

Contractible edges in 2-connected locally finite graphs

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Abstract

In this paper, we prove that every contraction-critical 2-connected infinite graph has no vertex of finite degree and contains uncountably many ends. Then, by investigating the distribution of contractible edges in a 2-connected locally finite infinite graph G , we show that the closure of the subgraph induced by all the contractible edges in the Freudenthal compactification of G is 2-arc-connected. Finally, we characterize all 2-connected locally finite outerplanar graphs nonisomorphic to K_3 as precisely those graphs such that every vertex is incident to exactly two contractible edges as well as those graphs such that every finite bond contains exactly two contractible edges.

Keywords: contractible edge, Hamilton cycle, outerplanar, infinite graph

1 Introduction

Since the pioneering work of Tutte [11] who proved that every 3-connected finite graph nonisomorphic to K_4 contains a contractible edge, a lot of research has been done on contractible edges in finite graphs. One may consult the survey paper by Kriesell [8] for details.

For any 2-connected graph nonisomorphic to K_3 , we have the well-known fact that every edge can either be deleted or contracted so that the resulting graph remains 2-connected. This immediately leads to the following result.

Theorem 1. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then the subgraph induced by all the contractible edges in G is 2-connected.*

Wu [12] investigated the distribution of contractible elements in matroids and extended Theorem 1 to simple 2-connected matroids. He also characterized all simple 2-connected matroids M having exactly $r(M) + 1$ contractible elements (where $r(M)$ is the rank of M) as those matroids isomorphic to a graphic matroid of an outerplanar Hamiltonian graph.

Theorem 2 (Wu [12]). *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then every vertex of G is incident to exactly two contractible edges if and only if G is outerplanar.*

On the other hand, only a few results were known for contractible edges in infinite graphs. For example, Mader [10] showed that every contraction-critical locally finite infinite graph has infinitely many triangles. Kriesell [9] provided a method of constructing contraction-critical k -connected infinite graphs ($k \geq 2$). In Section 3, we will prove that every contraction-critical 2-connected infinite graph contains vertices of infinite degree only and has uncountably many ends.

A natural way to extend Theorems 1 and 2 is to consider locally finite infinite graphs. Notice that Theorem 1 is no longer true as demonstrated by the infinite double ladder (the cartesian product of a double ray and K_2). The subgraph G_C induced by all the contractible edges is the disjoint union of two double rays and is not even connected. Interestingly, the situation changes dramatically by looking at the graph from a topological viewpoint as introduced by Diestel and Kühn [4, 5, 6]. By adding the two ends of the double ladder to G_C , the resulting closure $\overline{G_C}$ is a circle and is 2-arc-connected. In Section 4, we will prove that for every 2-connected locally finite infinite graph G , $\overline{G_C}$ is 2-arc-connected.

Returning to Theorem 2, the backward direction is straightforward. For the forward direction, by Theorem 1, G_C is spanning and 2-connected. Since every vertex is incident to exactly two contractible edges, G_C is a Hamilton cycle. Then it is easy to see that G is outerplanar. When extending to locally finite infinite graphs, we now need the non-trivial statement that if G is a 2-connected locally finite infinite graph such that every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This will be proved in Section 5. We will use it to prove an infinite analog of Theorem 2 for any 2-connected locally finite graph G nonisomorphic to K_3 . Also we will show that G is outerplanar if and only if every finite bond of G contains exactly two contractible edges.

2 Definitions

All basic graph-theoretical terminology can be found in Diestel [3]. Unless otherwise stated, all graphs considered in this paper can be finite or infinite. An edge of a k -connected graph is said to be *k -contractible* if its contraction results in a k -connected graph. Otherwise, it is called *k -non-contractible*. A k -connected graph in which every edge is k -non-contractible is called *contraction-critical k -connected*. We simply write 2-contractible as *contractible*. Let $G = (V, E)$ be a 2-connected graph. Denote the set of all contractible edges in G by E_C and the subgraph induced by all the contractible edges by $G_C := (V, E_C)$. Let X and Y be two disjoint subsets of V . An X - Y path P is a path such

that only the starting vertex of P lies in X and only the ending vertex of P lies in Y . Denote the set of all edges between X and Y by $E_G(X, Y)$. If X and Y form a partition of V , then $E_G(X, Y)$ is called a *cut*. A minimal non-empty cut is a *bond*. Denote the set of all edges incident to a vertex x by $E_G(x)$ and the set of all neighbors of x by $N_G(x)$. Define $N_G(X) := (\bigcup_{x \in X} N_G(x)) \setminus X$. A set S of k vertices is called a k -*separator* if $G - S$ is not connected.

Let G be a locally finite graph. A *ray* is a 1-way infinite path, a *double ray* is a 2-way infinite path, and the subrays of a ray or double ray are its *tails*. An *end* is an equivalence class of rays where two rays are *equivalent* if no finite set of vertices separates them. Denote the set of the ends by $\Omega(G)$. We define a topological space, denoted by $|G|$, on G together with its ends, which is known as the *Freudenthal compactification* of G as follows. View G as a 1-complex. Thus, every edge is homeomorphic to the unit interval. The basic open neighborhoods of a vertex x consists of a choice of half-open half edges $[xz)$, one for each incident edge xy , where z is any interior point of xy . For an end $\omega \in \Omega(G)$, we take as a basic open neighborhood the set of the form: $\hat{C}(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}(S, \omega)$, where $S \subseteq V$ is a finite set of vertices, $C(S, \omega)$ is the component of $G - S$ in which every ray from ω has a tail, $\Omega(S, \omega)$ is the set of all ends whose rays have a tail in $C(S, \omega)$, and $\mathring{E}(S, \omega)$ is the set of all interior points of edges between S and $C(S, \omega)$. Let H be a subgraph of G . Then the closure of H in $|G|$ is called a *standard subspace* and is denoted by \overline{H} . We say H *contains* a point x of $|G|$ if $x \in \overline{H}$.

Let X and Y be two topological spaces. A continuous map from the unit interval $[0, 1]$ to X is a *path* in X . A homeomorphic image of $[0, 1]$ in X is called an *arc* in X . This induces an ordering $<$ for the points in the arc. The images of 0 and 1 are the *endpoints* of the arc. An arc in X with endpoints x and y is called an x - y *arc*. A homeomorphic image of the unit circle in X is called a *circle* in X . A (*path*-)*component* of X is a maximal (*path*-)connected set in X . X is *2-connected* (*2-arc-connected*) if for all $x \in X$, $X \setminus x$ is connected (arc-connected). We say X can be *embedded* in Y if there exists an injective continuous function $\phi : X \rightarrow Y$ such that X is homeomorphic to $\phi(X)$ in the subspace topology of Y . Then ϕ is called an *embedding* of X in Y . Take Y to be \mathbb{R}^2 . A component of $\mathbb{R}^2 \setminus \phi(X)$ is called a *face* of $\phi(X)$ in \mathbb{R}^2 . A graph G is *planar* if G can be embedded in \mathbb{R}^2 . A graph G is *outerplanar* if there exists an embedding ϕ of G in \mathbb{R}^2 such that there is a face f of $\phi(G)$ in \mathbb{R}^2 whose boundary ∂f contains all the vertices of G . Chartrand and Harary [2] characterized outerplanar finite graphs as precisely those graphs that do not contain a $K_{2,3}$ - or K_4 - subdivision.

Suppose A is an arc in $|G|$ and x is a vertex in A . Then the vertex immediately before x in A if exists is denoted by x^- and the vertex immediately after x in A if exists is denoted by x^+ . An arc in $|G|$ is an ω -*arc* if the end ω is one of its endpoints and unless otherwise stated, it corresponds to the image of 1. Following Bruhn and Stein [1], we define the *end degree* of an end ω in G as the supremum over the cardinalities of sets of edge-disjoint rays in ω , and denote this number by $deg_G(\omega)$. In fact, they proved that this is equal to the supremum over the cardinalities of sets of edge-disjoint ω -arcs in $|G|$. For a subgraph H of G , define the degree of ω in H as the supremum over the cardinalities of sets of edge-disjoint ω -arcs in \overline{H} which is denoted by $deg_H(\omega)$.

3 Contraction-critical 2-connected infinite graphs

It is well-known that the only contraction-critical 2-connected finite graph is K_3 . However, there are infinitely many contraction-critical 2-connected infinite graphs as shown by the following construction due to Kriesell [9]. Define $G_0 := \emptyset$ and let G_1 be any 2-connected finite graph. Suppose we have constructed G_n such that $G_{n-1} \subsetneq G_n$. For each edge xy in $E(G_n) \setminus E(G_{n-1})$, add a new x - y path of length at least 2. The resulting graph is G_{n+1} . Repeat the process inductively. Then the graph $G := \bigcup_{i \geq 1} G_i$ is a contraction-critical 2-connected infinite graph. Note that G has no vertex of finite degree and has uncountably many ends. We will show that this holds in general for any contraction-critical 2-connected infinite graph. First, we state a fundamental fact about contractible edges in 2-connected graphs.

Lemma 3. *Let G be a 2-connected graph nonisomorphic to K_3 and e be an edge of G . Then $G - e$ or G/e is 2-connected.*

Now, we can develop some tools that will be used for the rest of the paper.

Lemma 4. *Let G be a 2-connected graph nonisomorphic to K_3 , and e and f be two non-contractible edges of G . Then f is a non-contractible edge of $G - e$.*

Proof. By Lemma 3, $G - e$ is 2-connected. Since $V(f)$ is a 2-separator of G , $V(f)$ is also a 2-separator of $G - e$ and f is a non-contractible edge of $G - e$. \square

Lemma 5. *Let G be a 2-connected graph nonisomorphic to K_3 and F be a finite subset of $E(G)$.*

- (a) *If $G - F$ is disconnected, then F contains at least two contractible edges.*
- (b) *If $G - F$ is connected but not 2-connected, then F contains at least one contractible edge.*

Proof. For (a), suppose F contains at most one contractible edge. Then by Lemma 3 and 4, we can delete all the non-contractible edges in F and the resulting graph is still 2-connected, a contradiction.

For (b), suppose all edges in F are non-contractible. Then by Lemma 3 and 4, we can delete all edges in F and $G - F$ is still 2-connected, a contradiction. \square

Lemma 6. *Let G be a 2-connected graph nonisomorphic to K_3 . Let $\{x, y\}$ be a 2-separator of G and C be a component of $G - x - y$. If $|E_G(x, C)|$ is finite, then $E_G(x, C)$ contains a contractible edge.*

Proof. Note that y is a cutvertex of $G - E_G(x, C)$. By Lemma 5(b), $E_G(x, C)$ contains a contractible edge. \square

Lemma 7. *Let G be a 2-connected graph nonisomorphic to K_3 and x be a vertex of G . Suppose all edges incident to x are non-contractible. Then*

- (a) x has infinite degree.
- (b) For any edge xy incident to x , every component of $G - x - y$ contains infinitely many neighbors of x .

Proof. For (a), Suppose x has finite degree. By applying Lemma 5(a) to $E_G(x)$, x is incident to at least two contractible edges, a contradiction.

For (b), let C be a component of $G - x - y$. By Lemma 6, $E_G(x, C)$ contains infinitely many edges. □

Theorem 8. *Let G be a contraction-critical 2-connected infinite graph. Then every vertex of G has infinite degree and G has uncountably many ends.*

Proof. By Lemma 7(a), every vertex of G has infinite degree.

Next, we will construct a rooted binary infinite tree T in G together with edges incident to each vertex of T with the following properties:

- (1) The root of T is denoted by x .
- (2) The vertices of T are denoted by $x_{n_1 n_2 \dots n_k}$ where $k \in \mathbb{N}$ and $n_i \in \{0, 1\}$ for $1 \leq i \leq k$. For $k = 0$, define $x_{n_1 n_2 \dots n_k} := x$.
- (3) Each vertex $x_{n_1 n_2 \dots n_k}$ of T is adjacent to two vertices $x_{n_1 n_2 \dots n_k 0}$ and $x_{n_1 n_2 \dots n_k 1}$ in T .
- (4) For each vertex $x_{n_1 n_2 \dots n_k}$ of T , there exists an edge $x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k}$ in G such that $y_{n_1 n_2 \dots n_k}$ does not lie in T .
- (5) The subtree of T rooted at $x_{n_1 n_2 \dots n_k}$ is defined as

$$T_{n_1 n_2 \dots n_k} := T[\bigcup_{i=0}^{\infty} \bigcup_{(m_1, m_2, \dots, m_i) \in \{0, 1\}^i} x_{n_1 n_2 \dots n_k m_1 m_2 \dots m_i}].$$

For fixed n_1, n_2, \dots, n_k , $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ separates $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ in G .

Each ray in T starting at x is of the form: $xx_{n_1}x_{n_1 n_2}x_{n_1 n_2 n_3} \dots$. Let $R := xx_{n_1}x_{n_1 n_2}x_{n_1 n_2 n_3} \dots$ and $Q := xx_{m_1}x_{m_1 m_2}x_{m_1 m_2 m_3} \dots$ be two distinct rays in T . Then there exists a smallest k such that $n_i = m_i$ for all $i \leq k$ and $n_{k+1} \neq m_{k+1}$. By property (5) above, $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ separates R and Q in G . Therefore, each ray in T starting at x belongs to a unique end of G , and G has uncountably many ends.

Now, it remains to construct T inductively. Let x be any vertex in G . Define $T_0 := (\{x\}, \emptyset)$. Choose any edge incident to x in G , say xy . Let C_0 and C_1 be any two components of $G - x - y$. Let x_0 be a neighbor of x in C_0 and x_1 be a neighbor of x in C_1 . Define $T_1 := (\{x, x_0, x_1\}, \{xx_0, xx_1\})$. Note that $N_G(C_0) \subseteq \{x, y\}$ and $N_G(C_1) \subseteq \{x, y\}$. Also, $G - C_0$ and $G - C_1$ are both connected.

Suppose we have constructed the rooted binary tree T_k where

$$V(T_k) := \bigcup_{i=0}^k \bigcup_{(n_1, n_2, \dots, n_i) \in \{0, 1\}^i} x_{n_1 n_2 \dots n_i} \text{ and}$$

$$E(T_k) := \bigcup_{i=0}^{k-1} \bigcup_{(n_1, n_2, \dots, n_i) \in \{0, 1\}^i} \{x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 0}, x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 1}\} \text{ such that}$$

- (i) each vertex $x_{n_1 n_2 \dots n_i}$ ($0 \leq i \leq k$) lies in a connected subgraph $C_{n_1 n_2 \dots n_i}$ of G (for $i = 0$, $x_{n_1 n_2 \dots n_i} := x$, $y_{n_1 n_2 \dots n_i} := y$ and $C_{n_1 n_2 \dots n_i} := G$),
- (ii) for each vertex $x_{n_1 n_2 \dots n_i}$ ($0 \leq i < k$), we have found an edge $x_{n_1 n_2 \dots n_i} y_{n_1 n_2 \dots n_i}$ that lies in $C_{n_1 n_2 \dots n_i}$ such that $C_{n_1 n_2 \dots n_i 0}$ and $C_{n_1 n_2 \dots n_i 1}$ are two components of $C_{n_1 n_2 \dots n_i} - x_{n_1 n_2 \dots n_i} - y_{n_1 n_2 \dots n_i}$ that are adjacent to $x_{n_1 n_2 \dots n_i}$,
- (iii) for fixed n_1, n_2, \dots, n_i ($1 \leq i \leq k$), $N_G(C_{n_1 n_2 \dots n_i}) \subseteq \bigcup_{j=0}^{i-1} \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$,
- (iv) for fixed n_1, n_2, \dots, n_i ($1 \leq i \leq k$), $G - C_{n_1 n_2 \dots n_i}$ is connected.

Now, for each vertex $x_{n_1 n_2 \dots n_k}$ in T_k , since it has infinite degree and $N_G(x_{n_1 n_2 \dots n_k}) \setminus C_{n_1 n_2 \dots n_k} \subseteq N_G(C_{n_1 n_2 \dots n_k})$ is finite by (iii), all but finitely many neighbors of $x_{n_1 n_2 \dots n_k}$ lie in $C_{n_1 n_2 \dots n_k}$. Let z be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k}$ and $B := C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z$. Suppose B is connected. Since $B' := G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), B and B' are the only two components of $G - x_{n_1 n_2 \dots n_k} - z$. By Lemma 7(b), B' contains infinitely many neighbors of $x_{n_1 n_2 \dots n_k}$ contradicting $N_G(x_{n_1 n_2 \dots n_k}) \setminus C_{n_1 n_2 \dots n_k} \subseteq N_G(C_{n_1 n_2 \dots n_k})$ which is finite by (iii). Therefore, B is not connected.

Note that at least one component of B is adjacent to $x_{n_1 n_2 \dots n_k}$. If not, then, by the 2-connectedness of G , each component of B has a neighbor in $N_G(C_{n_1 n_2 \dots n_k}) \subseteq G - C_{n_1 n_2 \dots n_k}$. By (iv), this implies $G - x_{n_1 n_2 \dots n_k} - z$ is connected, a contradiction. Suppose there are two components of B , say D and D' , that are both adjacent to $x_{n_1 n_2 \dots n_k}$. Then choose $y_{n_1 n_2 \dots n_k} := z$, $C_{n_1 n_2 \dots n_k 0} := D$ and $C_{n_1 n_2 \dots n_k 1} := D'$. Suppose only one component of B is adjacent to $x_{n_1 n_2 \dots n_k}$, say C . Each component of B other than C is adjacent to z by the connectedness of $C_{n_1 n_2 \dots n_k}$ and has a neighbor in $N_G(C_{n_1 n_2 \dots n_k}) \subseteq G - C_{n_1 n_2 \dots n_k}$ by the 2-connectedness of G . Denote the union of components of B other than C by C'' . Since $G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), $C''' := G[(G - C_{n_1 n_2 \dots n_k}) \cup C'']$ is connected. Hence, C and C''' are the only two components of $G - x_{n_1 n_2 \dots n_k} - z$ and $N_G(C) = \{x_{n_1 n_2 \dots n_k}, z\}$. Let z' be a neighbor of $x_{n_1 n_2 \dots n_k}$ in C . Then one component D of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$ contains z and C'' . Since $G - x_{n_1 n_2 \dots n_k} - z'$ is not connected, D cannot be the only component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$. Let D' be any component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$ other than D . Then D' lies in C and $N_G(D') \subseteq \{x_{n_1 n_2 \dots n_k}, z'\} \cup (N_G(C) - z) = \{x_{n_1 n_2 \dots n_k}, z'\}$. Now, choose $y_{n_1 n_2 \dots n_k} := z'$, $C_{n_1 n_2 \dots n_k 0} := D$ and $C_{n_1 n_2 \dots n_k 1} := D'$.

In both cases, $x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k}$ lies in $C_{n_1 n_2 \dots n_k}$, and $C_{n_1 n_2 \dots n_k 0}$ and $C_{n_1 n_2 \dots n_k 1}$ are two components of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ that are adjacent to $x_{n_1 n_2 \dots n_k}$. Let $x_{n_1 n_2 \dots n_k 0}$ be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k 0}$ and $x_{n_1 n_2 \dots n_k 1}$ be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k 1}$. Since $C_{n_1 n_2 \dots n_k 0} \subseteq C_{n_1 n_2 \dots n_k}$ and $C_{n_1 n_2 \dots n_k 1} \subseteq C_{n_1 n_2 \dots n_k}$, $N_G(C_{n_1 n_2 \dots n_k 0}) \subseteq N_G(C_{n_1 n_2 \dots n_k}) \cup \{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\} \subseteq \bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ and $N_G(C_{n_1 n_2 \dots n_k 1}) \subseteq N_G(C_{n_1 n_2 \dots n_k}) \cup \{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\} \subseteq \bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ by (iii). By the connectedness of $C_{n_1 n_2 \dots n_k}$, every component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ has a neighbor in $\{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\}$. For $n_{k+1} \in \{0, 1\}$, denote the union of the components of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ other than $C_{n_1 n_2 \dots n_k n_{k+1}}$ by $U_{n_{k+1}}$. Then $G[x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k} \cup U_{n_{k+1}}]$ is connected. Since $x_{n_1 n_2 \dots n_{k-1}} \in G - C_{n_1 n_2 \dots n_k}$, $x_{n_1 n_2 \dots n_{k-1}} x_{n_1 n_2 \dots n_k} \in E(G)$ and $G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), $G - C_{n_1 n_2 \dots n_k n_{k+1}} := G[(G - C_{n_1 n_2 \dots n_k}) \cup x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k} \cup U_{n_{k+1}}]$ is connected.

Define T_{k+1} where $V(T_{k+1}) := \bigcup_{i=0}^{k+1} \bigcup_{(n_1, n_2, \dots, n_i) \in \{0,1\}^i} x_{n_1 n_2 \dots n_i}$ and

$$E(T_{k+1}) := \bigcup_{i=0}^k \bigcup_{(n_1, n_2, \dots, n_i) \in \{0,1\}^i} \{x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 0}, x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 1}\}.$$

Finally, define $T := \bigcup_{k=0}^{\infty} T_k$. It is easy to see that T satisfies properties (1) through (4).

Let z_0 and z_1 be any vertices in $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ respectively. Then z_0 is of the form $x_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i}$ while z_1 is of the form $x_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_j}$. We have $z_0 = x_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i} \in C_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i} \subseteq C_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_{i-1}} \subseteq \dots \subseteq C_{n_1 n_2 \dots n_k 0}$ and $z_1 = x_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_j} \in C_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_j} \subseteq C_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_{j-1}} \subseteq \dots \subseteq C_{n_1 n_2 \dots n_k 1}$. Therefore, $T_{n_1 n_2 \dots n_k 0} \subseteq C_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1} \subseteq C_{n_1 n_2 \dots n_k 1}$. Since $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ contains both $N_G(C_{n_1 n_2 \dots n_k 0})$ and $N_G(C_{n_1 n_2 \dots n_k 1})$, it separates $C_{n_1 n_2 \dots n_k 0}$ and $C_{n_1 n_2 \dots n_k 1}$ in G and thus separates $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ in G . Hence, Property (5) holds for T and the proof is complete. \square

4 Subgraph induced by all the contractible edges

In this section, we will extend Theorem 1 to any 2-connected locally finite infinite graph G and prove that $\overline{G_C}$ is 2-arc-connected. Note that Lemma 5(a) implies that every vertex is incident to at least two contractible edges. Hence, G_C is spanning. Using the following two lemmas, it is easy to see that $\overline{G_C}$ is arc-connected.

Lemma 9 (Diestel [3]). *Let G be a locally finite graph. Then a standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of G of which it meets both sides.*

Lemma 10 (Diestel and Kühn [6]). *If G is a locally finite graph, then every closed connected subspace of $|G|$ is arc-connected.*

Theorem 11. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is arc-connected.*

Proof. Let F be any finite cut of G . By Lemma 5(a), F contains at least two edges in G_C . Hence, $\overline{G_C}$ is connected by Lemma 9. By Lemma 10, $\overline{G_C}$ is arc-connected. \square

Next, we prove that $\overline{G_C}$ is 2-connected.

Lemma 12. *Let G be a 2-connected locally finite infinite graph and x be a point of $|G|$. Suppose there is a partition (X, X') of $V(G \setminus x)$ such that $E_G(X, X')$ is non-empty and all edges in $E_G(X, X')$ are non-contractible. Then G contains a subdivision of a 1-way infinite ladder L consisting of two disjoint rays: $R := x_0 P_1 x_1 P_2 x_2 \dots$ and $R' := x'_0 P'_1 x'_1 P'_2 x'_2 \dots$ with the rungs of the ladder being $x_0 x'_0, x_1 x'_1, x_2 x'_2, \dots$, all of which are X - X' edges such that $x \notin \overline{L}$.*

Proof. Since G is 2-connected, $|E_G(X, X')| \geq 2$ unless x is a vertex and $|E_G(X, X')| = 1$. Consider any X - X' edge $x_0 x'_0$ that does not contain x . Let C be the component of $G - x_0 - x'_0$ containing x and C_1 be a component of $G - x_0 - x'_0$ not containing x .

Suppose we have constructed the finite ladder L_k consisting of two disjoint paths $R_k := x_0 P_1 x_1 P_2 x_2 \dots x_{k-1} P_k x_k$ and $R'_k := x'_0 P'_1 x'_1 P'_2 x'_2 \dots x'_{k-1} P'_k x'_k$ with the rungs of the

ladder being $x_0x'_0, x_1x'_1, \dots, x_kx'_k$, all of which are X - X' edges such that $L_k \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_k - x_k - x'_k]$ is connected. Let C_{k+1} be a component of $G - x_k - x'_k$ not containing x . Then $C_{k+1} \subseteq C_1$ and $C_{k+1} \cap L_k = \emptyset$. By applying Lemma 6 to $E_G(x_k, C_{k+1})$ and $E_G(x'_k, C_{k+1})$, there exist contractible edges $x_k y_{k+1}$ and $x'_k y'_{k+1}$ where $y_{k+1} \in C_{k+1}$ and $y'_{k+1} \in C_{k+1}$. Since $x \notin C_{k+1}$ and all edges in $E_G(X, X')$ are non-contractible, $y_{k+1} \in X$ and $y'_{k+1} \in X'$. Choose a path Q_{k+1} in C_{k+1} between y_{k+1} and y'_{k+1} . Then there exists an X - X' edge $x_{k+1}x'_{k+1}$ on Q_{k+1} such that $V(y_{k+1}Q_{k+1}x_{k+1}) \subseteq X$ and $x'_{k+1} \in X'$. Define $P_{k+1} := x_k y_{k+1} \cup y_{k+1} Q_{k+1} x_{k+1}$, $P'_{k+1} := x'_k y'_{k+1} \cup y'_{k+1} Q_{k+1} x'_{k+1}$, $R_{k+1} := R_k \cup P_{k+1}$, $R'_{k+1} := R'_k \cup P'_{k+1}$ and $L_{k+1} := L_k \cup P_{k+1} \cup P'_{k+1} \cup x_{k+1} x'_{k+1}$. Note that $L_{k+1} \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_{k+1} - x_{k+1} - x'_{k+1}]$ is connected.

Define $R := \bigcup_{k \geq 0} R_k$, $R' := \bigcup_{k \geq 0} R'_k$ and $L := \bigcup_{k \geq 0} L_k$. Then $L \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $x \notin \bar{L}$. \square

Theorem 13. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is 2-connected.*

Proof. Suppose $\overline{G_C}$ is not 2-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not connected. Let U and U' be two disjoint non-empty open sets in $|G|$ such that $\overline{G_C} \setminus x \subseteq U \cup U'$, $(\overline{G_C} \setminus x) \cap U \neq \emptyset$ and $(\overline{G_C} \setminus x) \cap U' \neq \emptyset$. Define $X := (\overline{G_C} \setminus x) \cap U \cap V(G)$ and $X' := (\overline{G_C} \setminus x) \cap U' \cap V(G)$. Since G_C is spanning, $X \cup X' = V(G \setminus x)$. Suppose U contains an interior point a of an edge bc of G_C . Then $\overline{G_C} \setminus x$ contains half edges $[ba]$ or $[ca]$ of bc . By the connectedness of half edge, U contains b or c . Suppose U contains an end ω of $|G|$. Then U contains a basic open neighborhood of ω , say $\hat{C}(S, \omega)$, and thus contains infinitely many vertices. The same arguments hold for U' . Therefore, both X and X' are non-empty. Since $G \setminus x$ is connected, $E_G(X, X')$ is non-empty.

Suppose x is a vertex or an end of G . Then all edges in $E_G(X, X')$ are non-contractible and (X, X') is a partition of $V(G \setminus x)$. Suppose x is an interior point of an edge e . Then all edges in $E_G(X, X')$ are non-contractible unless $e \in E_G(X, X') \cap E_C$. Note that, $E_G(X, X') - e$ is non-empty as G is 2-connected and every edge in $E_G(X, X') - e$ is non-contractible. Let $e = yy'$ where $y \in X$ and $y' \in X'$. Suppose $X = \{y\}$. By Lemma 5(a), since y is incident to at least two contractible edges, there is a contractible X - X' edge other than e , which is impossible. Therefore, $|X| \geq 2$. Now, all edges in $E_G(X - y, X')$ are non-contractible and $(X - y, X')$ is a partition of $V(G \setminus y)$. In both cases, by Lemma 12, G contains a subdivision of a 1-way infinite ladder L such that $x \notin \bar{L}$.

Let ω be the end of $|G|$ containing R and R' . Note that $\omega \neq x$. Since G_C is spanning, $\overline{G_C} \setminus x$ contains all the ends of $|G|$ except possibly x . Without loss of generality, assume $\omega \in U$. Since U is open, there exists a basic open neighborhood $\hat{C}(S, \omega) \subseteq U$. Since $x'_0, x'_1, x'_2, \dots \in X' \subseteq U'$ converge to ω , all but finitely many of them lie in $\hat{C}(S, \omega)$, contradicting $U \cap U' = \emptyset$. \square

Finally, we prove the main result of this section, namely, $\overline{G_C}$ is 2-arc-connected. This follows from a theorem by Georgakopoulos [7] concerning connected but not path-connected subspaces of locally finite graphs. Note that since $|G|$ is Hausdorff, path-connectedness is equivalent to arc-connectedness.

Theorem 14 (Georgakopoulos [7]). *Given any locally finite connected graph G , a connected subspace X of $|G|$ is path-connected unless it satisfies the following assertions:*

- (1) X has uncountably many path-components each of which consists of one end only;
- (2) X has infinitely many path-components that contain a vertex; and
- (3) every path-component of X contains an end.

Theorem 15. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is 2-arc-connected.*

Proof. Suppose $\overline{G_C}$ is not 2-arc-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not arc-connected. Note that $\overline{G_C} \setminus x$ is connected by Theorem 13. By Theorem 14, $\overline{G_C} \setminus x$ has uncountably many path-components each of which consists of one end only. Let ω and ω' be two such path-components of $\overline{G_C} \setminus x$. Since $\overline{G_C}$ is arc-connected by Theorem 11, there exists an arc A joining ω and ω' in $\overline{G_C}$. Now, x must lie in A for otherwise ω and ω' would lie in the same path-component of $\overline{G_C} \setminus x$. But the path-component of $\overline{G_C} \setminus x$ containing ω also contains $[\omega Ax)$, a contradiction. \square

5 Outerplanarity of 2-connected locally finite graphs

As mentioned in the introduction, in order to extend Theorem 2 to locally finite infinite graphs, we would like to prove that for any 2-connected locally finite infinite graph G , if every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This requires several lemmas listed below.

Lemma 16. *Let G be a locally finite graph. Then every arc in $|G|$ whose two endpoints are ends contains a vertex.*

Proof. Suppose A is an arc in $|G|$ whose two endpoints are ends ω_1 and ω_2 . Then there exists a finite set S of vertices such that $\hat{C}(S, \omega_1)$ and $\hat{C}(S, \omega_2)$ are distinct. By the connectedness of A , A contains a vertex of S . \square

Lemma 17. *Let G be a locally finite graph and ω be an end in $|G|$. Then every ω -arc A in $|G|$ contains a vertex, say z , and zA contains a ray starting with z .*

Proof. Denote the starting point of A by a . First, we show that A contains a vertex. If a is a vertex, then we are done. If a is an end, then it is true by Lemma 16. If a is an interior point of an edge xy , then by the connectedness of A , A contains x or y .

Let z be a vertex in A . By the connectedness of zA , zA contains an interior point of an edge incident to z , say zz_1 . Then the connectedness of zA implies zz_1 lies in zA . Repeat the above argument for z_1A and so on. We obtain a ray that starts with z and lies in zA . \square

Lemma 18. *Let G be a locally finite graph and ω be an end in $|G|$. Let A_1 and A_2 be two ω -arcs in $|G|$ that are disjoint except at ω . Then, for all finite subset S of $V(G)$, $\hat{C}(S, \omega)$ contains a subarc $A'_1\omega A'_2$ of $A_1\omega A_2$ and there is an A'_1 - A'_2 path in $C(S, \omega)$.*

Proof. Let x_1 be the last point of A_1 that lies in S and x_2 be the last point of A_2 that lies in S . By Lemma 17, x_1A contains a ray R_1 starting with x_1 and x_2A contains a ray R_2 starting with x_2 . Let y_1 be the neighbor of x_1 in R_1 and y_2 be the neighbor of x_2 in R_2 . Then $y_1A_1\omega A_2y_2$ lies in $\hat{C}(S, \omega)$. Also there is a y_1 - y_2 path in $C(S, \omega)$ which automatically contains a y_1A_1 - y_2A_2 path. \square

We also need a result on the characterization of a topological circle in $|G|$ in terms of its vertex and end degrees.

Lemma 19 (Bruhn and Stein [1]). *Let C be a subgraph of a locally finite graph G . Then \overline{C} is a circle if and only if \overline{C} is connected and every vertex and end of $|G|$ in \overline{C} has degree two in \overline{C} .*

Now, we can proceed with the proof.

Theorem 20. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . If every vertex of G is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle.*

Proof. Since G_C is spanning, $\overline{G_C}$ contains all vertices and ends of $|G|$. By Theorem 11, $\overline{G_C}$ is arc-connected. Obviously, every vertex of G has degree two in $\overline{G_C}$. Therefore, it remains to prove that every end of $|G|$ has degree two in $\overline{G_C}$.

Claim 21. *Let A be an arc in $\overline{G_C}$ and x be a vertex in A . Suppose that both x^- and x^+ exist in A . Let y be any neighbor of x other than x^- and x^+ . Then every x^- - x^+ arc in $|G|$ intersects $\{x, y\}$.*

Proof. Since xx^- and xx^+ are the only contractible edges incident to x , xy is non-contractible. Lemma 6 implies that $G - x - y$ has exactly two components, and x^- and x^+ lie in different components. By the connectedness of an arc, every x^- - x^+ arc in $|G|$ intersects $\{x, y\}$. \square

Claim 22. *Let ω be an end in $|G|$. Suppose A_1 and A_2 are two edge-disjoint ω -arcs in $\overline{G_C}$. Then A_1 and A_2 can intersect only at the ends of $|G|$ with the only possible exception being that the starting points of A_1 and A_2 are the same vertex.*

Proof. Obviously, A_1 and A_2 cannot intersect at an interior point of an edge. Suppose A_1 and A_2 intersect at a vertex x . If x is not the starting point for both A_1 and A_2 , then the degree of x in $\overline{G_C}$ is at least three, a contradiction. \square

Claim 23. *Let ω be an end in $|G|$. Suppose A_1 and A_2 are two edge-disjoint ω -arcs in $\overline{G_C}$ such that the starting points of A_1 and A_2 are distinct vertices. Then there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' unless $A_1 \cap A_2 = \{\omega\}$.*

Proof. Suppose $A_1 \cap A_2 \neq \{\omega\}$. Let ω' be the first point of A_2 that intersects A_1 . By Claim 22, ω' is an end in $|G|$ different from ω . Then $A_1\omega'$, $A_2\omega'$ and $\omega A_1\omega'$ are the required three ω' -arcs. \square

Claim 24. *Let ω be an end in $|G|$. Suppose there are three edge-disjoint ω -arcs in $\overline{G_C}$. Then there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' .*

Proof. Let A_1, A_2, A_3 be three edge-disjoint ω -arcs in $\overline{G_C}$. By Lemma 17, for each $i \in \{1, 2, 3\}$, A_i contains a ray R_i . Denote the first edge of R_i by $x_i y_i$. By Claim 22, y_1, y_2, y_3 are all distinct. Therefore, without loss of generality, we can assume that the starting points of A_1, A_2, A_3 are all distinct vertices. Consider A_1 and A_2 . If $A_1 \cap A_2 \neq \{\omega\}$, then the claim follows from Claim 23. Suppose $A_1 \cap A_2 = \{\omega\}$. If $A_2 \cap A_3 \neq \{\omega\}$, then again the claim follows from Claim 23. Therefore, suppose $A_2 \cap A_3 = \{\omega\}$. But then, A_1, A_2, A_3 are the desired three ω -arcs. \square

Claim 25. *For each end ω in $|G|$, $\deg_{G_C}(\omega) \leq 2$.*

Proof. Suppose there are three edge-disjoint ω -arcs in $\overline{G_C}$. By Claim 24, there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' . Denote these three ω' -arcs by A_1, A_2, A_3 . By Lemma 16, without loss of generality, we can assume A_1, A_2, A_3 start with vertices a_1, a_2, a_3 respectively.

By applying Lemma 18 to A_1 and A_2 with $S = \{a_1, a_2\}$, we obtain an $a_1^+ A_1 - a_2^+ A_2$ path P . Let $x_1 = P \cap A_1$, $x_2 = P \cap A_2$ and x be the neighbor of x_1 in Q . If P intersects A_3 , then interchange A_2 and A_3 . Therefore, without loss of generality, there is an $a_1^+ A_1 - a_2^+ A_2$ path P that does not intersect A_3 .

Now, apply Lemma 18 to $x_2 A_2$ and A_3 with $S = V(P)$. We obtain an $x_2^+ A_2 - A_3$ path Q not intersecting P . Let $y_2 = Q \cap x_2^+ A_2$ and $y_3 = Q \cap A_3$. By Claim 21, Q cannot intersect $A_2 x_2^-$, and Q cannot intersect both $A_1 x_1^-$ and $x_1^+ A_1$. Suppose $Q \cap A_1 x_1^- \neq \emptyset$. Let y be the first vertex of Q that lies in $A_1 x_1^-$. Then $x_1^- A_1 y Q y_2 A_2 \omega' A_1 x_1^+$ is an $x_1^- - x_1^+$ arc not intersecting $\{x_1, x\}$, contradicting Claim 21. Suppose $Q \cap x_1^+ A_1 \neq \emptyset$. Then there is an $x_1^+ A_1 - A_3$ subpath in Q not intersecting A_2 , and we interchange A_1 and A_2 . Therefore, without loss of generality, we can assume that there is an $x_2^+ A_2 - A_3$ path Q that does not intersect $P \cup A_1 \cup A_2 x_2^-$. Let u_2 be the neighbor of y_2 in Q and u_3 be the neighbor of y_3 in Q .

Finally, apply Lemma 18 to $x_1 A_1$ and $y_2 A_2$ with $S = V(P \cup Q)$. We obtain an $x_1^+ A_1 - y_2^+ A_2$ path R not intersecting $P \cup Q$. Let $z_1 = R \cap x_1^+ A_1$ and $z_2 = R \cap y_2^+ A_2$. By Claim 21, R cannot intersect $A_1 x_1^-$ and R cannot intersect $A_2 y_2^-$. Also, R cannot intersect both $A_3 y_3^-$ and $y_3^+ A_3$. Suppose $R \cap A_3 y_3^- \neq \emptyset$. Let z be the last vertex of R that lies in $A_3 y_3^-$. Then $y_3^- A_3 z R z_2 A_2 \omega' A_3 y_3^+$ is an $y_3^- - y_3^+$ arc not intersecting $\{y_3, u_3\}$, contradicting Claim 21. Suppose $R \cap y_3^+ A_3 \neq \emptyset$. Let z' be the first vertex of R that lies in $y_3^+ A_3$. Then $y_2^- A_2 x_2 P x_1 A_1 z_1 R z' A_3 \omega' A_2 y_2^+$ is an $y_2^- - y_2^+$ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 21. Therefore, $R \cap (A_1 x_1^- \cup A_2 y_2^- \cup A_3 \cup P \cup Q) = \emptyset$. But, $y_2^- A_2 x_2 P x_1 A_1 z_1 R z_2 A_2 y_2^+$ is an $y_2^- - y_2^+$ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 21. \square

Claim 26. *For each end ω in $|G|$, $\deg_{G_C}(\omega) = 2$.*

Proof. Let x be a vertex in G_C . Since $\overline{G_C}$ is arc-connected, there is an ω -arc A in $\overline{G_C}$ joining x to ω . Let y be the neighbor of x in A and a be an interior point of xy . Since $\overline{G_C}$ is 2-connected, $\overline{G_C} \setminus a$ is connected. Suppose $\overline{G_C} - xy$ is not connected. Then there exist two disjoint nonempty open sets U and V in $|G|$ such that $\overline{G_C} - xy \subseteq U \cup V$, $U \cap \overline{G_C} - xy \neq \emptyset$ and $V \cap \overline{G_C} - xy \neq \emptyset$. If $x, y \in U$, then $U \cup [x, a) \cup [y, a)$ and V are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is impossible. If $x \in U$ and $y \in V$, then $U \cup [x, a)$ and $V \cup [y, a)$ are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is also impossible. Therefore, $\overline{G_C} - xy$ is connected and is arc-connected by Lemma 10. Let A' be an x - ω arc in $\overline{G_C} - xy$. If $yA\omega \cap A'$ contains a vertex u , then u has degree at least three in G_C , a contradiction. Let ω' be the first point in $yA\omega \cap A'$ which is an end. If $\omega' \neq \omega$, then $\deg_{G_C}(\omega') \geq 3$ contradicting Claim 25. Therefore, $\omega' = \omega$ and we have $\deg_{G_C}(\omega) \geq 2$. By Claim 25, $\deg_{G_C}(\omega) = 2$. \square

We are now ready to prove the infinite analog of Theorem 2.

Theorem 27. *Let G be a 2-connected locally finite graph nonisomorphic to K_3 . Then the following are equivalent:*

- (1) *Every vertex of G is incident to exactly two contractible edges.*
- (2) *Every finite bond of G contains exactly two contractible edges.*
- (3) *G is outerplanar.*

Proof.

(2) \Rightarrow (1) Trivial.

(1) \Rightarrow (3) By Theorem 2, this is true for finite G . Therefore, assume G is infinite. Suppose every vertex of G is incident to exactly two contractible edges. By Theorem 20, $\overline{G_C}$ is a Hamilton circle. All edges in $E(G) \setminus E_C$ are chords of $\overline{G_C}$ and are non-contractible. Consider any chord xy of $\overline{G_C}$. Since every vertex of G is incident to exactly two contractible edges, by Lemma 6, $G - x - y$ consists of exactly two components C_1 and C_2 . Without loss of generality, assume that $x^+ \overline{G_C} y^- \subseteq C_1$ and $y^+ \overline{G_C} x^- \subseteq C_2$. Then there is no chord of $\overline{G_C}$ between $x^+ \overline{G_C} y^-$ and $y^+ \overline{G_C} x^-$. Hence, no chords of $\overline{G_C}$ are overlapping. Embed $\overline{G_C}$ in a circle of \mathbb{R}^2 and denote the embedding by ϕ . Now, draw every chord xy of $\overline{G_C}$ as a straight line segment between $\phi(x)$ and $\phi(y)$ in \mathbb{R}^2 . This shows that G is outerplanar.

(3) \Rightarrow (2) Let B be a finite bond of G between two components X and Y of $G - B$. By Lemma 5(a), B contains at least two contractible edges. Suppose B contains three contractible edges x_1y_1, x_2y_2, x_3y_3 such that $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Since X and Y are connected, there exists a path P in X joining x_1 to x_2 and a path Q in Y joining y_1 to y_2 . Let $C := P \cup x_1y_1 \cup Q \cup x_2y_2$. Obviously, $x_3y_3 \notin E(C)$. Take any x_3 - P path P' in X joining x_3 to P at x and any y_3 - Q path Q' in Y joining y_3 to Q at y . Let $R' := P' \cup x_3y_3 \cup Q'$. If $R' = x_3y_3$, then x_3y_3 is a chord of C . Since x_3y_3 is contractible, the two components of $C - x_3 - y_3$ are joined by a path, say R . Then $C \cup R \cup R'$ is a

K_4 -subdivision and G is not outerplanar. Suppose $R' \neq x_3y_3$. If both x - y paths in C are not edges, then $C \cup R'$ is a $K_{2,3}$ -subdivision and G is not outerplanar. If one of the two x - y paths in C is an edge, then without loss of generality, assume $x_2 = x$ and $y_2 = y$. Since x_2y_2 is contractible, the two components of $(C \cup R') - x_2 - y_2$ are joined by a path, say R . Then $C \cup R \cup R'$ is a K_4 -subdivision and G is not outerplanar. \square

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