Cofinitary nearly finitary matroids are l-nearly finitary for some $l \in \mathbb{N}$

Abstract. We characterise for cofinitary matroids $M$ the sets $F$ for which there are bases $B$ of $M$ and $B_{\text{fin}}$ of $M_{\text{fin}}$ such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \setminus B$.

The main result yields an alternative proof of Halin’s theorem.

1. Introduction

Matroids which are $l$-nearly finitary were introduced in [1]. The finitarisation of a matroid $M$ is the matroid $M_{\text{fin}}$ on the same ground set $E(M)$ as $M$ with circuit set $C(M_{\text{fin}}) = \{ C \in C(M) : C \text{ is finite} \}$. If for all bases $B$ of $M$ and all bases $B_{\text{fin}}$ of $M_{\text{fin}}$ such that $B \subseteq B_{\text{fin}}$ the set $B_{\text{fin}} \setminus B$ is finite, then $M$ is called nearly finitary.

If there is also a natural number $l \in \mathbb{N}$ which is an upper bound on the size of $B_{\text{fin}} \setminus B$, then $M$ is $l$-nearly finitary.

In [1] it was shown that every algebraic cycle matroid of an infinite graph and every topological cycle matroid of a 2-connected locally finite graph have the property that if they are nearly finitary then they are also $l$-nearly finitary for some natural number $l \in \mathbb{N}$. These proofs establish a connection between the existence of large families of pairwise disjoint rays in a graph and the property of the corresponding (topological or algebraic) cycle matroid to be ($l$-)nearly finitary. They then use the following theorem by Halin, which is an important theorem in infinite graph theory:

Theorem 1.1. [5, Halin] If an infinite graph $G$ contains $l$ (vertex-) disjoint rays for every $l \in \mathbb{N}$, then $G$ contains infinitely many disjoint rays.

Having shown the two propositions the authors of [1] made the following conjecture:

Conjecture 1.2. [1] Every nearly finitary matroid is $l$-nearly finitary for some natural number $l \in \mathbb{N}$.

In this paper it will be shown that Conjecture 1.2 holds for cofinitary matroids. The result can be used to prove Halin’s theorem by the connection established in [1] between families of pairwise disjoint rays in a graph $G$ and the algebraic cycle matroid. Via this connection, Conjecture 1.2 for cofinitary matroids can be seen as a matroidal analogue of Halin’s theorem.

2. Preliminaries

2.1. Infinite matroids. The definitions for infinite matroids (including minors and duality) used here are the ones from [4]. From that paper we will also need the following basic result about infinite matroids.

Lemma 2.1. [4] Let $B_1$ and $B_2$ be two bases of $M$. If $|B_1 \setminus B_2| < \infty$, then $|B_2 \setminus B_1| = |B_1 \setminus B_2|$.
Remark 2.2. A set $X \subseteq E(M)$ is spanning in $M$ if and only if its complement is independent in $M^*$ if and only if $X$ meets every cocircuit.

Definition 2.3. Let $X \subseteq E(M)$ be a subset of the ground set of $M$. Let $B_X$ be a base of $M|X$, $B_Y$ a base of $M - X$ and $B$ a base of $M$ contained in $B_X \cup B_Y$. Whether $(B_X \cup B_Y) \setminus B$ is finite does not depend on the choice of $B_X$, $B_Y$ and $B$. If $(B_X \cup B_Y) \setminus B$ is finite, then its size also does not depend on the choice of $B_X$, $B_Y$ and $B$ and is defined as the connectivity $\kappa_M(X)$ of $X$. If $(B_X \cup B_Y) \setminus B$ is infinite, then $\kappa_M(X)$ is defined as $\infty$. In particular $\kappa_M(X) = \kappa_M(E(M) \setminus X)$.

Lemma 2.4. The connectivity function is submodular, that is for all $X, Y \subseteq E(M)$ there holds

$$\kappa_M(X) + \kappa_M(Y) \geq \kappa_M(X \cup Y) + \kappa_M(X \cap Y).$$

Remark 2.5. Let $M$ be a matroid, $X \subseteq E(M)$ and $e$ an edge of $M$ not in $X$. Then $\kappa_M(X) - 1 \leq \kappa_M(X + e) \leq \kappa_M(X) + 1$, or in words: adding or removing an edge changes the connectivity of a set by at most one, and $\kappa_M(X) \leq \kappa_M(X - e) + 1$.

2.2. Matroids in graphs. The notation for graphs is the one of [5].

Definition 2.6. Let $G$ be a graph. The set of edge sets of finite cycles of $G$ is the set of circuits of a matroid $M_{FC}(G)$, called the finite cycle matroid of $G$. The set of edge sets of finite cycles of $G$ together with the set of edge sets of double rays of $G$ is the set of circuits of another matroid $M_{AC}(G)$, the algebraic cycle matroid, if $G$ does not contain a subdivision of the Bean graph.

Remark 2.7. Let $G$ be a graph for which the algebraic cycle matroid exists. Then $M_{AC}(G)_{\text{fin}} = M_{FC}$. If $G$ is locally finite then it does not contain a subdivision of the Bean graph, so $M_{AC}(G)$ exists. Furthermore in a locally finite graph $G$ the matroid $M_{AC}(G)$ is cofinitary.

3. Cofinitary nearly finitary matroids

In this section we will show that a nearly finitary cofinitary matroid is $l$-nearly finitary for some natural number $l \in \mathbb{N}$. The proof was inspired by the following observation, which holds for all (and not just cofinitary) matroids:

Lemma 3.1. Let $M$ be a matroid and let $B$ and $B_{\text{fin}}$ be bases of $M$ and $M_{\text{fin}}$ respectively such that $B \subseteq B_{\text{fin}}$. Let $F$ be contained in $B_{\text{fin}} \setminus B$ and let $F' \subseteq E(M)$ be a finite set. Then $\kappa_M(F') \geq |F \cap F'|$.

Proof. $F'$ is finite, so $F' \cap B_{\text{fin}}$ is independent in $M|F'$ and is thus contained in a base $B_1$ of $M|F'$. Since $B$ is a base of $M$, every edge of $F$ is spanned in $M$ by $B$, so $F \cap F'$ is spanned in $M$, $F'$ by $B \cap F' \subseteq B_1 \setminus F$. Hence $M.F'$ is spanned by $B_1 \setminus F$, so $\kappa_M(F') \geq |B_1 \setminus (B_1 \setminus F)| = |F \cap B_1| = |F \cap F'|$. \qed

Definition 3.2. Let $M$ be a matroid. Define $F(M)$ to contain all sets $F$ for which there are bases $B$ of $M$ and $B_{\text{fin}}$ of $M_{\text{fin}}$ such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \setminus B$. Let $G(M)$ consist of those sets $F$ for which $\kappa(F') \geq |F \cap F'|$ for all finite sets $F' \subseteq E(M)$.

So we just showed for every matroid $M$ that $F(M) \subseteq G(M)$. Sets contained in $G(M)$ are a lot easier to handle than sets contained in $F(M)$, because we do not have to consider corresponding pairs of bases all the time. For example we can
show that if $\mathcal{G}(M)$ contains two finite sets of different size, then the smaller one can be extended, which allows us to easily show that if $M$ is not $l$-nearly finitary for any $l \in \mathbb{N}$ then $\mathcal{G}(M)$ contains an infinite element:

**Lemma 3.3.** Let $M$ be a matroid and $F_1, F_2 \in \mathcal{G}(M)$ finite sets such that $F_1$ is strictly smaller than $F_2$. Then there is $e \in F_2 \setminus F_1$ such that $F_1 + e \in \mathcal{G}(M)$.

**Proof.** Let $G \subseteq F_2 \setminus F_1$ be a minimal set such that $\kappa_M(F') \geq |F_1| + 1$ for all finite sets $F' \subseteq E(M)$ containing $F_1 \cup G$. Such a minimal set exists because $F_2 \setminus F_1$ is a possible candidate and finite. By Remark 2.5, $\kappa_M(F_1) \leq |F_1|$ and thus $G$ contains at least one element. Suppose for a contradiction that it contains at least two elements $e_1 \neq e_2$. By the minimality of $G$ there are finite sets $S_j \subseteq E(M)$ such that $F_1 \cup G - e_j \subseteq S_j$ and $\kappa_M(S_j) \leq |F_1|$. The set $S_1 \cap S_2$ contains $F_1$, so its connectivity is at least $|F_1|$, and $S_1 \cup S_2$ contains $F_1 \cup G$, hence its connectivity is at least $|F_1| + 1$. So by the submodularity of the connectivity function

$$2|F_1| \geq \kappa_M(S_1) + \kappa_M(S_2) \geq \kappa_M(S_1 \cup S_2) + \kappa_M(S_1 \cap S_2) \geq |F_1| + 1 + |F_1|$$

which is the desired contradiction. Thus $G$ contains exactly one edge $e$ and for all finite sets $F' \subseteq E(M)$ it is true by Remark 2.5 that

$$\kappa_M(F') \geq \kappa_M(F' \cup F_1 + e) - |(F_1 + e) \setminus F'| \geq |F_1| + 1 - |(F_1 + e) \setminus F'| = |(F_1 + e) \cap F'|$$

and hence $F_1 + e \in \mathcal{G}(M)$.

**Lemma 3.4.** Let $M$ be a matroid which is not $l$-nearly finitary for any $l \in \mathbb{N}$. Then there is an infinite set $F \subseteq E(M)$ such that $\kappa(F') \geq |F' \cap F|$ for all finite sets $F' \subseteq E(M)$.

**Proof.** Let $F_0 = \emptyset$ and define recursively a nested family $(F_i)_{i \in \mathbb{N}}$ of finite sets contained in $\mathcal{G}(M)$ as follows: Suppose $F_i$ is already defined. By Lemma 3.1, $\mathcal{G}(M)$ contains a finite set which is bigger than $F_i$ and by Lemma 3.3 there is an edge $e \notin F_i$ such that $F_i + e$ is contained in $\mathcal{G}(M)$. Define $F_{i+1} = F_i + e$. Because $e$ being in $\mathcal{G}(M)$ is a finitary condition, $\bigcup_{i \in \mathbb{N}} F_i$ is also contained in $\mathcal{G}(M)$ and it is infinite.

**Remark 3.5.** For every matroid $M$ the set $\mathcal{G}(M)$ is the set of independent sets of a finitary matroid.

In order to prove Conjecture 1.2 for cofinitary matroids $M$ it suffices to show that in that case $\mathcal{G}(M) \subseteq F(M)$. To prove this is the same as to find for every $F \in \mathcal{G}(M)$ a family of circuits $(C_f)_{f \in F}$ such that $f \in C_f$ if and only if $f = g$ and such that $\bigcup_{f \in F} C_f$ does not contain finite circuits. If $M$ is countable, then it is possible to extend $F$ to such a family by adding finite pieces to the (future) circuits in a way such that in no step a finite circuit emerges and the unions of all the pieces belonging to the same circuit are indeed circuits. If $M$ has more than countably many edges, then it is necessary to complete the circuits by a compactness argument instead of adding countably many finite pieces. Lemmas 3.6 and 3.7 show that $F$ can be extended suitably by finite pieces.

**Lemma 3.6.** Let $M$ be a matroid and $P_1, P_2 \subseteq E(M)$ two disjoint sets of the same finite connectivity $n$ such that $E(M) \setminus (P_1 \cup P_2)$ is finite. Assume further that all sets $Z$ satisfying $P_1 \subseteq Z \subseteq E(M) \setminus P_2$ have connectivity at least $n$. Then there is a set $X$ which is a base of $M/P_1 - P_2$ as well as of $M - P_1/P_2$.
Proof. The proof is by induction on the size of \( E(M) \backslash (P_1 \cup P_2) \), which is denoted by \( Y \). Let \( B_1 \) be a base of \( M|P_1 \) and \( B'_1 \) a base of \( M, P \) which is contained in \( B_1 \). This implies that \( |B_1 \backslash B'_1| = n \). If \( Y \) is the empty set, then \( X = \emptyset \) meets the requirements of this lemma. If \( Y \) contains a unique element \( e \), then by Lemma 2.1 \( B_1 \cup B'_2 \) is a base of \( M \) if and only if \( B'_1 \cup B_2 \) is a base of \( M \). In this case let \( X = \emptyset \). Otherwise \( B_1 \cup B'_2 + e \) and \( B'_1 \cup B_2 + e \) are bases of \( M \), so let \( X = \{e\} \).

So assume \( |Y| \geq 2 \) and pick an edge \( e \in Y \). If for all sets \( Z \) with \( P_1 \subseteq Z \subseteq P_1 \cup Y - e \) we have that \( \kappa_{M-e}(Z) \geq n \), then by the induction hypothesis there is a set \( X' \subseteq Y - e \) which is a base both of \( (M-e) - P_1/P_2 \) and of \( (M-e)/P_1 - P_2 \). By Lemma 2.1 either \( B_1 \cup X' \cup B'_2 \) and \( B'_1 \cup X' \cup B_2 \) are bases of \( M \) or \( B'_1 \cup (X' + e) \cup B_2 \) and \( B_1 \cup (X' + e) \cup B'_2 \) are bases of \( M \). In the first case let \( X = X' \) and in the second case define \( X = X' + e \).

So consider the case that there is a set \( Z_1 \) such that \( P_1 \subseteq Z_1 \subseteq P_1 \cup Y - e \) and \( \kappa_{M-e}(Z_1) < n \). By \( |Y| \geq 2 \) either \( Z_1 \backslash P_1 \) or \( Y \backslash (Z_1 + e) \) is non-empty. Assume without loss of generality that \( Z_1 \backslash P_1 \) is non-empty and define \( Z_2 = E(M) \backslash Z_1 \). Then by Remark 2.1 and \( \kappa_M(Z_1) \geq n \) we have that \( \kappa_M(Z_1) = n \). By the induction hypothesis there are sets \( X_1 \subseteq Z_1 \backslash P_1 \) such that \( X_1 \) is a base of \( M, Z_1 - P_1 \) and of \( M|Z_1/P_1 \). Then \( X = X_1 \cup Z_2 \) is a base of \( M - P_1 \cup P_2 \) as well as of \( M/P_1 - P_2 \). \( \square \)

**Lemma 3.7.** Let \( M \) be a matroid and \( P_1, P_2 \subseteq E(M) \) two disjoint subsets of finite connectivity \( \kappa_M(P_1) = n_1 \) such that \( E(M) \backslash (P_1 \cup P_2) \) is finite. Assume further that \( n_1 \leq n_2 \) and that \( \kappa_M(Z) \geq n_1 \) for all sets \( Z \) satisfying \( P_1 \subseteq Z \subseteq E(M) \backslash P_2 \). Then there are bases \( B_1, B_2 \) of \( M/P_1 - P_2 \) and \( M - P_1/P_2 \) respectively such that \( B_2 \subseteq B_1 \) and \( |B_1 \backslash B_2| = n_2 - n_1 \).

Proof. The proof is by induction on the size of \( E(M) \backslash (P_1 \cup P_2) \), which is denoted by \( Y \). If \( n_1 = n_2 \), then we are done by Lemma 3.6 so assume \( n_2 > n_1 \). If \( Y \) contains exactly one edge \( e \), then \( n_1 + 1 = n_2 \), so let \( B_1 = \{e\} \) and \( B_2 = \emptyset \). Then \( B_1 \) and \( B_2 \) meet the requirements of this lemma.

So let \( Y \) contain at least two elements. If there is a set \( Z \) of connectivity \( n_1 \) such that \( P_1 \subseteq Z \subseteq P_1 \cup Y \) then by Lemma 3.6 there is a set \( X \) which is a base of \( M/P_1 - Z_2 \) as well as of \( M - P_1/Z_2 \). By the induction hypothesis there are bases \( B'_1 \) of \( M/Z_1 - P_2 \) and \( B'_2 \) of \( M - Z_1/P_2 \) such that \( B'_2 \subseteq B'_1 \) and \( |B'_1 \backslash B'_2| = n_2 - n_1 \). Then \( B_1 = X \cup B'_1 \) is a base of \( M/P_1 - P_2 \) and \( B_2 = X \cup B'_2 \) is a base of \( M - P_1/P_2 \). Also \( B_2 \subseteq B_1 \) and \( |B_1 \backslash B_2| = |B'_1 \backslash B'_2| = n_2 - n_1 \).

So assume that there is no such set \( Z \). Let \( e \) be an edge of \( Y \), then \( \kappa_M(P_1 + e) = n_1 + 1 \) and \( \kappa_M(Z) \geq n + 1 \) for all sets \( Z \) such that \( P_1 + e \subseteq Z \subseteq P_1 + Y \). By the induction hypothesis \( e \) is a base of \( (M/P_1)\{e\} \) and the empty set is a base of \( (M - P_1)\{e\} \). Also by the induction hypothesis there are bases \( B'_1 \) of \( M/(P_1 + e)/P_2 \) and \( B'_2 \) of \( M - (P_1 + e)/P_2 \) such that \( B'_2 \subseteq B'_1 \) and \( |B'_1 \backslash B'_2| = n_2 - (n_1 + 1) \). So \( B_1 = B'_1 + e \) is a base of \( M/P_1 - P_2 \), \( B_2 = B'_2 \) is a base of \( M - P_1/P_2 \), \( B_2 \subseteq B_1 \) and \( |B_1 \backslash B_2| = |B'_1 \backslash B'_2| + 1 = n_2 - n_1 \). \( \square \)

With these lemmas it is possible to prove \( G(M) \subseteq F(M) \) for cofinitary matroids.

**Lemma 3.8.** Let \( M \) be a cofinitary matroid and \( F \subseteq E(M) \) a set such that \( \kappa_M(F') \geq |F' \cap F| \) for all finite sets \( F' \subset E(M) \). Then there are bases \( B \) of \( M \) and \( B_{\text{fin}} \) of \( M_{\text{fin}} \) such that \( B \subseteq B_{\text{fin}} \) and \( F \subseteq B_{\text{fin}} \backslash B \).

Proof. The proof is by a compactness argument. Let \( Y = \{0, 1, 2\}^{E(M)} \) be a topological space with the product topology where each component carries the discrete
topology. Consider the following three types of closed subsets of $Y$:

\begin{align*}
Y_C &= \bigcup_{e \in C} \{ y \in Y : y(e) = 0 \} \quad \text{for a finite circuit } C \text{ of } M \\
Y_D &= \bigcup_{e \in D} \{ y \in Y : y(e) = 2 \} \quad \text{for a cocircuit } D \text{ of } M \\
Y_f &= \{ y \in Y : y(f) = 1 \} \quad \text{for every } f \in F
\end{align*}

If there is an $x \in Y$ which is contained in all of these sets, then the set \{ $e \in E(M) : x(e) = 2$ \} is spanning in $M$ (and hence contains a base $B$ of $M$), the set \{ $e \in E(M) : x(e) \neq 0$ \} is independent in $M_{\text{fin}}$ (and is thus contained in a base $B_{\text{fin}}$ of $M_{\text{fin}}$) and $B_{\text{fin}} \setminus B$ contains $F$. In order to show that there is such an $x$, it is enough to show that for each finite set $Y$ of closed subsets of the form $Y_C$, $Y_D$ or $Y_f$ their intersection is non-empty.

So let $Y$ be a finite set of closed subsets of the form $Y_C$, $Y_D$ or $Y_f$. Let $F' \subset F$ contain exactly those $f$ with $Y_f \in Y$ and let $R$ be the union of $F'$, the circuits $C$ such that $Y_C \in Y$ and the cocircuits $D$ such that $Y_D \in Y$. Then by Lemma 3.7 there are bases $B$ of $M/F'$ - $(E(M) \setminus R)$ and $B'$ of $M - F'/\{E(M) \setminus R\}$ such that $B'$ is a subset of $B$ and $|B \setminus B'| - \kappa_M(R) = \kappa_M(F')$. Define $B_2 = F' \cup B$ and $B'_2 = B \cup B'$. The set $B_2$ is a base of $M|R$, the set $B'_2$ is a base of $M.R$ and their difference contains $F'$. Thus every $x \in Y$ satisfying $x(e) = 0$ if $e \in R \setminus B_2$, $x(e) = 1$ if $e \in B_2 \setminus B'_2$ and $x(e) = 2$ if $e \in B'_2$ is contained in the intersection of the closed sets contained in $Y$. Thus $\bigcap Y$ is not empty. □

**Corollary 3.9.** Let $M$ be a nearly finitary cofinitary matroid. Then there is a natural number $l \in \mathbb{N}$ such that $M$ is $l$-nearly finitary.

**Proof.** By Lemma 3.8 and Lemma 3.4 □

4. THE CONNECTION TO HALIN’S THEOREM

The connection between Halin’s theorem and Conjecture 1.2 is that in a locally finite graph $G$ the infinite circuits of the algebraic cycle matroid are the double rays of that graph. If $G$ is connected, then there is a correspondence between large families of pairwise disjoint rays and pairs of bases of $M_{AC}$ and $(M_{AC})_{\text{fin}} = M_{FC}$ witnessing that $M$ is not $l$-nearly finitary for some large $l$. Lemma 4.1 is a more detailed version of the arguments already used in the corresponding proof in [1].

**Lemma 4.1.** Let $G$ be a locally finite connected graph containing a ray and $M$ its algebraic cycle matroid. Then $G$ contains a family $(R_i)_{i \in I}$ of vertex disjoint rays if and only if there are bases $B$ of $M$ and $B_{\text{fin}}$ of $M_{\text{fin}}$ containing $B$ such that $|B_{\text{fin}} \setminus B| \geq |I| - 1$.

**Proof.** Let $B \in \mathcal{B}(M)$ and $B_{\text{fin}} \in \mathcal{B}(M_{\text{fin}})$ be bases such that $B \subseteq B_{\text{fin}}$ and define $F = B_{\text{fin}} \setminus B$. Because $G$ is connected, $B_{\text{fin}}$ is the edge set of a spanning tree of $G$. Let $V_B$ be the set of components of the graph $G - (E \setminus B)$. Define a multigraph $G_B$ on vertex set $V_B$ with edge set $F$ such that the end vertices of $f \in F$ in $G_B$ are the components containing the original end vertices of $f$. Every path in $G_B$ can be extended by inserting finite paths to a path in $G$ whose edges are contained in $B_{\text{fin}}$. So also every finite cycle of $G_B$ can be extended to a cycle of $G$ whose edges are all contained in $B_{\text{fin}}$. Hence $G_B$ is a forest. Furthermore every two vertices of $G$ can
be joined by a path using only edges of $B_{\text{fin}}$, thus $G_B$ is a tree and in particular $|V_B| = |F| + 1$.

Let $v \in V_B$ be a vertex of $G_B$ which is an end vertex of an edge $e$ of $F$. Denote the end vertex of $e$ in $G$ which lies in $v$ by $v'$. The set $B + e \subseteq B_{\text{fin}}$ contains an infinite circuit, which is the edge set of a double ray. By deleting $e$ from the double ray we obtain two rays whose edges are all contained in $B$, one of these starting in $v'$. This ray is contained in $v$, so $v$ contains a ray. Thus every element of $V_B$ contains a ray (if $V_B$ has only one element then this is true by assumption). For each $v \in V_B$ let $R_v$ be a ray contained in $v$. Then $(R_v)_{v \in V_B}$ is a family of vertex-disjoint rays and $|B_{\text{fin}} \setminus B| = |F| = |V_B| - 1$.

To prove the other direction let $(R_i)_{i \in I}$ be a family of pairwise disjoint rays. Let $E_R$ be the join of the edge sets of the rays and $B \in B(M)$ a base containing $E_R$. Let $B_{\text{fin}}$ be a base of $M_{\text{fin}}$ containing $B$ and construct the tree $G_B$ as above. Then every ray $R_i$ is contained in a component of $V_B$ but because none of these components contain a double ray every two rays $R_i \neq R_j$ have to be contained in different components of $V_B$. This yields an injective function from $I$ to $V_B$ and thus $|B_{\text{fin}} \setminus B| = |V_B| - 1 \geq |I| - 1$. □

So for a given locally finite connected graph the fact that Halin’s theorem holds is equivalent to the fact that [Conjecture 1.2] holds for its algebraic cycle matroid, which is cofinitary.

**Lemma 4.2.** Let $G$ be a locally finite connected graph. If there are arbitrarily large finite families of pairwise vertex disjoint rays of $G$, then there is an infinite such family.

**Proof.** Let $M$ be the algebraic cycle matroid of $G$. Since $G$ is locally finite, $M$ exists and is cofinitary. By [Lemma 4.1] $M$ is not $l$-nearly finitary for any natural number $l$, so by [Corollary 3.9] there are bases $B$ of $M$ and $B_{\text{fin}}$ of $M_{\text{fin}}$ such that $B \subseteq B_{\text{fin}}$ and $B_{\text{fin}} \setminus B$ is infinite. Hence again by [Lemma 4.1] this implies that $G$ contains an infinite family of pairwise disjoint rays. □

In the last proof we used that $G$ is locally finite and connected. But because Halin’s theorem can be reduced to the case where $G$ is locally finite and connected, this re-proves Halin’s theorem from [Corollary 3.9].

**References**


