Masterarbeit im Studiengang Wirtschaftsmathematik

Characteristics of profiles

vorgelegt am: 09.06.2015

am Fachbereich Mathematik der Universität Hamburg

Name:	Philipp Eberenz
Matrikelnummer:	6232853
Erstgutachter:	Prof. Dr. Reinhard Diestel
Zweitgutachter:	Dr. Matthias Hamann

Contents

1 Introduction

Definition 1. *The unordered pair* (A, B) *is a separation of a graph* $G = (V, E)$ *if* $A \cup B = V$ *and* G *has no edge between* $A \backslash B$ *and* $B \backslash A$*. Clearly the latter is equivalent to saying that* $A \cap B$ *separates* A *from* B*. If both* $A \backslash B$ *and* $B \backslash A$ *are non-empty, the separation is proper. The number* $|A \cap B|$ *is the order of the separation* (A, B)*. Informally we think of* (A, B) *pointing towards* B *and away from* A *and call* B *the big side of* (A, B)*. We call* (B, A) *the inverse of the separation* (A, B)*.*

Definition 2. *Let* $(A, B), (C, D)$ *be two separations.*

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D$$

Note that this is equivalent to $(D, C) \leq (B, A)$*. If* $(A, B) \leq (C, D)$*, then we think of* (A, B) *pointing towards* (C, D) *and* (D, C)*, while* (C, D) *points away from* (A, B) *and* (B, A)*.*

Definition 3. *A set* P *of separations is consistent if it contains no two separations pointing away from each other: if* $(C, D) \leq (A, B) \in P$ *implies* $(D, C) \notin P$*. Note that this does not imply* $(C, D) \in P$*. It may also happen that* P *contains neither* (C, D) *nor* (D, C)*.*

Definition 4. *A Set* P *of separations of a Graph* G *is a* $profile$ *if it satisfies:*

- P *is consistent*
- *for all* $(A, B), (C, D) \in P : (B \cap D, A \cup C) \notin P$

Definition 5. *Let* G *be a graph. A profile* P *of* G *is a* k*-profile if all separations in* P *have order less than* k *and if for every separation* (A, B) *of* G *of order less than* k *either* $(A, B) \in P$ *or* $(B, A) \in P$*.*

In this master´s thesis we will deal with some characteristics of profiles. In the first section we will introduce a more 'abstract' notion of separations. In the following section we give another definition of profiles and we will show some general properties of profiles. The advantage of this new definition of profiles is, that it can be made symmetric. By doing so we get a weakened notion of profiles. In section six we will study some aspects of these weakened profiles. Some profiles, e.g. k-profiles, can be thought of as indicating highly connected parts of a graph. "There are a number of theorems in the structure theory of sparse graphs that assert a duality between high connectivity present somewhere in the graph and an overall tree strucuture [1,p.1]". Diestel and Oum proved in [1] a general duality theorem for width parameters in combinatorical structures such as graphs. Our aim in the fourth and fifth section of this master´s thesis is to proof that this general duality theorem can be applied to k-profiles. Diestel and Oum are working in [1] with a slightly weaker definition of consistent. We will cover the differences between their definition consistent and the definition of consistent in this paper in section seven. In this master´s thesis we will work with finite graphs and sets.

2 Abstract separations

Diestel and Oum are working in [1] with a more 'abstract' axiomatic definition of separations. As we will see separations of a graph are separations in terms of these 'abstract' separations. As we will work with these 'abstract' separations in most parts of this thesis, we will cover the definition of these separation in this section.

Definition 6. *A separation system* $(\vec{S}, \leq, *)$ *is a partially ordered set* $\vec{S}$ *with an order reversing involution* $*$. *Its elements are called oriented separations. When a given element of* $\vec{S}$ *is denoted as* $\vec{s}$, *its inverse* $\vec{s}^*$ *will be denoted as* $\overleftarrow{s}$ *and vice versa. The assumption that* $*$ *be order-reversing means that, for all* $\vec{r}, \vec{s} \in \vec{S}$,

$$\vec{r} \leq \vec{s} \Leftrightarrow \overleftarrow{r} \geq \overleftarrow{s}$$

Definition 7. *A separation is a set of the form* $\{\vec{s}, \overleftarrow{s}\}$, *and then denoted by* s. *We call* $\vec{s}$ *and* $\overleftarrow{s}$ *the orientations of* s. *The set of all such sets* $\{\vec{s}, \overleftarrow{s}\} \subseteq \vec{S}$ *will be denoted by* S. *If* $\vec{s} = \overleftarrow{s}$, *we call both* $\vec{s}$ *and* s *degenerate.*

As easy to be seen the separations defined in chapter one are separations in terms of the definition above.

Definition 8. *If there are binary operations* $\wedge$ *and* $\vee$ *on a separation system* $\vec{S}$, *such that* $\vec{r} \wedge \vec{s}$ *is the infimum and* $\vec{r} \vee \vec{s}$ *the supremum of* $\vec{r}, \vec{s}$ *in* $\vec{S}$, *we call* $(\vec{S}, \leq, *, \wedge, \vee)$ *a universe of (oriented) separations.*

As $*$ is order-reversing it satisfies De Morgan´s law [cf. 1, p.4]:

$$(\vec{r} \vee \vec{s})^* = (\overleftarrow{r} \wedge \overleftarrow{s})$$

"The oriented separations of a set V form such a universe: if $\vec{r} = (A, B)$ and $\vec{s} = (C, D)$, say, then $\vec{r} \vee \vec{s} := (A \cup C, B \cap D)$ and $\vec{r} \wedge \vec{s} := (A \cap C, B \cup D)$ are again oriented separations of V, and are the supremum and infimum of $\vec{r}$ and $\vec{s}$. Similarly, the oriented separations of a graph form a universe. Its oriented separations of order $< k$ for some fixed integer k, however, form a separation system inside this universe that may not itself be a universe with respect to $\vee$ and $\wedge$ as defined above." [1,p.4]

Definition 9. *A set* $O \subseteq \vec{S}$ *is consistent if there are no* $r, s \in S$ *with orientations* $\vec{r} < \vec{s}$ *such that* $\overleftarrow{r}, \vec{s} \in O$.[1]

Definition 10. *An orientation of a set* S *of separations is a set* $O \subseteq \vec{S}$ *that contains for every* $s \in S$ *exactly one of its orientations* $\vec{s}, \overleftarrow{s}$.

[1] This definition is slightly stronger than the definition of consistent in [1, p.5]. We will cover the differences of these two definitions and their impact on the proofs in Section seven. For now it is sufficient to know that every set of separations that is consistent by the **Definition 9** is consistent by the definition of consistent in [1, p.5].

Definition 11. *A nonempty set σ of oriented separations is a star of separations if they point towards each other: if $\vec{r} \leq \overleftarrow{s}$ for all distinct $\vec{r}, \vec{s} \in \sigma$.*

Lemma 1. *For every star σ that is not consistent there exists a separation r such that $\{\vec{r}, \overleftarrow{r}\} \subseteq \sigma$.*

Proof. Let σ be a non-consistent star. As σ is non-consistent there exists oriented separations $\vec{r}, \vec{s} \in \sigma$ such that $\overleftarrow{r} \leq \vec{s}$. Given that σ is a star it is also $\vec{s} \leq \overleftarrow{r}$. Thus it is $\vec{s} = \overleftarrow{r}$. □

Corollar 1. *Let O be an orientation of a set of separation. Then O cannot contain a non-consistent star.*

Proof. □

With the new characterisation of separations, which is used in [1], we have to adjust the definition of a k-profile.

Definition 12. *Let us call a real function $\vec{s} \mapsto |\vec{s}|$ on a universe $(\vec{U}, \leq, *, \vee, \wedge)$ of oriented separations an order function if it is non-negative, symmetric and submodular, that is, if $0 \leq |\vec{s}| = |\overleftarrow{s}|$ and*

$$|\vec{r} \vee \vec{s}| + |\vec{r} \wedge \vec{s}| \leq |\vec{r}| + |\vec{s}|$$

for all $\vec{r}, \vec{s} \in \vec{U}$ We then call $|s| := |\vec{s}|$ the order of s and of $\vec{s}$. For every positive integer k,

$$\vec{S_k} := \{\vec{s} \in \vec{U} : |\vec{s}| < k\}$$

is a separation system (though not necessarily a universe).

Definition 13. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$. A set $P \subseteq \vec{S}$ is a profile if it satisfies:*

- *P is consistent*
- $\forall \vec{r}, \vec{s} \in P : (\overleftarrow{r} \wedge \overleftarrow{s}) \notin P$ $\qquad (P)$

Definition 14. *A profile P is an S-profile if for all separations $\vec{r} \in P$ also $r \in S$ and if for every separation $s \in S$ either $\vec{s} \in P$ or $\overleftarrow{s} \in P$.*

Definition 15. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$ We say $\vec{S}$ is submodular if and only if for every $\vec{r}, \vec{s} \in \vec{S}$ either $(\vec{r} \vee \vec{s}) \in \vec{S}$ or $(\vec{r} \wedge \vec{s}) \in \vec{S}$.*

Lemma 2. *Every S_k is submodular.*

Proof. This follows directly of the submodularity of the order function of S_k. $\square$

Lemma 3. *Let S be a submodular set of separations and P be an S-profile. Then P is a consistent orientation of S.*

Proof. Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$. And let $P \subseteq \vec{S}$ be a profile. We have to show that P is consistent (by definition) and that P contains for every $s \in S$ exactly one of $\vec{s}$ and $\overleftarrow{s}$. As P is an S-profile P contains at least one of $\vec{s}$ and $\overleftarrow{s}$ for every $s \in S$. Let $\vec{s} \in P$. By (P) we know that $\overleftarrow{s} \wedge \overleftarrow{s} = \overleftarrow{s} \notin (P)$. Thus P cannot contain both $\vec{s}$ and $\overleftarrow{s}$. $\square$

3 Another definition of profiles

In this section we will give a definition of profiles, which can be made symmetric.

Definition 16. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$. A set of oriented separations $P \subseteq \vec{S}$ satisfies (P3) if and only if:*

$$\forall \vec{r}, \vec{s}, \in P \quad \forall \vec{t} \leq (\vec{r} \vee \vec{s}) : \overleftarrow{t} \notin P$$

Definition 17. *We call a universe $(\vec{U}, \leq, *, \wedge, \vee)$ of separations distributive if for every $\vec{r}, \vec{s}, \vec{t} \in \vec{U}$ the following holds:*

- $(\vec{r} \wedge \vec{s}) \vee \vec{t} = (\vec{r} \vee \vec{t}) \wedge (\vec{s} \vee \vec{t})$
- $(\vec{r} \vee \vec{s}) \wedge \vec{t} = (\vec{r} \wedge \vec{t}) \vee (\vec{s} \wedge \vec{t})$

We call it associative if for every $\vec{r}, \vec{s}, \vec{t} \in \vec{U}$ the following holds:

- $(\vec{r} \wedge \vec{s}) \wedge \vec{t} = \vec{r} \wedge (\vec{s} \wedge \vec{t})$
- $(\vec{r} \vee \vec{s}) \vee \vec{t} = \vec{r} \vee (\vec{s} \vee \vec{t})$

Theorem 1. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a distributive and associative universe of separations containing a submodular separation system $(\vec{S}, \leq, *)$. Let P be an orientation of $\vec{S}$. P satisfies (P3) if and only if P is consistent and satisfies (P).*

Proof. " $\Rightarrow$ "
Let P be an orientation of a submodular separation system satisfying (P3). Let $\vec{r}, \vec{s} \in P$, such that $\vec{r} \leq \vec{s}$. Then $\overleftarrow{r} \notin P$, as $\vec{r} \leq (\vec{s} \vee \vec{s}) = \vec{s}$. Thus P is consistent. Let $\vec{r}, \vec{s} \in P$ and $\overleftarrow{t} = (\overleftarrow{r} \wedge \overleftarrow{s})$ Then $\overleftarrow{t} \notin P$ as $\vec{t} = (\vec{r} \vee \vec{s})$.

We are now proving $\Leftarrow$.
Let P be an orientation of a submodular separation system $\vec{S}$. We assume that P is consistent and satisfies (P) but not $(P3)$.
Then there exist $\vec{r}, \vec{s}, \overleftarrow{t} \in P$ such that $\vec{t} \leq (\vec{r} \vee \vec{s})$. If $(\vec{t} \wedge \vec{r}), (\vec{t} \wedge \vec{s}) \in P$ (P) is violated, as
$(\vec{t} \wedge \vec{r})^* \wedge (\vec{t} \wedge \vec{s})^* = (\overleftarrow{t} \vee \overleftarrow{r}) \wedge (\overleftarrow{t} \vee \overleftarrow{s}) = \overleftarrow{t} \vee (\overleftarrow{r} \wedge \overleftarrow{s}) = \overleftarrow{t}$ given that $\vec{S}$ is distributive and $\overleftarrow{t} \geq (\overleftarrow{r} \wedge \overleftarrow{s})$. We are showing now that $(\vec{t} \wedge \vec{r}), (\vec{t} \wedge \vec{s}) \in P$. Let us assume that $(\vec{t} \wedge \vec{r}) \notin P$. Then either $(\vec{t} \wedge \vec{r})* = (\overleftarrow{t} \vee \overleftarrow{r}) \in P$ or $(\vec{t} \wedge \vec{r}) \notin \vec{S}$. As $\overleftarrow{r} \leq (\overleftarrow{t} \vee \overleftarrow{r})$ and $\vec{r} \in P$, it is $(\vec{t} \wedge \vec{r}) \notin \vec{S}$ by the consistency of P. This implies due to the submodularity of $\vec{S}$ that $(\vec{t} \vee \vec{r}) \in \vec{S}$. Given that $\vec{t} \leq (\vec{t} \vee \vec{r})$ and $\overleftarrow{t} \in P$, it is $(\vec{t} \vee \vec{r})^* = (\overleftarrow{t} \wedge \overleftarrow{r}) \in P$ by the consistency of P. Again due to the submodularity of $\vec{S}$ one of the following holds:

1. $(\overleftarrow{t} \wedge \overleftarrow{r}) \vee \overleftarrow{s} \in \vec{S}$

2. $(\overleftarrow{t}\wedge\overleftarrow{r})\wedge\overleftarrow{s}\in\overrightarrow{S}$

If 1. holds, then $((\overleftarrow{t}\wedge\overleftarrow{r})\vee\overleftarrow{s})^* = ((\overrightarrow{t}\vee\overrightarrow{r})\wedge\overrightarrow{s})\in P$, as $\overleftarrow{s}\le((\overleftarrow{t}\wedge\overleftarrow{r})\vee\overleftarrow{s})$. Thus $((\overleftarrow{t}\wedge\overleftarrow{r})\vee\overleftarrow{s})\wedge\overleftarrow{r}\notin P$, due to (P). But $((\overleftarrow{t}\wedge\overleftarrow{r})\vee\overleftarrow{s})\wedge\overleftarrow{r} = ((\overleftarrow{t}\wedge\overleftarrow{r})\wedge\overleftarrow{r})\vee(\overleftarrow{s}\wedge\overleftarrow{r}) = (\overleftarrow{t}\wedge\overleftarrow{r})\vee(\overleftarrow{s}\wedge\overleftarrow{r}) = (\overleftarrow{t}\wedge\overleftarrow{r})$. This follows from the fact that $(\overrightarrow{U},\le,*,\wedge,\vee)$ is distributive and that $(\overleftarrow{s}\wedge\overleftarrow{r})\le\overleftarrow{t}$ as well as $(\overleftarrow{s}\wedge\overleftarrow{r})\le\overleftarrow{r}$, hence $(\overleftarrow{s}\wedge\overleftarrow{r})\le(\overleftarrow{t}\wedge\overleftarrow{r})$. This is a contradiction to $(\overleftarrow{t}\wedge\overleftarrow{r})\in P$.

If 2 holds, then $(\overleftarrow{t}\wedge\overleftarrow{r})\wedge\overleftarrow{s}\in P$ as $\overrightarrow{t}\le((\overrightarrow{t}\vee\overrightarrow{r})\vee\overrightarrow{s}) = ((\overleftarrow{t}\wedge\overleftarrow{r})\wedge\overleftarrow{s})^*$. Due to (P) it is $(\overleftarrow{r}\wedge\overleftarrow{s})\notin P$. But $((\overleftarrow{t}\wedge\overleftarrow{r})\wedge\overleftarrow{s}) = (\overleftarrow{t}\wedge(\overleftarrow{r}\wedge\overleftarrow{s})) = (\overleftarrow{r}\wedge\overleftarrow{s})$ as $\overleftarrow{t}\ge(\overleftarrow{r}\wedge\overleftarrow{s})$. This is a contradiction. A similiar argument shows that $(\overrightarrow{t}\wedge\overrightarrow{s})\in P$, hence (P) is violated. □

By **Theorem 1** and **Lemma 3** a k-profile of a graph G is equivalent to an $\mathcal{F}$-avoiding orientation of S_k for the set $\mathcal{F}:=\{\{\overrightarrow{r},\overrightarrow{s},\overleftarrow{t}\}\subseteq 2^{S_k}:\overrightarrow{t}\le(\overrightarrow{r}\wedge\overrightarrow{s})\}$ as S_k is submodular and $\cap,\cup$ are distributive and associative. If $(\overrightarrow{U},\le,*,\wedge,\vee)$ is not distributive, **Theorem 1** is not true as the following example shows:

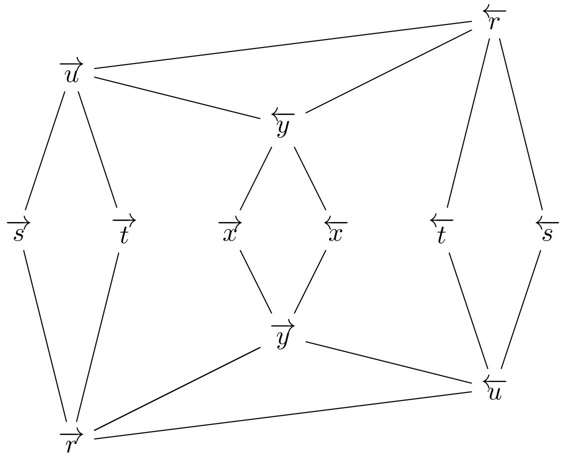

Figure 1: A universe of separations shown as a Hasse-diagramm

Theorem 2. *Let $(\overrightarrow{U}, \leq, *, \wedge, \vee)$ be the universe in figure 1 containing the separation system $\overrightarrow{S} = \overrightarrow{U} \backslash u$. Then $\overrightarrow{S}$ is a submodular separation system that has a consistent orientation O, which satisfies (P) but violates $(P3)$.*

Proof. It is $(\overleftarrow{x} \vee \overrightarrow{s}) \wedge \overrightarrow{t} = (\overrightarrow{u} \wedge \overrightarrow{t}) = \overrightarrow{t} \neq \overrightarrow{r} = (\overleftarrow{x} \wedge \overrightarrow{t}) \vee (\overrightarrow{s} \wedge \overrightarrow{t})$, thus $\overrightarrow{U}$ is not distributive.
As easy to be seen $\overrightarrow{S}$ has an order-reversing involution, hence $\overrightarrow{S}$ is a separation system. $\overrightarrow{S}$ is submodular, given that there exist no $\overrightarrow{a}, \overrightarrow{b} \in \overrightarrow{S}$ with $(\overrightarrow{a} \wedge \overrightarrow{b}), (\overrightarrow{a} \vee \overrightarrow{b}) \notin \overrightarrow{S}$.
Suppose it does, then $(\overrightarrow{a} \wedge \overrightarrow{b}) \in \{\overrightarrow{u}, \overleftarrow{u}\}$ as well as $(\overrightarrow{a} \vee \overrightarrow{b}) \in \{\overrightarrow{u}, \overleftarrow{u}\}$, given that $\overrightarrow{U}$ is a universe and the only separation of U that is not contained in S is u. It is $(\overrightarrow{a} \vee \overrightarrow{b}) \neq (\overrightarrow{a} \wedge \overrightarrow{b})$, otherwise both $\overrightarrow{a}$ and $\overrightarrow{b}$ would be an orientation of u, given that $(\overrightarrow{a} \wedge \overrightarrow{b}) \leq \overrightarrow{a} \leq (\overrightarrow{a} \vee \overrightarrow{b})$ The same is true for $\overrightarrow{b}$. This cannot be as $u \notin S$. Thus either $\overrightarrow{u} \leq \overrightarrow{a} \leq \overleftarrow{u}$ or $\overleftarrow{u} \leq \overrightarrow{a} \leq \overrightarrow{u}$. The same is true for $\overrightarrow{b}$. Therefore $a \in \{x, y\}$ and $b \in \{x, y\}$. But for all combination of orientations of the separations x and y even both the supremum and the infimum are again in S.
Let $O := \{\overrightarrow{r}, \overrightarrow{s}, \overrightarrow{t}, \overrightarrow{y}, \overleftarrow{x}\}$. Then O is an orientation of $\overrightarrow{S}$. Further O is consistent as for every $\overrightarrow{a} \in \overrightarrow{S}$, such that $\overrightarrow{a} \leq \overrightarrow{b} \in O$, $\overrightarrow{a} \in O$. Suppose O violates $(P2)$, hence there exist $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in O$ such that $\overrightarrow{c} = (\overleftarrow{a} \wedge \overleftarrow{b})$. Whenever $\overrightarrow{a}, \overrightarrow{b} \in O \backslash \overleftarrow{x}$ then $(\overleftarrow{a} \wedge \overleftarrow{b}) \in \{\overleftarrow{u}, \overleftarrow{a}, \overleftarrow{b}\}$. Thus one of $\overrightarrow{a}, \overrightarrow{b}$ has to be $\overleftarrow{x}$. In this case $(\overleftarrow{a} \wedge \overleftarrow{b}) \in \{\overrightarrow{x}, \overleftarrow{u}\}$, but neither $\overleftarrow{x} \in O$ nor $\overrightarrow{u} \in O$. Further O violates (P3), as $\{\overleftarrow{x}, \overrightarrow{s}, \overrightarrow{t}\} \subseteq O$, and $\overrightarrow{x} \leq (\overrightarrow{s} \vee \overrightarrow{t})$. □

3.1 Interesting properties of profiles

In this section we will cover some interestings properties of profiles.

Theorem 3. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a submodular separation system $(\vec{S}, \leq, *)$. Let O be an orientation of $\vec{S}$ with the following properties:*

- *O is consistent.*
- *Every star $\Sigma \subseteq O$ is subset of an $\vec{S}$-profile.*

Then O is an $\vec{S}$-profile.

Proof. Suppose that O is not an $\vec{S}$-profile. Then there exist $\vec{r}, \vec{s} \in O$, such that $(\overleftarrow{r} \wedge \overleftarrow{s}) \in O$. Due to the submodularity of $\vec{S}$ either $(\overleftarrow{r} \wedge \vec{s}) \in \vec{S}$ or $(\overleftarrow{r} \vee \vec{s}) \in \vec{S}$. If $(\overleftarrow{r} \wedge \vec{s}) \in \vec{S}$ then $(\overleftarrow{r} \wedge \vec{s}) \in O$ as O is consistent and $(\overleftarrow{r} \wedge \vec{s}) \leq \vec{s}$. But now $(\overleftarrow{r} \wedge \vec{s}), (\overleftarrow{r} \wedge \overleftarrow{s})$ and $\vec{r}$ form a star. This star has to be a subset of an $\vec{S}$-profile. This $\vec{S}$-profile P also contains either $\vec{s}$ or $\overleftarrow{s}$. In both cases P violates (P), hence P is not a profile. Therefore O is an $\vec{S}$-profile. In the case that $(\overleftarrow{r} \vee \vec{s}) \in \vec{S}$ the orientation O has to contain $(\overleftarrow{r} \vee \vec{s})^* = (\vec{r} \wedge \overleftarrow{s})$ as $(\vec{r} \wedge \overleftarrow{s}) \leq \vec{r}$ due to consistency of O. But now $(\vec{r} \wedge \overleftarrow{s}), (\overleftarrow{r} \wedge \overleftarrow{s}), \vec{s}$ form a star. This star has to be a subset of an $\vec{S}$-profile. This $\vec{S}$-profile P also contains either $\vec{r}$ or $\overleftarrow{r}$. In both cases P violates (P), hence P is not a profile. Therefore O is a $\vec{S}$-profile. □

For k-blocks we get a similar result.

Definition 18. *Given $k \in \mathbb{N}$, a set I of at least k vertices of a graph G is $(< k)$-inseparable if no set W of fewer than k vertices of G separates any two vertices of $I \backslash W$ in G. A maximal $(< k)$-inseparable set is a k-block.*

Definition 19. *A set of separations is nested if each of them is comparable with every other or its inverse. Thus, two nested separations are either comparable, or point towards each other, or point away from each other. Two separations that are not nested are said to cross.*

Definition 20. *Let S, P be two sets of separation. We say $S \leq P$, if a function $f : S \mapsto P$ exists, such that $(A, B) \leq f(A, B)$ for every $(A, B) \in S$.*

Definition 21. *Let us say a separation (A, B) lies on the small side of an oriented separation (C, D) if there exists an orientation (A, B) of this separation such that $(A, B) \leq (C, D) \Leftrightarrow (D, C) \leq (B, A)$.*
Let us say a separation (A, B) lies on the big side of an oriented separation (C, D) if there exists an orientation (A, B) of this separation such that $(C, D) \leq (A, B) \Leftrightarrow (B, A) \leq (D, C)$

Definition 22. *A set of S oriented separations points towards a k-block if the k-block is contained in the big side of every separation $(A, B) \in S$*

Theorem 4. *Let G be a Graph and S a nested set of separations of G. Let O be an orientation of S with the following properties:*

- *O is consistent.*
- *Every star $\Sigma \subseteq O$ points towards a k-block.*

Then every subset $S \subseteq O$ points towards a k-block.

Proof. Let $\Sigma \subseteq O$ be a maximal star (with respect to **Definition 20**). This star Σ points towards a k-block R. We now show that every other separation $(A, B) \in O$ points towards this k-block as well. Given that S is nested every separation in S is lying either on the small side of a separation of Σ or on the big side of all separations of Σ. Let $(A, B) \in O \backslash \Sigma$ be a separation, which lies on the small side of a separation $(C, D) \in \Sigma$. Then it is $(A, B) \leq (C, D)$, because otherwise O would not be consistent. Thus (A, B) points to the k-block R as well. Let $(A, B) \in O \backslash \Sigma$ be a separation, which lies on the big side of all separations $(C, D) \in \Sigma$. As Σ is a maximal star, $\Sigma \cup (A, B)$ is not a star. Hence for at least one of the separations $(C, D) \in \Sigma$ holds $(C, D) < (A, B)$. Let $P := \{(C, D) \in \Sigma : (C, D) < (A, B)\}$. Then $(A, B) \cup \Sigma \backslash P$ is a bigger star than Σ, as $P < (A, B)$, hence $\Sigma < (A, B) \cup \Sigma \backslash P$. This is a contradiction to the maximality of Σ. Therefore no separation $(A, B) \in O$ lies on the big side of all separations $(C, D) \in \Sigma$. □

4 Terminology for the general duality theorem in [1]

Diestel and Oum proved in [1] the following duality theorem:

Theorem 5. [*cf*.1, *Theorem* 4.4] *Let* $(\vec{U}, \leq, *, \wedge, \vee)$ *be a universe of separations containing a separation system* $(\vec{S}, \leq, *)$. *Let* $\mathcal{F} \subseteq 2^{\vec{S}}$ *be a standard set of stars. If* $\vec{S}$ *is* $\mathcal{F}$*-separable, exactly one of the following assertions holds:*

- *There exists an S-tree over* $\mathcal{F}$.
- *There exists an* $\mathcal{F}$*-tangle of S.*

To understand this theorem we need some more definitions, which we will give in the following section.

4.1 Further definitions

Definition 23. *Given a separation system* $(\vec{S}, \leq, *)$ *and a set* $\mathcal{F} \subseteq 2^{\vec{S}}$, *let us call an orientation O of S an* $\mathcal{F}$*-tangle if it is consistent and avoids* $\mathcal{F}$, *that is* $2^O \cap \mathcal{F} = \emptyset$.

Definition 24. *A separation* $\vec{r} \in \vec{S}$ *is trivial in* $\vec{S}$, *and* $\overleftarrow{r}$ *is co-trivial, if there exists* $s \in S$ *such that* $\vec{r} < \vec{s}$ *as well as* $\vec{r} < \overleftarrow{s}$

Note that if $\vec{r}$ is trivial in $\vec{S}$ then so is every $\vec{r'} \leq \vec{r}$. If $\vec{r}$ is trivial, witnessed by $\vec{s}$, then $\vec{r} < \vec{s} < \overleftarrow{r}$ by De Morgan´s law. Hence if $\vec{r}$ is trivial, then $\overleftarrow{r}$ cannot be trivial. $[cf.1, p.4]$

Definition 25. *We say* $\mathcal{F}$ *forces a separation* $\vec{r} \in \vec{S}$ *if* $\{\overleftarrow{r}\} \in \mathcal{F}$ *or r is degenerate. We call* $\mathcal{F}$ *standard if it forces every trivial separation in* $\vec{S}$.

Definition 26. *Let* $\vec{r} \leq \vec{s_0}$ *be some non-trivial elements of a separation system* $(\vec{S}, \leq, *)$ *contained in some universe* $(\vec{U}, \leq, *, \wedge, \vee)$ *and* $\vec{S}_{\geq \vec{r}}$ *be the set of all separations* $s \in S$ *that have an orientation* $\vec{s} \geq \vec{r}$. *Then*

$$f\downarrow^{\vec{r}}_{\vec{s_0}}(\vec{s}) := (\vec{s} \vee \vec{s_0})$$

$$f\downarrow^{\vec{r}}_{\vec{s_0}}(\overleftarrow{s}) := (\vec{s} \vee \vec{s_0})^* = (\overleftarrow{s} \wedge \overleftarrow{s_0})$$

defines a shifting map $f\downarrow^{\vec{r}}_{\vec{s_0}} : \vec{S}_{\geq \vec{r}} \to \vec{U}$.

Definition 27. *Let us say that* $\vec{s_0} \in \vec{S}$ *is linked to* $\vec{r} \in \vec{S}$ *if* $\vec{s_0} \geq \vec{r}$ *and every* $\vec{s} \geq \vec{r}$ *in* $\vec{S}$ *satisfies* $\vec{s} \vee \vec{s_0} \in \vec{S}$. *Let us call* $\vec{S}$ *separable if for every two non-trivial* $\vec{r}, \overleftarrow{r'} \in \vec{S}$, *such that* $\vec{r} \leq \vec{r'}$ *there exists an* $s_0 \in S$ *with an orientation* $\vec{s_0}$ *linked to* $\vec{r}$ *and its inverse* $\overleftarrow{s_0}$ *linked to* $\overleftarrow{r'}$. *Let us say that a separation* $\vec{s_0} \in \vec{S}$ *is* $\mathcal{F}$*-linked to a non-trivial* $\vec{r} \in \vec{S}$ *if* $\vec{s_0}$ *is linked to* $\vec{r}$ *and the image under* $f \downarrow^{\vec{r}}_{\vec{s_0}}$ *of any star* $\sigma \subseteq \vec{S}_{\geq \vec{r}}$ *in* $\mathcal{F}$ *that has an element* $\vec{s} \geq \vec{r}$ *is again in* $\mathcal{F}$. *We say that* $\vec{S}$ *is* $\mathcal{F}$*-separable if for all non-trivial* $\vec{r}, \overleftarrow{r'} \in \vec{S}$ *such that* $\vec{r} \leq \overleftarrow{r'}$ *there exists an* $s_0 \in S$ *with an orientation* $\vec{s_0}$ *that is* $\mathcal{F}$*-linked to* $\vec{r}$ *and such that* $\overleftarrow{s_0}$ *is* $\mathcal{F}$*-linked to* $\overleftarrow{r'}$.

Definition 28. *Let* S *be a set of separations. An* $S-tree$ *is a pair* (T, α) *of a tree* T *with at least one edge and a function* $\alpha : \vec{E}(T) \to \vec{S}$ *from the set*

$$\vec{E}(T) := \{(x, y) : \{x, y\} \in E(T)\}$$

of the orientations of its edges to $\vec{S}$ *such that*

- *for every edge* xy *of* T, *if* $\alpha(x, y) = \vec{s}$ *then* $\alpha(y, x) = \overleftarrow{s}$.

It is an S*-tree over* $\mathcal{F} \subseteq 2^{\vec{S}}$ *if, in addition,*

- *for every node* t *of* T *we have* $\alpha(\vec{F_t}) \in \mathcal{F}$, *where*

$$\vec{F_t} := \{(x, t) : xt \in E(T)\}$$

5 Applying the general duality theorem from [1] to S_k-profiles

In this section we proof that we can apply **Theorem 5** to S_k-profiles. Therefore we have to show that S_k-profiles are $\mathcal{F}$-tangles, at which $\mathcal{F}$ has to be a standard set of stars. Furthermore we also have to proof, for this particular $\mathcal{F}$, that the set of all separations, which have order $< k$, is $\mathcal{F}$-separable.

Lemma 4. [1, *Lemma* 5.2] *"Every S_k, as in **Definition 12** is separable."*

Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations and $\alpha : \vec{U} \mapsto \mathbb{N}$ be an order function. Let $\vec{S_k} := \{\vec{s} \in \vec{U} : |\vec{s}| < k\}$ denote all separations in $\vec{U}$ that have an order $< k$. Let us define $\mathcal{F}^0 := \{\{\vec{r}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\overleftarrow{r} \wedge \vec{s})\} \subseteq 2^{S_k} : r, s \in S_k\}$. As easy to be seen $\mathcal{F}^0$ is a set of stars. Furthermore S_k-profiles are $\mathcal{F}^0$-tangles, as the following results show.

Theorem 6. *Any consistent $\mathcal{F}^0$-avoiding orientation O of S_k is a S_k-profile.*

Proof. Let O be a consistent F^0-avoiding orientation of S_k. Suppose that O is not an S_k-profile. Then O violates (P), as O is consistent. Thus there exist $\vec{r}, \vec{s}, \vec{t} \in O$ with $\vec{t} = (\overleftarrow{r} \wedge \overleftarrow{r})$. As S_k is submodular either $(\overleftarrow{r} \wedge \vec{s}) \in S_k$ or $(\overleftarrow{r} \vee \vec{s}) \in S_k$. If $(\overleftarrow{r} \wedge \vec{s}) \in S_k$ then due to the consistency of O it is $(\overleftarrow{r} \wedge \vec{s}) \in O$, as $\overleftarrow{s} \leq (\overleftarrow{r} \wedge \vec{s})^* = (\vec{r} \vee \overleftarrow{s})$ and $\vec{s} \in O$. But now $\{\vec{r}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\overleftarrow{r} \wedge \vec{s})\} \subseteq O$ and $\{\vec{r}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\overleftarrow{r} \wedge \vec{s})\} \in \mathcal{F}^0$, hence O is not avoiding $\mathcal{F}^0$. If $(\overleftarrow{r} \vee \vec{s}) \in S_k$ then due to the consistency of O it is $(\overleftarrow{r} \vee \vec{s})^* = (\vec{r} \wedge \overleftarrow{s}) \in O$ as $\overleftarrow{r} \leq (\overleftarrow{r} \vee \vec{s})$ and $\vec{r} \in O$. But now $\{\vec{s}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\vec{r} \wedge \overleftarrow{s})\} \subseteq O$ and $\{\vec{s}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\vec{r} \wedge \overleftarrow{s})\} \in \mathcal{F}^0$, hence O is not avoiding $\mathcal{F}^0$. □

Theorem 7. *Any S_k-profile O avoids $\mathcal{F}^0$.*

Proof. Let O be a S_k-profile. Suppose that O has a subset σ with $\sigma \in \mathcal{F}^0$. Then σ has the form $\{\vec{r}, (\overleftarrow{r} \wedge \overleftarrow{s}), (\overleftarrow{r} \wedge \vec{s})\}$. It is $s \in S_k$, as $\sigma \in \mathcal{F}^0$, hence O has to contain either $\overleftarrow{s}$ or $\vec{s}$. If $\vec{s} \in O$ then $\{\vec{s}, \vec{r}, (\overleftarrow{r} \wedge \overleftarrow{s})\} \subseteq O$, hence O is not an S_k profile as it violates (P). A similar argument shows that O violates (P) if $\overleftarrow{s} \in O$. □

From **Theorem 6** and **Theorem 7** follows that S_k-profiles are indeed $\mathcal{F}^0$-tangles. To apply **Theorem 5** to S_k-profiles we have to prove that S_k is $\mathcal{F}^0$-separable.

Definition 29. *$\mathcal{F}$ is closed under shifting if whenever $\vec{s_0} \in \vec{S}$ is linked to some non-trivial $\vec{r} \leq \vec{s_0}$ it is even $\mathcal{F}$-linked to $\vec{r}$.*

Lemma 5. [1, *Lemma* 4.2.] *If $\vec{S}$ is separable and $\mathcal{F}$ is closed under shifting, then $\vec{S}$ is $\mathcal{F}$-separable.*

Proof. □

We would like $\mathcal{F}^0$ to be closed under shifting, thus S_k would be $\mathcal{F}^0$-separable by **Lemma 5**. Unfortenately $\mathcal{F}^0$ is not closed under shifting.
If $\vec{s_0} \in \vec{S_k}$ is linked to some non-trivial $\vec{r} \le \vec{s}$ and $\sigma = \{\vec{t}, (\overleftarrow{t} \wedge \overleftarrow{u}), (\overleftarrow{t} \wedge \vec{u})\} \in \mathcal{F}^*$ be a star with $(\overleftarrow{t} \wedge \overleftarrow{u}) \ge \vec{r}$ then $f \downarrow^{\vec{r}}_{\vec{s_0}} (\sigma)$ need not be in $\mathcal{F}^0$ again. But we can assure that it does by making $\mathcal{F}^0$ bigger. We are doing this recursive. We start with $\mathcal{F}^0$ and define

$$\mathcal{F}^n := \{\{(\vec{a} \vee \vec{x}), (\vec{b} \wedge \overleftarrow{x}), (\vec{c} \wedge \overleftarrow{x})\} \subseteq 2^{\vec{S_k}} : \{\vec{a}, \vec{b}, \vec{c}\} \in \mathcal{F}^{n-1}, x \in S_k\} \ .$$

We are iterating this process until $\bigcup_{i=0}^{n-1} \mathcal{F}^i = \bigcup_{i=0}^{n} \mathcal{F}^i$. As S_k is finite this iteration will end after finite many steps.

Finally let $\mathcal{F}^* := \bigcup \mathcal{F}^n$.

Lemma 6. *$\mathcal{F}^n$ is a set of stars for every integer n .*

Proof. We are showing this by induction. For $n = 0$ this is clear. Let $n > 0$ and $\sigma = \{(\vec{a} \vee \vec{x}), (\vec{b} \wedge \overleftarrow{x}), (\vec{c} \wedge \overleftarrow{x})\} \in \mathcal{F}^n$. It is $(\vec{b} \wedge \overleftarrow{x}) \le (\overleftarrow{c} \vee \vec{x}) = (\vec{c} \wedge \overleftarrow{x})^*$, as $(\vec{b} \wedge \overleftarrow{x}) \le \vec{b} \le \overleftarrow{c} \le (\overleftarrow{c} \vee \vec{x})$, as $\{\vec{a}, \vec{b}, \vec{c}\} \in \mathcal{F}^{n-1}$, hence they form a star. The same argument shows $(\vec{c} \wedge \overleftarrow{x}) \le (\overleftarrow{b} \vee \vec{x}) = (\vec{b} \wedge \overleftarrow{x})^*$
Furthermore it is $(\vec{a} \vee \vec{x}) \le (\overleftarrow{b} \vee \vec{x}) = (\vec{b} \wedge \overleftarrow{x})^*$, given that $\vec{a} \le \overleftarrow{b}$, as $\{\vec{a}, \vec{b}, \vec{c}\} \in \mathcal{F}^{n-1}$, hence they form a star. Doing so for every missing combination of two separations in σ yields the claim.

□

As a result of **Lemma 6** we get that the set $\mathcal{F}^*$ is a set of stars
Now we proof that S_k-profiles are $\mathcal{F}^*$-tangles.

Theorem 8. *Any consistent $\mathcal{F}^*$-avoiding orientation O of S_k is an S_k-profile*

Proof. This follows directly from **Theorem 6**. □

Theorem 9. *Any S_k-profile avoids $\mathcal{F}^*$.*

Proof. Let O be an S_k-profile. Suppose O does not avoid $\mathcal{F}^*$. Then O contains a subset that is an element of some $\mathcal{F}^n$. Let m denote the smallest integer such that O contains a subset of some $\mathcal{F}^n$. Let $\sigma \in \mathcal{F}^m$ denote this subset. If $m = 0$ then we are done (**Theorem 7**), hence $m > 0$. We now show that O also contains a subset that is an element of $\mathcal{F}^{m-1}$ contradicting our choice of m. We know σ is of the form $\{(\vec{a} \vee \vec{x}), (\vec{b} \wedge \overleftarrow{x}), (\vec{c} \wedge \overleftarrow{x})\}$. As $\sigma \in \mathcal{F}^m$, we also know that $\{a, b, c, x\} \in S_k$. As $\vec{a} \le (\vec{a} \vee \vec{x})$ and O is consistent O has to contain $\vec{a}$. Further O contains $\vec{b}$. Suppose not. Then O contains $\overleftarrow{b}$. Due to the consistency O also contains $\vec{x}$ as $\vec{x} \le (\vec{a} \vee \vec{x})$. Thus O violates (P), hence is not a profile, as $\{\vec{x}, \overleftarrow{b}, (\vec{b} \wedge \overleftarrow{x})\} \in O$. The same argument shows that $\vec{c} \in O$, hence O contains an element of $\mathcal{F}^{m-1}$. □

Theorem 5 requires that the set $\mathcal{F}$ is standard. Therefore it has to contain $\overleftarrow{r}$ for every $r \in S_k$ such that $\overrightarrow{r}$ is trivial. We are making $\mathcal{F}^*$ standard by adding $\overleftarrow{r}$ for every $r \in S_k$ such that $\overrightarrow{r}$ is trivial.

$\mathcal{F}^{**} := \mathcal{F}^* \cup \{\{\overleftarrow{r}\} : r \in S_k, \overrightarrow{r} \textit{ is trivial}\}$

Theorem 10. *S_k-profiles are $\mathcal{F}^{**}$-tangles.*

Proof. This follows directly from **Theorem 8** and **Theorem 9** and the fact that for every consistent orientation O of S it is $\overleftarrow{r} \notin O$ if $\overrightarrow{r} \in S$ is small, as $\overrightarrow{r} \leq \overleftarrow{r}$. Given that every trivial separation is a small separation O does not contain $\overleftarrow{r}$ if $\overrightarrow{r}$ is trivial. □

The last thing we have to show to apply **Theorem 5** to S_k-profiles is that S_k is $\mathcal{F}^{**}$-separable. By **Lemma 5** it is sufficient if we proof that $\mathcal{F}^{**}$ is closed under shifting, as we already know that S_k is separable (**Lemma 4**).

Theorem 11. *$\mathcal{F}^{**}$ is closed under shifting.*

Proof. Let $\overrightarrow{s_0} \in \overrightarrow{S_k}$ be linked to some non-trivial $\overrightarrow{r} \leq \overrightarrow{s}$. Let $\sigma = \{\overrightarrow{t}, \overrightarrow{u}, \overrightarrow{v}\} \in \mathcal{F}^{**}$ be a star with $\sigma \subseteq \overrightarrow{S}_{\geq \overrightarrow{r}}$. Then $\sigma \in \mathcal{F}^*$, as $\overrightarrow{r}$ is non-trivial. Let $\overrightarrow{r} \leq \overrightarrow{t}$ and n be an integer, such that $\sigma \in \mathcal{F}^n$. We now have to show that $f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\sigma) \in \mathcal{F}^{**}$. It is $f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\overrightarrow{t}) = (\overrightarrow{s_0} \vee \overrightarrow{t})$ as $\overrightarrow{r} \leq \overrightarrow{t}$ and

$$f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\overrightarrow{u}) = (\overleftarrow{u} \vee \overrightarrow{s_0})^* = (\overrightarrow{u} \wedge \overleftarrow{s_0})$$

$$f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\overrightarrow{v}) = (\overleftarrow{v} \vee \overrightarrow{s_0})^* = (\overrightarrow{v} \wedge \overleftarrow{s_0})$$

as $\overrightarrow{r} \leq \overrightarrow{u}^*$ and $\overrightarrow{r} \leq \overrightarrow{v}^*$. Given that $\overrightarrow{s_0}$ is linked to $\overrightarrow{r}$, hence $f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\sigma) \subseteq \overrightarrow{S_k}$ and given that $\overrightarrow{s_0} \in \overrightarrow{S_k}$ it is $f \downarrow^{\overrightarrow{r}}_{\overrightarrow{s_0}} (\sigma) = \{(\overrightarrow{t} \vee \overrightarrow{s_0}), (\overrightarrow{u} \wedge \overleftarrow{s_0}), (\overrightarrow{v} \wedge \overleftarrow{s_0})\} \in \mathcal{F}^{n+1}$. □

6 Weak tangles

In this section we introduce a weakened definition of profiles and we will show some properties of these weakened profiles. For this whole section we will work again with separations of a graph G.

Definition 30. *Call an unordered triple* $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *of separations bad if* $\{h, i, j\} = \{1, 2, 3\}$ *exist, such that:*
$(B_h, A_h) \leq (A_i \cup A_j, B_i \cap B_j)$

This triples are exactly the set of separations, which violate the property (P3). Profiles of a graph can be seen as sets of separations, which does not contain bad triples (**Theorem1**). Bad triples can be made symmetrical in the following way.

Definition 31. *Call an unordered triple* $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *of separations awful if* $\{h, i, j\} = \{1, 2, 3\}$ *exist, such that:*
$(B_h, A_h) \leq (A_i \cup A_j, B_i \cap B_j)$ *holds for every* $\{h, i, j\} = \{1, 2, 3\}$.

Definition 32. *A Set* P *of separations of a Graph* G *is a weak-tangle if:*

- P *is consistent*
- P *contains no awful triple*

A weak-tangle P *is a weak-k-tangle if all separations in* P *have order less than* k *and if for every separation* (A, B) *of* G *of order less than* k *either* $(A, B) \in P$ *or* $(B, A) \in P$.

6.1 Awful stars

Definition 33. *Call a bad triple* S*, that is also a star, a bad star. Call an awful triple* S*, that is also a star, an awful star.*

In this section we will study some properties of bad an awful stars. The main result in this section gives us a characterisation for awful stars in an orientation of a graph G, that contains no 3-separation. (A separation (A, B), such that $G \backslash (A \cap B)$ consists of at least three components).

Lemma 7. *Let* $S = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *be a bad star with* $A_3 \supseteq (B_1 \cap B_2)$ *and* $B_3 \subseteq (A_1 \cup A_2)$*. Then* $A_3 = (B_1 \cap B_2)$ *and* $B_3 = (A_1 \cup A_2)$.

Proof. It is $A_3 \supseteq (B_1 \cap B_2)$. As S is also a star it is $A_3 \subseteq B_1$ and $A_3 \subseteq B_2$, hence $A_3 \subseteq (B_1 \cap B_2)$, therefore $A_3 = (B_1 \cap B_2)$. Similar we can show that $B_3 = A_1 \cup A_2$. □

Corollar 2. *Let* $S = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *be an awful star. Then* $A_h = B_j \cap B_k$ *and* $B_h = A_i \cup A_j$ *for every* $\{h, i, j\} = \{1, 2, 3\}$.

Proof. □

Lemma 8. *Let* $S = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *be a bad star (with* $A_3 \supseteq (B_1 \cap B_2)$*,* $B_3 \subseteq (A_1 \cup A_2)$*). Then*

- $A_1 \backslash A_3 = B_2 \backslash A_3$
- $A_2 \backslash A_3 = B_1 \backslash A_3$

Proof. "$A_1 \backslash A_3 \subseteq B_2 \backslash A_3$"

Let $x \in A_1 \backslash A_3$. As S is a star $A_1 \subseteq B_2$, hence $x \in B_2 \backslash A_3$.

"$B_2 \backslash A_3 \subseteq A_1 \backslash A_3$"

We may assume that $B_2 \backslash A_3 \neq \emptyset$, as otherwise it would be $A_1 \backslash A_3 = B_2 \backslash A_3 = \emptyset$, given that $A_1 \subseteq B_2$. Let $\emptyset \neq x \in B_2 \backslash A_3$. With $A_3 = (B_1 \cap B_2)$ follows $x \notin B_1 \backslash A_3$, hence $x \in A_1 \backslash A_3$ as (A_1, B_1) is a separation. This shows $A_1 \backslash A_3 = B_2 \backslash A_3$. With a similar argument we show $A_2 \backslash A_3 = B_1 \backslash A_3$. □

The next theorem implies that every awful star contains either a 3-separation or the awful star is of the form $\{(X, Y), (Z, V), (Y, X)\}$ with $Z = (X \cap Y)$.

Theorem 12. *Let* $S = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ *be an awful star. Then either* $G \backslash (A_3 \cap B_3)$ *consists of at least three components or* S *is of the form* $\{(X, Y), (Z, V), (Y, X)\}$ *with* $Z = (X \cap Y)$*.*

Proof. Let $A_1 \backslash A_3 \neq \emptyset$, $A_2 \backslash A_3 \neq \emptyset$ and $A_3 \backslash B_3 \neq \emptyset$ and let xy be an edge with $x \in A_1 \backslash A_3$ and $y \in A_2 \backslash A_3 = B_1 \backslash A_3$ (**Lemma 8**). Then either $x \in (A_1 \cap B_1) \backslash A_3$ or $y \in (A_1 \cap B_1) \backslash A_3$, as (A_1, B_1) is a separation. It is $x \in A_1 \backslash A_3 = B_2 \backslash A_3$, hence $x \notin B_1 \backslash A_3$, as $A_3 = (B_1 \cap B_2)$. Thus $x \notin (A_1 \cap B_1) \backslash A_3$. With a similar argument we can show that $y \notin (A_1 \cap B_1) \backslash A_3$. This is a contradiction. Therefore there exists no edge between $A_1 \backslash A_3$ and $A_2 \backslash A_3$. There exists no edge between $A_3 \backslash B_3$ and $A_1 \backslash A_3$ or $A_2 \backslash A_3$, as (A_3, B_3) is a separation and $A_1 \backslash A_3, A_2 \backslash A_3 \subseteq B_3 \backslash A_3$, hence $G \backslash (A_3 \cap B_3)$ consist of at least three components.

Now we are proving that if at least one of $A_1 \backslash A_3, A_2 \backslash A_3, A_3 \backslash B_3$ is empty, S has got the form $\{(X, Y), (Z, V), (Y, X)\}$ with $Z = (X \cap Y)$.

Let $A_3 \backslash B_3 = \emptyset$. Then $A_3 \subseteq B_3$, hence $B_3 = V$, as (A_3, B_3) is a separation. With **Lemma 7** follows, $A_1 = (B_2 \cap B_3) = (B_2 \cap V) = B_2$, $A_2 = (B_1 \cap B_3) = (B_1 \cap V) = B_1$ and $A_3 = (B_1 \cap B_2) = (A_1 \cap B_1) = (A_2 \cap B_2)$. We get a star of the form $\{(A_1 \cap B_1, V), (A_1, B_1), (B_1, A_1)\} = \{(A_2 \cap B_2, V), (A_2, B_2), (B_2, A_2)\}$. Now let $A_2 \backslash A_3 = \emptyset$ or $A_1 \backslash A_3 = \emptyset$. Without loss of generality let $A_2 \backslash A_3 = \emptyset$. With Lemma **Lemma 8** follows $A_2 \backslash A_3 = B_1 \backslash A_3 = \emptyset$, hence $B_1 \subseteq A_3$. As S is a star $B_1 \supseteq A_3$ holds, hence $B_1 = A_3$. Given that S is an awful star with **Corollar 2** follows $A_1 = (B_2 \cap B_3) = (A_3 \cup A_1) \cap B_3 = (B_1 \cup A_1) \cap B_3 = (V \cap B_3) = B_3$, hence $(A_3, B_3) = (B_1, A_1)$. Further $B_2 = (A_1 \cup A_3) = (B_3 \cup A_3) = V$ and $A_2 = (B_3 \cap B_1) = (A_1 \cap B_1) = (A_3 \cap B_3)$. □

6.2 Canonical tree-decomposition for weak-tangles

In [2] Carmesin, Diestel, Hamann, and Hundertmark constructed a tree decomposition for a given graph that distinguishes all their k-profiles for a fixed integer k. An interesting question to ask is, whether there exists such a tree-decomposition for weak-tangles. In this section we will give a counterexample.

Definition 34. *A separation (A,B) is nested with a separation (C,D), written as $(A, B)||(C, D)$, if it is $\leq$-comparable with either (C,D) or (D,C). Since*

$$(A, B) \leq (C, D) \Leftrightarrow (D, C) \leq (B, A)$$

, the relation $||$ is reflexive and symmetric. Two separations that are not nested are said to cross.

Definition 35. *A separation (A,B) is nested with a set S of separations, written as $(A, B)||S$, if $(A, B)||(C, D)$ for every $(C, D) \in S$. A set S of separations is nested with a set S' of separations, written as $S||S'$, if $(A, B)||S'$ for every $(A, B) \in S$; then also $(C, D)||S$ for every $(C, D) \in S'$.*

Definition 36. *A set of separation is called nested if every two of its elements are nested.*

Definition 37. *A separation (A,B) separates a set $X \subseteq V$ if X meets both $A \backslash B$ and $B \backslash A$. Given a set S of separations, we say that X is S-inseparable if no separation in S separates X. An S-block of G is a maximal S-inseparable set of vertices.*

Definition 38. *A tree decomposition of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V}= (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for every node of T, such that:*

- $V(G) = \bigcup_{t \in T} V_t$;
- *for every edge $e \in G$ there exists a $t \in T$ such that both ends of e lie in V_t;*
- $V_{t1} \cap V_{t3} \subseteq V_{t2}$ *whenever t_2 lies on the $t_1 - t_3$ path in T.*

The sets V_t in such a tree-decomposition are its parts. Their intersections $V_t \cap V_{t'}$ for edges tt' of the decoposition tree T are the adhesion sets of $(T, \mathcal{V})$; their maximum size is the adhesion of $(T, \mathcal{V})$. Deleting an oriented edge $\overrightarrow{e} = t_1, t_2$ of T divides $T - e$ into two components $T_1 \ni t_1$ and $T_2 \ni t_2$. Then $(\bigcup_{t \in T_1} V_t, \bigcup_{t \in T_2} V_t)$ is a separation of G with separator $V_{t_1} \cap V_{t_2}$ [4, Lemma 12.3.1]; we say our edge $\overrightarrow{e}$ induces this separation.

Definition 39. *A node $t \in T$ is a hub node if the corresponding part V_t is the separator of a separation induced by an edge of T at t. If t is a hub node, we call V_t a hub.*

The separations induced by a tree-decomposition $(T, \mathcal{V})$ are nested. Conversely Carmesin, Diestel, Hamann, and Hundertmark proved in [3] that every nested separation system is induced by some tree-decomposition.

Theorem 13. *[3, Theorem 4.8] Every nested proper separation system N is induced by a tree-decomposition $(T, \mathcal{V})$ of G such that:*

- *every N-block of G is a part of the decompositions;*
- *every part of the decomposition is either an N-block of G or a hub.*

This theorem implies that we can construct a tree-decomposition that distinguishes all weak-k-tangles, whenever we can find a nested set N of separations which distinguishes all weak-k-tangles.

Definition 40. *A separation (A, B) distinguishes two sets P,P' of separations if $(A, B) \in P \backslash P'$ and $(B, A) \in P' \backslash P$, or vice versa.*

Only proper separations can distuinguish two consistent sets of separations [cf. 2, p.6], as a consistent set of separations never contains a separation of the form (V, A). A consistent set of separations containing (V, A) must not contain the inverse of (A, V), which is (V, A), as $(A, V) \leq (V, A)$.
We show now that there exist graphs, such that their weak-k-tangles cannot be separated by a nested set of separations.

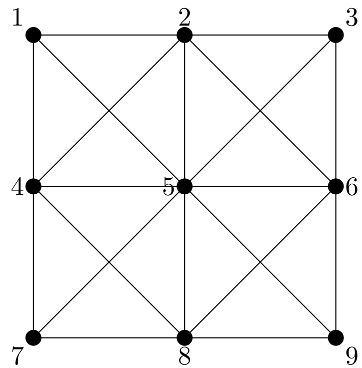

Figure 2: A graph, whose weak-4-tangles cannot be distigiushed by a nested set of separations

Let G be the graph shown in figure 2 and let S be the set of all the separations of G of order < 4. Let $A, B, C, D \subseteq V(G)$ denote the following vertex sets: $A := \{1, 2, 3, 4, 5, 6\}, B := \{4, 5, 6, 7, 8, 9\}, C := \{1, 2, 4, 5, 7, 8\}$,$D := \{2, 3, 5, 6, 8, 9\}$. It is easily checked that the unordered pairs $(A, B), (C, D), (B \cap C, A \cup D)$, $(B \cup C, A \cap D), (B \cap D, A \cup C), (B \cup D, A \cap C)$ are the only proper separations of order < 4 of G. Let us define the following separation sets:

- $S_1 := \{(A, B), (C, D), (B \cap C, A \cup D), (A \cap D, B \cup C), (B \cap D, A \cup C), (A \cap C, B \cup D)\} \cup \{(X, V) : (X, V) \in S, |X| < 4\}$
- $S_2 := \{(B, A), (C, D), (B \cap C, A \cup D), (A \cap D, B \cup C), (B \cap D, A \cup C), (A \cap C, B \cup D)\} \cup \{(X, V) : (X, V) \in S, |X| < 4\}$

- $S_3 := \{(A,B),(D,C),(B\cap C, A\cup D),(A\cap D, B\cup C),(B\cap D, A\cup C),$ $(A\cap C, B\cup D)\}\cup\{(X,V) : (X,V)\in S, |X|<4\}$

The following results show that S_1, S_2 and S_3 each are a weak 4-tangle of G. But we cannot distinguish them by a nested set of separations.

Lemma 9. *Let P be a consistent set of separations of a graph G. Let (X,V) be a separation of G. Then $P\cup(X,V)$ is still consistent.*

Proof. Let P be a consistent set of separations of a graph $G=(V,E)$ and (X,V) a separation with $X\subset V$. As P is consistent P does not contain any separation of the Form (V,Y) with $Y\subset V$, hence $(V,X)\not\leq(A,B)\in P$. Let us assume there exists a separation $(C,D)\in P$, such that $(D,C)\leq(X,V)$. Then $C=V$, hence P is not consistent, given that no consistent set of separations can contain a separation of the form (V,Z) with $Z\subseteq V$. □

Lemma 10. *The separation sets S_1, S_2, S_3 are weak-4-tangles.*

Proof. We are going to prove this for S_1. The proofs for S_2 and S_3 are analog. For every separation (X,Y) of order <4 of G, the separation set S_1 contains exactly one of its orientations. Therefore S_1 is an orientation of the set of all separations of G that have order <4. Further S_1 is consistent.

Suppose not, then there exist separations $(Y_1,X_1),(X_2,Y_2)\in S_1$ such that $(X_1,Y_1)\leq(X_2,Y_2)$. By **Lemma 9** and the fact that $\sigma := \{(B\cap C, A\cup D),$ $(A\cap D, B\cup C),(B\cap D, A\cup C),(A\cap C, B\cup D)\}$ is a consistent star (**Corollar 1**) either $(Y_1,X_1)\in\{(A,B),(C,D)\}$ or $(X_2,Y_2)\in\{(A,B),(C,D)\}$. They cannot both be contained in $\{(A,B),(C,D)\}$, as (A,B), and (C,D) cross each other. Let $(A,B)=(Y_1,X_1)$, then $(X_2,Y_2)\in\sigma$, by **Lemma 9** and the fact that $(A,B),(C,D)$ cross each other. But for every separation $(E,F)\in\sigma$ it is either $(E,F)\leq(A,B)$ or $\{(A,B),(E,F)\}$ form a consistent star, hence $(X_1,Y_1)=(B,A)\not\leq(E,F)=(X_2,Y_2)$, contradicting our assumption. If $(X_2,Y_2)=(A,B)$ then again $(Y_1,X_1)\in\sigma$. And equally for every seperation $(E,F)\in\sigma$ it is either $(E,F)\leq(A,B)$ or $\{(A,B),(E,F)\}$ form a consistent star, hence $(X_1,Y_1)=(F,E)\not\leq(A,B)=(X_2,Y_2)$ contradicting our assumption. The same arguments hold if either $(Y_1,X_1)=(C,D)$ or $(X_2,Y_2)=(C,D)$. S_1 does not contain any awful triple. By **Theorem 12** it does not contain an awful star, as it contains no 3-separation and it is consistent. An awful triple cannot contain a separation of the form (X,V), as the union of the small sides of the two other separations would have to cover the whole graph, but S_1 does not contain any such two separations. Hence every awful triple has to contain (A,B) and (C,D). The small side of the third separation (A_3,B_3) of the awful triple has to satisfy $(B\cap D\subseteq A_3)$. The only separation satisfying this condition is $(B\cap D, A\cup C)$, but $(A\cup C)\cap B\not\subseteq C$, hence $\{(A,B),(C,D),(B\cap D, A\cup C)\}$ is no awful triple. This proves the claim.

□

Theorem 14. *Let G be the graph shown in figure 2. The weak-4-tangles cannot be distiguished by a nested set of separations.*

Proof. Let N be a set that distinguishes all the weak-4-tangles of G. Given that S_1, S_2 and S_3 are weak k-tangles N has to distinguish them. The weak-4-tangles S_1 and S_2 differ only in the orientation of the separation (A, B). Thus N has to contain an orientation of the separation (A, B). The weak-4-tangles S_1 and S_3 differ only in the orientation of the separation (C, D). Thus N has to contain an orientation of the separation (C, D). As (A, B) and (C, D) cross, N is not nested. □

7 New definition of consistent

In [1] Diestel and Oum are using a weaker definition of consistent as the one in **Definition 9**. To distinguish the two definitions of consistent we will say that a set of separations is weak consistent if it is consistent by the definition of consistent in [1, p.5].

Definition 41. *[cf.1, p.5] Let $(\vec{S}, \leq, *)$ be a separation system. A set $O \subseteq \vec{S}$ is weak consistent if there are no distinct $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in O$.*

7.1 Differences between consistent and weak consistent

In this section we will cover the differences of these two definitions and their impact on the proofs in this paper.

Theorem 15. *Let $(\vec{S}, \leq, *)$ be a separation system and let $O \subseteq \vec{S}$ be a consistent set of oriented separations. Then O is also weak consistent.*

Proof. As O is consistent there exist no $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in O$. Thus O contains no distinct $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in O$. □

Definition 42. *Let us call an oriented separation $\vec{r}$ maximal with respect to a set of oriented separations P if and only if*

$$\forall \vec{s} \in P, \text{ such that } r \neq s \text{ either } \vec{s} \leq \vec{r} \text{ or } \vec{r} \text{ and } \vec{s} \text{ are incomparable.}$$

Let us call a separation r maximal with respect to P if both orientations $\vec{r}$ and $\overleftarrow{r}$ are maximal with respect to P.

Lemma 11. *Let S be a set of separations. Every weak consistent orientation O of S has to contain $\vec{r}$ for every $r \in S$ such that $\vec{r}$ is trivial.*

Proof. As $\vec{r}$ is trivial, there exists $s \in S$, such that $\vec{r} < \vec{s}$ and $\vec{r} < \overleftarrow{s}$, hence $\overleftarrow{r} \notin O$, given that O has to contain one orientation of s. □

Theorem 16. *Every consistent set of oriented separations $\vec{O}$, that is not a weak consistent set of oriented separations, contains either the inverse of a small separation $\vec{s}$, such that s is maximal with respect to $\vec{O}$ or there exists a separation r, such that $\{\vec{r}, \overleftarrow{r}\} \subseteq \vec{O}$ and r is maximal with respect to $\vec{O}$.*

Proof. Let $\vec{O}$ be a set of oriented separations that is weak consistent but not consistent. As $\vec{O}$ is not consistent there exist separations r, s with orientations $\vec{r} \leq \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in \vec{O}$. As $\vec{O}$ is weak consistent $r = s$. Thus either $\vec{s} = \vec{r}$ or $\vec{s} = \overleftarrow{r}$.
If $\vec{s} = \overleftarrow{r}$ then $\vec{r} \leq \overleftarrow{r}$, hence $\vec{r}$ is small. Suppose r is not maximal with respect to $\vec{O}$. Then there exists $\vec{t} \in \vec{O}$ such that either $\vec{r} \leq \vec{t}$ or $\overleftarrow{r} \leq \vec{t}$

and $r \neq t$. In both cases $\vec{O}$ is not weak consistent given that O contains $\overleftarrow{r}$ but it is $\vec{r} \leq \overleftarrow{r} \leq \vec{t}$ or $\vec{r} \leq \vec{t}$.
If $\vec{s} = \vec{r}$, then $\{\vec{r}, \overleftarrow{r}\} \subseteq \vec{O}$. If either $\vec{r}$ or $\overleftarrow{r}$ is not maximal with respect to $\vec{O}$, then there exists a separation $t \neq r$ with $\vec{t} \in \vec{O}$ such that $\vec{r} < \vec{t}$ or $\overleftarrow{r} < \vec{t}$. Thus $\vec{O}$ is not weak consistent as it contains $\vec{r}, \overleftarrow{r}$ *and* $\vec{t}$. □

Theorem 17. *Let $\vec{O}$ be a weak consistent set of oriented separations, that contains neither the inverse of a small separation $\vec{s}$, such that s is maximal with respect to $\vec{O}$ nor there exists a separation r, such that $\{\vec{r}, \overleftarrow{r}\} \subseteq \vec{O}$ and r is maximal with respect to $\vec{O}$, then $\vec{O}$ is a consistent set of oriented separations.*

Proof. Suppose not, then there exist separations r, s with orientations $\vec{r} \leq \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in \vec{O}$. Given that O is weak consistent it is $r = s$. Thus either $\vec{s} = \vec{r}$ or $\vec{s} = \overleftarrow{r}$.
If $\vec{s} = \overleftarrow{r}$ then $\vec{r} \leq \overleftarrow{r}$, hence $\vec{r}$ is small. Suppose r is not maximal with respect to $\vec{O}$. Then there exists $\vec{t} \in \vec{O}$ such that either $\vec{r} \leq \vec{t}$ or $\overleftarrow{r} \leq \vec{t}$ and $r \neq t$. In both cases $\vec{O}$ is not weak consistent as it contains $\overleftarrow{r}$ and $\vec{r} \leq \overleftarrow{r} \leq \vec{t}$ or $\vec{r} \leq \vec{t}$.
If $\vec{s} = \vec{r}$, then $\{\vec{r}, \overleftarrow{r}\} \subseteq \vec{O}$. If either $\vec{r}$ or $\overleftarrow{r}$ is not maximal with respect to $\vec{O}$, then there exists a separation $t \neq r$ with $\vec{t} \in \vec{O}$ such that $\vec{r} < \vec{t}$ or $\overleftarrow{r} < \vec{t}$. Thus $\vec{O}$ is not weak consistent as it contains $\vec{r}, \overleftarrow{r}$ *and* $\vec{t}$. □

Definition 43. *Let $(\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$. A set $P \subseteq \vec{S}$ is a weak profile if it satisfies:*

- *P is weak consistent*
- $\forall \vec{r}, \vec{s} \in P : (\overleftarrow{r} \wedge \overleftarrow{s}) \notin P$ $\qquad (P)$

Definition 44. *A weak profile P is a weak S-profile if for all separations $\vec{r} \in P$ also $r \in S$ and if for every separation $s \in S$ either $\vec{s} \in P$ or $\overleftarrow{s} \in P$.*

Definition 45. *Let G be a graph. A weak profile P of G is a weak k-profile if all separations in P have order less than k and if for every separation (A, B) of G of order less than k either $(A, B) \in P$ or $(B, A) \in P$.*

Corollar 3. *Every weak profile $\vec{P}$, that is not a profile, contains the inverse of a small separation $\vec{s}$, such that s is maximal with respect to $\vec{P}$*

Proof. This follows directly from **Theorem 16** and **Lemma 3**. □

For k-profiles, such that $k > 2$, both definitions of consistent are equivalent.

Lemma 12. *Let G be a graph. The small separations of G are exactly the separations of the form (A, V) with $A \subseteq V$.*

Proof. "$\Rightarrow$"
Let (A, B) be a small separation of G. Then $(A, B) \leq (B, A)$. Thus $B = V$, as $A \subseteq B$ and (A, B) is a separation.
"$\Leftarrow$"
It is $(A, V) \leq (V, A)$, hence (A, V) is a small separation. □

Theorem 18. *Let G be a graph and let P be a weak k-profile of G, such that $k > 2$, then P is a k-profile.*

Proof. Suppose not, then P contains a separation of the form (V, A) by **Lemma 12**. Given that (V, A) and (A, V) are maximal with respect to P (**Corollar 3**), the set A contains at least two vertices. Thus we can find $X \neq Y \neq A \subseteq V$, such that $X \cup Y = A$. As P is a weak k-profile it has to contain (X, V) and (Y, V), as a weak k-profile cannot contain two distinct separations of the form (V, Z), with $Z \subseteq V$.

If a weak k-profile P does contain two distinct separations (V, Z_1) and (V, Z_2) they both have to be maximal with respect to P by **Corollar 3**. By (P) it is $(Z_1 \cap Z_2, V) \notin P$. Given that $|Z_1 \cap Z_2| < k$ the weak-k-profile P has to contain $(V, Z_1 \cap Z_2)$. But as P is weak consistent $(V, Z_1 \cap Z_2) \notin P$, given that either $(V, Z_1) < (V, Z_1 \cap Z_2)$ or $(V, Z_2) < (V, Z_1 \cap Z_2)$, since $Z_1 \neq Z2$. Thus either (V, Z_1) or (V, Z_2) is not maximal with respect to P. This is a contradiction.

By (P) it is $(V, X \cup Y) = (V, A) \notin P$. Hence X=Y=A. This is a contradiction. □

7.2 Impact on the proofs if we require only weak consistency instead of consistency

If we require only weak consistency in our proofs whenever consistency is required, some of the results will be false. But in some cases only the proof gets a bit more difficult.

Lemma 13. *Let S be a submodular set of separations and P be a weak S-profile. Then P is a weak consistent orientation of S.*

Proof. This follows directly from the proof of **Lemma 3**, if we replace consistent by weak consistent. □

Lemma 1 changes to the following: Stars are weak consistent: if $\overleftarrow{r}, \overrightarrow{s}$ lie in the same star we cannot have $\overrightarrow{r} < \overrightarrow{s}$, since also $\overrightarrow{s} \leq \overrightarrow{r}$ by the star property. If we consider the new definition of consistent **Theorem 1** is not true anymore. Let G=(V,E) be a graph and $S := \{(A, V), (V, A)\}$ a set of separations. Then $P := \{(V, A)\}$ is an orientation of S. As easy to be seen P is a weak profile, but not a profile. S is clearly submodular. But it is $(A, V) \leq (V \cup V, A \cap A) = (V, A)$. Hence to satisfy (P3) P must not contain the separation (V, A), but it does.

Definition 46. *Let* $(\vec{U}, \leq, *, \wedge, \vee)$ *be a universe of separations containing a submodular separation system*$(\vec{S}, \leq, *)$. *Let* $\vec{r}, \vec{s} \in \vec{S}$.
Then $[\vec{r} \wedge \vec{s}] := \{(\vec{r} \wedge \vec{s}), (\vec{r} \wedge \vec{s})^*\}$ *and* $[\vec{r} \vee \vec{s}] := \{(\vec{r} \vee \vec{s}), (\vec{r} \vee \vec{s})^*\}$.

With the following Lemma we can fix some of our proofs in this master´s thesis.

Lemma 14. *Let* $(\vec{U}, \leq, *, \wedge, \vee)$ *be a universe of separations containing a submodular separation system* $(\vec{S}, \leq, *)$. *Let O be a weak consistent orientation of S. Whenever* $\vec{r}, \vec{s}, (\overleftarrow{r} \wedge \overleftarrow{s}) \in O$ *and* $[\vec{r} \wedge \overleftarrow{s}] \in S$ *then* $(\vec{r} \wedge \overleftarrow{s}) \in O$.

Proof. Suppose not. Then $(\vec{r} \wedge \overleftarrow{s})^* = (\overleftarrow{r} \vee \vec{s}) \in O$ given that O is an orientation of S. As O is weak consistent and $(\vec{r} \wedge \overleftarrow{s}) \leq \vec{r}$, it is $[\vec{r} \wedge \overleftarrow{s}] = r$. Thus $(\overleftarrow{r} \vee \vec{s}) = \vec{r}$ as O contains $\vec{r}$. With $\vec{s} \leq (\overleftarrow{r} \vee \vec{s}) = \vec{r}$ follows $(\vec{s} \vee \vec{r}) = \vec{r}$, hence $(\overleftarrow{r} \wedge \overleftarrow{s}) = \overleftarrow{r}$. This is a contradiction as O cannot contain both $\overleftarrow{r}$ and $\vec{r}$. □

Theorem 19. *Let* $(\vec{U}, \leq, *, \wedge, \vee)$ *be a universe of separations containing a submodular separation system* $(\vec{S}, \leq, *)$. *Let O be an orientation of* $\vec{S}$ *with the following properties:*

- *O is weak consistent.*
- *Every star* $\Sigma \subseteq O$ *is subset of a weak* $\vec{S}$*-profile.*

Then O is a weak $\vec{S}$*-profile.*

Proof. The proof of **Theorem 3** together with **Lemma 14**. □

Theorem 20. *Let G be a Graph and S a nested set of separations of G. Let O be an orientation of S with the following properties:*

- *O is weak consistent.*
- *Every star* $\Sigma \subseteq O$ *points towards a k-block.*

Then every subset $S \subseteq O$ *points towards a k-block.*

Proof. The proof of **Theorem 4**. □

Definition 47. $[cf.1, p.11]$ *Given a separation system* $(\vec{S}, \leq, *)$ *and a set* $\mathcal{F} \subseteq 2^{\vec{S}}$, *let us call an orientation O of S a weak* $\mathcal{F}$*-tangle if it is weak consistent and avoids* $\mathcal{F}$, *that is,* $2^O \cap \mathcal{F} = \emptyset$.

In section five we proved that S_k-profiles are $\mathcal{F}^{**}$-tangles. Weak S_k-profiles are equally weak $\mathcal{F}^0$-tangles, but unfortunately they are not weak $\mathcal{F}^{**}$-tangles, as the following results show.

Theorem 21. *Any weak consistent $\mathcal{F}^0$-avoiding orientation O is a weak S_k-profile.*

Proof. The proof of **Theorem 6** together with **Lemma 14**. □

Theorem 22. *Any weak S_k-profile O avoids $\mathcal{F}^0$.*

Proof. The proof of **Theorem 7**. □

Theorem 21 and **Theorem 22** show that weak S_k-profiles are indeed weak $\mathcal{F}^0$-tangles.

Theorem 23. *Any weak consistent $\mathcal{F}^*$-avoiding orientation O of S_k is a weak S_k-profile.*

Proof. This follows directly from **Theorem 21**. □

The converse of **Theorem 23** fails, as the following example shows.

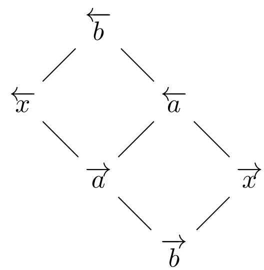

Figure 3: A universe of separations

Theorem 24. *Let $(\overrightarrow{U}, \leq, *, \wedge, \vee)$ be the universe of separations as in figure 3. Let $\alpha : \overrightarrow{U} \to \mathbb{N}$ be a function with $\overrightarrow{r} \mapsto 1$. Then α is an order function of $\overrightarrow{U}$ and $P := \{\overleftarrow{a}, \overrightarrow{x}, \overrightarrow{b}\}$ is a weak S_2-profile that contains a star $\sigma \in \mathcal{F}^*$.*

Proof. α is nonnegative, symmetric and submodular, hence α is an order function of $\overrightarrow{U}$. It can easily be checked, that P is weak consistent and satisfies (P), hence P is a weak S_2-profile. Let $\sigma := \{\overleftarrow{a}, \overrightarrow{b}\}$. It is $\sigma \subseteq P$ and $\sigma \in \mathcal{F}^*$, as $\{\overleftarrow{a}, \overrightarrow{b}\} = \{\overrightarrow{a} \vee \overrightarrow{x}, \overrightarrow{x} \wedge \overleftarrow{x}\}$ and $\{\overrightarrow{a}, \overrightarrow{x}\} = \{\overrightarrow{a}, (\overleftarrow{a} \wedge \overleftarrow{x}), (\overleftarrow{a} \wedge \overrightarrow{x})\} \in \mathcal{F}^0$. □

8 Open Questions

In this master´s thesis we managed to proof that the duality theorem (**Theorem 5**) can be applied to k-profiles. A question we have not answered in this paper is, whether the duality theorem can be applied to profiles wich are induced by k-blocks as well.
Given that every profile, which is induced by a k-block, is still a profile our choice of $\mathcal{F}$ has at least to contain $\mathcal{F}^{**}$. But this set of stars is not sufficient as there exist profiles, which are not induced by a k-block ([cf. 2, p.5]). A good extension for $\mathcal{F}^{**}$ might be the set of all stars σ, such that the intersection of all the big sides of the separations in σ has less than k vertices. The problem here is to show that every profile that is not induced by a k-block contains such a star. Furthermore these stars have to be closed under shifting or at least can be made closed under shifting.

In section 6 we introduced weak k-tangles which are slightly weaker than k-profiles. A natural question to ask is whether the duality theorem can be applied to weak k-tangles as well. To do that we have to find a set of stars $\mathcal{F}$ such that weak k-tangles are $\mathcal{F}$-tangles. This is probably not the case as the following example might suggest.

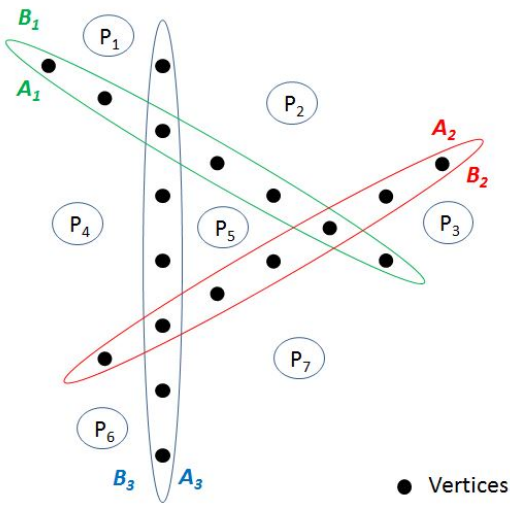

Figure 4: A graph with all its proper separations of order < 8 that have a minimal order

Let G be a graph as in figure 4. That means the proper separations of G of order < 8 that have a minimal order are exactly the separations shown in figure 4, whereas a proper separation (A,B) has a minimimal order if there exists no

proper separation $(A_1, B) \neq (A, B)$ such that $(A_1, B) \leq (A, B)$ nor a proper separation $(A, B_1) \neq (A, B)$ such that $(A, B) \leq (A, B_1)$. The P_i denote k-blocks such that $k > 7$. Let O be an orientation of all the separations of G which have order < 8 such that O contains $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ and for every other separation (X, Y) the orientation such that P_5 lies in the big side of (X, Y). The separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ form an awful triple. Suppose weak k-tangles are $\mathcal{F}$-tangles. If O is consistent (which it should be), then O contains a star $\sigma \in \mathcal{F}$. This star σ has to contain exactly one of the separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$. It cannot contain two of these separations given that they cross each other. If σ would not contain any of them σ would be subset of the profile P which is induced by the k-block P_5. Thus weak k-tangles would not be $\mathcal{F}$-tangles given that no profile contains an awful triple. Without loss of generality (G is symmetric) let (A_1, B_1) be this separations.
Let O_1 be equal to P with the exception that O_1 contains (A_1, B_1) instead of (B_1, A_1). Then O_1 contains every possible choice of σ, given that both O_1 and P orientate all separations (X, Y) but $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ in a way, such that P_5 lies in the big side of (X, Y). Suppose O_1 is consistent (which it should be), then O_1 has to contain an awful triple if weak k-tangles are $\mathcal{F}$-tangles. Otherwise O_1 would be a weak 8-tangle which would contain a star $\sigma \in \mathcal{F}$.
This awful triple has to contain the separation (A_1, B_1), given that P is a profile and differs in O_1 only in the orientation of the separation (A_1, B_1). The remaining two separations $(X_1, Y_1), (X_2, Y_2)$ in this awful triple have to satisfy $B_1 \subseteq (X_1 \cup X_2)$. That means the union of their small sides cover the big side of (A_1, B_1). It is only possible to find such two separations if one of them is the separation $(B_1 \cap A_3, A_1 \cup B_3)$, as the small side of every other separation covers only one of the k-blocks P_1, P_2 and P_3. But to cover B_1 the union of the small sides of (X_1, Y_1) and (X_2, Y_2) has to cover all three of them. Without loss of generality let $(X_1, Y_1) = (B_1 \cap A_3, A_1 \cup B_3)$. Then (X_2, Y_2) has to satisfy, $X_2 \supseteq (A_1 \cup B_3) \cap B_1$. But for all possible choices of (X_2, Y_2) that satisfy $X_2 \cup (B_1 \cap A_3) \supseteq B_1$ (the possible choices are $(B_3, A_3), (B_1 \cap B_3, A_1 \cup A_3)$, $(A_1 \cap B_3, B_1 \cup A_3)$) it is $(Y_2 \cap (A_1 \cup B_3)) \not\subseteq A_1$. Thus O_1 contains no awful triple. This shows that there exist weak tangles which do not avoid $\mathcal{F}$ for every choice of $\mathcal{F}$, hence weak tangles are no $\mathcal{F}$-tangles.

The difficulties in the approach above are to show that O and O_1 are consistent. Further it has to be shown that there exists a graph which satisfies all the properties needed. At last we have to proof that we covered all possible choices of (X_2, Y_2).

References

[1] Diestel, R., Oum, S.: *Unifying duality theorems for width parameters in graphs and matroids I. Weak and strong duality.* , 2015.

[2] Carmesin, J, Diestel, R., Hamann, M., Hundertmark F.: *Canonical tree-decompositions of finite graphs I. Existence and algorithms*, 2014.

[3] Carmesin, J, Diestel, R., Hamann, M., Hundertmark F.: *Connectivity and tree structure in finite graphs.* Combinatorica, 34(1):1-35, 2014.

[4] Diestel, R.:*Graph Theory.* Springer, 4th edition, 2010

Erklärung

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt. Die Arbeit habe ich vorher nicht in einem anderen Prüfungsverfahren eingereicht. Die eingereichte schriftliche Fassung entspricht genau der auf dem elektronischen Speichermedium.

Hamburg, der 09.06.2015

Philipp Eberenz