.o.g. $|M| \geq 3$. Let $(T, (R_v)_{v \in V(T)})$ be the tree decomposition from Theorem 2.4. By Corollary 2.6, there exists some $v \in V(T)$ such that $(\phi_v(E), \phi_v(E^c))$ is a 2-separation of M_v. We may assume that M_v is a circuit, the case that it is a cocircuit is dual. Then clearly there is no cycle in M restricted to $\phi_v(E)$ and by definition of the torso M restricted to E is not connected. □

For the rest of this section, let G be a connected simple graph-like space representing a connected matroid M. By Proposition 8.1 G is then also 2-connected.

Going from separations of the graph-like space to those of the matroid is easier than vice versa, but unfortunately it cannot always be done uniquely, since an edge in the separator may be placed on either side.

Lemma 8.3. *If (A, B) is a 2-separation (A, B) of G, $(E(G[A]), (E(G[A]))^c)$ is a 2-separation of M. Moreover, if (A, B) is good, then so is at least one of $(E(G[A]), (E(G[A]))^c)$ and $(E(G[B]), (E(G[B]))^c)$.*

Proof. Let $a \in A \setminus B$ and $b \in B \setminus A$ be arbitrary vertices and let v, w be the vertices of $A \cap B$. Since G is 2-connected, there are pseudo-arcs from a to v and w avoiding w or v, respectively and similarly for b, so $E(G[A])$ and $(E(G[A]))^c$ both contain at least two edges. Now let B_1 and B_2 be bases of M restricted to these two sets respectively. Since M is connected, $B_1 \cup B_2$ contains a pseudo-cycle C. Let e be any edge on C and let x and y be its endvertices. W.l.o.g. e is contained in B. Now it suffices to prove that $B_1 \cup B_2 - e$ is independent. If not, then there exists a pseudo-cycle D whose edges are contained in this set. W.l.o.g. we may assume that x lies between y and v on the part of C in B. Then there are pseudo-arcs f_1 from x to v and f_2 from w to y contained in C and B. There is also a pseudo-arc g from v to w on D contained in B. Concatenating f_1, g and f_2 gives a pseudo-path from x to y using only edges of $B_2 - e$, which contains a pseudo-arc by Proposition 2.10. Then B_2 contains a cycle, a contradiction.

Let us now prove the second assertion. Since (A, B) is good, by Proposition 7.2 one of A and B meets just one component of $G \setminus (A \cap B)$, w.l.o.g. A. Let $e_1, e_2 \in (E(G[B]))^c)$. Applying Corollary 3.4 to the set of separating vertices $\{x, y\}$ and $V(e_1)$ or $V(e_2)$ in $G[A]$ respectively, we obtain pseudo-lines L_1 and L_2 from x to y. Let D be a result of applying Lemma 2.13 to L_1 and L_2, x and y as well as e_1 and e_2 in $G[B]$. If D is a pseudo-cycle, then since only x and y are incident with edges not in $(E(G[B]))^c)$, $E(D)$ must be a cycle in M contracted to $(E(G[B]))^c)$. If D is a pseudo-line, then since $G[A]$ is connected by Lemma 7.1, $E(D)$ must be a cycle in M contracted to $(E(G[B]))^c)$. Thus, this matroid is connected. By Proposition 7.2 we may choose F to be one of A and B such that $G[F]$ is 2-connected, preferring A. Let $e_1, e_2 \in E(G[F])$ be given. Then applying Corollary 3.4 to $V(e_1)$ and $V(e_2)$ in $G[F]$ gives a cycle containing e_1 and e_2, so M restricted to $E(G[F])$ is connected. If x and y are adjacent, by Proposition 4.3 both $G[A]$ and $G[B]$ are connected, so in either case $E(G[F])$ is one of the sides of $(E(G[B]), (E(G[B]))^c)$. Thus $(E(G[B]), (E(G[B]))^c)$ is good by Proposition 8.2. □

The other direction requires a bit more work. First, we observe that for good 2-separations of matroids, unlike for arbitrary ones, the set of vertices incident with edges of one of the sides is connected in the graph-like space. As it will turn out, we only need to consider these vertices to find our separation of G, the rest will then fall into place.

Corollary 8.4. *If (E, E^c) is a good 2-separation of M, then for any vertices $v, w \in V(E)$ there exists a pseudo-arc from v to w using only edges of E and similarly for E^c.*

Proof. By Lemma 8.2 we may assume w.l.o.g. that M restricted to E is connected. Then the assertion for E follows from the the fact that any two edges of E are contained in a common cycle using only edges of E. To prove

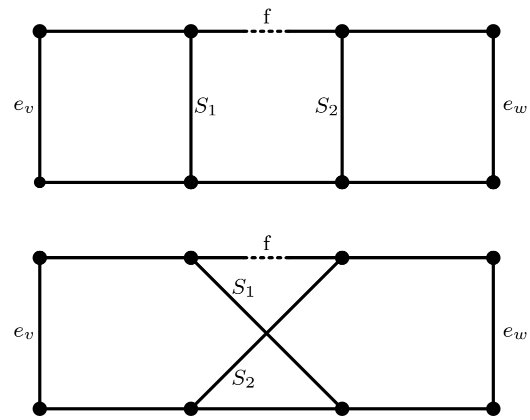

Figure 4: The two cases in the proof of Corollary 8.4

the assertion for E^c let $v, w \in V(E^c)$ be given. Let e_v and e_w be edges of E^c adjacent to v and w respectively and let C be a circuit containing e_v and e_w. Let L_1 and L_2 be the two pseudo-lines which are the components of $C - e_v - e_w$. If L_1 or L_2 contains no element of E, then we are done. Otherwise choose some $e_i \in E(L_i) \cap E$ for $i \in \{1, 2\}$. Since M restricted to E is connected, there is some cycle C' contained in E including e_1 and e_2. Let S_1' and S_2' be the two pseudo-lines from an endvertex of e_1 to an endvertex of e_2 which are the components of $C' - e_1 - e_2$. By Lemma 2.12 we can obtain a pseudo-line S_i for $i \in \{1, 2\}$ meeting L_1 and L_2 once each and only in its endvertices by restricting S_i to its segment from its last vertex a_i in L_1 to its first vertex b_i after that in L_2.

Let f be some edge between a_1 and a_2 on L_1. Let $F = E(L_1) \cup E(L_2) \cup E(S_1) \cup E(S_2) + e_v + e_w - f$. Since any cycle contained in F meets S_1 or S_2 in an edge, $F \cap E^c$ is independent in E^c. Since any cycle contained in F contains e_v or e_w, $F \cap E$ is independent in E. Thus there is some edge g such that $F - g$ is independent.

W.l.o.g. a_1 occurs before a_2 on L_1. If b_1 occurs before b_2 on L_2, then the cycle obtained from e_1 and S_1 by connecting them with segments of L_1 and L_2 and the analogous one containing e_2 and S_2 are edge-disjoint cycles contained in F, so one of them avoids g. If b_1 occurs after b_2 on L_2, then the cycle obtained from e_1 and S_1 by connecting them with segments of L_1 and L_2, the analogous one containing e_2 and S_2 and the unique cycle in F containing both S_1 and S_2 are three cycles in F with empty intersection, so one of them avoids g. In both cases this contradicts the independence of $F - g$, finishing the proof. $\square$

This is helpful for showing that there are exactly two vertices incident with edges of both sides of such a separation.

Lemma 8.5. *Let (E, E^c) be a good 2-separation of M. Then $|V(E) \cap V(E^c)| = 2$.*

Proof. Let us first show that $V(E) \cap V(E^c)$ has at most two vertices. Otherwise let v_1, v_2 and v_3 be three such vertices. By Corollary 8.4 there is a pseudo-arc f_1 from v_1 to v_2 using only edges of E and a pseudo-arc from v_3 to v_1, from which we can obtain a pseudo-arc f_2 from v_3 onto f_1 by Lemma 2.12. We can define g_1 and g_2 similarly for E^c. Since the set of edges of f_1 and f_2 is acyclic, it can be extended to a base B_1 of M restricted to E. Similarly we obtain a base B_2 of M restricted to E^c containing the edges of g_1 and g_2. Now there exists some edge e such that $B_1 \cup B_2 - e$ is a base of M. W.l.o.g. $e \in B_1$. Then $B_1 - e$ still contains a pseudo-arc between two of $\{v_1, v_2, v_3\}$ and between these two vertices there is also a pseudo-arc with edges in B_2, contradicting the fact that $B_1 \cup B_2 - e$ is independent.

In addition $V(E) \cap V(E^c)$ has at least two vertices, since there exists some cycle in M contained in $B_1 \cup B_2$. $\square$

In light of this, it seems clear that these two vertices should be the separator, with the sides containing at least all the vertices incident with an edge of that sides. Some vertices might be left over, however, so these need to be distributed. Luckily, it turns out that this can be done uniquely.

Lemma 8.6. *Given a good 2-separation (E, E^c) of M, let $A = V(E)$ and $B = V(E^c)$. Then there exists a unique good 2-separation $(C.D)$ of G with $A \subseteq C$, $B \subseteq D$ and $A \cap B = C \cap D$.*

Proof. By Lemma 8.5 $|A \cap B| = 2$. Therefore $A \setminus B$ and $B \setminus A$ must be nonempty, since in a simple graph-like space two edges will always induce at least three vertices.

Now we will show that $A \cap B$ separates A from B. If not, let f be a pseudo-arc from A to B avoiding $A \cap B$. Then by Corollary 8.4 there exist pseudo-arcs g_1 and g_2' from the vertices of $A \cap B$ onto f using only edges of E and h_1 and h_2' from the vertices of $A \cap B$ onto f using only edges of E^c. Let g_2 be the part of g_2' until the infimum of points from g_1 and let h_2 be defined analogously. The edges of f that lie in E together with those of g_1 and those of g_2 are acyclic, so they can be extended to a base B_1' of E.

Similarly, we obtain a base B'_2 of E^c. Thus there must be some edge d such that $B'_1 \cup B'_2 - d$ is independent. Then d must clearly lie on f and both g_1 and h_1 must meet f on different sides of d. The same however, holds for g_2 and h_2, so there must be a pseudo-cycle in $B'_1 \cup B'_2 - d$, a contradiction.

By Proposition 8.2 w.l.o.g. M contracted to E is connected. Thus for any two edges $e_1, e_2 \in E$ there exists a pseudo-cycle containing e_1 and e_2 not meeting both vertices of $A \cap B$ or a pseudo-line with set of endvertices $A \cap B$ containing both e_1 and e_2. In particular $A \cap B$ does not separate any two vertices of A. Let C' be the component of $G \setminus (A \cap B)$ meeting A. We define $(C, D) = (C' \cup A \cap B, C'^c)$.

By Proposition 8.2 one of M restricted to E or E^c is connected. We will only consider the case where E is connected, the other one is analogous. By Proposition 7.2 it now suffices to show that C is 2-connected. By Lemma 7.1 C is 1-connected, so assume (I, J) to be a 1-separation of C. Since G restricted to E is connected, for any two edges $e_1, e_2 \in E$ there is a pseudo-cycle containing both. Thus A is completely contained in I or J. In particular, this holds for $A \cap B$, so by Corollary 4.5 (I, J) must be improper.

Finally, uniqueness follows from the fact that any component of $G \setminus (A \cap B)$ must be incident with an edge by the connectedness of G. □

To make things a bit nicer, we sum up our findings so far with two functions.

Corollary 8.7. *There exist a homomorphism $f : g_2(M) \to g_2(G)$ and a function $g : g_2(G) \to g_2(M)$ satisfying the following conditions:*

1. *$f(g(s)) = s$ for every $s \in g_2(G)$*
2. *For any $(E, E^c) \in g_2(M)$, if $f((E.E^c)) = (A, B)$ for some $(A, B) \in g_2(G)$, then $E(A) \supseteq E$ and $E(B) \supseteq E^c$*

Proof. Let f be given by Lemma 8.6. Then the second condition is obvious. By uniqueness we have $f(s^*) = f(s)^*$ for any $s \in g_2(M)$. To prove that f is a homomorphism, let $(E, E^c) \leq (F, F^c)$ in $g_2(M)$ be given. Then the components of $G \setminus (V(E) \cap V(E^c))$ contained in the first component of $f((E, E^c))$ do not meet $V(F^c)$. But since each component is incident with an edge, they must then be contained in the first component of $f((F, F^c))$. Thus $f((E, E^c)) \leq f((F, F^c))$ and f is a homomorphism. Let g be any function as given by Lemma 8.3. Let $s \in g_2(G)$ be arbitrary. Then s is a possible choice when applying Lemma 8.6 to $g(s)$. By uniqueness $s = f(g(s))$. □

For the rest of this section, we fix functions f and g as in Corollary 8.7. For a 2-block V_o of G let $E^*(V_o)$ be the set of edges induced by V_o, whose set of endvertices is not the separator of any good 2-separation. Fix a tree decomposition $D = (T, (R_v)_{v \in V(T)})$ as in Theorem 2.4, where we take $V(T)$ to consist of the consistent orientations of the good 2-separations of M. Putting it all together, we obtain a correspondence result as desired.

Proposition 8.8. *G is 2-connected and there exists a unique function h from the set of orientations defining 2-blocks of G to $V(T)$ such that $f[h(o)] = o$. Moreover, h is injective, $im(h) = \{t \in V(T); \forall a \neq b \in g_2(M) f(a) = f(b) \implies |\{a,b\} \cap t| \neq 1\}$, $E^*(V_o) \subseteq h(o)$ and the set of circuits of the torso of V_o is the same as that of the torso of the part associated with $h(o)$ (up to renaming added edges).*

Proof. By Proposition 8.1 G is 2-connected. Let V_o be a 2-block of G. We define h by $h(o) = \{s \in g_2(M); f(s) \in o\}$. This is a consistent orientation because f is a homomorphism. We have $f[h[o]] \subseteq o$ by definition and the reverse inclusion because g is a right-inverse of f. If h' differs from h on o, then for $s \in h'(o) \setminus h(o)$ we have $f(s) \notin o$, which proves uniqueness. Furthermore h is injective because f is surjective.

Clearly, any $t \in V(T)$ contained in the image of h must satisfy the condition in the assertion, the other inclusion follows from $h(\{s \in g_2(G); g(s) \in t\}) = t$ for any such $t \in V(T)$. The second condition in Corollary 8.7 gives $E^*(o) \subseteq h(o)$ for any o defining a 2-block.

Finally, if C is a circuit of the torso of a 2-block V_o, then replacing the added edges in C with their associated pseudo-arcs gives a circuit of G, which witnesses that the equivalent edge set is a cycle in the torso of $h(o)$. Conversely, if C is a cycle of the torso of $h(o)$, then there exists some pseudo-cycle C' in G meeting $f(V_o)$ exactly in $h(o) \cap C$. By Lemma 5.7 we may assume that the pseudo-arcs contained in C' are those chosen for the construction of the torso. But then replacing these pseudo-arcs in C' with their equivalent edges gives a pseudo-cycle as required. □

Our translation of separations also gives us a version of Proposition 2.3 for graph-like spaces inducing matroids.

Corollary 8.9. *There exists no $\omega + 1$-chain of good 2-separations of G. Furthermore, if C is an ω-chain of good 2-separations of G, then $G[V_C]$ contains no edge.*

Proof. Since there exist at most two comparable separations with the same separator, by Proposition 2.3 and Lemma 8.7 to establish both statements it suffices to prove that for any chain C in $g_2(G)$ such that no two separations have the same separator, $g(C)$ is a chain in M of the same length.

So let C be such a chain. Since there exist at most two comparable separations with the same separator, we may assume that no two separations in C have the same. Let $(A_1, B_1), (A_2, B_2) \in C$ with $(A_1, B_1) < (A_2, B_2)$ be given and define $(E_i, E_i^c) = g((A_i, B_i))$ for $i \in \{1, 2\}$. To prove that $(E_1, E_1^c) \leq (E_2, E_2^c)$, let $e \in E_2$ be arbitrary. Then $e \in E[A_2]$ by the second condition of Corollary 8.7. Since (A_1, B_1) and (A_2, B_2) have different separators, $e \notin E(B_1)$. Thus $e \in E_1$. But since f is a left inverse of g, g is injective and we are done. □

9 Reconstruction

Our goal for this section is to reconstruct a graph-like space from its 2-block decomposition. For this to make sense, we first need to specify what information we will use for this reconstruction.

Call a pair of a tree-like space T and a family of simple graph-like spaces $(G_t)_{t \in V(T)}$ satisfying the following conditions a *skeleton*:

1. If $x \in G_s \cap G_t$ for $s, t \in V(T)$, then x is a vertex in both spaces.
2. There are no two $e, f \in \bigcup_{t \in T} E(G_t)$ that connect the same vertices
3. $G_s \cap G_t \subseteq G_x$ for all $s, t, x \in V(T)$ with $x \in L_T(s, t)$

It is *short* if on any pseudo-line L in T the supremum of any infinite chain of different vertices is an endvertex and for any such supremum v the space G_v has at most one vertex.

Given a proper tree-like decomposition $D = (T, (V_t)_{t \in V(T)})$ of adhesion $< \aleph_0$ of some graph-like space we can obtain a skeleton with the same tree-like space by taking the induced space of every part and arbitrarily deleting edges contained in multiple parts from all but one of them. We will call any skeleton obtained in this way an induced skeleton of D. This will be the data used for our proof of reconstruction for compact spaces. In particular we can obtain such a skeleton from our decomposition into 2-blocks and as we will see later, this skeleton will be short. Thus we can limit ourselves to considering only short skeletons.

To simplify notation, let us fix for this section a natural number $k \leq 1$, a compact k-connected graph-like space G and a short skeleton $S = (T, (G_t)_{t \in V(T)})$ induced by a proper tree-like decomposition of G of adhesion $\leq k$. Our first goal now is identify from this skeleton the topology that is supposed to agree with the original one. It is optimal for us to declare as open as few sets as possible while still defining a graph-like space, since this will make it more likely that all these sets were open in the original space and for a compact space this is all that needs to be checked. Write $V(S)$ for the set $\bigcup_{t \in V(T)} V(G_t)$, $E(S)$ for the set $\bigcup_{t \in V(T)} E(G_t)$ and $\bigcup S$ for $\bigcup_{t \in V(T)} G_t$. For any vertex t of T and any direction d at t let C_d^t be the union of all those G_s such that $s \neq t$ occurs in any element of d. Let τ^* be the set of all subsets U of $\bigcup S$ satisfying the following conditions:

1. The intersection of U with G_t is open in G_t for all $t \in V(T)$.

2. If U contains $G_t \neq \emptyset$ for some limit point t then there exists some edge e of T such that U contains all G_s where s lies in the same component of $T - e$ as t.

3. For any $t \in V(T)$ the set of all adjacent directions d at t such that U meets $G_t \cap G_{t^d}$, but does not contain C_t^d is finite.

Let τ be the topology on $\bigcup S$ generated by τ^*. Call this space $R'(S)$.

By showing that witnesses in G still qualify in $R'(S)$, we can prove that this is a graph-like space.

Lemma 9.1. *The space $R'(S)$ together with vertex set $V(S)$, edge set $E(S)$ and maps as in the G_t the edge was contained in is a simple graph-like space.*

Proof. Since G is a graph-like space, all conditions not referring to the topology are trivially satisfied. Furthermore, if U is open in $(0,1)$ and e is an edge occurring in some G_a with $a \in V(T)$, then $t_e^{G_a}(U) \in \tau^*$. Thus the maps restricted to $(0,1)$ are still open.

Now it remains to prove that for any two vertices $v_1, v_2 \in V(S)$ there exists a $(\{v_1\}, \{v_2\})$-witness in $R(D)$. Let (X, Y) be a $(\{v_1\}, \{v_2\})$-witness in G. Now it suffices to show that $X, Y \in \tau^*$. The first condition is clearly satisfied.

For the second condition let v be the unique vertex of G_t for some limit point t and let d be the unique direction at t such that $v \in V(t, d)$. If $C_s^{d'}$ always meets both X and Y for every $s \in V(T)$ and direction d' with $s \neq t$ and $v \in V(s, d')$, then, since $G[V(t,d)]$ is connected by Lemma 5.3, there must always be some edge with endpoints in $V(t, d)$ that is not contained in $X \cup Y$. But since all vertices besides v will eventually not be contained in $V(t, d)$, there must be infinitely many edges not contained in $X \cup Y$, which together with X and Y form an open cover of G with no finite subcover, a contradiction.

To prove the third condition, assume that it fails for one of them, w.l.o.g. X at some $t \in V(T)$. Then there is an infinite set D of directions d such that X does not contain C_t^d, but meets $G_t \cap G_{t^d}$. To deduce a contradiction as before it now suffices to prove that there is a family of distinct edges $(e_d)_{d \in D}$ not contained in $X \cup Y$.

So let $d \in D$ be given. By assumption both X and Y meet C_t^d. Since C_t^d is connected by Lemma 5.3, $X \cup Y$ can not cover it. Thus there must be some edge e_d not contained in $X \cup Y$. But since every edge is contained in G_s for just one $s \in V(T)$, all e_d for $d \in D$ are distinct. □

We will write $R(S)$ for the graph-like space in Lemma 9.1, although, as we will show next, it is really just G.

Theorem 9.2. $R(S) = G$

Proof. Since G is compact and $R(S)$ is Hausdorff it suffices to show that any open set in $R(S)$ is open in G. So let $U \in \tau^*$ be given and assume for a contradiction that there exists a net $(x_d)_{d \in D}$ in $G \setminus U$ converging to some $x \in U$. Clearly D is unbounded. If x occurs in G_t only for some limit point t, then by the second condition there exists some vertex s of T and some adjacent direction r at s such that U^c is completely contained in C_s^r. But by Lemma 5.3 this space is closed, a contradiction.

Otherwise the set of $t \in V(T)$ with $x \in G_t$ that are not limits points together with the edges between them in T forms a graph-theoretical tree T'. Let $\mathcal{U}$ be an ultrafilter on D. For each edge e of T' and endvertex a of e let D_e^a be the set of all those $d \in D$ such that $x_d \in G_s$, where s is contained in the same same component of $T - e$ as a. Let o be the set of all those orientations of edges e pointing towards a, where $D_e^a \in \mathcal{U}$. Clearly this defines a consistent orientation. By Lemma 2.2 o points to a vertex or an end. If it points to an end, then x must also be contained in the limit point corresponding to that end, so the edge that we obtain from the second condition for tree-like decompositions should have been oriented differently. Thus it points to a vertex t.

But the first condition implies that no subnet of $(x_d)_{d \in D}$ can be contained in G_t, so t must be a leaf. Therefore we may assume w.l.o.g. that $(x_d)_{d \in D}$ does not meet G_t and only meets C_t^r for directions r such that $x \notin G_t \cap G_{t^r}$. Now Lemma 5.3 implies that $(x_d)_{d \in D}$ cannot have a subnet in C_t^r for any direction r at t, so we may assume that it meets each such direction at most once. By the third condition for tree-like decompositions we may also assume that $(x_d)_{d \in D}$ does not meet any C_t^r for any direction r such that $G_t \cap G_{t^r} \subseteq U$.

Let $r(d)$ be the direction r' at t such that $x_d \in C_t^{r'}$, let $X_d = C_t^{r(d)}$ and let y_d be some vertex of $G_t \cap G_{t^{r(d)}}$. By Lemma 2.18 $\lim_{\mathcal{U}}(\{y_d\})_{d \in D}$ has a unique element y, which is contained in $G_t \setminus U$ by Lemma 5.2 and the first condition. By Lemma 2.17 $\lim_{\mathcal{U}}(X_d)_{d \in D}$ also contains $x \in U$, so it has at least two elements. But by Lemma 5.3 the X_d are closed and connected, so this contradicts Corollary 2.20. □

For this to be useful, we still need to check that the 2-block decomposition satisfies the requirements.

Corollary 9.3. *Let G be a compact 2-connected graph-like space and let S be an induced skeleton of its 2-block decomposition. Then $R(S) = G$.*

Proof. By Theorem 9.2 all that remains to be checked is that S is short. Since by Theorem 2.8 G induces a matroid, by Corollary 8.9 there are no $\omega + 1$-chains of good separations. Thus there can not be any more edges after the supremum v of an infinite chain of vertices, so Lemma 2.7 implies that v is an endvertex.

Now we need to show that the part V_v associated with v has just one vertex. So assume that a and b are different vertices of V_v. Then there exist disjoint open sets U and W of G covering $V(G)$ with $a \in U$ and $b \in W$. Since G is compact, $U \cup W$ contains all inner points of all but finitely many edges. Let F be the finite set of other edges and let L be a nontrivial pseudo-line in T ending at v. Since any set of comparable separations with the same separator has size at most two, any edge of F is induced by the parts at only finitely many elements of L. Furthermore, by Corollary 8.9, $G[V_v]$ contains no edges. Thus by the second and third condition for tree-like decompositions there exists some vertex $x \neq v$ of L such that for any vertex y in the segment of L from x to v no endpoint of an edge of F is included in V_y. Let z_1 and z_2 be the vertices of $V_x \cap V_{x'}$, where x' is the successor of x toward v. Let f be a pseudo-arc from a to b avoiding z_1. Since the image of f is topologically connected, f must use an edge of F and therefore one of its endvertices r. But then the parts of f before and after r must both meet z_2, a contradiction. □

The requirement that G be compact can not be weakened to just having it induce a matroid, even if stability is assumed. More formally, there exist two 2-connected stable graph-like space inducing matroids which have the same 2-block decomposition and whose torsos of these 2-blocks may be chosen to be identical. In particular, this can happen because we can not tell the difference between the multitude of possible ways topologies of graphs may behave at vertices of infinite degrees if each part contains only finitely many of their incident edges and these are not chosen for the torso.

Example 9.4. Let $V = \omega \times 2$ and let $E = E_0 \cup E_1 \cup E'$, where $E_i = \{((0,i),(a,b)); a \in \omega, a > 0, b \in 2\}$ for $i \in 2$ and $E' = \{((a,0),(a,1)); a \in \omega, a > 0\}$. Let $Z = \mathcal{P}(E_0) \times \mathcal{P}(E_1)$. Define E_v^1 to be the set of all $e \in E$ whose first component is v and define E_v^2 analogously. Let τ be the set of open intervals of edges in E and let $\tau_{(a,b)}$ for some $(a,b) \in V$ with $a > 0$ be the set $\{\{(a,b)\} \cup (0,\epsilon) \times E^1_{(a,b)} \cup (1-\epsilon,1) \times E^1_{(a,b)}; \epsilon \in (0,1]\}$. Furthermore, for $A \subseteq E_0$ let $\tau^A_{(0,0)} = \{\{(0,0)\} \cup (0,\epsilon) \times (E_1 \setminus A) \cup \{(x,a); a \in A, x \in (0,\epsilon^*_a)\}; \epsilon \in (0,1], \epsilon^* \in (0,1]^A\}$ and analogously define $\tau^B_{(0,1)}$ for some $B \subseteq E_1$. For each $(A,B) \in Z$ let $T_{(A,B)}$ be the topological space induced by the union of τ, $\tau_{(a,b)}$ for all $(a,b) \in V$ with $a > 0$, $\tau^A_{(0,0)}$ and $\tau^B_{(0,1)}$.

Since for any partition (P,Q) of V we can find a (P,Q)-witness in T_z for any $z \in Z$ by taking the union over $p \in P$ and $q \in Q$ respectively of sets from the subbasis containing that vertex and the midpoint of no edge, T_z together with vertex set V, edge set E and the trivial open maps forms a graph-like space G_z. The space G_z is simple and 2-connected because (V,E) is a simple, 2-connected graph and stable because the only pseudo-arcs in it are finite paths. Furthermore it represents the finite cycle matroid of (V,E), since as before its pseudo-cycles are all finite and any partition of the vertices induces a topological cut, which contains exactly the crossing edges of the partition. Since the 2-separations of G_z are the same as those of $G_{z'}$ for any z, z' in Z, they have the same 2-block decomposition.

Let $G_1 = G_{(E_0,E_1)}$ and $G_2 = G_{(E_0^*,E_1^*)}$, where $E_i^* = \{((0,i),(a,0)); a \in \omega, a > 0\}$ for $i \in 2$. Since E_0 and E_0^* differ by infinitely many edges we can choose ϵ_a for $a \in E_0 \setminus E_0^*$ such that 0 is their closure to obtain an open set in G_2 such that the length of edges at $(0,0)$ is not bounded below. But since for any open set of G_1 these are bounded below, their topologies are different. To obtain equal torsos for G_1 and G_2, it suffices to choose the pseudo-lines defining them as the unique ones avoiding E_0^* and E_1^*. Indeed, then the torsos of the center are identical by construction and in those of the leaves every vertex is incident with only finitely many edges, so both have the topology of the associated finite graph.

This result is not dependent on the specific choice of torsos given. Indeed, if the choice of torso for the center vertex can be given by some total ordering on all pseudo-arcs which occur in the set of graph-like spaces constructed, there always exists an edge not in the highest priority pseudo-arc for every leaf and we can thus obtain two graph-like spaces $G_z, G_{z'}$ as above with the

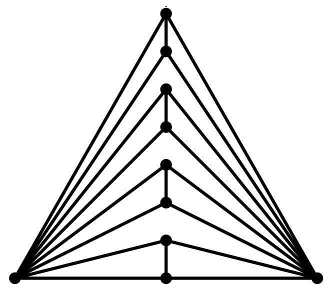

Figure 5: The graph from Example 9.4

same torso, where at least one component of z differs from that component of z' by infinitely many edges and which are thus different.

10 Summary

The main objective of this thesis was to find an equivalent to the known decompositions of graphs and matroids along their 2-separations for graph-like spaces. To this end we first observed some useful properties of pseudo-arc connectivity that made it a suitable choice for defining our separations, including a Menger-type result. Based on the prior work in [9], we described how to obtain a tree-like decomposition from any nested set of separations of a graph-like space. We then applied this first to the simpler problem of finding an analogue for blocks and found no major difficulties in constructing a decomposition of a connected graph-like space into maximally 2-connected parts in this way.

For 2-blocks things did not turn out quite so nicely. For any simple, 2-connected graph-like space G we can still use the same procedure as before to find a decomposition along its 2-separations, but we need G to be stable if we want the torso of every part to be 3-connected, a pseudo-cycle or a bond. Furthermore, if G represents a matroid then each part of our 2-block decomposition corresponds to a part of the known decomposition for matroids. Finally, if G is compact then it can be reconstructed knowing only the decomposition tree-like space and the topologies of each part, but this is not necessarily possible otherwise even if G is stable and represents a matroid.

References

[1] Elad Aigner-Horev, Reinhard Diestel, and Luke Postle. *The structure of 2-separations of infinite matroids.* 2015. URL: `https://arxiv.org/abs/1201.1135`.

[2] Bruhn et al. *Axiom for infinite Matroids.* 2010. URL: `https://arxiv.org/abs/1003.3919`.

[3] Nathan Bowler, Johannes Carmesin, and Robin Christian. *Infinite graphic matroids Part I.* 2013. URL: `https://arxiv.org/abs/1309.3735`.

[4] Nathan Bowler, Johannes Carmesin, and Robin Christian. *Infinite graphic matroids Part II.* in preparation.

[5] Henning Bruhn and Paul Wollan. *Finite connectivity in infinite Matroids.* 2011. URL: `https://arxiv.org/abs/1101.5621`.

[6] Reinhard Diestel. *Abstract separation systems.* 2017. URL: `https://arxiv.org/pdf/1406.3797.pdf`.

[7] Reinhard Diestel. *Graph Theory.* 2002.

[8] Reinhard Diestel. *Tree Sets.* 2017. URL: `https://arxiv.org/pdf/1406.3797.pdf`.

[9] Pascal Gollin and Jakob Kneip. *Tree-like spaces.* in preparation.

[10] Neil Hindman and Dona Strauss. *Algebra in the Stone-Ĉech Compactification: Theory and Applications.* 1998.

[11] R. Bruce Richter. "Decomposing infinite 2-connected graphs into 3-connected components". In: *The Electronic Journal of Combinatorics* (2004).

[12] Carsten Thomassen and Antoine Vella. "Graph-like continua, augmenting arcs, and Mengers theorem". In: *Combinatorica* (2008).

Eidesstattliche Erklärung

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