Tangles determined by majority vote

Christian Elbracht

Masterarbeit

Fakultät für Mathematik, Informatik und
Naturwissenschaften der Universität Hamburg

September 2017

Gutachter:
Prof. Dr. Reinhard Diestel
Prof. Dr. Nathan Bowler

Versicherung

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt. Die Arbeit habe ich vorher nicht in einem anderen Prüfungsverfahren eingereicht. Die eingereichte schriftliche Fassung entspricht genau der auf dem elektronischen Speichermedium.

1 Introduction

For as long as the connectivity of graphs has been studied, people have considered different types of highly connected substructures that may occur in a graph, for example highly connected subgraphs, blocks, highly connected (topological) minors and large grids.

The simplest example of such a structure is a k-connected subgraph for some large integer k. A more general structure that may be investigated are the k-blocks. Whereas we require for the vertex set X of a k-connected subgraph that the induced graph $G[X]$ is k-connected, for X to be a k-block we only require X to be "k-connected in our graph G". More precisely a k-block is a maximal set $X \subseteq V$ of at least k vertices, such that no two vertices $v, w \in X$ can be separated from each other in the graph G by less than k vertices. In particular, every k-connected subgraph extends to a k-block in our graph G. The opposite direction however is far from true: There are graphs G containing k-blocks X such that $G[X]$ is not even connected. Consider for example a graph G consisting of $r \geq k$ isolated vertices that are pairwise joined by k pairwise disjoint independent paths of length two. Then this set of r vertices is a k-block: There are $k+1$ disjoint paths between any two of this vertices. However the subgraph induced by this vertex set is not even connected and does note extend to even a 3-connected subgraph of G.

Another type of "highly connected" substructure, the k-connected topological minors, also extends to k-blocks in this way. Indeed, the branch vertices of any topological minor cannot be separated from each other by a set of less then k vertices and hence extend to a k-block. Instead of considering topological minors we might also consider ordinary minors and ask whether our graph contains a k-connected minor. A graph containing

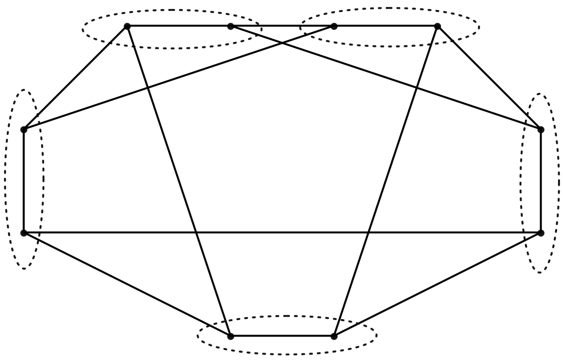

Figure 1: The dotted sets define the branch sets of a K_5.

a k-connected minor does not necessarily contain a k-block; consider for example the graph in Figure 1. This graph has a K_5 as minor but does not contain a 4-block, as any two vertices that are not adjacent can be separated from each other by 3 other vertices.

Large ($(n \times n)$, say) grids are another substructure we would like to consider as "highly connected". These grids do not give rise to a k-block for $k \geq 5$, but it is still reasonable to declare them as a highly connected substructure, as we can, for example, not separate them into two parts of roughly equal size by deleting a small number of vertices.

Different though they may appear, all these substructures have one thing in common: They define an orientation of the separations of order less than k, as follows. Given a graph $G = (V, E)$, a *separation* in G is a unordered pair $\{A, B\}$ such that $A \cup B = V$ and there is no edge from $A \setminus B$ to $B \setminus A$. The *order* of such a separation is the size $|A \cap B|$ of $A \cap B$. By declaring one side, for example A, of such a separation to be *small* and the other site, for example B, to be *big*, we can orient these separations. We write (A, B) for the oriented separation with A as small side and B as big side and think of this separation as pointing towards B. If we orient all separations of order less than k in such a way that no two separations point away from each other (so there are no two oriented separations (A, B) and (C, D) such that $D \subseteq A$ and $B \subseteq C$), we call such an orientation *consistent*.

Let us now see how the various highly connected substructures mentioned earlier orient the separations of small order of a graph.

Let X be a k-block and $\{A, B\}$ a separation of order less than k. If X had a vertex v in $A \setminus B$ and a vertex w $B \setminus A$, the set $A \cap B$ would be a set of order less than k that separates these vertices from each other in G. As no two vertices in a k-block can be separated from each other by such a set, therefore there is exactly one side, for example B, of the separation $\{A, B\}$ such that $X \subseteq B$. By orienting all separations in this direction a k-block induces a consistent orientation of the separations of order less than k.

A k-connected ordinary minor $H \preceq G$ also induces such an orientation. Indeed, let $\{A, B\}$ be a separation of order less than k. Every k-connected minor has at least k branch sets, therefore at least one of these branch sets X does not meet the set $A \cap B$. As every branch set is connected, therefore there is at least one branch set X and one side B of $\{A, B\}$ such that X lies completely in $B \setminus A$. If there was another branch set Y that lay completely in $A \setminus B$, these two branch sets could then be separated from each other in G by the set $A \cap B$. Let S be the set of all the vertices of H whose branch set meets $A \cap B$. Let us suppose that there is a path P between the vertices associated with X and Y in $H - S$. As branch sets are connected, this would imply that there is a path between X and Y in G whose vertex set is a subset of the union of the branch sets associated with the vertices in P. Especially, this path would therefore not meet $A \cap B$, contradicting the fact that $A \cap B$ separates X from Y in G. Thus there cannot be such a path, so the set S separates the vertices associated with X and Y from each other. As S is of order less than k, this therefore contradicts the fact that H is k-connected. Therefore, given any k-connected minor, there is always exactly one side B of any separation $\{A, B\}$ of order less than k such that at least one branch set X of the minor satisfies $X \subseteq B \setminus A$. Orienting every separation of order less than k in this direction again defines a consistent orientation of the separations of order less than k. Thus, if we say that a vertex of H lives in one side B of a separation if and only if the branch set assosiated with this vertex meets B, as for k-blocks, the minor lives completely in one side of every separation.

Let H be an $(n \times n)$-grid. The way in which H as subgraph of G induces such an orientation is less obvious. Given a separation $\{A, B\}$ of G, the vertex set X of H (and even an arbitrary subset Y of X of size at least k) does not need to lie entirely in one side of this separation. Namely it is possible, if for example $G = H$, that there is for every subset Y of X of size at least k a separation $\{A, B\}$ of order less than k (if k is > 5), such that Y meets $A \setminus B$ as well as $B \setminus A$. In contrast to the k-connected minors there also is no obvious definition of when a grid "lives completely on one side" of a separation that allows us to orient every separation in that direction, as a grid does not have something like branch-sets. However, if $n > k$, a grid $H \subseteq G$ with vertex set X induces a consistent orientation of the separations of order less than k by orienting every separation $\{A, B\}$ so that $|X \cap A| < |X \cap B|$. Usually, if n is large enough, the size of $X \cap B$ will be considerably larger than the size of $X \cap A$. If for example n is at least $2k$, we have $|X \cap B| > 7\,|X \cap A|$ for every separation (A, B) of order less than k.

The consistent orientations of all the separations of order less than k that are induced by any of the highly connected substructures we considered (k-connected subgraphs, minors or topological minors, as well as k-blocks and large grids) are all examples of *tangles*. Formally, a k-tangle in a graph is a consistent orientation of all its separations of order $< k$ that satisfies an additional consistency requirement made not of the pairs but of the triples of its oriented separations. (See Section 2 for a formal definition). We might consider tangles in general as an abstract definition of a highly connected substructure. It is not known whether every tangle is induced by a "concrete" highly connected substructure like k-blocks or grids. In fact, we do not even know whether there is, for a given tangle, any set of vertices, highly connected or not, that induces this tangle in the following sense:

Problem 1.1. *Let G be a graph and $k \in \mathbb{N}$. If $\mathcal{T}$ is a k-tangle in G, is there always a vertex set X such that $|X \cap A| < |X \cap B|$ for every separation $(A, B) \in \mathcal{T}$?*

When a tangle $\mathcal{T}$ and a set X of vertices of a graph G are related as in Problem 1.1, we say that X determines $\mathcal{T}$ *by majority vote*. Problem 1.1 is the guiding question of this dissertation, we were not able to answer it in general, but solved some special cases. If k is at most 3, the answer follows from a Theorem of Tutte proved in the 1980s ([1],[8]), in which he showed that we can decompose, in a tree-like way, every 2-connected graph into parts that are either 3-connected or circles.

Theorem 1.2. *Let G be a graph. If $k \leq 3$, then for every k-tangle $\mathcal{T}$ in G there is a set X of vertices such that $|X \cap A| < |X \cap B|$ for every separation $(A, B) \in \mathcal{T}$.*

We will prove this theorem later, without using Tutte's Theorem.

The main result of this dissertation is that for any given integer k, under a mild assumption on the order of our graph, $(k+1)$-tangles in k-connected graphs are determined by majority vote:

Theorem 1.3. *Let k be an integer, G a k-connected graph with at least $4k$ vertices and $\mathcal{T}$ a $(k+1)$-tangle in G. Then there is a set X of vertices such that $|X \cap A| < |X \cap B|$ for every separation $(A, B) \in \mathcal{T}$.*

The structure of this dissertation is as follows: In Section 2 we give the basic definitions of and review some basic facts about separations and tangles. In Section 3 we present some background theory, namely, two theorems required for our first approach to solve Problem 1.1, which we present in Section 4. Sections 5 to 7 contain the proof of Theorem 1.3. The facts presented in Section 5 are also valid in arbitrary graphs, whereas the facts presented in Section 6 and 7 require our graph to be k-connected.

2 Basic definitions and facts

Given a graph $G = (V, E)$, a *separation* is a set $\{A, B\}$ such that $A \cup B = V$ and there is no edge from $A \setminus B$ to $B \setminus A$. The oriented pairs (A, B) and (B, A) are the *orientations* of $\{A, B\}$, we call them *oriented separations*. We say that A is the *small* or *left side* and that B is the *big* or *right side* of (A, B). The *proper* left/small (respectively right/big) side of (A, B) is $A \setminus B$ (respectively $B \setminus A$). The *inverse* of (A, B) is (B, A). The *order* $|\{A, B\}|$ is the size $|A \cap B|$ of $A \cap B$, the order $|(A, B)|$ of an oriented separation is defined accordingly. We call an oriented separation (A, B) *small* if $A \subseteq B$, this is equivalent to $B = V$. Given two oriented separations (A, B) and (C, D), defining

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D$$

results in a partial ordering on the set of all oriented separations. Given two separations $\{A, B\}$ and $\{C, D\}$, we say that $\{A, B\}$ and $\{C, D\}$ are *nested* if there are orientations (A, B) and (C, D) such that $(A, B) \leq (C, D)$. Otherwise $\{A, B\}$ and $\{C, D\}$ *cross*. Two oriented separations (A, B) and (C, D) are nested if $\{A, B\}$ and $\{C, D\}$ are nested, otherwise they cross. Given two oriented separations (A, B) and (C, D), we define the *infimum* $(A, B) \wedge (C, D)$ as the oriented separation $(A \cap C, B \cup D)$, the *supremum* $(A, B) \vee (C, D)$ is the oriented separation $(A \cup C, B \cap D)$.

Note that our order-function is *submodular*, so given two separations (A, B) and (C, D) it is

$$|(A, B)| + |(C, D)| \geq |(A, B) \vee (C, D)| + |(A, B) \wedge (C, D)|$$

For $k \in \mathbb{N}$ let S_k be the set of all separations of order strictly less than k in G and let $\mathcal{S}$ be the set of all separations in G. Given any set of separations S, we denote by $\vec{S}$ the set of all orientations of separations in S (so $\vec{S}_k$ is the set of all orientation of separations in S_k). An *orientation* $O \subseteq \vec{S}$ *of* S is a set of oriented separations such that for all $\{A, B\} \in S_k$ exactly one of (A, B) and (B, A) lies in O. A set $\vec{N}$ of oriented separations is said to be *consistent* if there are no two separations $(A, B), (C, D) \in \vec{N}$ such that $(B, A) \leq (C, D)$. So, a set of separations is consistent, if there are no two separations pointing away from each other. An orientation O of S_k is therefore consistent if for all oriented separations in O all smaller ones lie in O as well, so if $(C, D) \leq (A, B)$ and $(A, B) \in O$ then $(C, D) \in O$ as well. A *tangle* $\mathcal{T}$ of order k or a *k-tangle* in G is a consistent orientation of S_k fullfilling

$$(A, B), (C, D), (E, F) \in \mathcal{T} \Rightarrow G[A] \cup G[C] \cup G[E] \neq G \tag{1}$$

For a separation $(A, B) \in \mathcal{T}$ we think of A as the small and B as the big side of (A, B), so in a tangle no three small sides cover the whole graph. Especially, every k-tangle contains all small separations (A, V) of order less than k.

A separation $\{A, B\}$ *distinguishes* two tangles $\mathcal{T}$ and $\mathcal{T}'$ in G if $(A, B) \in \mathcal{T}$ and $(B, A) \in \mathcal{T}'$.

Given a vertex set $X \subseteq V$, we say that X *determines* an oriented separation (A, B) if $|A \cap X| < |B \cap X|$. Given a tangle $\mathcal{T}$, we say that X *determines* $\mathcal{T}$ if X determines every separation $(A, B) \in \mathcal{T}$.

We say that a separation (A, B) is *semi-small* if there are separations $(C, D) \leq (A, B)$ and $(E, F) \leq (A, B)$ both of order less than $|(A, B)|$ such that $G[C] \cup G[E] \cup G[B] = G$. Thus, if a tangle of order $> |(A, B)|$ contains (C, D) and (E, F) it must also contain (A, B).

Given a separation (A, B) of order k, if $|A \setminus B| < \frac{k}{2}$, the separation (A, B) lies in any $(k + 1)$-tangle:

Lemma 2.1. *Let G be a graph and $\mathcal{T}$ a $(k+1)$-tangle in G. Let (A, B) be a separation of order at most k in G. If $|A \setminus B| < \frac{k}{2}$, it is $(A, B) \in \mathcal{T}$.*

Proof. Let $B_1 \sqcup B_2 = A \cap B$ be a partition of $A \cap B$ into two sets such that $|B_1|, |B_2| \leq \lceil \frac{k}{2} \rceil$. Then $((A \setminus B) \cup B_1, V)$ and $((A \setminus B) \cup B_2, V)$ are both small separations of order at most k, so they both lie in $\mathcal{T}$. As $G[(A \setminus B) \cup B_1] \cup G[(A \setminus B) \cup B_2] \cup G[B] = G$, it is therefore $(B, A) \notin \mathcal{T}$, so $(A, B) \in \mathcal{T}$ as claimed. □

If a set X determines a separation (A, B), the set X also determines every separation $(C, D) \leq (A, B)$:

Lemma 2.2. *Let G be a graph and (A, B) a separation that is determined by a set X of vertices. Then every separation (C, D) such that $(C, D) \leq (A, B)$ is determined by X as well.*

Proof. As (A, B) is determined by X, it is $|X \cap A| < |X \cap B|$. By definition of $\leq$ for separations, it is $C \subseteq A$ and $B \subseteq D$, so it is

$$|X \cap C| \leq |X \cap A| < |X \cap B| \leq |X \cap D|$$

So (C, D) is indeed determined by X. □

If a consistent set $\vec{N}$ of oriented separations is nested, we know that the intersection of all the big sides of separations in $\vec{N}$ is not empty. If the set $\vec{N}$ is a subset of a tangle, it is then determined by this intersection:

Lemma 2.3. *Let G be a graph and $\vec{N}$ a set of oriented separations in G that is nested and consistent. Then the intersection*

$$X := \cap_{(A,B)\in\vec{N}} B$$

of the big sides of these separations is not empty. If $\vec{N}$ is additionally a subset of a k-tangle $\mathcal{T}$, the size of X is at least k, so X determines every separation in $\vec{N}$.

Proof. Let (A, B) be a $\leq$-maximal separation in $\vec{N}$. We claim that $A \cap B \subseteq X$. Otherwise, let $x \in (A \cap B) \setminus X$. Then there is a separation $(C, D) \in \vec{N}$ such that $x \notin D$, so $x \in C \setminus D$. Thus $A \not\subseteq D$ and therefore, as $\vec{N}$ is consistent and nested, it is $(A, B) < (C, D)$ contradicting the maximality of (A, B).

Let us now suppose that $\vec{N}$ is a subset of a k-tangle $\mathcal{T}$ and that X is of order at most k. Let $\vec{M} := \{(A_i, B_i) | 1 \leq i \leq n\}$ be the set of $\leq$-maximal separations in $\vec{N}$. As it is $A_i \cap B_i \subseteq X$ for all i, we know that the separation $(\tilde{A_2}, \tilde{B_2}) := (A_1, B_1) \vee (A_2, B_2)$ is of order less than k, as $\tilde{A_2} \cap \tilde{B_2} \subseteq X$. So it is $(\tilde{A_2}, \tilde{B_2}) \in \mathcal{T}$. Accordingly, the recursively defined separations

$$(\tilde{A_i}, \tilde{B_i}) := (A_i, B_i) \vee (\tilde{A_{i-1}}, \tilde{B_{i-1}})$$

lie in $\mathcal{T}$ as well. But, as $\tilde{B_n} = \cap_{i=1}^{n} B_i$ and the (A_i, B_i) are all the $\leq$-maximal separations in $\vec{N}$, it is $\tilde{B_n} = X$. Especially, $V \setminus X \subseteq \tilde{A_n} \setminus \tilde{B_n}$. But, as $(X, V) \in \mathcal{T}$, this contradicts the definition of a tangle, as $G[X] \cup G[\tilde{A_n}] = G$.

So, given a separation $(A, B) \in \vec{N}$, it is $X \subseteq B$ and $|X| > |A \cap B|$, so X indeed determines (A, B). □

3 Background Theory

In this section we will introduce two theorems: The Tangle-Tree-Theorem proved by Carmesin, Diestel, Hamann and Hundertmark in [5] and the Strong Duality Theorem proved by Diestel and Oum in [2]. The first Theorem basicly tells us that, given any graph G, there is a nested set of separations distinguishing, for a given integer k, all k-tangles in G. The Duality Theorem basicly gives us, in a general setup, a characterisation of graphs not containing a tangle. We will use this theorems both together in section 4 for an aproach to solve Problem 1.1.

But for making this precise, we first need to phrase both theorems and the definitions required for them. Both theorems are stated in a more general setup than the one we will refer to, but the version stated here will be general enough for our purposes. Let us start with the Tangle-Tree-Theorem from [5].

We say that a separation $\{A, B\}$ distinguishes two tangles $\mathcal{T}$ and $\mathcal{T}'$ *efficently*, if $\{A, B\}$ distinguishes $\mathcal{T}$ and $\mathcal{T}'$ and is of minimal order with this property, so for every separation $\{C, D\}$ with $|(C, D)| < |(A, B)|$ there is an orientation (C, D) of $\{C, D\}$ such that $(C, D) \in \mathcal{T} \cap \mathcal{T}'$. Given a set of tangles $\mathcal{T}^\circ$, we say that a separation is $\mathcal{T}^\circ$*-essential* if it efficently distinguishes some pair of tangles in $\mathcal{T}^\circ$. A set of separations N *distinguishes* a set of tangles $\mathcal{T}^\circ$ (*efficently*), if there is for every pair $\mathcal{T}, \mathcal{T}'$ of tangles in $\mathcal{T}^\circ$ a separation $\{A, B\}$ in N that distinguishes $\mathcal{T}$ and $\mathcal{T}'$ (efficently). A nested set of separations is *canonical* if the algorithm we used to construct this set is invariant under the automorphisms of the graph G. The Tangle-Tree-Theorem from [5] now states the following:

Theorem 3.1 (Tangle-Tree-Theorem)**.** *Given a graph G and a set $\mathcal{T}^\circ$ of k-tangles in G, there is a canonical nested set N of $\mathcal{T}^\circ$-essential separations of order $< k$ that distinguishes all the tangles in $\mathcal{T}^\circ$ efficently.*

We can especially distinguish all tangles of order k by a nested set of separations.

The other theorem we will use is the Duality-Theorem by Diestel and Oum. Therefore we will now refer to this theorem as well as some definitions from [2]. As we only want to use the Duality Theorem for separations of graphs, we will phrase these definitions and the theorem only in the generality required for this and not in the generality Diestel and Oum stated it.

A small separation $(A, V) \in \vec{S}_k$ is *trivial*, if there is another small separation $(B, V) \in \vec{S}_k$ such that $(A, V) < (B, V)$.

Given any set of not oriented separations S, we say that a non-empty set $\tau \subseteq \vec{S}$ is a *star of separations*, if any two separations in τ point towards each other, so $(A, B) \leq (D, C)$ for all $(A, B), (C, D) \in \tau$.

Given any set of not oriented separations S, an S*-tree* (T, α) is a pair of a non-trivial tree T together with a map α, mapping the set $\vec{E}$ of all orientations of edges of T (so $\vec{E} := \{(x, y) | xy \in E(T)\}$) into the set $\vec{S}$ of all orientations of separations in S such that this map is order-reversing (so if $\alpha(x, y) = (A, B)$ then $\alpha(y, x) = (B, A)$).

We say that (T, α) is an S-tree *over* a set of stars $\mathcal{F} \subseteq 2^{\vec{S}}$ (so every element of $\mathcal{F}$ is a star of separations), if given a vertex $t \in V(T)$ of T, the set $\alpha(\vec{F}_t)$ is in $\mathcal{F}$, where $\vec{F}_t := \{(x, t) | xt \in E(T)\}$ is the set of all oriented edges in T pointing towards t.

We call such a set of stars $\mathcal{F} \subseteq 2^{\vec{S}}$ *standard* if $\{(V, A)\} \in \mathcal{F}$ for every trivial separation $(A, V) \in \vec{S}$.

Given a non-trivial element (A, B) of $\vec{S}$, let $S_{\geq(A,B)}$ be the set of all separations $\{C, D\}$ fullfilling either $(A, B) \leq (C, D)$ or $(A, B) \leq (D, C)$. Given a separation $(A, B) \leq (A_0, B_0) \in \vec{S}$ and a separation $(C, D) \in \vec{S}_{\geq(A,B)} \setminus \{(A, B), (B, A)\}$, let us define:

$$f \downarrow_{(A_0,B_0)}^{(A,B)} ((C, D)) := (C, D) \vee (A_0, B_0) \text{ if } (C, D) \geq (A, B) \text{ and}$$
$$f \downarrow_{(A_0,B_0)}^{(A,B)} ((C, D)) := (C, D) \wedge (B_0, A_0) \text{ if } (D, C) \geq (A, B)$$

As for every separation $\vec{S}_{\geq(A,B)} \ni (C, D) \notin \{(A, B), (B, A)\}$ it is either $(A, B) \leq (C, D)$ or $(A, B) \leq (D, C)$, this is well defined. We set $f \downarrow_{(A_0,B_0)}^{(A,B)} ((A, B)) := (A_0, B_0)$ and $f \downarrow_{(A_0,B_0)}^{(A,B)} ((B, A)) := (B_0, A_0)$.

Given a separation $(A_0, B_0) \in \vec{S}$, we say that (A_0, B_0) is *linked* to $(C, D) \in \vec{S}$ if $(A_0, B_0) \geq (C, D)$ and it is $(E, F) \vee (A_0, B_0) \in \vec{S}$ for all $(D, C) \neq (E, F) \geq (C, D)$.

A set $\vec{S}$ of separations is said to be *separable* if, given any two separations (A, B) and (D, C) in $\vec{S}$ such that $(A, B) \leq (C, D)$, there is a separation (A_0, B_0) such that (A_0, B_0) is linked to (A, B) and (B_0, A_0) is linked to (D, C).

Given a set of stars $\mathcal{F} \subseteq 2^{\vec{S}}$, we say that a separation $(A_0, B_0) \in \vec{S}$ is $\mathcal{F}$-linked to $(C, D) \in \vec{S}$, if (A_0, B_0) is linked to (C, D) and, given a star $\tau \subseteq S_{\geq(C,D)}$ not containing (D, C), the image of τ under $f \downarrow_{(A_0,B_0)}^{(C,D)}$ is again in $\mathcal{F}$.[1]

We say that $\vec{S}$ is $\mathcal{F}$*-separable* if for any two separations $(A, B), (D, C) \in \vec{S}$ such that $(A, B) \leq (C, D)$ it is either $\{(B, A)\} \in \mathcal{F}$, or $\{(C, D)\} \in \mathcal{F}$, or there exists a separation $(A_0, B_0) \in \vec{S}$ such that (A_0, B_0) is $\mathcal{F}$-linked to (A, B) and (B_0, A_0) is $\mathcal{F}$-linked to (D, C).

A set $\mathcal{F}$ of stars is said to be *closed under shifting* if, given any separation (A_0, B_0) that is linked to a separation $(A, B) \leq (A_0, B_0)$, it is either $\{(B, A)\} \in \mathcal{F}$ or (A_0, B_0) is $\mathcal{F}$-linked to (A, B). The following Lemma by Diestel and Oum tells us that this property can be used to show that a set of separations is $\mathcal{F}$-separable:

Lemma 3.2. *If $\vec{S}$ is separable and $\mathcal{F}$ is closed under shifting, then $\vec{S}$ is $\mathcal{F}$-separable.*

Finally, given a set S of not oriented separations and a set $\mathcal{F}$ of stars, a consistent orientation O of S is a $\mathcal{F}$*-tangle* if it avoids $\mathcal{F}$, so if $\tau \not\subseteq O$ for any star τ in $\mathcal{F}$.

Diestel and Oum's Strong Duality Theorem now states the following:

[1] Diestel and Oum showed, that the map $f \downarrow_{(A_0,B_0)}^{(A,B)}$ preserves the ordering $\leq$ on $\vec{S}_{\geq(A,B)} \setminus \{(D, C)\}$, thus the image of τ under this map is always again a star.

Theorem 3.3 (Strong Duality Theorem)**.** *Let S be a set of separations and $\mathcal{F} \subseteq 2^{\vec{S}}$ a standard set of stars. If $\vec{S}$ is $\mathcal{F}$-separable, exactly one of the following assertions holds:*

(i) There exists an S-tree over $\mathcal{F}$.

(ii) There exists an $\mathcal{F}$-tangle of S.

4 The structural approach

Given any tangle $\mathcal{T}$ of order k in G, we know by the Tangle-Tree-Theorem, that there is a star of separations distinguishing $\mathcal{T}$ from every other k-tangle in G, namely the following corollary is true:

Corollary 4.1. *Let G be a graph, k an integer and $\mathcal{T}$ a tangle of order k in G. Then there is a star of separations $\vec{D} \subseteq \mathcal{T}$ such that $\vec{D} \nsubseteq \mathcal{T}'$ for any tangle $\mathcal{T}' \neq \mathcal{T}$ of order k.*

Proof. By Theorem 3.1, there is a nested set N of separations distinguishing all tangles of order k in G efficently. Let $\vec{N}$ be the orientation of N induced by $\mathcal{T}$. Let $\vec{D}$ be the set of $\leq$- maximal separations in $\vec{N}$. Then $\vec{D}$ is a star of separations, as $\vec{D}$ is nested and there are no two separations $(A, B), (C, D) \in \vec{D}$ such that $(A, B) < (C, D)$. We claim that $\vec{D}$ is the desiered set. Let $\mathcal{T}' \neq \mathcal{T}$ be a tangle of order k. Then there is a separation $(A, B) \in \vec{N}$ such that $(B, A) \in \mathcal{T}'$. If $(A, B) \notin \vec{D}$, there is a separation $(A, B) \leq (C, D) \in \vec{D}$ witnessing this. But then, as $(D, C) \leq (B, A) \in \mathcal{T}'$, it is $(D, C) \in \mathcal{T}'$, so it is indeed $(C, D) \notin \mathcal{T}'$. □

The general idea of this structural approach to prove the existence of a majority set for any given tangle now is as follows: Given a graph G and a k-tangle $\mathcal{T}$ in G, let $\vec{D}$ be a star of separations fullfilling the properties from corollary 4.1. Let

$$X := \cap_{(A,B)\in\vec{D}} B$$

be the intersetion of all big sides of separations in $\vec{D}$. We then know by Lemma 2.3 that X determines all separations in $\vec{D}$. So by Lemma 2.2, every separation (C, D) that is less or equal to a separation $(C, D) \leq (A, B) \in \vec{D}$ is also determined by X. Let $(Y, Z) \in \mathcal{T}$ be a $\leq$-maximal separation in $\mathcal{T}$ that is not less or equal to one in $\vec{D}$. As $\vec{D}$ distinguishes $\mathcal{T}$ from every other tangle $\mathcal{T}'$, we know that there is no tangle containing every separation in $\vec{D}$ and (Z, Y). As the Duality-Theorem of Diestel and Oum is stated in a really general setup, we will be able to use it to get a tree-like structure, which we can use to construct the separation (Y, Z) from small separations and separations less or equal to one in $\vec{D}$.

For our application of this theorem, let S be the set S_k of separations of order less than k. We now want to define a set $\mathcal{F} \subseteq 2^{\vec{S}_k}$ of stars such that an $\mathcal{F}$-tangle would be a tangle containing (Z, Y) and every separation in $\vec{D}$, as we know that such a tangle cannot exist.

Thus we then get an S_k-tree over this set $\mathcal{F}$. We therefore construct our set $\mathcal{F}$ as follows: Let

$$\mathcal{F}_P := \{\{(A,B),(C,D),(E,F)\}|G[A]\cup G[C]\cup G[E] = G\}$$

$$\mathcal{F}_T := \{\{(B,A)\}|\exists (C,D)\in \vec{D} : (A,B)\le (C,D)\}$$

Let $\mathcal{F}'_P \subseteq \mathcal{F}_P$ be the set of stars in $\mathcal{F}_P$. Set

$$\mathcal{F} := \mathcal{F}'_P \cup \mathcal{F}_T \cup \{\{(C,D)\}|(Y,Z)\le (C,D)\} \cup \{\{(V,A)\}|A\subseteq V\}$$

This is then a set of stars, we will check that $\vec{S}_k$ is $\mathcal{F}$-separable, as $\mathcal{F}$ is obviously standard.

Let us first show that this set $\mathcal{F}$ is closed under shifting. The proof of this is similar to the one given by Diestel and Oum for applying there Duality Theorem on k-tangles in graphs:

Lemma 4.2. *$\mathcal{F}$ is closed under shifting.*

Proof. Let (A_0,B_0) be linked to a separation $(A,B)\le (A_0,B_0)$ and suppose that $\{(B,A)\}$ is not a star in $\mathcal{F}$. Let τ be a star in $\vec{S}_{\ge(A,B)}$. If τ is not a star in $\mathcal{F}'_P$, it is $\tau = \{(C,D)\}$ for a separation $(C,D)\in \vec{S}_{\ge(A,B)}$. But this implies that $(C,D)\not\le (B,A)$, as otherwise, by the definition of $\mathcal{F}$, the set $\{(B,A)\}$ would also be a star in $\mathcal{F}$. Thus, the only possibility is $(A,B)\le (C,D)$. But then $f\downarrow^{(A,B)}_{(A_0,B_0)}(\tau) = \{(C,D)\vee (A_0,B_0)\}$, which is, as $(C,D)\le (C,D)\vee(A_0,B_0)$, again a star in $\mathcal{F}$.

Thus τ must be a star in $\mathcal{F}'_P$, so

$$\tau = \{(A_1,B_1),(A_2,B_2),(A_3,B_3)\}\subseteq \vec{S}_{\ge(A,B)}$$

Without loss of generality, let $(A,B)\le (A_1,B_1)$. Then

$$f\downarrow^{(A,B)}_{(A_0,B_0)}(\tau) = \{(A_1\cup A_0, B_1\cap B_0),(A_2\cap B_0, B_2\cup A_0),(A_3\cap B_0,B_3\cup A_0)\}$$

And as $f\downarrow^{(A,B)}_{(A_0,B_0)}(\tau)$ is a star and it is $G[B_1\cap B_0]\cup G[B_2\cup A_0]\cup G[B_3\cup A_0] = G$, it is therefore $f\downarrow^{(A,B)}_{(A_0,B_0)}(\tau)\in \mathcal{F}'_P$. □

We know by Diestel and Oum ([2], Lemma 5.1) that $\vec{S}_k$ is separable.

Lemma 4.3 ([2], Lemma 5.1)**.** *$\vec{S}_k$ is separable.*

Thus we know by Lemma 3.2 that $\vec{S}_k$ is $\mathcal{F}$-separable. So we can apply the Strong Duality Theorem with this set $\mathcal{F}$, so there is either an S_k-tree over $\mathcal{F}$ or an $\mathcal{F}$-tangle $\mathcal{T}'$ of S_k.

Let us now show that every $\mathcal{F}$-tangle is also a k-tangle, thus the existence of such an $\mathcal{F}$-tangle would contradict the definition of $\vec{D}$. The proof of this is similar to the one Diestel and Oum used in Lemma 5.2 of [2].

Lemma 4.4. *Any $\mathcal{F}$-tangle $\mathcal{T}$ of S_k in G is a tangle of order k.*

Proof. An $\mathcal{F}$-tangle $\mathcal{T}$ of S_k in G is a consistent orientation of S_k that avoids $\mathcal{F}$. So, if $\mathcal{T}$ would not be a k-tangle, it would violate (1), so there would be separations $(A, B), (C, D)$ and $(E, F) \in \mathcal{T}'$ such that $G[A] \cup G[C] \cup G[E] = G$. If $(A, B), (C, D), (E, F)$ are nested, our consistent orientation would not be $\mathcal{F}$-avoiding, so they are not.

If $(A, B) \leq (C, D)$, say, we can replace (C, D) by (A, B). Thus we may assume that $(A, B), (C, D), (E, F)$ are chosen such that this is not the case.

As these separations do not form a nested set, at least two of this three separations cross, let us assume without loss of generality that (A, B) and (C, D) cross. Then either the separation $(A \cap D, B \cup C) \leq (A, B)$ or the separation $(B \cap C, A \cup D) \leq (C, D)$ also lies in $\vec{S}_k$ by submodularity, thus if we replace one of the separations (A, B) and (C, D) with the smaller separation $(A \cap D, B \cup C)$ or $(B \cap C, A \cup D)$, our set of separations remain a set in $\mathcal{F}_P$ and we reduce the amount of crossings between these three separations. Thus, if we choose $(A, B), (C, D), (E, F)$ such that the amount of crossings between these three separations is minimal, we must have chosen them as a nested set of separations, and therefore $\mathcal{T}$ would not be $\mathcal{F}$-avoiding. Thus $\mathcal{T}'$ is indeed a tangle. □

We can now use all this Lemmas together with the Strong-Duality-Theorem to prove the existence of an $\mathcal{F}$-tree that can be used to generate (Y, Z).

Theorem 4.5. *Let G be a graph, $\mathcal{T}$ a tangle of order k in G and let $\vec{D} \subseteq \mathcal{T}$ be any star of separations separating $\mathcal{T}$ from every other k-tangle, so $\vec{D} \not\subseteq \mathcal{T}'$ for any tangle $\mathcal{T}' \neq \mathcal{T}$ of order k in G. Let (Y, Z) be a $\leq$-maximal separation in $\mathcal{T} \setminus \vec{D}$. Then there is an S_k-tree (T, α) over the set $\mathcal{F}$ defined above containing a leaf t_0 such that $\alpha(F_{t_0}) = \{(Y, Z)\}$.*

Proof. By Lemma 4.2, 4.3 and 3.2 we can apply the Strong-Duality-Theorem 3.3 with the sets S_k and $\mathcal{F}$. Thus there is either an S_k-tree over $\mathcal{F}$ or an $\mathcal{F}$-tangle of S_k. If there would be an $\mathcal{F}$-tangle, by Lemma 4.4 this $\mathcal{F}$-tangle $\mathcal{T}'$ would be a tangle of order k. But $\mathcal{T}'$ contains (Z, Y) whereas $\mathcal{T}$ contains (Y, Z), so they are not the same tangle. By construction there would therefore be a separation $(A, B) \in \vec{D}$ such that $(A, B) \in \mathcal{T}$ and $(B, A) \in \mathcal{T}'$, contradicting the fact that $\mathcal{T}'$ does not contain a star in $\mathcal{F}_T$.

So there must be an S_k-tree (T, α) over $\mathcal{F}$. If we orient every separation associated to an edge of this S_k-tree in the same direction as $\mathcal{T}$ orients this separation, we get a vertex t of T such that $\alpha(F_t)$ is a subset of $\mathcal{T}$. As $\mathcal{T}$ is a consistent orientation that avoids

$$\mathcal{F}_P' \cup \mathcal{F}_T \cup \{\{(C, D)\} | (Y, Z) < (C, D)\}$$

(by maximality of (Y, Z)), this vertex t must suffice $\alpha(F_t) = \{(Y, Z)\}$. So this vertex must be a leaf. □

Let us consider this tree as rooted in this leaf t_0 satisfying $\alpha(F_{t_0}) = \{(Y, Z)\}$ and let us think of every edge $\vec{e}$ as oriented away from the root. Then all images $\alpha(\vec{e}) = (C, D)$ in the tree satisfy $(Z, Y) \le (C, D)$, especially for every leaf $t \neq t_0$ it is $\alpha(F_t) \in \mathcal{F}_T$ or $\alpha(F_t) = \{(V, A)\}$ for a separation (V, A) of order less than k. So, if $\vec{e}$ is the edge pointing towards t, the separation $\alpha(\vec{e}) = (C, D)$ satisfies $X \subseteq C$.

So the separation (Y, Z) can be build along this tree: Starting at the leaves, we know that every separation forms a star in $\mathcal{F}$ together with the at most two separations above it, so the three small sides of these separations together cover the whole graph.

Unfortunately, if $\{(A, B), (C, D), (E, F)\} \in \mathcal{F}'_P$ is a star in $\mathcal{F}'_P$ and it is $X \subseteq B$ and $X \subseteq D$ this does not imply that $X \subseteq E$ (see Figure 2).

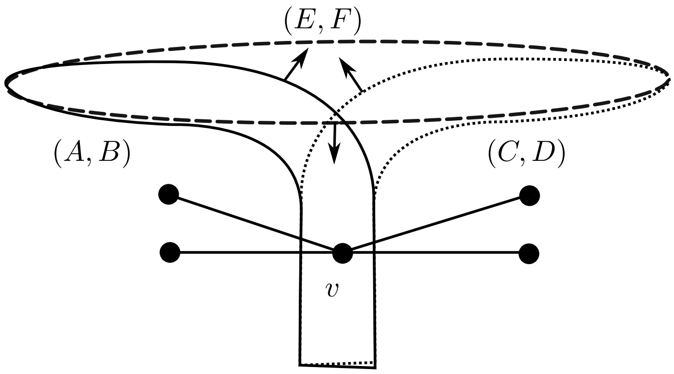

Figure 2: The separations $(A, B), (C, D), (E, F)$ form a star in $\mathcal{F}$, the vertex v lies in B and D but not in E.

We could therefore not hope to use this construction to show that $X \subseteq Y$. As additionally the height of our tree T is not bounded, we could not even guarantee that the majority of X still lies in Y.

Thus, if this construction should be of any use for proving the existence of a majority set, there are only two possibilities. Either we find a vertex set $\tilde{X} \subseteq V$ (maybe a subset of X) that is still a majority set for all the separations in $\vec{D}$ as well as all the small separations and has the following additional propoerty: If $\{(A, B), (C, D), (E, F)\} \in \mathcal{F}_P$ is a star in $\mathcal{F}_P$ and $\tilde{X}$ is a majority set for (A, B) and (C, D) then X is a majority set for (F, E) as well.

The other possibility is to try is to choose the set $\vec{D}$ in a clever way. One might try, for example, to choose $\vec{D}$ in such a way that X is of minimal size $|X|$. Then we know that for every star $\{(A, B), (C, D), (E, F)\} \in \mathcal{F}_P$ such that $(A, B), (C, D) \in \vec{D}$, we can not reduce the size of X by replacing (A, B) and (C, D) in $\vec{D}$ by (F, E).

Thus we may then be able to show that every such star $\{(A, B), (C, D), (E, F)\} \in \mathcal{F}_P$ fullfilles $X \subseteq E$. Unfortunately, in general most of (or even all of) the separations related to the leaves of the tree T are small separations, which are not effected by our choice of the set $\vec{D}$. Consider for example an arbitrary long path $P = v_1 v_2 \dots v_n$ (see Figure 4), were only v_1 and v_n have other neighbours in our graph G.

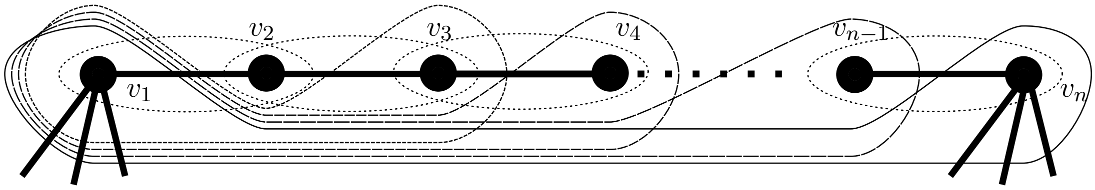

Figure 3: The separation separating the long path is build along small separations.

Consider the following stars of separations of order 2:

$$F_1 := \{(\{v_1, v_2\}, V), (\{v_2, v_3\}, V), (V \setminus \{v_2\}, \{v_1, v_2, v_3\})\}$$
$$F_2 := \{(\{v_1, v_2, v_3\}, V \setminus \{v_2\}), (\{v_3, v_4\}, V), (V \setminus \{v_2, v_3\}, \{v_1, v_2, v_3, v_4\})\}$$
$$\vdots$$
$$F_{n-2} := \{(\{v_1, \dots, v_{n-1}\}, V \setminus \{v_2, \dots, v_{n-2}\}), (\{v_{n-1}, v_n\}, V), (V \setminus \{v_2, \dots v_{n-1}\}, \{v_1, \dots v_n\})\}$$

These stars are the stars assigned to the inner vertices (all the vertices that are not leaves) of an S_k-tree over $\mathcal{F}$ that we may have used to construct the separation $(V \setminus \{v_2, \dots v_{n-1}\}, \{v_1, \dots v_n\})$. Thus, for beeing a majority set for $(V \setminus \{v_2, \dots v_{n-1}\}, \{v_1, \dots v_n\})$, we must make sure that our majority set X is not the set $\{v_1, \dots v_n\}$. As all the separations in the stars F_i do not distinguish any tangle, we were not able to achieve this by only choosing the set $\vec{D}$ in a clever way.

We were therefore not able to prove the existence of a majority set with this approach, even if we assume our graph G to be $k-1$-connected.

Our second approach however does solve the problem for these connected graphs, as long as they have enough vertices.

5 The second approach in arbitrary graphs

For this approach, the majority set we construct does not necessary lie in the center of a star. Instead, we try to find some sets in G that are a majority set for a (as large as possible) collection of separations.

But first let us try to reduce the problem: Let $M \subseteq \mathcal{T}$ be the set of maximal separations in $\mathcal{T}$, so the set of all $(A, B) \in \mathcal{T}$ such that for all separations (C, D) with $(A, B) \leq (C, D)$ it is $(C, D) \notin \mathcal{T}$. Our first Lemma will show us that we can restrict ourselves on the set of these maximal separations, as a set X determining M also determines $\mathcal{T}$:

Lemma 5.1. *Let G be a graph, $X \subseteq V$ a set of vertices, $\mathcal{T}$ a tangle in G and M the set of maximal separations in $\mathcal{T}$. Then X determines every separation in M if and only if it determines $\mathcal{T}$.*

Proof. The if direction is obvious, for the only if direction let $(A, B) \in \mathcal{T} \setminus M$, we show that (A, B) is determined by X. As $(A, B) \notin M$, there is a separation $(C, D) \in M$ such that $(A, B) < (C, D)$. As X determines (C, D), the set X also determines (A, B) by Lemma 2.2. □

Thus, we can restrict ourselves on the set M. This is useful as we can especially use the submodularity to get some information about crossing separations in M:

Lemma 5.2. *Let $\mathcal{T}$ be a $k+1$-tangle in G and M the set of maximal separations in $\mathcal{T}$. If a separation $(A, B) \in M$ crosses a separation (C, D) in $\mathcal{T}$, the order of the separation $(A, B) \wedge (C, D)$ is less than $\min\{|(A, B)|, |(C, D)|\}$, so especially, $|(A, B) \wedge (C, D)| < k$.*

Proof. As (A, B) is in M and therefore maximal in $\mathcal{T}$, the separation $(A, B) \vee (C, D)$ cannot be in $\mathcal{T}$, as (C, D) crosses (A, B) and therefore $(A, B) \vee (C, D) > (A, B)$. So, as a $k+1$-tangle is a consistent orientation of $\vec{S}_{k+1}$, either the separation $(A, B) \vee (C, D)$ is not in $\vec{S}_{k+1}$ or its inverse, namely the separation $(B, A) \wedge (D, C) = (B \cap D, A \cup C)$ is in $\mathcal{T}$. But this second case is not possible, as $G[A] \cup G[C] \cup G[B \cap D] = G$ contradicting property (1) of a tangle.

So $(A, B) \vee (C, D) \notin \vec{S}_{k+1}$, and therefore by the definition of $\vec{S}_{k+1}$ the order of the separation $(A, B) \vee (C, D)$ is at least $k + 1$. Thus it is especially

$$|(A, B) \vee (C, D)| > |(A, B)| \text{ and } |(A, B) \vee (C, D)| > |(C, D)|.$$

So we know by submodularity that the order of $(A, B) \wedge (C, D)$ is less than $\min\{|(A, B)|, |(C, D)|\}$. □

We are now ready to start our proof of Theorem 1.2, that for tangles of order at most 3 there is always a majority set. The main reason why this is true is that, if k is at most 2, we can make some minor adujstments on the set of k-separations to make it nested. If this set is nested, the intersection of all big sides is then not empty, thus can be used as majority set. For $k < 2$, the only adjustment we have to do is to reduce ourselves on a connected graph G. For $k = 2$, we then reduce ourselves on a 2-connected component. The reason why we can do this is basicly that we know that we can decompose a connected graph into pieces, that are either 2-connected or to small, to contain a 3-tangle. Before we make this precise, let us state two Lemmas needed in the proof:

Lemma 5.3. *If G is a connected graph and $\mathcal{T}$ a tangle of order 2 in G, then the set M of $\leq$-maximal separations in $\mathcal{T}$ is nested.*

Proof. Let $\{A, B\}$ be an arbitrary separation of order 1 in a connected graph. We claim that $G[A]$ and $G[B]$ are connected: Otherwise there are two vertices $v, w \in A$, say, such that there is no path from v to w in $G[A]$. Let x be the vertex in $A \cap B$. As G is connected, there is a path $P = v_0 v_1 v_2 \dots v_m$, with $v = v_0$ and $w = v_m$ in G. As P is not a path in $G[A]$, there are vertices v_i in P, that are not vertices in A. Let i_0 be the smallest index such that $v_{i_0} \notin A$ and i_1 the largest index such that $v_{i_1} \notin A$. As $v_{i_0 - 1}, v_{i_1 + 1} \in A$ and $v_{i_0}, v_{i_1} \notin A$, it is $v_{i_0 - 1} = x = v_{i_1 + 1}$, contradicting the definition of a path. Thus, P is a path in $G[A]$ and $G[A]$ is therefore connected.

Let us now suppose that M is not nested, so there are separations $(A, B), (C, D) \in M$ that cross. As G is connected, these separations are separations of order 1, let v be the vertex in $A \cap B$ and let w be the vertex in $C \cap D$. Let us first consider the case that $v \neq w$.

As two oriented separations are nested if and only if the unordered separations they induce are nested, it is enough to consider the case that $w \in A \setminus B$ and $v \in D \setminus C$. As $G[C]$ is connected and $v \notin C$, it is either $C \subseteq A$ or $C \subseteq B$. As $w \in A \setminus B$, only $C \subseteq A$ is possible. Accordingly, it is $B \subseteq D$, thus $\{A, B\}$ and $\{C, D\}$ are nested.

If $v = w$, the separation $(A, B) \vee (C, D)$ is of order 1 as well. As therefore $(A, B) \vee (C, D) \in \mathcal{T}$, it is, by the maximality of (A, B) and (C, D) in $\mathcal{T}$, either $(A, B) \vee (C, D) = (A, B)$ or $(A, B) \vee (C, D) = (C, D)$. Let us assume without loss of generality that $(A, B) \vee (C, D) = (A, B)$. But then $A \cup C = A$ and $B \cap D = B$, so $C \subseteq A$ and $B \subseteq D$. Therefore $(C, D) \leq (A, B)$ and these separations are nested as well. □

Whereas the last Lemma gave us information about separations of order 1 in connected graphs, the next Lemma will give us information about separations of order 2 in 2-connected graphs:

Lemma 5.4. *Let G be a 2-connected graph and $\mathcal{T}$ a tangle of order 3 in G. If (A, B) and (C, D) are $\leq$-maximal separations in $\mathcal{T}$ that are not small, then (A, B) and (C, D) are nested.*

Proof. If (A, B) and (C, D) are such separations that are not nested, it is either $A \not\subseteq D$ or $C \not\subseteq B$. Thus we may assume without loss of generality that $A \not\subseteq D$, so there is a vertex v in $A \setminus D$. Additionally, we know by Lemma 5.2, that the separation $(A, B) \wedge (C, D)$ is of order less then 2, so, as G is a 2-connected graph, this separation must be small. It is therefore $|A \cap C| \leq 1$ and $B \cup D = V$. As $v \notin D$, it is $v \in C$ and therefore $\{v\} = A \cap C$. Additionally, as $v \in V = B \cup D$, it is $v \in B$. So $(C \cap D) \cap A = \emptyset$.

Let w be the other vertex in $A \cap B$, so $A \cap B = \{v, w\}$. Then the separation $(A, B) \wedge (D, C)$ is of order at most 1 as well, as its separator is

$$(A \cap D) \cap (B \cup C) = (A \cap B \cap D) \cup (A \cap D \cap C) = (\{v, w\} \cap D) \cup \emptyset \subseteq \{w\}$$

Therefore, as G is two connected, this separation is small, so it is

$$A \cap D = (A \cap D) \cap (B \cup C) \subseteq \{w\}$$

and therefore

$$A = (A \cap C) \cup (A \cup D) \subseteq \{v\} \cup \{w\} = \{v, w\}$$

So it is $A \subseteq B$ and the separation (A, B) is small, contradicting the assumption. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. If k is 0, the assertion is obvious, as every 1-tangle represents one connected component. Thus we may assume that k is at least 1, so especially $\mathcal{T}$ induces a 2-tangle T_2 in G. Additionally we may assume without loss of generality that the graph G is connected. By Lemma 5.3 and Lemma 2.3, we know that the intersection $X := \bigcap_{(A,B) \in T_2} B$ of all the big sides of separations in T_2 is a set of order at least 2. We claim that $G[X]$ is either 2-connected or a K_2.

Given 2 vertices $x, y \in X$, there is either an edge between x and y or there are two disjoint paths between x and y in G. Otherwise, by Mengers Theorem, there would be a separation (A, B) of order 1 such that $x \in A \setminus B$ and $y \in B \setminus A$, say. As this separation must be oriented by the tangle T_2, this would contradict the assumption that $\{x, y\} \subseteq X$. Given two disjoint paths between x and y, all vertices on these paths lie in X as well, as there is no separation of order 1 separating any of these vertices from the set $\{x, y\}$. Thus, the two disjoint paths between x and y lie in $G[X]$. Together, this implies the fact stated above: If $G[X]$ contains at least 3 vertices, it is 2-connected, as there are 2 disjoint paths in $G[X]$ between any two not adjacent vertices. If $G[X]$ contains only 2 vertices, it is a K_2. If $\mathcal{T} = \mathcal{T}_2$, this would already imply the assumption, as then X fullfilles $|X \cap A| < |X \cap B|$ for every separation $(A, B) \in \mathcal{T}$.

Let us now suppose that $\mathcal{T}$ is a 3-tangle and the set X does not meet the required property. Suppose that $G[X]$ is a K_2, with vertex set $\{v_1, v_2\}$, say. let K_{v_2} be the component of $G - v_1$ containing v_2 and let K_{v_1} be the component of $G - v_2$ containing v_1. Then $(V \setminus K_{v_2}, K_{v_2} \cup \{v_1\})$ and $(V \setminus K_{v_1}, K_{v_1} \cup \{v_2\})$ are separations of order 1 that lie in $\mathcal{T}$. Thus, as $(\{v_1, v_2\}, V)$ is a separation in $\mathcal{T}$ as well, this contradicts the definition of a tangle. Therefore $G[X]$ is not a K_2, so it is 2-connected.

We can then restrict ourselves on the Graph $G' := G[X]$. Every separation (A, B) of order 2 in $\mathcal{T}$ induces a separation of G' via $(A', B') := (A \cap X, B \cap X)$. One can easily check that the set $\mathcal{T}'$ of all separations induced by separations in $\mathcal{T}$ is consistent and fullfilles (1) – for example there are no two separations (A, B) and $(C, D) \in \mathcal{T}$ such that $(A', B') = (D', C')$: Otherwise let (A, B) and (C, D) both be chosen $\leq$-maximal. If (A', B') would be a separation of order 1, this would contradict the choice of X. Thus (A', B') is a separation of order 2, so it is $A \cap B = A' \cap B' = C' \cap D' = C \cap D$. As $G[A] \cup G[C] \neq G$, there is a vertex $v \notin A \cup C$, so $v \in (B \setminus A) \cap (D \setminus C)$. Let K be the connected component of $G - X$ containing v. By the way we have chosen X, there is a unique vertex $x_v \in X$ such that $N(K) = \{x_v\}$ in G. And, as (A, B) and (C, D) are separations in G, it is $x_v \in B \cap D$. Thus, as $A' \cup C' = X$, it is $x_v \in A \cap B$ or $x_v \in C \cap D$, let us assume without loss of generality, that $x_v \in A \cap B$. But, as X lies in the big side of every separation of order 1 in $\mathcal{T}$, it is $(V(K) \cup \{x_v\}, V \setminus V(K)) \in \mathcal{T}$, thus we know that $(A \cup V(K), B \setminus V(K)) \in \mathcal{T}$ as well, contradicting the maximality of (A, B). So there are indeed no two separations (A, B) and $(C, D) \in \mathcal{T}$ such that $(A', B') = (D', C')$.

We know that, given a vertex v not in X, there is a unique vertex $x_v \in X$ separating v from X. So there is, given a separation $\{E, F\}$ of G', a separation $\{A, B\}$ of order at most 2 in G such that $\{A', B'\} = \{E, F\}$: Let A consist of E together with all vertices v fullfilling $x_v \in A$ and let B consist of F together with $V \setminus A$.

So the set $\mathcal{T}'$ is a tangle of order 3 in G'. Thus, by Lemma 5.4 and Lemma 2.3, we know that the intersection $X' := \cap_{(A',B') \in \mathcal{T}'} B'$ of all the big sides of separations in $\mathcal{T}'$ is a set of order at least 3.

Especially, this set is a majority set for $\mathcal{T}'$ and therefore for $\mathcal{T}$ as well: Given any separation $(A, B) \in \mathcal{T}$ we know that $|A \cap X'| = |A' \cap X'| < |B' \cap X'| = |B \cap X'|$ □

Thus, the general Theorem is true at least for small values of k.

The next Lemma gives us some information about crossing separations in M and additionally justifies the definition of semi-small separations:

Lemma 5.5. *If $(A, B) \in M$ is a separation that is not semi-small and crosses a separation $(C, D) \in \mathcal{T}$, then the separation $(A, B) \wedge (D, C)$ is not of order less than $|(A, B)|$.*

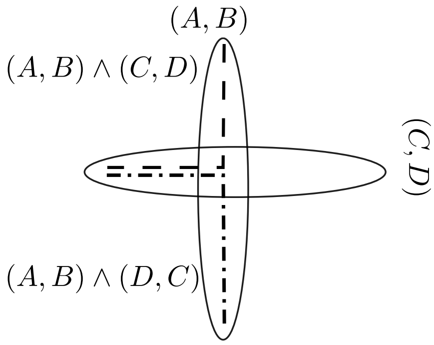

Figure 4: The separation $(A, B) \wedge (C, D)$ is of order less than $|(A, B)|$, so $|(A, B) \wedge (D, C)| \geq |(A, B)|$.

Proof. The separation $(A, B) \wedge (C, D)$ is of order less than $|(A, B)|$ by Lemma 5.2. Thus, if $(A, B) \wedge (D, C)$ is also of order less than $|(A, B)|$, the separation (A, B) would be semi-small, as $G[A \cap C] \cup G[A \cap D] \cup G[B] = G$ and $(A, B) \wedge (C, D), (A, B) \wedge (D, C) \leq (A, B)$. □

Semi-small separations behave a bit like small separations, that is why we call them semi-small: One essential property of small separations is that every tangle contains all small separations in our graph G. Semi-small separations however, lie not necessarily in all tangles. But, if (A, B) is a semi-small separation of order k and $\mathcal{T}$ a tangle of order $(k+1)$ containing (A, B), then there is no other $(k+1)$-tangle inducing the same k-tangle as $\mathcal{T}$ and containing (B, A). So, the semi-small separations are not charateristic for a specific tangle, they do not minimal distinguish two tangles:

Lemma 5.6. *Let $\mathcal{T}$ be a $k+1$-tangle in G containg a semi-small separation (A, B) of order k. Let T_k be the tangle induced from $\mathcal{T}$ on S_k. Then every $k+1$-tangle $\mathcal{T}'$ that induces T_k also includes (A, B).*

Proof. As (A, B) is semi-small, there are two separations $(C, D), (E, F) < (A, B)$ of order less than $|(A, B)|$ such that $G[B] \cup G[C] \cup G[E] = G$. As $\mathcal{T}'$ induces T_k, it is $(C, D), (E, F) \in \mathcal{T}'$. So, $(B, A) \in \mathcal{T}'$ would violate (1), therefore it is $(A, B) \in \mathcal{T}'$. □

Especially, if every maximal separation in a tangle $\mathcal{T}$ is semi-small, this tangle is completely induced by a tangle of smaller order. Therefore it is reasonable to focus on tangles including maximal separations that are not semi-small. In such tangles we can then find a set X, that is nearly a majority set, except of the fact that there will be one separation $(A, B) \in M$ only fullfilling $|A \cap X| = |B \cap X|$:

Lemma 5.7. *Let G be a graph, $\mathcal{T}$ a tangle of order $k+1$ and $(A, B) \in \mathcal{T}$ a maximal separation that is not semi-small. Let $X := A \cap B$. Then for every separation $(C, D) \in \mathcal{T}$ it is $|C \cap X| < |D \cap X|$ or $C \cap D = X$ and $(C, D) \leq (A, B)$.*

Proof. Let $(A, B) \neq (C, D) \in \mathcal{T}$. If $(C, D) \leq (A, B)$, the assertion is obvious. By Lemma 5.1, we may therefore assume that (C, D) is maximal in $\mathcal{T}$. Thus, if (C, D) is nested with (A, B), it is $(C, D) \leq (B, A)$ as (A, B) is a maximal separation in $\mathcal{T}$. So in this case it is $D \supseteq A$ and therefore $X \subseteq D$. So, the assertion is true as long as $X \not\subseteq C$, so let us suppose, that this is not the case.

In a maximal separation (A, B) the set $B \setminus A$ must be connected: Otherwise there is a connected proper subset K of $B \setminus A$, but then the separation $(V \setminus K, K \cup X)$ cannot be in $\mathcal{T}$ by the maximality of (A, B). Thus $(K \cup X, V \setminus K)$ is in $\mathcal{T}$, but then, by (1), the separation $(A \cup K, B \setminus K)$ must be in $\mathcal{T}$, contradicting the maximality of (A, B). Thus $G[B \setminus A]$ is connected, so $B \setminus A \subseteq C$ and therefore $(A, B) = (D, C)$, contradicting the definition of a tangle.

So we may assume that (A, B) and (C, D) cross. Then by Lemma 5.2, the separation $(A, B) \wedge (C, D)$ is small and of order less than $|(A, B)|$.

We then know by Lemma 5.5 that the separation $(A, B) \wedge (D, C)$ cannot be of order less than $|(A, B)|$ as well, as this would imply that (A, B) is semi-small. So it is

$|(A,B) \wedge (C,D)| < |(A,B) \wedge (D,C)|$ and therefore by the definition of the order function it is

$$|(A \cap C) \cap (B \cup D)| < |(A \cap D) \cap (B \cup C)|$$

And therefore

$$|X \cap (C \setminus D)| = |A \cap B \cap (C \setminus D)| < |A \cap B \cap (D \setminus C)| = |X \cap (D \setminus C)|$$

So it is indeed $|X \cap C| < |X \cap D|$. □

This Lemma guarantees the existence of a majority set for some types of tangles:

First, given a tangle $\mathcal{T}$ of order k, if there is an integer $l > k$ and two different tangles $\mathcal{T}'$ and $\mathcal{T}''$ of order l both inducing $\mathcal{T}$ and efficently distinguished by a separation of order $l-1$, there is a majority set for $\mathcal{T}$: There exists a maximal, not semi-small separation (A,B) in $\mathcal{T}' \setminus \mathcal{T}$, so by Lemma 5.7, there is a majority set for every separation (C,D) in $\mathcal{T}'$ not fullfilling $C \cap D = A \cap B$. Especially, this is a majority set for every separation in $\mathcal{T}$, as every separation in $\mathcal{T}$ is of order less than $l-1$.

Also, if there are two maximal not semi-small separations (A,B) and (C,D) in our given tangle $\mathcal{T}$ such that $A \cap B$ and $C \cap D$ are disjoint, we get by Lemma 5.7 that there is a majority set for $\mathcal{T}$: In this case the union of $A \cap B$ and $C \cap D$ is the desiered set, as $A \cap B$ is a majority set for every separation in M except of (A,B) and $C \cap D$ is a majority set for every separation in M except of (C,D).

If we consider $(k+1)$-tangles in a k-connected graph, we will be able to use this Lemma as one part of the construction of a majority set for our tangle.

6 $(k+1)$-tangles in k-connected graphs

From now on let G be a k-connected graph, we consider the set S_{k+1} of separations of order at most k. As G is k-connected, every separation of order less than k must be small. Thus, Lemma 5.2 implies that, if a maximal separation $(A,B) \in \mathcal{T}$ crosses a separation $(C,D) \in \mathcal{T}$, the separation $(A,B) \wedge (C,D)$ must be a small separation, as it is a separation of order less than k. Especially, there are no vertices lying properly on the small side of two different maximal separations of $\mathcal{T}$.

Also, in a k-connected graph we get an additional property of the semi-small separations which will allow us to concentrate our efforts for constructing the majority set on the set of not semi-small separations:

Lemma 6.1. *If G is a k-connected graph and (A,B) a semi-small separation of order k in G, it is $|A \setminus B| < \frac{k}{2}$*

Proof. Let $(C, D) \leq (A, B)$ and $(E, F) \leq (A, B)$ be the separations of order less than k witnessing that (A, B) is semi-small, so $G[B] \cup G[C] \cup G[E] = G$. As (C, D) and (E, F) are of order less than k, they are small separations, so $D = F = V$ and $|C|, |E| < k$.

As G is k-connected, we know that every vertex in $A \cap B$ has a neighbour in $A \setminus B$ (as there would be otherwise a separation of order less than k that is not small). Thus, as $G[B] \cup G[C] \cup G[E] = G$, we know that every vertex in $A \cap B$ must lie in at least one of the sets C and E. If one vertex v in $A \setminus B$ would lie in C but not in E, say, the set $C \setminus \{v\}$ would be a set of order less than k that separates v from G, contradicting the fact that G is k-connected. Thus every vertex in $A \setminus B$ lies in both sets, C and E. Therefore it is:

$$k + 2|A \setminus B| = |A \cap B| + 2|A \setminus B| \leq |C| + |E| < 2k$$

And therefore $|A \setminus B| < \frac{k}{2}$. $\square$

This Lemma implies that every semi-small separation (A, B) in our tangle $\mathcal{T}$ is determined by any set X of order at least $2k$, as for a semi-small separation (A, B) it is $|A \setminus B| < \frac{k}{2}$. Therefore, for constructing a majority set, we restrict ourselves on the set $\bar{M}$ of all maximal separations of $\mathcal{T}$ that are not semi-small, so

$$\bar{M} = M \setminus \{(A, B) \in \vec{S}_{k+1} | (A, B) \text{ semi-small}\},$$

and would like to find a set X of order at least $2k$ that determines all of $\bar{M}$.

For this, we will distinguish the separations in $\bar{M}$ into two classes, the *deep* separations and the *flat* separations.

We say that a vertex $v \in V$ is *deep with respect to* $(A, B) \in \bar{M}$ if $v \in D \setminus C$ for all $(C, D) \in \bar{M} \setminus \{(A, B)\}$, so v is deep if $\{v\}$ determines all separations in $\bar{M}$ except of (A, B). A separation (A, B) is *deep* if there exists a vertex v that is deep with respect to (A, B), otherwise (A, B) is *flat*.

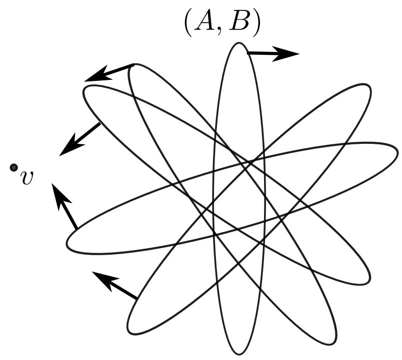

Figure 5: The vertex v is deep with respect to (A, B).

The picture of deep vertices is that they are lying deep inside the side $A \setminus B$, so that they cannot be touched from any other separation $(C, D) \in \bar{M}$.

As G is k-connected we get an at the first glance surprising result about a class of separations that must be deep, namely we will show that, given a separation $(A, B) \in \bar{M}$, all but at most k vertices in $A \setminus B$ are deep with respect to (A, B). The reason why this works is mainly because there are no vertices lying properly in the small side of two separations. Thus, roughly speaking, if a vertex is not deep, it lies in the interior of an other separation. But this other separation must cross the original separation (A, B) and this cannot happen arbitrary wild, as these separations have to interact with each other as well. Precisely the proof is as follows:

Lemma 6.2. *Let $(A, B) \in \bar{M}$, then all but at most k vertices in $A \setminus B$ are deep with respect to (A, B).*

Proof. We construct an injective function f from the set of all non deep vertices W in $A \setminus B$ to $A \cap B$. As $|A \cap B| \leq k$, we then get the claimed result.

For every non deep vertex in $A \setminus B$ we have a separation $(C, D) \in \bar{M}$ witnessing that this vertex is not deep, namely $v \notin D \setminus C$, so $v \in C$. By Lemma 5.2 the separation $(A, B) \wedge (C, D)$ is small, so there is no vertex in $(A \cap C) \setminus (B \cup D)$. Especially, v therefore lies in $C \cap D$. Pick one such separation (C, D) witnessing that v is not deep for every vertex v in W, let $F \subseteq \bar{M}$ be the set of all these witnesses and let $W_{(C,D)}$ for $(C, D) \in F$ be the set of vertices for which (C, D) was picked as witness.

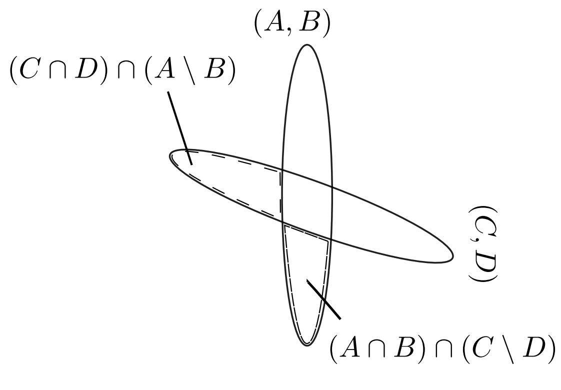

Figure 6: The set $(C \cap D) \cap (A \setminus B)$ has at most the same order as $(A \cap B) \cap (C \setminus D)$.

Let $(C, D) \in F$, then $W_{(C,D)} \subseteq (C \cap D) \cap (A \setminus B)$. We show that

$$|(C \cap D) \cap (A \setminus B)| \leq |(A \cap B) \cap (C \setminus D)| \tag{2}$$

By Lemma 5.5, as the separation (C, D) is not semi-small, the separation $(B, A) \wedge (C, D)$ is of order at least k. So, descriptive, (2) is true as $(B, A) \wedge (C, D)$ has at least the same order as (C, D), so there are at least as many vertices in the interior of $(B, A) \wedge (C, D)$ that are not in the interior of (C, D) as vice versa. For the formal proof we calculate:

$$\begin{aligned} & |B \cap C \cap (A \cup D)| \geq |C \cap D| \\ \Rightarrow & |(B \cap C \cap D) \sqcup ((B \cap C \cap A) \setminus D)| \geq |(C \cap D \cap B) \sqcup ((C \cap D \cap A) \setminus B)| \\ \Rightarrow & |(B \cap C \cap A) \setminus D| \geq |(C \cap D \cap A) \setminus B| \\ \Rightarrow & |(C \cap D) \cap (A \setminus B)| \leq |(A \cap B) \cap (C \setminus D)| \end{aligned}$$

So (2) is true indeed. For every vertex $v \in W_{(C,D)}$ pick one vertex in $(A \cap B) \cap (C \setminus D)$ as image of v under f such that now two vertices in $W_{(C,D)}$ get the same image.

We claim, that this f is indeed injecitve: Otherwise there are two vertices $v, w \in W$ such that $f(v) = f(w)$. By construction their witnesses (C_v, D_v) and (C_w, D_w) are different (as we constructed f such that all vertices in $W_{(C,D)}$ have different images). But then, by construction, $f(v) \in C_v \setminus D_v$ and $f(w) \in C_w \setminus D_w$. So there is a vertex in $C_v \setminus D_v \cap C_w \setminus D_w = (C_v \cap C_w) \setminus (D_v \cup D_w)$, contradicting Lemma 5.2, as this implies that $(C_v, D_v) \wedge (C_w, D_w)$ is not a small separation.

So f is indeed injective, and therefore there are at most k vertices in $A \setminus B$ that are not deep. □

7 Constructing a majority set

The picture we had in mind for our construction of a majority set is the one in figure 7.

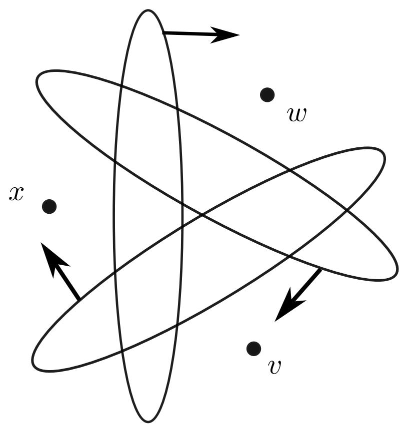

Figure 7: The three vertices v, w, x together form a majority set for the three separations.

If we find vertices that lie inside the small side of only one separation, we could pick such vertices for different separations resulting in a majority set for all separations. This inspired the definition of deep vertices. There are two problems we have to face with here: First we need to make sure that our majority set is large enough, as we do not know anything about the relation between deep vertices and semi-small separations. The other problem is that we cannot guarantee that there are enough, namely at least 3, deep separations to make this work. Facing this two problems, our strategy for finding a majority set is as follows:

If the set V is not a majority set, we can show that there is a deep separation (A, B), if G contains enough vertices. We then would like to pick two sets $X_1 \subseteq A \setminus B$ and $X_2 \subseteq B \setminus A$ such that $|X_1| = |X_2| - 1$, where the set X_1 will be a set of deep vertices with respect to (A, B). We then would like to set $X = X_1 \sqcup X_2$. No matter how we choose these sets X_1, X_2, we then determine (A, B). Also, for every other separation (C, D) in $\bar{M}$, we know that the whole set X_1 lies in its proper big side, thus this separation can only be not determined correctly if all but at most one vertex of X_2 lies on the proper small side of (C, D). By picking the set X_2 in a clever way, we can make sure that this is not the case, so all other separations in $\bar{M}$ are also determined by X.

As the last step, we then may add a set we got by Lemma 5.7 to our set X to make sure that it is of size at least $2k$, so all semi-small separations in M are then determined by X as well.

For constructing these sets X_1 and X_2, we shall need to distinguish some cases. As already said above, if there are no deep separations at all we will be able to pick the whole set V:

Lemma 7.1. *Let G be a k-connected graph with more than $3k$ vertices and let $\mathcal{T}$ be a $(k+1)$-tangle in G. If there are no deep separations in $\mathcal{T}$, the set $X := V$ is a majority set for $\mathcal{T}$.*

Proof. By Lemma 5.1, it suffices to show that X determines every separation in M. As there are no deep separations in M, by Lemma 6.1 and Lemma 6.2 it is $|A \setminus B| \leq k$ for every separation $(A, B) \in M$. But it is $|A \cap B| \leq k$ and

$$|A \setminus B| + |A \cap B| + |B \setminus A| = |V| > 3k$$

and therefore $|B \setminus A| > k \geq |A \setminus B|$, so X determines (A, B). □

Thus, we may assume that there are deep separations in $\mathcal{T}$. The clue is now to find a suitable set X_2. To have a chance at least to pick this set X_2 such that X then determines every separation in $\bar{M}$, it is necessary that, given a separation $(C, D) \in \bar{M}$, we can find some vertices in $B \setminus A$ that do not lie in $C \setminus D$, as otherwise this approach would be hopeless. Luckly, this is indeed the case, as the small sides of at most three separations in a tangle cannot cover the whole graph:

Lemma 7.2. *Let G be a k-connected graph, $\mathcal{T}$ a $(k+1)$-tangle and $k \geq 3$. If (A,B) and (C,D) are separations in $\mathcal{T}$, then there are at least two vertices $v,w \in (B \setminus A) \cap D$.*

Proof. If $(B \setminus A) \cap D = \emptyset$, this would imply that $G = G[A] \cup G[C]$: Let $e = vw$ be an edge not in $G[A]$, then without loss of generality, it is $v \in B \setminus A$. Thus, it is $v \in C \setminus D$ and therefore, as (C,D) is a separation, it is $w \in C$ as well. So $e \in G[C]$. But $G = G[A] \cup G[C]$ is a contradiction to the fact that $\mathcal{T}$ is a tangle. Thus $|B \setminus A \cap D| \geq 1$. Suppose now, that there is only one vertex v in $(B \setminus A) \cap D$.

If the neighbourhood $N(v)$ of v is a subset of $A \cap B$, then $(\{v\} \cup (A \cap B), V \setminus \{v\})$ would be a separation of order k in $\mathcal{T}$ that (as $k \geq 3$), by Lemma 2.1, lies in $\mathcal{T}$. But it is $G[A] \cup G[C] \cup G[\{v\} \cup A \cap B] = G$: If $e = wu$ is an edge not adjacent to v and not in $G[A]$, then as above it is without loss of generality $w \in C \setminus D$ and therefore $e \in G[C]$. If $w = v$ say, then $e \in G[\{v\} \cup A \cap B]$ as $N(v) \subseteq A \cap B$. But this contradicts property (1) of $\mathcal{T}$.

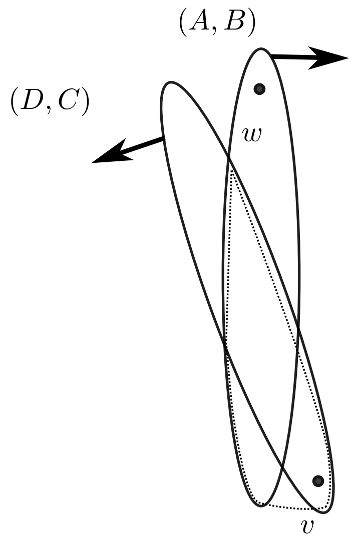

Figure 8: If $v \in C \cap D$, the dotted line gives us a small separation.

So the only possibility is $N(v) \not\subseteq A \cap B$ and therefore, as $v \in B \setminus A$, it is $N(v) \cap C \setminus D \neq \emptyset$, so $v \in C$ (see Figure 8). As G is k-connected and $(B \setminus A) \setminus \{v\} \subseteq C \setminus D$, there is at least one vertex $w \in A \cap B \cap C$, as $\{v\} \cup (C \cap A \cap B)$ separates $B \setminus (A \cup \{v\})$ from the rest of the graph. So $(\{v\} \cup (A \cap B) \setminus \{w\}, V)$ is a small separation and therefore in $\mathcal{T}$. But as above $G[A] \cup G[C] \cup G[\{v\} \cup (A \cap B) \setminus \{w\}] = G$ contradicting the definiton of a tangle.

Thus there are at least two vertices in $(B \setminus A) \cap D$. $\square$

Thus, if we can pick the whole set $B \setminus A$ as our set X_1, we will indeed determine all separations in $\bar{M}$. This is the case if there is at least one separation (A, B), for which there are sufficently many deep vertices, namely at least $|B \setminus A| - 1$:

Lemma 7.3. *Let G be a graph, k an integer and $\mathcal{T}$ a tangle of order $(k+1)$ in G. Let $\bar{M}$ be defined as above and let $(A, B) \in \bar{M}$. Let $V_D := \{v \in A \setminus B | v \text{ deep w.r.t. } (A, B)\}$ be the set of all deep vertices with respect to (A, B). If $|V_D| \geq |B \setminus A| - 1$, there is a set X determining all separations in $\mathcal{T}$.*

Proof. Let $X_2 := B \setminus A$ and let $X_1 \subseteq V_D$ be an arbitrary subset of the set of deep vertices such that $|X_1| = |B \setminus A| - 1$. We show that $X' := X_1 \sqcup X_2$ determines every separation in $\bar{M}$: X' determines (A, B) as $X' \cap A = X_1$ and $X \cap B = X_2$ and by construction $|X_1| = |X_2| - 1$. For $(C, D) \neq (A, B) \in \bar{M}$ it is, by the definition of deep vertices, $X_1 \subseteq D \setminus C$. By Lemma 7.2, there are at least two vertices in $X_2 \setminus (C \setminus D)$. Thus $|X' \cap (C \setminus D)| \leq |X_2| - 2$. Therefore

$$\left|X' \cap (C \setminus D)\right| \leq |X_2| - 2 < |X_1| \leq \left|X' \cap (D \setminus C)\right|$$

so X' determines (C, D). As by Lemma 5.7 the set $A \cap B$ determines every separation $(A, B) \neq (C, D) \in M$, the set $X := X' \cup A \cap B$ also determines every separation in $\bar{M}$. Additionally it is $|X'| \geq k$: Otherwise it is $|X_2| \leq \frac{k}{2}$, thus, by Lemma 2.1, the separation (B, A) would lie in $\mathcal{T}$ contradicting the definition of a tangle. So

$$|X| = \left|X'\right| + |A \cap B| \geq 2k.$$

Therefore, X determines every semi-small separation as well. So every separation in M is determined by X, by Lemma 5.1 the set X therefore determines every separation in $\mathcal{T}$. □

So, from now on we may assume that there is no such separation. If there is additionally only one deep separation, we have no chance to use our approach $X := X_1 \cup X_2$, as it may be the case that for any subset X_2 of $B \setminus A$ having the correct size, there is one separation $(C, D) \in M$ such that $X_2 \subseteq C \setminus D$. Thus in this case we have to use the same approach as in the case of no deep separations at all, namely that if V is large enough, it will then automaticly determine $\mathcal{T}$:

Lemma 7.4. *If a $(k+1)$-tangle $\mathcal{T}$ in a k-connected graph G with more than $4k$ vertices contains exactly one deep separation (A, B), there is a set $X \subseteq V$ that determines every separation in $\mathcal{T}$.*

Proof. If there are at least $|B \setminus A| - 1$ many deep vertices with respect to (A, B), the assertion is true by Lemma 7.3. Otherwise by Lemma 6.2 it is $|A \setminus B| - k < |B \setminus A|$, so there are more than k vertices in $B \setminus A$: As G contains more than $4k$ vertices it is $|A \setminus B| + |A \cap B| + |B \setminus A| = |V| > 4k$ and therefore $2\,|B \setminus A| > 2k$. Thus, we can pick a set X of order at least $3k + 1$ that determines (A, B): For example let X consist of

$A \cap B$ and $B \setminus A$ together with a subset of $A \setminus B$ of order $|B \setminus A| - 1$. Then X determines every separation in $\mathcal{T}$: By Lemma 5.1, we only need to check the separations in M, every separation except of (A, B) in M is either semi-small or flat, thus for any such separation (C, D) it is $|C \setminus D| \leq k$ and therefore $|X \cap C| < |X \cap D|$. Thus X indeed determines every separation in $\mathcal{T}$. □

If we have at least two deep separations, we can return to the strategy mentioned above. Let (A, B) and (C, D) be tow such deep separations. We know by Lemma 7.2, that we can pick X_2 such that X also determines (C, D). Additionally, we know that there is a vertex that is deep with respect to (C, D). By making sure that this vertex also lies in X_2, we then know that all other separations in $\bar{M}$ are determined as well: There are more than $\frac{|X|}{2}$ vertices in X that are deep with respect to (A, B) or (C, D), thus all other separations in $\bar{M}$ are then determined automaticly:

Lemma 7.5. *If a $(k+1)$-tangle $\mathcal{T}$ in a k-connected graph G with more than $4k$ vertices contains at least two deep separations and k is at least 3, there is a set $X \subseteq V$ that determines every separation in $\mathcal{T}$.*

Proof. If there is a deep separation (A, B) such that

$$|\{v \in V | v \text{ deep w.r.t. } (A, B)\}| \geq |B \setminus A|,$$

the assertion would be true by Lemma 7.3, so we may assume that this is not the case. Additionally we may assume that there is a deep separation (A, B) such that $|B \setminus A| \leq |A \setminus B|$: If this is not the case, the set $X := V$ would determine every separation in $\mathcal{T}$. Every deep separation would then be determined by assumption and given a separation (C, D) in M that is not deep, we know by Lemma 6.2 that $|C \setminus D| \leq k$, and thus, as $|V| > 4k$, it is $|D \setminus C| > 2k$. So by Lemma 5.1, X would determine every separation in $\mathcal{T}$. Thus we may assume that there is a deep separation (A_1, B_1) such that $|B_1 \setminus A_1| \leq |A_1 \setminus B_1|$. Let (A_2, B_2) be another deep separation.

Let V_D be the set of deep vertices with respect to (A_1, B_1). As $|V| > 4k$, by Lemma 6.2 together with the two unequations shown above there are more than $\frac{k}{2}$ vertices in V_D:

$$\begin{aligned} &4k < |V| = |A_1 \setminus B_1| + |A_1 \cap B_1| + |B_1 \setminus A_1| \leq k + 2|A_1 \setminus B_1| \\ \Rightarrow &3k < 2|A_1 \setminus B_1| \leq 2(k + |V_D|) \\ \Rightarrow &\frac{k}{2} < |V_D| \end{aligned}$$

By Lemma 7.2 there are at least two vertices in $B_1 \setminus A_1$ that are not in $A_2 \setminus B_2$. Let $v \neq w$ be two such vertices. Additionally, as (A_2, B_2) is a deep separation, there is a vertex x that is deep with respect to (A_2, B_2). By the definiton of deep, this vertex lies in $B_1 \setminus A_1$ as well and is different from v and w. As k is at least 3, there are at least 2 vertices in V_D. As $|B_1 \setminus A_1| > |V_D| + 1$, we can pick an arbitrary subset $X_2 \subseteq B_1 \setminus (A_1 \cup \{v, w, x\})$

of size $|V_D| - 2$. We claim that the set $X' := V_D \cup X_2 \cup \{v, w, x\}$ determines every separation in $\bar{M}$.

By construction, X' determines (A_1, B_1) as $|X' \cap (A_1 \setminus B_1)| = |X' \cap (B_1 \setminus A_1)| - 1$. The separation (A_2, B_2) is determined by X' as V_D, by the definition of deep, is a subset of $B_2 \setminus A_2$ and $v, w \notin A_2 \setminus B_2$. Thus

$$\big|X' \cap (A_2 \setminus B_2)\big| \leq |X_2 \cup \{x\}| = |V_D| - 1 \leq \big|X' \cap (B_2 \setminus A_2)\big| - 1$$

Last, all other separations in $\bar{M}$ are determined by X' as, given such a separation (C, D), it is $V_D \cup \{x\} \subseteq D \setminus C$ by the definition of deep. As $|V_D \cup \{x\}| > \frac{|X|}{2}$, the set X indeed determines (C, D). Thus, as $|X'| \geq k$ and $X' \cap (A_1 \cap B_1) = \emptyset$, as in the proof of Lemma 7.3, the set $X := X' \cup (A \cap B)$ determines every separation in $\mathcal{T}$. □

Thus, putting all this cases together we can now proof our main theorem:

Proof of Theorem 1.3. If k is at most 2, the assertion is true by Theorem 1.2. If G contains no deep separations, the assertion is true by Lemma 7.1. If G contains exactly one deep separation, the assertion is true by Lemma 7.4. If k is at least 3 and G contains at least two deep separations, the assertion is true by Lemma 7.5. □

So, for which graphs G and tangles $\mathcal{T}$ of order k do we know that they have a majority set?

First all tangles of order at most 3 have a majority set by Theorem 1.2. Then, all k-tangles in $(k-1)$-connected graphs have a majority set by Theorem 1.3. Also, if a k-tangle in a not necessarily $(k-1)$-connected graph induces a k-tangle in a $(k-1)$-connected subgraph, there is a majority set for this tangle.

Additionally, if there are two different tangles $\mathcal{T}'$ and $\mathcal{T}''$ of the same order $l > k$ both inducing $\mathcal{T}$, there is a majority set by Lemma 5.7. We also get a majority set by 5.7 if there are two maximal, non-semi-small separations with disjoint interior in $\mathcal{T}$.

And, if the tangle $\mathcal{T}$ is induced by a block of G, the set of vertices of this block is a majority set.

References

[1] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.

[2] R. Diestel and S. Oum. Unifying duality theorems for width parameters, I. Weak and strong duality. arXiv:1406.3797v4, March 2015.

[3] J. Carmesin, R. Diestel, F. Hundertmark and M. Stein. Connectivity and tree structure in finite graphs. *Combinatorica*, 34(1):1–35, 2014.

[4] N. Robertson and P.D. Seymour. Graph minors. X. Obstructions to tree- decomposition. *J. Combin. Theory (Series B)*, 52:153–190, 1991.

[5] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. Canonical tree- decompositions of finite graphs I. Existence and algorithms. *J. Combin. Theory Ser. B*, 116:1-24, 2016

[6] R. Diestel, J. Erde and Ph. Eberenz. Duality theorems for blocks and tangles in graphs, *SIAM Journal on Discrete Mathematics*, 31(3):1514-1528, 2017

[7] J. Erde. Refining a Tree-Decomposition which Distinguishes Tangles. arXiv:1512.02499v2, March 2017.

[8] W. T. Tutte. *Graph Theory.* Addison-Wesley, 1984.