COFINITARY NEARLY FINITARY MATROIDS ARE l-NEARLY FINITARY FOR SOME $l \in \mathbb{N}$

Abstract. We characterise for cofinitary matroids M the sets F for which there are bases B of M and B_{fin} of M_{fin} such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \backslash B$. The main result yields an alternative proof of Halin's theorem.

1. Introduction

Matroids which are l-nearly finitary were introduced in [1]. The *finitarisation* of a matroid M is the matroid M_{fin} on the same ground set $E(M)$ as M with circuit set $\mathcal{C}(M_{\text{fin}}) = \{C \in \mathcal{C}(M)\colon\ C \text{ is finite}\}$. If for all bases B of M and all bases B_{fin} of M_{fin} such that $B \subset B_{\text{fin}}$ the set $B_{\text{fin}} \backslash B$ is finite, then M is called *nearly finitary*. If there is also a natural number $l \in \mathbb{N}$ which is an upper bound on the size of $B_{\text{fin}} \backslash B$, then M is *l-nearly finitary*.

In [1] it was shown that every algebraic cycle matroid of an infinite graph and every topological cycle matroid of a 2-connected locally finite graph have the property that if they are nearly finitary then they are also l-nearly finitary for some natural number $l \in \mathbb{N}$. These proofs establish a connection between the existence of large families of pairwise disjoint rays in a graph and the property of the corresponding (topological or algebraic) cycle matroid to be (l-)nearly finitary. They then use the following theorem by Halin, which is an important theorem in infinite graph theory:

Theorem 1.1. [5, Halin] *If an infinite graph G contains l (vertex-) disjoint rays for every $l \in \mathbb{N}$, then G contains infinitely many disjoint rays.*

Having shown the two propositions the authors of [1] made the following conjecture:

Conjecture 1.2. [1] *Every nearly finitary matroid is l-nearly finitary for some natural number $l \in \mathbb{N}$.*

In this paper it will be shown that Conjecture 1.2 holds for cofinitary matroids. The result can be used to prove Halin's theorem by the connection established in [1] between families of pairwise disjoint rays in a graph G and the algebraic cycle matroid. Via this connection, Conjecture 1.2 for cofinitary matroids can be seen as a matroidal analogue of Halin's theorem.

2. Preliminaries

2.1. **Infinite matroids.** The definitions for infinite matroids (including minors and duality) used here are the ones from [4]. From that paper we will also need the following basic result about infinite matroids.

Lemma 2.1. [4] *Let B_1 and B_2 be two bases of M. If $|B_1 \backslash B_2| < \infty$, then $|B_2 \backslash B_1| = |B_1 \backslash B_2|$.*

Remark 2.2. *A set $X \subseteq E(M)$ is spanning in M if and only if its complement is independent in M^* if and only if X meets every cocircuit.*

Definition 2.3. [2] Let $X \subseteq E(M)$ be a subset of the ground set of M. Let B_X be a base of $M|X$, B_Y a base of $M - X$ and B a base of M contained in $B_X \cup B_Y$. Whether $(B_X \cup B_Y) \backslash B$ is finite does not depend on the choice of B_X, B_Y and B. If $(B_X \cup B_Y) \backslash B$ is finite, then its size also does not depend on the choice of B_X, B_Y and B and is defined as the *connectivity* $\kappa_M(X)$ of X. If $(B_X \cup B_Y) \backslash B$ is infinite, then $\kappa_M(X)$ is defined as ∞. In particular $\kappa_M(X) = \kappa_M(E(M) \backslash X)$.

Lemma 2.4. [2] *The connectivity function is submodular, that is for all $X, Y \subseteq E(M)$ there holds*

$$\kappa_M(X) + \kappa_M(Y) \geq \kappa_M(X \cup Y) + \kappa_M(X \cap Y).$$

Remark 2.5. *Let M be a matroid, $X \subseteq E(M)$ and e an edge of M not in X. Then $\kappa_M(X) - 1 \leq \kappa_M(X + e) \leq \kappa_M(X) + 1$, or in words: adding or removing an edge changes the connectivity of a set by at most one, and $\kappa_M(X) \leq \kappa_{M-e}(X) + 1$.*

2.2. **Matroids in graphs.** The notation for graphs is the one of [5].

Definition 2.6. [3][6] Let G be a graph. The set of edge sets of finite cycles of G is the set of circuits of a matroid $M_{FC}(G)$, called the *finite cycle matroid* of G. The set of edge sets of finite cycles of G together with the set of edge sets of double rays of G is the set of circuits of another matroid $M_{AC}(G)$, the *algebraic cycle matroid*, if G does not contain a subdivision of the Bean graph.

Remark 2.7. [3] *Let G be a graph for which the algebraic cycle matroid exists. Then $M_{AC}(G)_{fin} = M_{FC}$. If G is locally finite then it does not contain a subdivision of the Bean graph, so $M_{AC}(G)$ exists. Furthermore in a locally finite graph G the matroid $M_{AC}(G)$ is cofinitary.*

3. Cofinitary nearly finitary matroids

In this section we will show that a nearly finitary cofinitary matroid is l-nearly finitary for some natural number $l \in \mathbb{N}$. The proof was inspired by the following observation, which holds for all (and not just cofinitary) matroids:

Lemma 3.1. *Let M be a matroid and let B and B_{fin} be bases of M and M_{fin} respectively such that $B \subseteq B_{fin}$. Let F be contained in $B_{fin} \backslash B$ and let $F' \subseteq E(M)$ be a finite set. Then $\kappa_M(F') \geq |F \cap F'|$.*

Proof. F' is finite, so $F' \cap B_{\text{fin}}$ is independent in $M|F'$ and is thus contained in a base B_1 of $M|F'$. Since B is a base of M, every edge of F is spanned in M by B, so $F \cap F'$ is spanned in $M.F'$ by $B \cap F' \subseteq B_1 \backslash F$. Hence $M.F'$ is spanned by $B_1 \backslash F$, so $\kappa_M(F') \geq |B_1 \backslash (B_1 \backslash F)| = |F \cap B_1| = |F \cap F'|$. □

Definition 3.2. Let M be a matroid. Define $\mathcal{F}(M)$ to contain all sets F for which there are bases B of M and B_{fin} of M_{fin} such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \backslash B$. Let $\mathcal{G}(M)$ consist of those sets F for which $\kappa(F') \geq |F \cap F'|$ for all finite sets $F' \subseteq E(M)$.

So we just showed for every matroid M that $\mathcal{F}(M) \subseteq \mathcal{G}(M)$. Sets contained in $\mathcal{G}(M)$ are a lot easier to handle than sets contained in $\mathcal{F}(M)$, because we do not have to consider corresponding pairs of bases all the time. For example we can show that if $\mathcal{G}(M)$ contains two finite sets of different size, then the smaller one can be extended, which allows us to easily show that if M is not l-nearly finitary for any $l \in \mathbb{N}$ then $\mathcal{G}(M)$ contains an infinite element:

Lemma 3.3. *Let M be a matroid and $F_1, F_2 \in \mathcal{G}(M)$ finite sets such that F_1 is strictly smaller than F_2. Then there is $e \in F_2 \backslash F_1$ such that $F_1 + e \in \mathcal{G}(M)$.*

Proof. Let $G \subseteq F_2 \backslash F_1$ be a minimal set such that $\kappa_M(F') \geq |F_1| + 1$ for all finite sets $F' \subseteq E(M)$ containing $F_1 \cup G$. Such a minimal set exists because $F_2 \backslash F_1$ is a possible candidate and finite. By Remark 2.5, $\kappa_M(F_1) \leq |F_1|$ and thus G contains at least one element. Suppose for a contradiction that it contains at least two elements $e_1 \neq e_2$. By the minimality of G there are finite sets $S_j \subseteq E(M)$ such that $F_1 \cup G - e_j \subseteq S_j$ and $\kappa_M(S_j) \leq |F_1|$. The set $S_1 \cap S_2$ contains F_1, so its connectivity is at least $|F_1|$, and $S_1 \cup S_2$ contains $F_1 \cup G$, hence its connectivity is at least $|F_1| + 1$. So by the submodularity of the connectivity function

$$2|F_1| \geq \kappa_M(S_1) + \kappa_M(S_2) \geq \kappa_M(S_1 \cup S_2) + \kappa_M(S_1 \cap S_2) \geq |F_1| + 1 + |F_1|$$

which is the desired contradiction. Thus G contains exactly one edge e and for all finite sets $F' \subseteq E(M)$ it is true by Remark 2.5 that

$$\kappa_M(F') \geq \kappa_M(F' \cup F_1 + e) - |(F_1 + e) \backslash F'| \geq |F_1| + 1 - |(F_1 + e) \backslash F'| = |(F_1 + e) \cap F'|$$

and hence $F_1 + e \in \mathcal{G}(M)$. □

Lemma 3.4. *Let M be a matroid which is not l-nearly finitary for any $l \in \mathbb{N}$. Then there is an infinite set $F \subseteq E(M)$ such that $\kappa(F') \geq |F' \cap F|$ for all finite sets $F' \subseteq E(M)$.*

Proof. Let $F_0 = \emptyset$ and define recursively a nested family $(F_i)_{i \in \mathbb{N}}$ of finite sets contained in $\mathcal{G}(M)$ as follows: Suppose F_i is already defined. By Lemma 3.1 $\mathcal{G}(M)$ contains a finite set which is bigger than F_i and by Lemma 3.3 there is an edge $e \notin F_i$ such that $F_i + e$ is contained in $\mathcal{G}(M)$. Define $F_{i+1} = F_i + e$. Because being in $\mathcal{G}(M)$ is a finitary condition, $\bigcup_{i \in \mathbb{N}} F_i$ is also contained in $\mathcal{G}(M)$ and it is infinite. □

Remark 3.5. *For every matroid M the set $\mathcal{G}(M)$ is the set of independent sets of a finitary matroid.*

In order to prove Conjecture 1.2 for cofinitary matroids M it suffices to show that in that case $\mathcal{G}(M) \subseteq \mathcal{F}(M)$. To prove this is the same as to find for every $F \in \mathcal{G}(M)$ a family of circuits $(C_f)_{f \in F}$ such that $f \in C_g$ if and only if $f = g$ and such that $\bigcup_{f \in F} C_f$ does not contain finite circuits. If M is countable, then it is possible to extend F to such a family by adding finite pieces to the (future) circuits in a way such that in no step a finite circuit emerges and the unions of all the pieces belonging to the same circuit are indeed circuits. If M has more than countably many edges, then it is necessary to complete the circuits by a compactness argument instead of adding countably many finite pieces. Lemmas 3.6 and 3.7 show that F can be extended suitably by finite pieces.

Lemma 3.6. *Let M be a matroid and $P_1, P_2 \subseteq E(M)$ two disjoint sets of the same finite connectivity n such that $E(M) \backslash (P_1 \cup P_2)$ is finite. Assume further that all sets Z satisfying $P_1 \subseteq Z \subseteq E(M) \backslash P_2$ have connectivity at least n. Then there is a set X which is a base of $M/P_1 - P_2$ as well as of $M - P_1/P_2$.*

Proof. The proof is by induction on the size of $E(M)\backslash(P_1 \cup P_2)$, which is denoted by Y. Let B_i be a base of $M|P_i$ and B_i' a base of $M.P_i$ which is contained in B_i. This implies that $|B_i \backslash B_i'| = n$. If Y is the empty set, then $X = \emptyset$ meets the requirements of this lemma. If Y contains a unique element e, then by Lemma 2.1 $B_1 \cup B_2'$ is a base of M if and only if $B_1' \cup B_2$ is a base of M. In this case let $X = \emptyset$. Otherwise $B_1 \cup B_2' + e$ and $B_1' \cup B_2 + e$ are bases of M, so let $X = \{e\}$.

So assume $|Y| \geq 2$ and pick an edge $e \in Y$. If for all sets Z with $P_1 \subseteq Z \subseteq P_1 \cup Y - e$ we have that $\kappa_{M-e}(Z) \geq n$, then by the induction hypothesis there is a set $X' \subset Y - e$ which is a base both of $(M-e) - P_1/P_2$ and of $(M-e)/P_1 - P_2$. By Lemma 2.1 either $B_1 \cup X' \cup B_2'$ and $B_1' \cup X' \cup B_2$ are bases of M or $B_1' \cup (X'+e) \cup B_2$ and $B_1 \cup (X'+e) \cup B_2'$ are bases of M. In the first case let $X = X'$ and in the second case define $X = X' + e$.

So consider the case that there is a set Z_1 such that $P_1 \subseteq Z_1 \subseteq P_1 \cup Y - e$ and $\kappa_{M-e}(Z_1) < n$. By $|Y| \geq 2$ either $Z_1 \backslash P_1$ or $Y \backslash (Z_1 + e)$ is non-empty. Assume without loss of generality that $Z_1 \backslash P_1$ is non-empty and define $Z_2 = E(M) \backslash Z_1$. Then by Remark 2.5 and $\kappa_M(Z_1) \geq n$ we have that $\kappa_M(Z_1) = n$. By the induction hypothesis there are sets $X_i \subseteq Z_i \backslash P_i$ such that X_i is a base of $M.Z_i - P_i$ and of $M|Z_i/P_i$. Then $X = X_1 \cup X_2$ is a base of $M - P_1/P_2$ as well as of $M/P_1 - P_2$. □

Lemma 3.7. *Let M be a matroid and $P_1, P_2 \subseteq E(M)$ two disjoint subsets of finite connectivity $\kappa_M(P_i) = n_i$ such that $E(M)\backslash(P_1 \cup P_2)$ is finite. Assume further that $n_1 \leq n_2$ and that $\kappa_M(Z) \geq n_1$ for all sets Z satisfying $P_1 \subseteq Z \subseteq E(M)\backslash P_2$. Then there are bases B_1, B_2 of $M/P_1 - P_2$ and $M - P_1/P_2$ respectively such that $B_2 \subseteq B_1$ and $|B_1 \backslash B_2| = n_2 - n_1$.*

Proof. The proof is by induction on the size of $E(M)\backslash(P_1 \cup P_2)$, which is denoted by Y. If $n_1 = n_2$, then we are done by Lemma 3.6, so assume $n_2 > n_1$. If Y contains exactly one edge e, then $n_1 + 1 = n_2$, so let $B_1 = \{e\}$ and $B_2 = \emptyset$. Then B_1 and B_2 meet the requirements of this lemma.

So let Y contain at least two elements. If there is a set Z of connectivity n_1 such that $P_1 \subsetneq Z \subsetneq P_1 \cup Y$ then by Lemma 3.6 there is a set X which is a base of $M/P_1 - Z_2$ as well as of $M - P_1/Z_2$. By the induction hypothesis there are bases B_1' of $M/Z_1 - P_2$ and B_2' of $M - Z_1/P_2$ such that $B_2' \subset B_1'$ and $|B_1' \backslash B_2'| = n_2 - n_1$. Then $B_1 = X \cup B_1'$ is a base of $M/P_1 - P_2$ and $B_2 = X \cup B_2'$ is a base of $M - P_1/P_2$. Also $B_2 \subseteq B_1$ and $|B_1 \backslash B_2| = |B_1' \backslash B_2'| = n_2 - n_1$.

So assume that there is no such set Z. Let e be an edge of Y, then $\kappa_M(P_1 + e) = n_1 + 1$ and $\kappa_M(Z) \geq n + 1$ for all sets Z such that $P_1 + e \subseteq Z \subseteq P_1 \cup Y$. By the induction hypothesis e is a base of $(M/P_1)|\{e\}$ and the empty set is a base of $(M - P_1).\{e\}$. Also by the induction hypothesis there are bases B_1' of $M/(P_1 + e) - P_2$ and B_2' of $M - (P_1 + e)/P_2$ such that $B_2' \subset B_1'$ and $|B_1' \backslash B_2'| = n_2 - (n_1 + 1)$. So $B_1 = B_1' + e$ is a base of $M/P_1 - P_2$, $B_2 = B_2'$ is a base of $M - P_1/P_2$, $B_2 \subseteq B_1$ and $|B_1 \backslash B_2| = |B_1' \backslash B_2'| + 1 = n_2 - n_1$. □

With these lemmas it is possible to prove $\mathcal{G}(M) \subseteq \mathcal{F}(M)$ for cofinitary matroids.

Lemma 3.8. *Let M be a cofinitary matroid and $F \subseteq E(M)$ a set such that $\kappa_M(F') \geq |F' \cap F|$ for all finite sets $F' \subset E(M)$. Then there are bases B of M and B_{fin} of M_{fin} such that $B \subseteq B_{fin}$ and $F \subseteq B_{fin} \backslash B$.*

Proof. The proof is by a compactness argument. Let $Y = \{0, 1, 2\}^{E(M)}$ be a topological space with the product topology where each component carries the discrete topology. Consider the following three types of closed subsets of Y:

$$Y_C = \bigcup_{e \in C} \{y \in Y\colon\ y(e) = 0\} \qquad \text{for a finite circuit } C \text{ of } M$$

$$Y_D = \bigcup_{e \in D} \{y \in Y\colon\ y(e) = 2\} \qquad \text{for a cocircuit } D \text{ of } M$$

$$Y_f = \{y \in Y\colon\ y(f) = 1\} \qquad \text{for every } f \text{ in } F$$

If there is an $x \in Y$ which is contained in all of these sets, then the set $\{e \in E(M)\colon\ x(e) = 2\}$ is spanning in M (and hence contains a base B of M), the set $\{e \in E(M)\colon\ x(e) \neq 0\}$ is independent in M_{fin} (and is thus contained in a base B_{fin} of M_{fin}) and $B_{\text{fin}} \backslash B$ contains F. In order to show that there is such an x, it is enough to show that for each finite set $\mathcal{Y}$ of closed subsets of the form Y_C, Y_D or Y_f their intersection is non-empty.

So let $\mathcal{Y}$ be a finite set of closed subsets of the form Y_C, Y_D or Y_f. Let $F' \subset F$ contain exactly those f with $Y_f \in \mathcal{Y}$ and let R be the union of F', the circuits C such that $Y_C \in \mathcal{Y}$ and the cocircuits D such that $Y_D \in \mathcal{Y}$. Then by Lemma 3.7 there are bases B of $M/F' - (E(M)\backslash R)$ and B' of $M - F'/(E(M)\backslash R)$ such that B' is a subset of B and $|B\backslash B'| = \kappa_M(R) - \kappa_M(F')$. Define $B_2 = F' \cup B$ and $B_2' = \emptyset \cup B'$. The set B_2 is a base of $M|R$, the set B_2' is a base of $M.R$ and their difference contains F'. Thus every $x \in Y$ satisfying $x(e) = 0$ if $e \in R\backslash B_2$, $x(e) = 1$ if $e \in B_2\backslash B_2'$ and $x(e) = 2$ if $e \in B_2'$ is contained in the intersection of the closed sets contained in $\mathcal{Y}$. Thus $\bigcap \mathcal{Y}$ is not empty. □

Corollary 3.9. *Let M be a nearly finitary cofinitary matroid. Then there is a natural number $l \in \mathbb{N}$ such that M is l-nearly finitary.*

Proof. By Lemma 3.8 and Lemma 3.4. □

4. The connection to Halin's theorem

The connection between Halin's theorem and Conjecture 1.2 is that in a locally finite graph G the infinite circuits of the algebraic cycle matroid are the double rays of that graph. If G is connected, then there is a correspondence between large families of pairwise disjoint rays and pairs of bases of M_{AC} and $(M_{AC})_{\text{fin}} = M_{FC}$ witnessing that M is not l-nearly finitary for some large l. Lemma 4.1 is a more detailed version of the arguments already used in the corresponding proof in [1].

Lemma 4.1. *Let G be a locally finite connected graph containing a ray and M its algebraic cycle matroid. Then G contains a family $(R_i)_{i\in I}$ of vertex disjoint rays if and only if there are bases B of M and B_{fin} of M_{fin} containing B such that $|B_{fin}\backslash B| \geq |I| - 1$.*

Proof. Let $B \in \mathcal{B}(M)$ and $B_{\text{fin}} \in \mathcal{B}(M_{\text{fin}})$ be bases such that $B \subseteq B_{\text{fin}}$ and define $F = \mathcal{B}_{\text{fin}}\backslash B$. Because G is connected, B_{fin} is the edge set of a spanning tree of G. Let V_B be the set of components of the graph $G - (E\backslash B)$. Define a multigraph G_B on vertex set V_B with edge set F such that the end vertices of $f \in F$ in G_B are the components containing the original end vertices of f. Every path in G_B can be extended by inserting finite paths to a path in G whose edges are contained in B_{fin}. So also every finite cycle of G_B can be extended to a cycle of G whose edges are all contained in B_{fin}. Hence G_B is a forest. Furthermore every two vertices of G can be joined by a path using only edges of B_{fin}, thus G_B is a tree and in particular $|V_B| = |F| + 1$.

Let $v \in V_B$ be a vertex of G_B which is an end vertex of an edge e of F. Denote the end vertex of e in G which lies in v by v'. The set $B + e \subseteq B_{\text{fin}}$ contains an infinite circuit, which is the edge set of a double ray. By deleting e from the double ray we obtain two rays whose edges are all contained in B, one of these starting in v'. This ray is contained in v, so v contains a ray. Thus every element of V_B contains a ray (if V_B has only one element then this is true by assumption). For each $v \in V_B$ let R_v be a ray contained in v. Then $(R_v)_{v \in V_B}$ is a family of vertex-disjoint rays and $|B_{\text{fin}} \backslash B| = |F| = |V_B| - 1$.

To prove the other direction let $(R_i)_{i \in I}$ be a family of pairwise disjoint rays. Let E_R be the join of the edge sets of the rays and $B \in \mathcal{B}(M)$ a base containing E_R. Let B_{fin} be a base of M_{fin} containing B and construct the tree G_B as above. Then every ray R_i is contained in a component of V_B but because none of these components contain a double ray every two rays $R_i \neq R_j$ have to be contained in different components of V_B. This yields an injective function from I to V_B and thus $|B_{\text{fin}} \backslash B| = |V_B| - 1 \geq |I| - 1$. □

So for a given locally finite connected graph the fact that Halin's theorem holds is equivalent to the fact that Conjecture 1.2 holds for its algebraic cycle matroid, which is cofinitary.

Lemma 4.2. *Let G be a locally finite connected graph. If there are arbitrarily large finite families of pairwise vertex disjoint rays of G, then there is an infinite such family.*

Proof. Let M be the algebraic cycle matroid of G. Since G is locally finite, M exists and is cofinitary. By Lemma 4.1, M is not l-nearly finitary for any natural number l, so by Corollary 3.9 there are bases B of M and B_{fin} of M_{fin} such that $B \subseteq B_{\text{fin}}$ and $B_{\text{fin}} \backslash B$ is infinite. Hence again by Lemma 4.1 this implies that G contains an infinite family of pairwise disjoint rays. □

In the last proof we used that G is locally finite and connected. But because Halin's theorem can be reduced to the case where G is locally finite and connected, this re-proves Halin's theorem from Corollary 3.9.

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