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Masterarbeit

Decomposing Edge-Colored Infinite Graphs into Monochromatic Paths and Cycles

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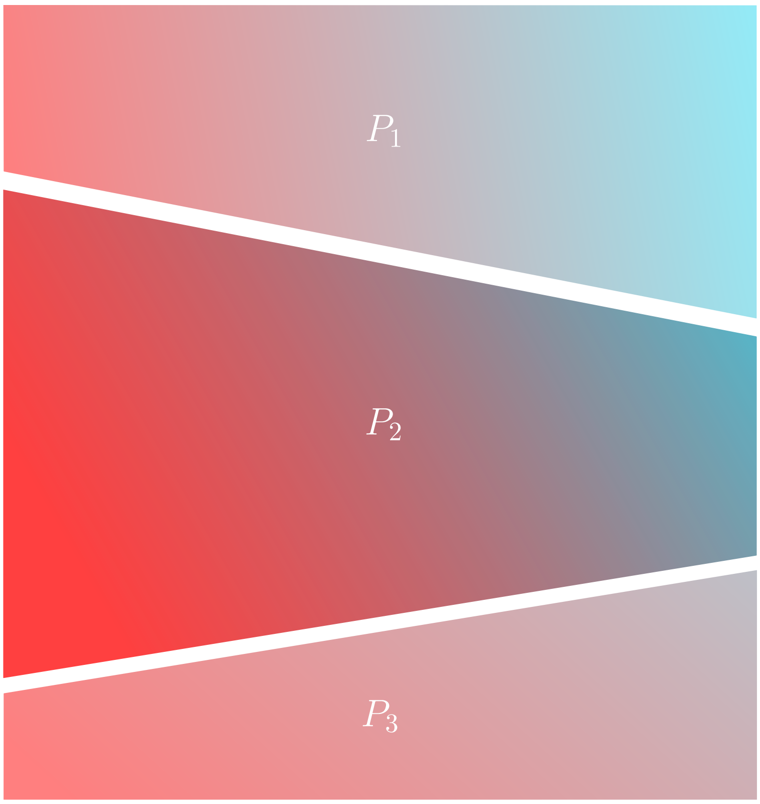

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Abstract

In the 1970s, Richard Rado showed that the vertex set of an edge-colored complete graph of countably infinite order with $r \in \mathbb{N}$ many colors has a partition into vertex sets of monochromatic paths of different colors. He asked whether this remains true for uncountable complete graphs and generalized paths. In 2016, Daniel Soukup answered this in the affirmative and conjectured that a similar result should hold for complete bipartite graphs with bipartition classes of the same infinite cardinality and up to $2r - 1$ monochromatic generalized paths of not necessarily different colors allowed — Soukup already confirmed this in the countably infinite case. As one of our main result of this thesis, we prove the smallest uncountable case of Soukup's conjecture, i.e., we show that the vertex set of an r-edge-colored complete bipartite graph with bipartition classes of size $\aleph_1$ admits a partition into $2r-1$ monochromatic generalized paths. Furthermore, we study analog statements for countably infinite graphs, where this time we partition the vertex set into more general classes of locally finite graphs than just paths.

Contents

1 Introduction

In 1978, R. Rado published the following result by P. Erdős:

Theorem 1.1 (P. Erdős, [9])**.** *Every* 2*-edge-colored, complete, countably infinite graph has a partition of its vertex set into two monochromatic paths of different colors*[1]*.*

Here, paths are trivial graphs, finite paths, or 1-way infinite paths. Rado generalized this theorem to finite edge-colorings, i.e., edge-colorings which use only finitely many colors:

Theorem 1.2 (R. Rado, [9])**.** *Every finitely edge-colored, complete, countably infinite graph has a partition of its vertex set into monochromatic paths of different colors.*

Since then, numerous authors have studied questions. Rado was interested in further results as well. He asked whether Theorem 1.2 remains true for complete graphs of arbitrary infinite order and the following notion of generalized paths:

Definition 1.3 (R. Rado, [9])**.** [2] Let G be a graph, P a set of vertices, and $\preceq$ be a well-order on P. The pair $\langle P, \preceq\rangle$ is a *generalized path* in G iff for every vertex $p \in P$ the set

$$\{q \colon q \prec p\} \cap N_G(p)$$

of $\preceq$-down-neighbors is cofinal in $\{q \colon q \prec p\}$. (Cf. Figure 1.)

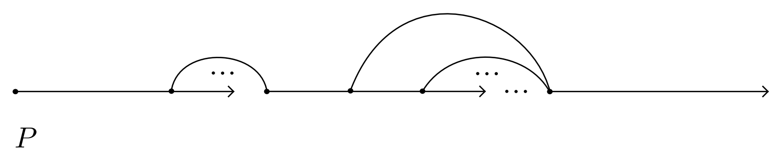


Figure 1: Generalized Path.

If the situation is clear, we will write P instead of $\langle P, \preceq\rangle$. The well-order $\preceq$ on P will be referred to as $\preceq_P$.

Only recently, M. Elekes, D. Soukup, L. Soukup, and Z. Szentmiklóssy proved in [6] that Theorem 1.2 remains true for 2-edge-coloured complete graphs of order $\aleph_1$ and D. Soukup gave a proof of the full statement shortly after:

Theorem 1.4 (D. Soukup, [11])**.** *Every finitely edge-colored, complete, infinite graph has a partition of its vertex set into monochromatic generalized paths of different colors.*

[1] In this thesis, we allow partition classes to be empty.

[2] R. Diestel and I. Leader have studied so called T-graphs and generalized paths are exactly the T-graphs for which T is an ordinal (cf. [4]).

At the end of his paper, D. Soukup conjectures that a similar statement holds for complete bipartite infinite graphs:

Conjecture 1.5 (D. Soukup, [11]). *Every r-edge-colored complete bipartite graph with bipartition classes of some fixed infinite size has a partition its vertex set into $2r-1$ monochromatic generalized paths.*

In [10] D. Soukup gives a proof of the countably infinite case. We prove the first uncountable case:

Theorem 1.6. *Let r be a positive integer. Every r-edge-colored complete bipartite graph with bipartition classes of size $\aleph_1$ has a partition of its vertex set into $2r-1$ monochromatic generalized paths.*

The crux of the proof of Theorem 1.4 was to recursively constructed and reconstructed generalized paths using *nice* chains of countable *elementary submodels*. We will adopt this technique.

Now, turning to partitions of the vertex set of a graph into the vertex sets of cycles, a prominent result in the spirit of Theorem 1.2 is the following, which verifies a conjecture of J. Lehel:

Theorem 1.7 (S. Bessy, S. Thomassé, [1]). *Every 2-edge-colored, complete, finite graph has a partition of its vertex set into the vertex sets of two, differently colored, monochromatic graphs that are cycles, single edges, or trivial graphs.*

Evidently, this statement cannot be generalized directly to infinite complete graphs, because a finite number of finite cycles cannot cover the whole vertex set of an infinite graph. R. Diestel asked, whether the statement becomes true again when "cylce" is substituted by his notion of topological circle. We prove the following:

Theorem 1.8. *Every r-edge-colored, complete, countably infinite graph has a partition of its vertex set into the vertex sets of monochromatic hamiltonian graphs of different colors.*[3]

Furthermore, we prove a related statement on complete bipartite graphs with countably infinite bipartition classes:

Theorem 1.9. *Every r-edge-colored complete bipartite graph with countably infinite bipartition classes has a partition of its vertex set into the vertex sets of $2r-1$ monochromatic hamiltonian graphs.*

At the beginning of this thesis, we clarify the most basic definitions and nomenclature for our theorems and proofs. Most of the notation is standard and intuitive. However, it is important to be precise at this point, because of the model-theoretic techniques that are used. In Section 3, results on inseparability are listed. The reader may skip this section and come back whenever statements are needed. Theorem 1.8 and Theorem

[3]Trivial graphs are *hamiltonian* in this thesis.

1.9 are discussed in Section 4. The output hamiltonian graphs are either trivial graphs, cycles, or 1-way infinite ladders. In Section 5, we discuss Theorem 1.6 which is the main result of this thesis. Section 4 and Section 5 can be considered separately. However, the discussion of cycle decompositions is a good introduction to the proof of Theorem 1.6. We conclude this thesis with two open problems and notes on *locally finite* edge-colorings. These are edge-colorings, which possibly use infinitely many colors, but only finitely many in the neighborhood of every vertex.

It is assumed that the kind reader is knowledgeable about infinite graph theory and set theory. All the proofs that use model-theoretic arguments are included in an additional subsection so that large parts of this thesis can be understood without knowledge about elementary submodels.

2 Notation

Mostly we follow the established textbooks *Graph Theory* by Diestel [3] and *Set Theory* by K. Kunen [7]. Everything not mentioned can be found there. For edge-colorings we mainly adopt the notation that Soukup et al. use in [6].

Definition 2.1. Let $\leq$ be a partial order on a set A, let $B \subseteq A$ and let $y, z \in A$.

☞ $y{\downarrow} := \{x \in A \colon x < y\}$.

☞ B is an *initial segment* of A iff $\forall a \in A, b \in B[a \leq b \to a \in B]$.

☞ $[y, z) := \{x \in A \colon y \leq x < z\}$.

Definition 2.2. Let $\leq$ be a well-order on a set W and $x \in W$.

☞ x is a $\leq$*-successor* iff W has a smallest element above x.

☞ x is a $\leq$*-limit* iff it is no $\leq$-successor. If x is the smallest limit above y, then we refer to x by $y + \omega$.

The orders in Definition 2.1 and Definition 2.2 will always belong to generalized paths or *von Neumann ordinals*.

Definition 2.3. Let x be a set.

☞ $S(x) := x \cup \{x\}$.

☞ x is *transitive* iff $\forall y \in x[y \subseteq x]$.

☞ x is a (*von Neumann*) *ordinal* iff it is transitive and well-ordered by $\subseteq$.

Now, we list some helpful notation for graphs:

Definition 2.4. Let X be a set, $G = \langle V, E\rangle$ a graph, $A, B \subseteq V$ disjoint sets of vertices and $M \subseteq E$.

☞ $G[A, B] := G[A \sqcup B] \backslash (E(A) \sqcup E(B))$.

☞ $G[X] := G[V \cap X]$.

☞ $N_G[A] := \bigcap\{N_G(a) \colon a \in A\}$.

☞ The graph G *covers* A iff $A \subseteq V(G)$.

☞ A *copy* of G is a pair $\langle G', \phi\rangle$ where G' is a graph and ϕ is a graph isomorphism between G and G'.

☞ M is a *matching* iff the edges in M are pairwise non-adjacent.

☞ M is a *perfect matching* iff M is a matching and every vertex is incident to an edge in M.

☞ If M is a matching and $xy \in M$, then x is the *matching partner* of y and vice versa.

Definition 2.5. Let $G = \langle V, E \rangle$ be a graph and κ a cardinal. An *edge-coloring* or κ-*edge-coloring* of G is a function $c\colon E \to \kappa$.

The cardinal κ will be a positive integer for the main part of this thesis. However, in Section 6, we give some notes on ω-edge-colorings of graphs.

Definition 2.6. Let $G = \langle V, E \rangle$ be a graph, $v \in V$ a vertex, A a set of vertices, $c\colon E \to \kappa$ an edge-coloring, and $i \in \kappa$.

- ☞ The elements of κ are the *colors*.
- ☞ An edge $e \in E$ *has color* i if and only if $c(e) = i$.
- ☞ A subgraph $H \subseteq G$ *has color* i if and only if $c(E(H)) = \{i\}$.
- ☞ A subgraph $H \subseteq G$ is *monochromatic* (with color j) if and only if it has color j for some $j \in \kappa$.
- ☞ The coloring c is *finite* if and only if κ is finite.
- ☞ $G_i := \langle V, c^{-1}(i) \rangle$.
- ☞ $G_{\neq i} := \langle V, E \backslash c^{-1}(i) \rangle$.
- ☞ $N_G(v, i) := N_{G_i}(v)$.
- ☞ $N_G[A, i] := N_{G_i}[A] = \bigcap \{N_G(a, i)\colon a \in A\}$.

3 Inseparability

In this section, we discuss highly connected sets of vertices — those which are $<\kappa$-*inseparable* for some infinite cardinal κ:

Definition 3.1. Let κ be a cardinal number and G a graph. A set U of vertices is $<\kappa$-*inseparable* in G iff no two distinct vertices in U can be separated by a set of less than κ many vertices.

Three notations for $<\kappa$-inseparability can be found in the literature: Soukup et al. call it κ-linked in [6] and D. Soukup calls it κ-unseparable in [11]. We adobt the notation that Carmesin et al. use in [2].

Countable $<\aleph_0$-inseparability sets of vertices are easily seen to be covered by a generalized path: If Y is a countable $<\aleph_0$-inseparable set of vertices in a given graph, then recursively define a generalized path by adding in each step the next vertex of some fixed enumeration of Y (which can be done since Y is $<\aleph_0$-inseparable). However, $\aleph_1$ sized sets of vertices which are $<\aleph_1$-inseparability cannot always be covered by a generalized path: D. Soukup shows in [11] that there is a graph G satisfying that $N_G[F]$ is uncountable for every finite set F of vertices and has no uncountable generalized path. The following lemma is a Menger type characterisation of $<\kappa$-inseparability:

Lemma 3.2. *Let κ be a cardinal and G a graph. A set U of vertices is $<\kappa$-inseparable iff each two distinct vertices $u, u' \in U$ are linked by κ many independent paths.*

Proof. First, suppose that U is $<\kappa$-inseparable. Fix distinct vertices $u, u' \in U$ and let $\mathcal{P}$ be a maximal set of independent u–u'-paths in G. Then $\mathcal{P}$ has size at least κ: otherwise the set $\{V(P) \colon P \in \mathcal{P}\}$ would be a separator contradicting the $<\kappa$-inseparability of U.

Now, assume that for each two distinct vertices $u, u' \in U$ there is a set $\mathcal{P}$ of κ many independent u–u'-paths. Again, fix vertices $u, u' \in U$. Then every less than κ sized set X that is included in $V(G) \setminus \{u, u'\}$ meets less than κ many of paths in $\mathcal{P}$. Hence u and u' are part of the same connected component of $G - X$. □

We will need κ-*uniform* ultrafilters in order to obtain $<\kappa$-inseparable sets of vertices — these are special free ultrafilters:

Definition 3.3. [8, p.144] Let A be non-empty set. A filter $\mathcal{F}$ is κ-*uniform* iff every set $X \in \mathcal{F}$ has size at least κ.

For infinite κ it is always possible to find a κ-uniform ultrafilter on a given κ sized set:

Lemma 3.4. *[8, p.144] If κ is an infinite cardinal and A is a set of size κ, then there is a κ-uniform ultrafilter on A.*

Proof. Let κ be an infinite cardinal and A a set of size κ. Consider the *(generalized) Fréchet filter*

$$\mathcal{F} := \{X \in \mathcal{P}(A) \colon |A \setminus X| < \kappa\}$$

on A. Let us check that this is indeed a filter. The ground set A is contained in $\mathcal{F}$ since $A \setminus A = \emptyset$ has size κ and $\emptyset$ is not part of $\mathcal{F}$ since $A \setminus \emptyset = A$ has size κ. If X and Y are elements of $\mathcal{F}$, then

$$A \setminus (X \cap Y) = (A \setminus X) \cup (A \setminus Y)$$

and $(A \setminus X) \cup (A \setminus Y)$ has size less than κ (using that κ is infinite). Hence $A \setminus (X \cap Y)$ has size less than κ, in other words, the intersection $X \cap Y$ is contained in $\mathcal{F}$. Finally, if X is an element of $\mathcal{F}$ and Y is a subset of A that includes X, then $|A \setminus Y| \leq |A \setminus X| < \kappa$ and it follows that Y is contained in $\mathcal{F}$.

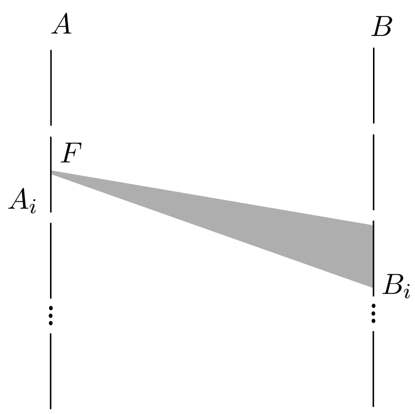
Extending $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ yields the desired κ-uniform ultrafilter. Indeed, if X is a subset of A that has size less than κ, then $A \setminus X$ is contained in $\mathcal{F}$ which itself is a subset of $\mathcal{U}$. Thus, X is not contained in $\mathcal{U}$. □

We often want to say that a set of vertices is nearly joined complete to another one:

Definition 3.5 (cf. [11])**.** Let κ be a cardinal, G a graph, and B a set of vertices. A set A of vertices is *κ-complete* in B iff $B \setminus N_G(a)$ has size less than κ for every vertex $a \in A$.

Every κ-complete set of vertices (more precisely, every set of vertices that is κ-complete in some other set of vertices) in an r-edge-colored graph has a partition with partition classes that are κ-inseparable for different colors:

Lemma 3.6 ([6])**.** *Let κ be a cardinal, G a graph, B a set of κ many vertices, and A a set of vertices that is κ-complete in B. Moreover, let $c \colon E \to \{0, \ldots, r-1\}$ be an edge-coloring. Then there is a partition $\{A_i \colon i < r\}$ of A satisfying that for every finite partition $\{B_i \colon i \in J\}$ of B there is an index $j \in J$ such that $N_G[F, i] \cap B_j$ has size κ for every finite $F \subseteq A_i$ and every $i < r$.*



Proof. Fix a κ-uniform ultrafilter $\mathcal{U}_B$ of B (possible by Lemma 3.4) and a partition $\{B_i \colon i \in J\}$ of B. We find for every vertex $a \in A$ a unique color $i = i(a)$ such that $N_G(a, i)$ is contained in $\mathcal{U}_B$. Let A_i be the set of vertices whose i colored neighborhood lies in the ultrafilter, i.e. $A_i := \{a \in A \colon i(a) = i\}$. Consider the partition $\{A_i \colon i < r\}$ of A. Now, it holds that for every color i and every finite $F \subseteq A_i$ the set $N_G[F, i]$ is an element of the ultrafilter $\mathcal{U}_B$. This is because

$$N_G[F, i] = \bigcap\{N_G(v, i) \colon v \in F\},$$

i.e. it can be written as a finite intersection of filter elements.

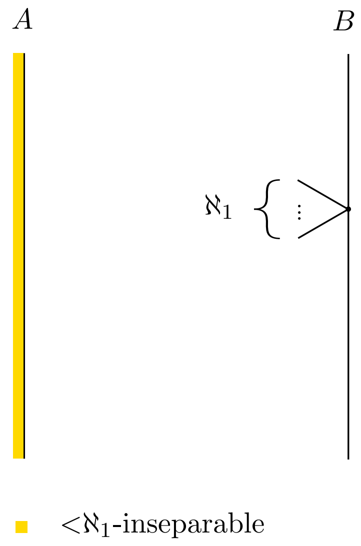
Finally, if j is the unique index satisfying that B_j is contained in the ultrafilter $\mathcal{U}_B$, then for each color i the set $N_G[F, i] \cap B_j$ is contained in $\mathcal{U}_B$ as well. Thus, also using that $\mathcal{U}_B$ is κ-uniform, the intersection $N_G[F, i] \cap B_j$ has size κ for every finite $F \subseteq A_i$ and every color $i < r$. □

The next lemmas give criteria of sets of vertices being $<\aleph_1$-inseparable. The statements are extracted from proofs written by Soukup et al. in [6] or D. Soukup in [11].

Lemma 3.7. *Let G be graph, A a set of vertices and $A_1, A_2 \subseteq A$. Moreover, suppose that A_1 and A_2 are both $<\aleph_1$-inseparable, the intersection $A_1 \cap A_2$ has size $\aleph_1$, and $A = A_1 \cup A_2$. Then A is $<\aleph_1$-inseparable.*

Proof. Let a_1 and a_2 be distinct vertices that lie in the intersection $A_1 \cap A_2$ and let X be a countable set of vertices that is included in $V(G) \setminus \{a_1, a_2\}$. We show that a_1 and a_2 belong to the same connected component of $G - X$. If both a_1 and a_2 are contained in either A_1 or A_2, then this follows by the $<\aleph_1$-inseparability of A_1 and A_2 respectively. So suppose that $a_1 \in A_1$ and $a_2 \in A_2$ or vice versa. By symmetry we may assume that $a_1 \in A_1$ and $a_2 \in A_2$. Since the intersection $A_1 \cap A_2$ is uncountable, we find a vertex $\tilde{a} \in (A_1 \cap A_2) \setminus X$. By our assumption that A_1 and A_2 are $<\aleph_1$-inseparable, it follows that a_i and $\tilde{a}$ are part of the same component of $G - X$ for $i = 1, 2$. By transitivity, the vertices a_1 and a_2 are part of the same component of $G - X$ as well. □

Lemma 3.8. *Let G be a graph and A a $<\aleph_1$-inseparable set of vertices that has size $\aleph_1$. Moreover, suppose that B is a set of vertices that is disjoint from A and satisfying that every vertex in B has $\aleph_1$ many neighbors in A. Then the union $A \cup B$ is $<\aleph_1$-inseparable.*

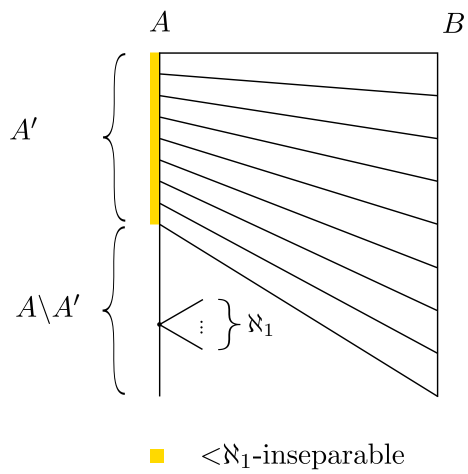


Proof. Let us fix distinct vertices $v_1, v_2 \in A \cup B$ and a countable set X that is included in $V(G)\setminus\{v_1, v_2\}$. We show that X does not separate v_1 and v_2 in G. If both v_1 and v_2 lie A, then X does not separate them because A is $<\aleph_1$-inseparable.

Now, suppose that $v_1 \in A$ and $v_2 \in B$ or vice versa. By symmetry, we may assume that $v_1 \in A$ and $v_2 \in B$. Then v_2 has $\aleph_1$ many neighbors in A and thus we find a vertex $\tilde{a} \in A\setminus X$ that is linked to v_2 by an edge. The previous case shows that X does not separate a_0 and v_1. Moreover, X does not separate $\tilde{a}$ and v_2 by the choice of $\tilde{a}$. Transitivity implies that X does not separate v_1 and v_2.

Finally, suppose that both v_1 and v_2 lie in B. We find a vertex $\tilde{a} \in A\setminus X$ (using that A is uncountable). By the previous case, the set X neither separates v_1 nor v_2 from $\tilde{a}$. Transitivity completes our proof. □

Lemma 3.9. *Let G be graph and A, B disjoint sets of vertices. Moreover, suppose that there is a set $A' \in [A]^{\aleph_1}$ of vertices such that A' is $<\aleph_1$-inseparable, $G[A', B]$ has a perfect matching, and every vertex in $A \backslash A'$ has $\aleph_1$ many neighbors in B. Then A is $<\aleph_1$-inseparable.*



Proof. Let us fix a perfect matching M of $G[A', B]$, distinct vertices $a_1, a_2 \in A$, and a countable set $X \subseteq V(G)\backslash\{a_1, a_2\}$. We show that X does not separate a_1 and a_2 in G. If both vertices a_1 and a_2 lie in A', then X does not separate them because A' is $<\aleph_1$-inseparable.

Now, suppose that either $a_1 \in A\backslash A'$ and $a_2 \in A'$ or vice versa. By symmetry, we may assume that $a_1 \in A\backslash A'$ and $a_2 \in A'$. Let B' consist of those neighbors of a_1 in B that are not part of X. Then B' has size $\aleph_1$ because X is countable and a_1 has $\aleph_1$ many neighbors in B. The set

$$A'' := \{a \in A' \colon \exists b \in B'[ab \in M]\}$$

of matching partners of vertices in B' has size $\aleph_1$ as well. Hence $A''\backslash X$ is non-empty and we find a vertex $a_0 \in A''\backslash X$. Let $b_0 \in B'$ be the matching partner of a_0. By the first case (both vertices lie in A'), the set X does not separate a_0 and a_2. Moreover, the path $a_1 b_0 a_0$ shows, that a_0 and a_1 belong to the same component of $G - X$. Now, transitivity implies that X does not separate a_1 and a_2.

Finally, suppose that both a_1 and a_2 lie in $A\backslash A'$. Since X is countable and A' has size $\aleph_1$, we find a vertex $\tilde{a}$ in $A'\backslash X$. By the previous case, X neither separates the vertex a_1 nor the vertex a_2 from $\tilde{a}$ in G. Again, transitivity implies that X does not separate a_1 and a_2. □

We conclude this section with a lemma that will help us to find a situation as in Lemma 3.9:

Lemma 3.10. *Let $G = \langle A \sqcup B, E\rangle$ be a graph with bipartition classes A, B both of size $\aleph_1$ and suppose that $N_G[F]$ is uncountable for every finite $F \subseteq A$. Then there are disjoint sets $B', B'' \subseteq B$ of vertices such that the intersection $N_G[F] \cap B'$ is uncountable for every finite $F \subseteq A$ and $G[A, B'']$ has a perfect matching.*

Proof. Let us fix an enumeration $\langle a_\alpha \colon \alpha < \omega_1\rangle$ of A and denote $A_\alpha := \{a_\xi \colon \xi \leq \alpha\}$. Simultaneously define countable sets $B_\alpha \subseteq B$ and distinct vertices $b_\alpha \in B$ satisfying the following for $\alpha < \omega_1$ (cf. Figure: 3):

(1) $a_\alpha b_\alpha$ is an edge of G.

(2) b_α is not contained in B'_α.

(3) $B_\xi \subsetneq B_\alpha$ whenever $\xi < \alpha$.

(4) $N_G[F] \cap (B'_\alpha \backslash B_\xi)$ is non-empty for every finite $F \subseteq A_\alpha$ and every ordinal $\xi < \alpha$.

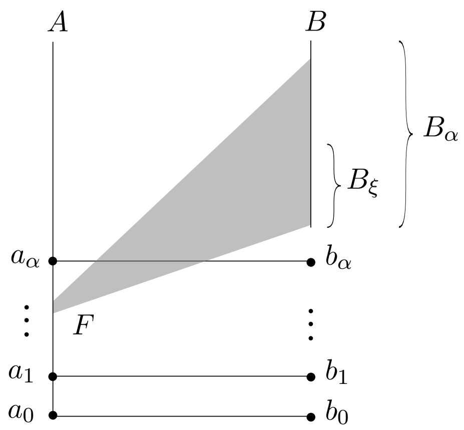


Figure 2: The Construction of b_α and B_α.

We begin with $B_0 := \emptyset$ and an arbitrary neighbor b_0 of a_0. Now, assume that B_α and b_α have been define for $\alpha < \beta$. If β is a limit ordinal, then let $B_\beta := \bigcup\{B_\alpha \colon \alpha < \beta\}$ and b_β an arbitrarily vertex in

$$N_G(a_\beta)\backslash(B_\beta \cup \{b_\alpha \colon \alpha < \beta\}).$$

Otherwise, if $\beta = \alpha + 1$, then we define B_β by adding for every finite $F \subseteq A_\beta$ a vertex in the common neighborhood of the vertices in F that is distinct from everything defined so far to B_α, i.e. a vertex in the set

$$N_G[F]\backslash(B_\alpha \cup \{b_\xi \colon \xi < \beta\}).$$

As before, choose b_β arbitrarily in

$$N_G(a_\beta)\backslash(B_\beta \cup \{b_\alpha \colon \alpha < \beta\}).$$

Consider the disjoint sets $B' := \bigcup\{B_\alpha \colon \alpha < \omega_1\}$ and $B'' := \{b_\alpha \colon \alpha < \omega_1\}$. By condition (1), the graph $G[A, B'']$ has a perfect matching, namely the set $\{a_\alpha b_\alpha \colon \alpha < \omega_1\}$. We want to show that $N_G[F] \cap B'$ is uncountable for every finite $F \subseteq A$. For this purpose, it suffices to show that if $X \subseteq V(G)$ is a countable set of vertices and $F \subseteq A$ is finite, then $(N_G[F] \cap B')\backslash X$ is non-empty. Fix such sets F and X. We find an ordinal α such that $F \subseteq A_\alpha$ and $X \cap B' \subseteq B_\alpha$. By condition (3), we have that $N_G[F] \cap (B_{\alpha+1} \backslash B_\alpha)$ is non-empty. Hence $(N_G[F] \cap B')\backslash X$ is non-empty as well. □

4 Circle Decomposition for $K_{\aleph_0}$ and $K_{\aleph_0,\aleph_0}$

Many well known theorems that deal with cycles in finite graphs are false for infinite graphs. However, there is loophole in the case of locally finite graphs: often theorems become true again when "cycle" is replaced by an appropriate notion of *topological circle*. The topological space $|G|$ considered here is a certain compactification of G together with its ends. For example, the outer double ray (fat) of the 1-way infinite latter seen below defines a topological circle that 'meets' the unique end of the ladder.

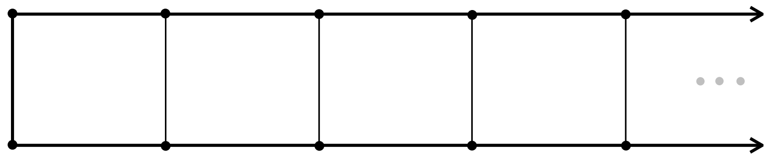


Figure 3: 1-Way Infinite Ladder.

The reader should keep in mind that this circle defines a *Hamilton circle* and that the 1-way infinite ladder is *hamiltonian*. We will not discuss these definitions in detail. For an introduction to the topological approach to locally finite graphs and cycles see Diestel and Sprüssel's survey [5].

The purpose of this section is to prove an appropriate version of Lehel's conjecture for infinite graphs (cf. Theorem 1.8). Furthermore, we prove a similar result for bipartite graphs (cf. Theorem 1.9). The proof of Theorem 1.8 imitates those of Theorem 1.2 that can be found in [6]. In contrast, our proof of Theorem 1.9 has some more changes to its template (Daniel Soukup's theorem that every r-colored complete bipartite graph with bipartition classes of countably infinite size has a partition into $2r-1$ paths). Originally, all the decomposition graphs are built simultaneously in one step. Instead, we will first simultaneously define disjoint, monochromatic, hamiltonian graphs of different colors that cover a bipartition class of the given bipartite graph. Afterwards, we throw away one of the hamiltonian graphs and simultaneously define hamiltonian graphs picking up everything still uncovered. Technically, it is also possible to simultaneously built the decomposition graphs in one single step. However, our strategy is a good preparation for Section 5.

Lemma 4.1. *Let G be a graph and F a finite set of vertices. If $N_G[F]$ is infinite, then F is covered by finite or trivial subgraphs of G.*

Proof. Fix $F' \subseteq N_G[F]$ and note that $G[F, F']$ is complete bipartite. If F contains less then two vertices, then it is clearly covered by a trivial graph. Otherwise, the graph $G[F, F']$ is hamiltonian and it follows that G is covered by a cycle. □

Next, we introduce a class of graphs, which will help to construct 1-way infinite ladders:

Definition 4.2. ☞ Let n be an odd positive integer. A *ladder fragment* (of size n) is a finite path $v_0 \dots v_{2n}$ with additional edges $v_{2i}v_{2i+3}$ for $i < n-1$ (cf. Figure 4). The vertices v_{2n-2} and v_{2n} are the *connection points* of the ladder fragment.

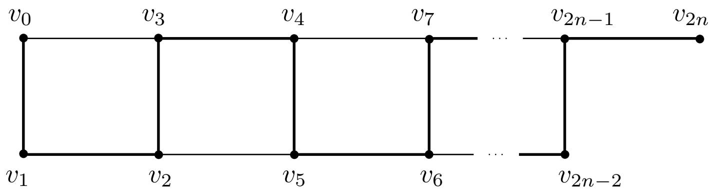


Figure 4: Ladder Fragment of size n. The path $v_0 \dots v_{2n}$ is indicate fat.

☞ Let G be a graph. A ladder fragment $L \subseteq G$ with connection points v_{2n-2} and v_{2n} has an *extension* iff there are vertices $v_{2n+1}, v_{2n+2} \in V(G)\backslash V(L)$ such that $v_{2n-2}v_{2n+1}$, $v_{2n}v_{2n+1}$ and $v_{2n+1}v_{2n+2}$ are edges of G (cf. Figure 5). The graph

$$L + E(v_{n+1}, \{v_{n-2}, v_n, v_{n+2}\})$$

is an extension of L.

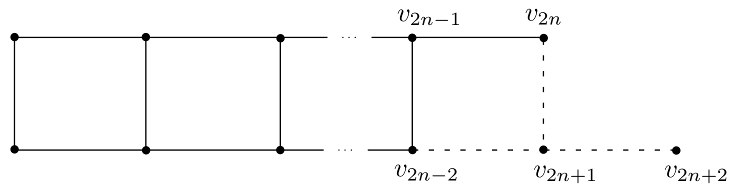


Figure 5: Extending a ladder Fragment.

☞ Let $L_0, \dots, L_n$ be ladder fragments satisfying that L_{i+1} is an extension of L_i for $i < n$. Then L_n *is an extension* of L_0.

The following lemma does the most work for the proof of Theorem 1.8:

Lemma 4.3. *Let $G = \langle V, E\rangle$ be a countable graph and suppose that A and B are sets of vertices satisfying that A is $\aleph_0$-complete in B and B is countably infinite. Moreover, let $c\colon E \to \{0, \dots, r-1\}$ be an edge-coloring. Then A can be covered by disjoint monochromatic graphs that are trivial, cycles, or 1-way infinite ladders of different colors.*

Proof. By Lemma 3.6, we find a partition $\{A_i : i < r\}$ of A such that $N_G[F, i]$ is countably infinite for every finite $F \subseteq A_i$ and by Lemma 4.1, we may assume that every A_i is infinite. Let us fix an enumerations $\langle a_n : n < \omega\rangle$ of A. By recursion on n, we define simultaneously sequences $\langle L_i^n : n < \omega\rangle$ satisfying the following:

(1) L_i^n is a ladder fragment of size $n+1$ in G_i and has connection points in A_i.

(2) L_i^n has color i.

(3) L_i^{n+1} is an extension of L_i^n.

(4) L_i^n and L_j^n have disjoint vertex sets for different colors i and j.

(5) The vertex a_n is contained in $W_n := \bigcup\{V(L_i^n)\colon i < r\}$.

Once the L_i^n are defined we are done. Indeed, for each color i let L_i be the graph $\bigcup\{L_i^n\colon n < \omega\}$. By condition (1) and condition (3), the graph L_i is a 1-way infinite ladder for $i < r$ and by condition (2), L_i has color i for $i < r$. Moreover, by condition (4), all the L_i have disjoint vertex sets for $i < r$ and by condition (5), the union $\bigcup\{V(L_i)\colon i < r\}$ covers A.

In the recursion base construct disjoint ladder fragments L_i^0 for colours $i < r$ that satisfy the conditions (1)-(5). This can be done by recursion on i using that $N_G[F, i]$ is infinite for every finite $F \subseteq V_i$ and every color $i < r$ (a ladder fragment of size one is just a finite path of length three and the end vertices of the path are exactly the connection points of the ladder fragment).

Now, assume that L_i^n has already been defined. We find pairwise distinct vertices a_i^{n+1} that are contained in $A_i \backslash W_n$ for $i < r$; if possible choose a_i^{n+1} as the vertex a_{n+1}. Let F_i^{n+1} consist of the connection points of L_i^n and the vertex a_i^{n+1} for $i < r$. Using the second part of (1), we find a vertex b_i^{n+1} that is contained in $N_G[F_i^{n+1}, i]$. Since all the sets A_i are infinite, we can choose the vertices b_i^{n+1} pairwise distinct. Letting

$$L_i^{n+1} := L_i^n + E(F_i^{n+1}, b_i^{n+1})$$

completes our construction (cf. Figure 6).

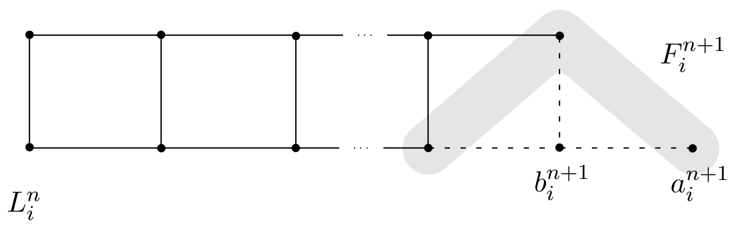


Figure 6: Construction of L_i^{n+1}.

□

Corollary 4.4. *Let $G = \langle V, E\rangle$ be a graph and suppose that A and B are sets of vertices satisfying that A is $\aleph_0$-complete in B, that A is countable, and that B is infinite. Moreover, let $c\colon E \to \{0, \ldots, r-1\}$ be an edge-coloring. Then A can be covered by the vertex sets of disjoint monochromatic graphs that are trivial, finite paths, or 1-way infinite paths of different colors.*

Proof. Every cycle contains a spanning path and every 1-way infinite ladder contains a 1-way infinite path. □

Theorem 1.8. *Every r-edge-colored, complete, countably infinite graph has a partition of its vertex set into the vertex sets of monochromatic hamiltonian graphs of different colors.*[4]

Proof of Theorem 1.8. Apply Lemma 4.3 to the graph G for $A = B = V$. □

Theorem 1.8 is optimal because sometimes r trivial graph, finite cycles, or 1-way infinite ladders are needed to partition the whole vertex set:

Example 4.5. (1) Let $G = \langle V, E\rangle$ be a countably infinite complete graph and fix vertices $v_0, \ldots, v_{r-2} \in V$. Assign color i to the edges in $E(v_i, V \backslash \{v_0, \ldots v_{r-2}\})$ for $i < r - 1$ and color $r - 1$ to the rest (cf. Figure 7).

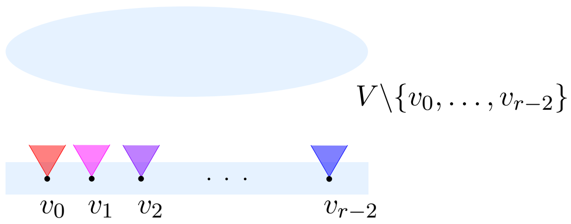


Figure 7: Edge-Coloring of G.

Suppose that $\mathcal{H}$ is a set of monochromatic trivial graphs, cycles, or 1-way infinite ladders in G such that $\bigcup\{V(H) : H \in \mathcal{H}\}$ is a partition of V. Since all graphs in $\mathcal{H}$ have finite maximum degree, infinitely many of the vertices in V must belong to a graph $H \in \mathcal{H}$ that has color $r - 1$. Using that graphs from $\mathcal{H}$ are connected, it follows that the vertices v_i cannot belong to H. Moreover, v_i and v_j must belong to different graphs from the set $\mathcal{H}$ whenever i and j are different colours. Hence $\mathcal{H}$ has size at least r.

(2) There is an edge-coloring such that all the decomposition graphs in Theorem 1.8 must be 1-way infinite ladders. Indeed, let $G = \langle V, E\rangle$ be a countably infinite complete graph. By recursion on n, define a partition $\{V_n : n < \omega\}$ such that $|V_{<n}| < |V_n| - (r - 2)$ where $V_{<n}$ is the set $\bigcup\{V_m : m < n\}$. For every $i < r$, assign an edge vw with the color i if $v \in V_n$, $w \in V_m$ with $m \leq n$ and $m \equiv i \pmod r$. Let $\mathcal{H}$ be a set of r monochromatic trivial graphs, cycles, or 1-way infinite ladders in G such that $\bigcup\{V(H) : H \in \mathcal{H}\}$ is a partition of V. Suppose for a contradiction that $\mathcal{H}$ contains a finite graph H and let j be the color of H. Choose N large enough such that $V(H)$ is included in $V_{<N}$ and let N' be a positive integer of size at least

[4] Trivial graphs are *hamiltonian* in this thesis.

N with $N' \equiv j \pmod r$. All the vertices in $V_{N'}$ must belong to graphs in $\mathcal{H}$ that have a color different to j. Estimating the summed degrees of vertices in $V_{N'}$ with respect to the graphs in $\mathcal{H}$ to that they belong generates a contradiction to the choice of the set $V_{N'}$.

Before we proof Theorem 1.9, we need just one more lemma, which helps us to pick up certain vertices with 1-way infinite ladders:

Lemma 4.6. *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph with bipartition classes A and B both countably infinite, $b \in B$ a vertex of infinite degree, and L a ladder fragment with connection points in A. Moreover, suppose that $N_G[F]$ is infinite for every finite $F \subseteq A$. Then there is an extension L' of L that contains b and has connection points in A.*

Proof. Let a_0 and a_1 be the connection points of L. Since b has infinite degree, we find neighbors a_2, a_3 and a_4 of b that are not part of L. Moreover, using that $N_G[F]$ is infinite for every finite $F \subseteq A$, we find vertices

$$b_0 \in N_G[\{a_0, a_1, a_2\}] \backslash (V(L) \cup \{b\})$$
$$b_1 \in N_G[\{a_1, a_2, a_3\}] \backslash (V(L) \cup \{b, b_0\}).$$

Letting

$$L' := L + E(b_0, \{a_0, a_1, a_2\}) + E(b_1, \{a_1, a_2, a_3\}) + E(b, \{a_2, a_3, a_4\})$$

yields an extension of L containing b with connection points a_3 and a_4 that are contained in A (cf. Figure 8).

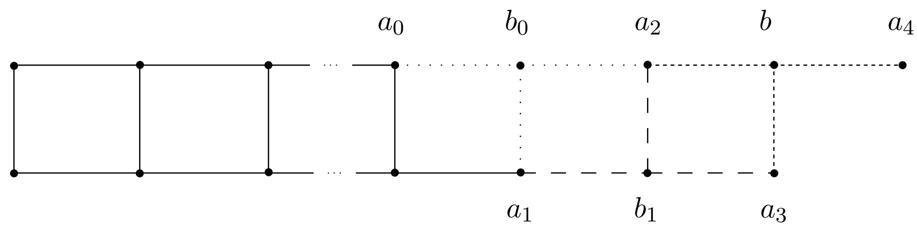


Figure 8: Construction of L'.

□

Theorem 1.9. *Every r-edge-colored complete bipartite graph with countably infinite bipartition classes has a partition of its vertex set into the vertex sets of $2r-1$ monochromatic hamiltonian graphs.*

Proof of Theorem 1.9. By Lemma 4.3, we find disjoint graphs $C_0, \dots, C_{r-1}$ such that each C_i is a trivial graph, a cycles, or a 1-way infinite ladder and has color i. Our plan is to keep $r-1$ of these graphs for the partition of V and to simultaneously define r new decomposition graphs that pick up all the vertices still being uncovered.

Let A_i be the intersection $A \cap V(C_i)$, let B_i be the intersection $B \cap V(C_i)$ for $i < r$, and let B_r be the set $B \backslash \bigcup\{B_i \colon i < r\}$. Consider the partition $\{A_i \colon i < r\}$ of A and the partition $\{B_i \colon i \leq r\}$ of B (cf. Figure 9). A look at the proof of Lemma 4.3 shows that $\{A_i \colon i < r\}$ appeared through the application of Lemma 3.6. Hence we find an index

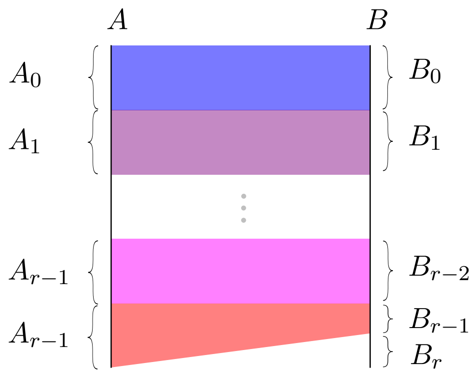


Figure 9: Cover A with Hamiltonian Graphs

j such that $N_G[F,i] \cap B_j$ is infinite for every finite $F \subseteq A_i$ and every $i < r$. We may assume that j is either the index $r-1$ or the index r and that B_{r-1} is infinite. Note that the intersection $N_G[F,i] \cap (B_{r-1} \sqcup B_r)$ is infinite for every finite $F \subseteq A_i$ and every $i < r$. Consider the partition $\{B', B''\}$ of $B_{r-1} \sqcup B_r$ where

$$B' := \{b \in B_{r-1} \sqcup B_r \colon |N_G(b, r-1) \cap A_{r-1}| = \aleph_0\}$$
$$B'' := (B_{r-1} \sqcup B_r) \backslash B'.$$

Observe that B'' is $\aleph_0$-complete in A_{r-1} referring to the graph $G_{\neq r-1}$. By Lemma 3.6, we find a partition $\{B''_i \colon i < r-1\}$ of B'' such that $N_G[F,i] \cap A_{r-1}$ is infinite for every finite $F \subseteq B''_i$ and every $i < r$. We may assume that all the B''_i are infinite (cover finite sets B''_i by finite cycles using Lemma 4.1). Let us fix an enumeration $\langle a_n \colon n < \omega \rangle$ of A_{r-1}, an enumeration $\langle b'_n \colon n < |B'| \rangle$ of B' and an enumeration $\langle b''_n \colon n < \omega \rangle$ of B''. By recursion on n we simultaneously define sequences $\langle L^n_i \colon n < \omega \rangle$ for $i < r$ satisfying the following:

(1) L_i^n is a ladder fragment of size $n+1$ whenever $i < r-1$.

(2) L_i^n has color i.

(3) L_i^n has connection points in A_{r-1}.

(4) L_i^{n+1} is an extension of L_i^n.

(5) All the L_i^n have pairwise disjoint vertex sets.

(6) $W_n := \bigcup\{V(L_i^n) \colon i < r\}$ contains the vertices a_n, b'_n and b''_n where $b'_n := b_0$ for $n \geq |B'|$.

Once the L_i^n are defined we are done. Indeed, $\{V(L_i) \colon i < r\}$ will be a partition of $A_{r-1} \sqcup B_{r-1} \sqcup B_r$ into 1-way infinite ladders of different colors where L_i is the 1-way infinite ladder $\bigcup\{L_i^n \colon n < \omega\}$ and as a consequence the vertex sets of the monochromatic hamiltonian graphs

$$L_0, \dots, L_{r-1}, C_0, \dots, C_{r-2}$$

will define a suitable partition of $V(G)$.

The ladder fragments L_i^0 can be defined by recursion on $i < r-1$ using that $N_G[F,i] \cap A_{r-1}$ is infinite for every finite $F \subseteq B''_i$. For the definition of L_i^n, let

$$W^0_{<r-1} := \bigcup\{V(L_i^0) \colon i < r-1\}.$$

If $B' \backslash W^0_{<r-1}$ is empty, then all the vertices in B' have already been picked up by previously defined ladder fragments and L^0_{r-1} can be defined using that the intersection $N_G[F, r-1] \cap (B' \sqcup B'')$ is infinite for every finite $F \subseteq A_{r-1}$.

Otherwise, if $B' \backslash W^0_{<r-1}$ is non-empty, then choose $\tilde{a}_0$ minimal in $A_{r-1} \backslash W^0_{<r-1}$ with respect to the enumeration of A_{r-1} and choose $\tilde{b}'_0$ minimal in $B' \backslash W^0_{<r-1}$ with respect to the enumeration of B'. Fix a ladder fragment $\tilde{L}^0_{r-1}$ of size one that does not meet $W^0_{<r} \cup \{\tilde{b}'_0\}$, has $\tilde{a}_0$ as one of its connection point, and is monochromatic with color $r-1$. By Lemma 4.6 applied to the graph

$$G_{r-1}[A_{r-1}, B_{r-1} \sqcup B_r] - W^0_{<r},$$

the vertex $\tilde{b}'_0$, and the ladder fragment $\tilde{L}^0_{r-1}$, we find an extension L^0_{r-1} of $\tilde{L}^0_{r-1}$ satisfying the conditions (1)-(6).

Now, suppose that L_i^n has already been defined for $i < r$. By recursion on $i < r-1$, define extensions L_i^{n+1} of L_i^n satisfying (1)-(6) as in the proof of Theorem 1.8. For the construction of L^{n+1}_{r-1}, let us write

$$W^{n+1}_{<r-1} := \bigcup\{V(L_i^{n+1}) \colon i < r\}.$$

If $B' \backslash (W^{n+1}_{<r-1} \cup L^n_{r-1})$ is empty, then all the vertices in B' have already been picked up by previously defined ladder fragments and as before L^{n+1}_{r-1} can be defined using that $N_G[F, r-1] \cap (B' \sqcup B'')$ for every finite $F \subseteq A_{r-1}$.

Otherwise, if $B'\backslash(W^{n+1}_{<r-1} \cup L^n_{r-1})$ is non-empty, then choose $\tilde{a}_{n+1}$ minimal in the set $A_{r-1}\backslash(W^{n+1}_{<r-1} \cup L^n_{r-1})$ with respect to the enumeration of A_{r-1} and $\tilde{b}'_{n+1}$ minimal in the set $B'\backslash(W^{n+1}_{<r-1} \cup L^n_{r-1})$ with respect to the enumeration of B'. By Lemma 4.6 applied to the graph

$$G_{r-1}[A_{r-1}, B_{r-1} \sqcup B_r] - W^{n+1}_{<r},$$

the vertex $\tilde{b}'_{n+1}$, and the ladder fragment L^n_{r-1}, we find an extension L^{n+1}_{r-1} of L^n_{r-1} satisfying (1)-(6). □

Note that at most two of the hamiltonian graphs we found have the same color. Furthermore, Theorem 1.9 is optimal because sometimes $2r-1$ non-empty decomposition graphs are needed:

Example 4.7 ([10])**.** Let $G = \langle A \sqcup B, E \rangle$ be a complete bipartite graph with bipartition classes A and B both of size $\aleph_0$. Let us fix a partition $\{A_i \colon i < r\}$ of A and a partition $\{B_i \colon i < r\}$ of B such that $A_1, \ldots, A_{r-1}$ are singletons and every other partition class is countably infinite. Assign an edge ab with color i whenever $a \in A_{i_1}$, $b \in B_{i_2}$, and $i \equiv i_1 + i_2 \pmod r$. Suppose that $\mathcal{H}$ is a set of monochromatic trivial graphs, cycles, or 1-way infinite ladders in G such that $\bigcup\{V(H) \colon H \in \mathcal{H}\}$ is a partition of $V(G)$. Since all graphs in $\mathcal{H}$ are connected and by the definition of our coloring, it follows that the $r-1$ vertices that lie in singleton partition classes are part of different trivial graph from $\mathcal{H}$. Moreover, vertices that are contained in different partition classes of the partition of B must lie in different graphs from $\mathcal{H}$ as well. Hence $\mathcal{H}$ has size at least $2r-1$.

5 Path Decomposition for $K_{\aleph_1,\aleph_1}$

5.1 The Decomposition Theorem

In this section, we prove one of our main result: Theorem 1.6. The rough structure will remind the reader to Theorem 1.9. First, we show that bipartition class can be covered by disjoint monochromatic generalized paths of different colors. In addition, we ensure that one of these generalized paths P is X-*robust* for some set X of uncountable size. X-robust informally means that deleting vertices from X does not destroy the generalized path. In a second step, we substitute P by disjoint monochromatic generalized paths, whose union contains the vertices of P and all the vertices that were not covered yet.

The section is organized as follows: After clarfying our notation, we state two main lemmas for the proof of Theorem 1.6 and show how they imply the theorem. In Subsection 5.1, we give an overview of elementary submodels and discuss some of their properties that are crucial for our proof. Finally, in a second subsection we prove the two main lemmas.

Definition 5.1 ([11])**.** Let P and Q be generalized paths.

- ☞ Q *extends* P iff P is an initial segment of Q.
- ☞ Suppose that P and Q are disjoint and that $N_G(q_0)$ is cofinal in P where $q_0 := \min(Q)$. Then $P^\frown Q$ is the generalized path $\langle P \sqcup Q, P \times Q \cup \preceq_P \cup \preceq_Q \rangle$. If $Q = \{q_0\}$ is a singleton, then we simply write $P^\frown q_0$ instead of $P^\frown \{q_0\}$.
- ☞ Suppose that $p_1, p_2 \in P$ with $p_1 \prec_P p_2$. The *restriction* $P \restriction [p_1, p_2)$ to $[p_1, p_2)$ is the generalized path $\langle [p_1, p_2), \preceq_P \cap [p_1, p_2)^2 \rangle$ in G.
- ☞ The ordinal type$(P; \preceq_P)$ is the *order type* of P.

An important example of a graph, whose vertex set defines a generalized path is the graph $H_{\mu,\mu}$:

Definition 5.2 ([6])**.** Let μ be an ordinal. The bipartite graph $H_{\mu,\mu}$ has bipartition classes $\mu \times \{0\}$ and $\mu \times \{1\}$. Two vertices $\langle \alpha, 0 \rangle$ and $\langle \beta, 1 \rangle$ are adjacent iff $\alpha \leq \beta$. We call $\mu \times \{0\}$ the *main class* of $H_{\mu,\mu}$. The $H_{\mu,\mu}$*-order* $\preceq_{H_{\mu,\mu}}$ is defined by letting $\langle \alpha, i \rangle \preceq_{H_{\mu,\mu}} \langle \beta, j \rangle$ whenever $\alpha \leq \beta$ and $j \leq i$. (Cf. Figure 10.)

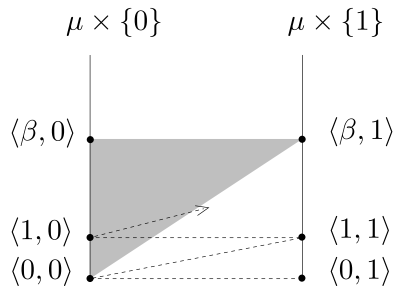


Figure 10: The graph $H_{\mu,\mu}$ and its partial order (dashed).

It is easily seen, that the $H_{\mu,\mu}$-order defines a well-order, which together with the vertex set of $H_{\mu,\mu}$ defines a generalized path (in each graph G that has $H_{\mu,\mu}$ as a subgraph). In this thesis, we will always have $\mu = \omega$ or $\mu = \omega_1$. For both of these two cases, it holds that the generalized path $\langle H_{\mu,\mu}, \preceq_{H_{\mu,\mu}} \rangle$ has order type μ.

Definition 5.3. Let G be a graph. Let μ be an ordinal and $\langle H, \phi \rangle$ a copy of $H_{\mu,\mu}$ in G. The *main class* of H is the set $\phi(\mu \times \{0\})$. Moreover, $\preceq_H$ is the partial ordering defined by letting $v \preceq_H w$ whenever $\phi^{-1}(v) \preceq_{H_{\mu,\mu}} \phi^{-1}(w)$ for $v, w \in H$.

The next lemma will be used in situations where we want cover a set of vertices by a generalized path.

Lemma 5.4 (cf. [11]). *Let $G = \langle A \sqcup B, E \rangle$ be a bipartite graph with bipartition classes A and B. Suppose that A is $\aleph_1$-complete in B, that A has size $\mu \leq \aleph_1$ and B has size $\aleph_1$ (cf. Figure 11). Then there is a subgraph $H \subseteq G$ that is a copy of $H_{\mu,\mu}$ with main class A.*

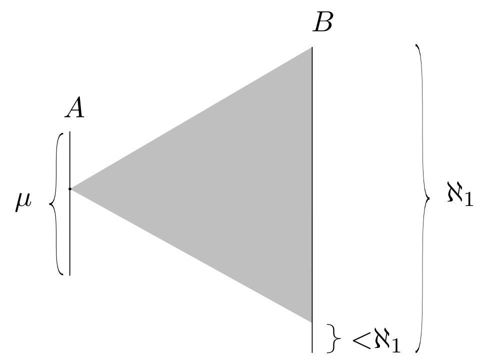


Figure 11: The assumptions for Lemma 5.4.

Proof. Let us fix an enumeration $\{a_\alpha \colon \alpha < \mu\}$ of A and for $\alpha < \mu$, let A_α be the set $\{a_\xi \in A \colon \xi \leq \alpha\}$. By the equation

$$N_G[A_\alpha] = B \setminus \bigcup \{B \setminus N_G(a_\xi) \colon \xi \leq \alpha\},$$

each of the sets $N_G[A_\alpha]$ for $\alpha < \mu$ must lie cocountable in B (using that the set $\bigcup\{B\backslash N_G(a_\xi)\colon \xi \leq \alpha\}$ is countable as a countable union of countable sets).

By recursion on α, we find distinct vertices $b_\alpha \in N_G[A_\alpha]$ for $\alpha < \mu$. Let B' consist of all these vertices, that is to say, $B' := \{b_\alpha\colon \alpha < \mu\}$. Since by construction $A_\alpha \subseteq N(b_\alpha)$ for every $\alpha \leq \mu$, the induced subgraph $H := G[A \sqcup B']$ has a spanning subgraph isomorphic to $H_{\mu,\mu}$ with main class A where $\langle \alpha, 0\rangle$ is mapped to a_α and $\langle \alpha, 1\rangle$ is mapped to b_α under the isomorphism for $\alpha < \mu$. □

Definition 5.5. Let G be a graph and $X \subseteq V(G)$. A generalized path P is X*-robust* iff for every $X' \subseteq X$, there is a well-order $\preceq_{P\backslash X'}$ on $P\backslash X'$ such that the following holds:

- $\langle P\backslash X', \preceq_{P\backslash X'}\rangle$ is a generalized path.
- $\langle P, \preceq_P\rangle$ and $\langle P\backslash X', \preceq_{P\backslash X'}\rangle$ have the same order type.
- $\min_{\preceq_P}(P) = \min_{\preceq_{P\backslash X'}}(P\backslash X')$.

If $X = \{x\}$ is a singleton, then we just say that P is x-robust. (Cf. Figure 12.)

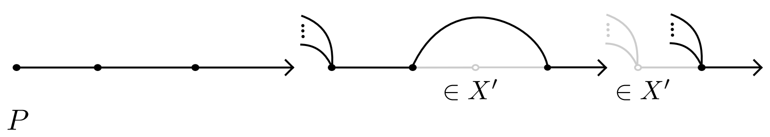


Figure 12: X-robust generalized path.

If P is X-robust and we want to cover the vertices that are contained in P plus some other vertices by r disjoint generalized paths, then it suffices to construct further $r-1$ disjoint generalized paths that meet P only in a subset of X. The rest of P will still be a generalized path and can be added to our collection.

For the recursive constructions of generalized paths it will help to have local conditions that imply X-robustness — the next definitions will turn out to be useful.

Definition 5.6 ([11])**.** Let G be a graph and A a set of vertices. A generalized path P is *concentrated* on A iff

$$A \cap P \restriction [p_1, p_2) \cap N_G(p_2)$$

is non-empty whenever p_2 is a $\preceq_P$-limit and $p_1 \prec_P p_2$. (Cf. Figure 13.)

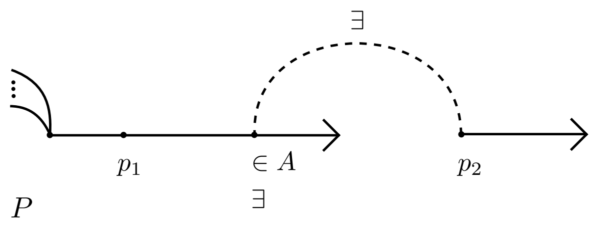


Figure 13: Generalized path P that is concentrated on A.

Definition 5.7. Let G be a graph, P a generalized path and X a set of vertices.

☞ $S \subseteq P$ is a *standard interval* of P iff $S = P \restriction [p, p+\omega)$ for some $\preceq_P$-limit p.

☞ $X \subseteq V(G)$ is *scattered* on P iff X meets every standard interval S in at most one vertex and if X meets S in x, then x is not the first vertex on S, that is, $x \neq \min(S)$.

☞ P is *locally X-robust* iff for every $x \in X$ every standard interval S of P is x-robust.

Note that standard intervals are generalized paths of order type ω by definition.

Lemma 5.8. *Let $G = \langle V, E\rangle$ be a graph, A, X sets of vertices and P a locally X-robust generalized path of order type ω_1 that is concentrated on $A \backslash X$. Moreover, suppose that X is scattered on P. Then P is X-robust.*

Proof. Fix a set $X' \subseteq X$. Let $\mathcal{S}$ denote the set of all standard intervals of P. Then $\mathcal{S}$ partitions P. Moreover, let us write $S(p)$ for the unique standard interval of P that contains p for each $p \in P$. We define a well-order $\preceq_{\mathcal{S}}$ on $\mathcal{S}$ by letting $S_1 \preceq_{\mathcal{S}} S_2$ whenever $v_1 \preceq_P v_2$ for two vertices $v_1 \in S_1$ and $v_2 \in S_2$. Clearly, the well-order $\preceq_{\mathcal{S}}$ does not depend on the choice of $v_i \in S_i$ for $i = 1, 2$.

Let $\mathcal{S}' := \{S' \colon S \in \mathcal{S}\}$ where $S' := S \backslash X'$. Since X is scattered on P and P is locally X-robust we find for every $S' \in \mathcal{S}'$ a well-order $\preceq_{S'}$ such that $\langle S', \preceq_{S'}\rangle$ is a generalized path of order type ω and the first vertex on S' coincides with the first vertex on S.

Now, let $\lhd$ be the lexicographic product of $\preceq_{\mathcal{S}}$ and $\preceq_P$ on $\mathcal{S} \times P$ and let $\preceq_{P'}$ on P' by letting $p_1 \preceq_{P'} p_2$ iff $\langle S(p_1), p_1\rangle \lhd \langle S(p_2), p_2\rangle$ for each two vertices $p_1, p_2 \in P'$.

By definition, $\langle P', \preceq_{P'}\rangle$ and $\langle \mathcal{S} \times P', \lhd \cap (\mathcal{S} \times P')^2\rangle$ are order isomorphic and the latter partial order is a well-order (here we use that lexicographic products of two well-orders are well-orders and that subsets of well-ordered sets are well-ordered). Hence $\preceq_{P'}$ is a well-order on P'.

Next we will show that $\langle P', \preceq_{P'}\rangle$ is a generalized path. For this purpose let p_2 be a vertex on P'. If p_2 is the $\preceq_{P'}$-successor, then $N_G(p_2) \cap \{p_2\}{\downarrow}$ is cofinal in $\{p_2\}{\downarrow}$ because $S(p_2)'$ is a generalized path. Now, suppose that p_2 is a $\preceq_{P'}$-limit. Fix a vertex p_1 in the down-closure $\{p_2\}{\downarrow}$ (with respect to $\preceq_{P'}$). Note that p_2 is also a $\preceq_P$-limit and that $p_1 \prec_P p_2$. Since P is concentrated on $A \backslash X$, we find a vertex a that is contained in

$$P \restriction [p_1^*, p_2) \cap (A \backslash X) \cap N_G(p_2)$$

where p_1^* is the vertex

$$p_1^* := \max\{r \in S(p_1)' \colon r \preceq_{S(p_1)'} p_1\} + 1,$$

where the maximum is taken with respect to the well-order $\prec_{S(p_1)'}$. Note that

$$\{r \in S(p_1)' \colon r \preceq_{S(p_1)'} p_1\}$$

is a finite set and thus this maximum indeed exists. Furthermore, note that p_1^* lies in the standard interval $S(p_1)'$. By the definition of $\preceq_{P'}$, this vertex a satisfies $p_1 \prec_{P'} a \prec_{P'} p_2$.

P' has order type ω_1 since P is still uncountable after deleting X' and X is scattered on P. The first vertex on P' coincides with the first vertex on P by the definition of $\preceq_{P'}$ and because X is scattered on P. $\square$

Definition 5.9. Let $G = \langle V, E \rangle$ be a graph and A a set of vertices. A generalized path P is *strong* on A iff there is a set $X \in [A \cap P]^{\aleph_1}$ satisfying the following:

- ➛ P is locally X-robust.
- ➛ P is concentrated on $A \setminus X$.
- ➛ X is scattered on P.

Lemma 5.10 (First Main Lemma)**.** *Let G be a bipartite graph with bipartition classes A, B both of size $\aleph_1$ and Y a $<\aleph_1$-inseparable set of vertices. Moreover, suppose that A is included in Y. If G has a generalized path that is strong on A, then it also has a generalized path that is strong on A and covers Y. (Cf. Figure 14.)*

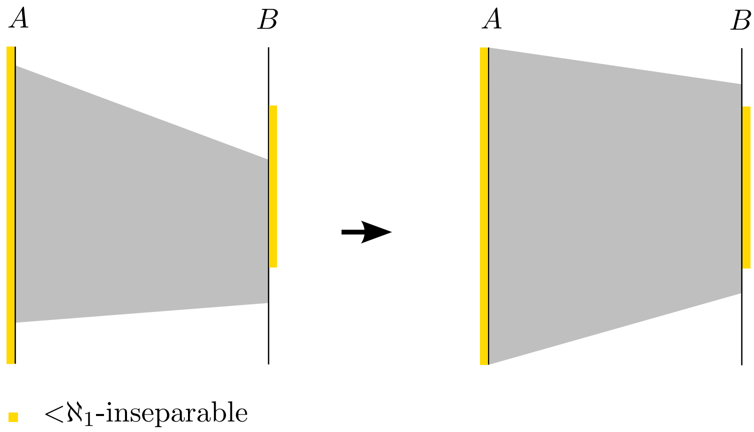


Figure 14: Blowing up a generalized path.

Lemma 5.11 (Second Main Lemma). *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph with bipartition classes A, B, both of size $\aleph_1$ and suppose that A is $\aleph_1$-complete in B. Moreover, let $c\colon E \to \{0, \dots, r-1\}$ be an edge-coloring of G. Then there is a partition $\{A_i : i < r\}$ of A, disjoint sets $B_0, \dots, B_{r-1} \subseteq B$, well-orders $\preceq_{A_0 \sqcup B_0}, \dots, \preceq_{A_{r-1} \sqcup B_{r-1}}$, and distinct colors $i_0, \dots, i_{r-1}$ satisfying the following:*

- *A_0 is $<\aleph_1$-inseparable in $G_{i_0}[A_0, B_0]$.*
- *$\langle A_0 \sqcup B_0, \preceq_{A_0 \sqcup B_0}\rangle$ is a generalized path of order type ω_1 in the graph G_{i_0}, that is strong on A.*
- *$\langle A_j \sqcup B_j, \preceq_{A_j \sqcup B_j}\rangle$ is a generalized path in the graph G_{i_j} for $j = 1, \dots, r-1$.*

  (Cf. Figure 15.)

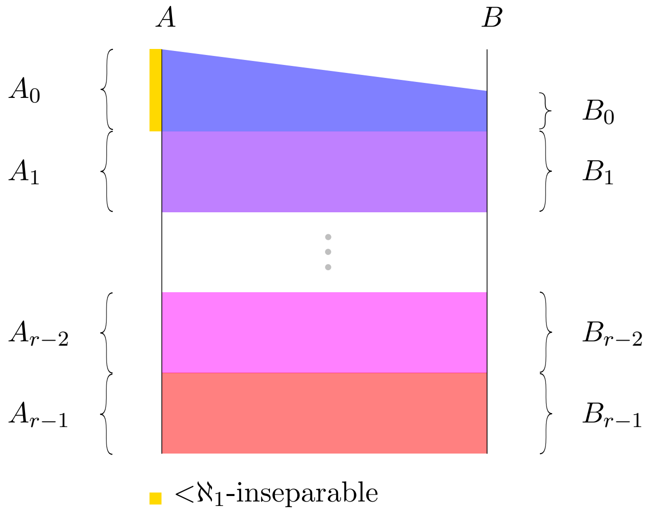


Figure 15: Covering A by disjoint generalized paths of different colors.

Theorem 1.6. *Let r be a positive integer. Every r-edge-colored complete bipartite graph with bipartition classes of size $\aleph_1$ has a partition of its vertex set into $2r-1$ monochromatic generalized paths.*

Proof. Let $G = \langle A \sqcup B, E\rangle$ be a complete bipartite graph with bipartition classes A, B and let $c\colon E \to \{0, \dots, r-1\}$ be an edge-coloring. First, apply Lemma 5.11 and find a partition $\{A_i : i < r\}$ of A, disjoint set of vertices $B_0, \dots, B_{r-1} \subseteq B$, well-orders $\preceq_{A_i \sqcup B_i}$ for $i < r$, and distinct colors $i_0, \dots, i_{r-1}$ as in the lemma. By symmetry, we may assume that $i_j = j$ for $j < r$.

Let P_i be the generalized path $P_i := \langle A_i \sqcup B_i, \preceq_{A_i \sqcup B_i}\rangle$ for $0 < i < r$. Let $\tilde{B}_0$ be the superset of B_0 that consists of the vertices in B_0 and all the vertices that are not covered yet by generalized paths, that is to say $\tilde{B}_0 := B \setminus \bigcup\{B_i : 1 \leq i < r\}$. Consider the partition $\{B', B''\}$ of $\tilde{B}_0$ where B' is the set of vertices from $\tilde{B}_0$ sending uncountably many 0 colored edges to A_0, i.e.,

$$B' := \{b \in \tilde{B}_0 \colon |N_G(b, 0) \cap A_0| = \aleph_1\}$$

and $B'' := \tilde{B}_0 \backslash B'$. Then $Y := B' \cup A_0$ is $<\aleph_1$-inseparable in $G_0[A_0, \tilde{B}_0]$ by Lemma 3.8. Apply Lemma 5.10 to $G_0[A_0, \tilde{B}_0]$ and Y in order to find a generalized path P_0' of order type ω_1 that is strong on A_0 and covers Y.

By Lemma 5.8 and the definition of strong, we find a set $X \in [A_0 \cap P]^{\aleph_1}$ such that P_0' is X-robust. Moreover, since $\tilde{B}_0 \backslash P_0'$ is included in B'', every vertex in $\tilde{B}_0 \backslash P_0'$ sends at most countably many 0-colored edges to A_0; in particular to X. Hence every vertex in $\tilde{B}_0 \backslash P_0'$ sends cocountably many edges to X that have not color 0. By Lemma 5.11 and Corollary 4.4 respectively applyed with reversed roles of A and B (the set $\tilde{B}_0 \backslash P_0'$ takes the role of A and X takes the role of B), we find monochromatic disjoint generalized paths $P_r, \ldots, P_{2r-2}$ in $G_{\neq 0}[X, \tilde{B}_0 \backslash P_0']$ that cover $\tilde{B}_0 \backslash P_0'$. Let us define

$$P_0 := P_0' \backslash \bigcup \{P_i \colon r \leq i < 2r - 1\}.$$

Since P_0' is X-robust, we find a well-order $\preceq_{P_0}$ such that $\langle P_0, \preceq_{P_0} \rangle$ is a generalized path. Now, $\{P_i \colon i < 2r - 1\}$ is a partition of $V(G)$ into monochromatic generalized paths. □

Note that at most two of the generalized paths that we have found are of the same color. Furthermore, Theorem 1.6 is optimal in the sense that there are colorings such that $2r - 1$ decomposition generalized paths are needed. Indeed, exchange all the partition classes of size $\aleph_0$ by partition classes of size $\aleph_1$ in Example 4.7. The argumentation is the same because graphs that are induced by generalized paths are connected.

5.2 Elementary Submodels

In this subsection, we discuss *elementary submodels* and provide some lemmas that help us in subsection 5.2. Elementary submodels are used in infinite combinatorics as follows: If a graph G is contained in $H(\theta)$ (for some big cardinal θ), then Downward Löwenheim-Skolem-Tarski Theorem guarantees the existence of a countable elementary submodel M of $H(\theta)$ that contains G. Often properties of G can be modeled within M. One says, that the properties of G are *reflected* from $H(\theta)$ to M. If this is the case, the intersection $G \cap M$ will have these properties as well (possibly in a weakened form). The fact that M is countable makes it then possible to use results on countable graphs.

It is also common to use increasing $\in$-chains of countable elementary submodels that cover G, so that every element of G can be considered in large enough submodels of the chain.

We begin this subsection with the definition of elementary submodels and list some facts. Afterwards, we discuss properties that are needed for the proofs of the two main lemmas. Finally, we show how *nice* chains of elementary submodels are generated.

More information on elementary submodels can be found in [7]. For an introduction to the use of elementary submodels in infinite combinatorics see [12].

Definition 5.12 ([7, p. 87])**.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for some lexicon $\mathcal{L}$. We say that $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$ written $\mathfrak{A} \preceq \mathfrak{B}$, if the following holds:

(1) $\mathfrak{A} \subseteq \mathfrak{B}$.

(2) $\mathfrak{A} \models \varphi[\sigma]$ iff $\mathfrak{B} \models \varphi[\sigma]$, for every formula φ and every assignment σ.

Theorem 5.13 (**Downward Löwenheim-Skolem-Tarski Theorem**, [7, p. 88])**.** *Let $\mathfrak{B}$ be a structure for some lexicon $\mathcal{L}$. Moreover, let κ be a cardinal satisfying that* $\max(\mathcal{L}, \aleph_0) \leq \kappa \leq |B|$ *and let $S \in [B]^{<\kappa}$. Then $\mathfrak{B}$ has an elementary submodel $\mathfrak{A}$ with $S \subseteq A$ and $|A| = \kappa$.* □

It can be recommended to look up the proof of Theorem 5.13 as it takes some of the magic out of the statement.

Definition 5.14 ([7, p. 93])**.** Let $\mathfrak{A}$ be a structure for some lexicon $\mathcal{L}$, $P \subseteq A$, and k a positive integer. $S \subseteq A^k$ is *definable* with parameters in A^k iff there is a formula $\varphi(\vec{x}, \vec{y})$ and $\vec{b} \in P$ such that $S = \{\vec{x} \colon \mathfrak{A} \models \varphi(\vec{x}, \vec{b})\}$. An element $a \in A$ is definable with parameters in P iff $\{a\}$ is definable with parameters in P. If P is empty, then we simply say that S (and a respectively) is definable.

Lemma 5.15. *Suppose that θ is an uncountable cardinal and that M is a countable elementary submodel of $H(\theta)$. If $a \in H(\theta)$ is definable with parameters in M, then a is contained in M.*

Proof. Let $\vec{b} \in M$ and $\varphi(x, \vec{y})$ be a formula such that $x = a$ iff $H(\theta) \models \varphi(x, \vec{b})$. Since $a \in H(\theta)$, we have that $H(\theta) \models \exists x \varphi(x, \vec{b})$ and by elementarity $M \models \exists x \varphi(x, \vec{b})$. So there is some $a' \in M$ with $M \models \varphi(a', \vec{b})$. Again, by elementarity, $H(\theta) \models \varphi(a', \vec{b})$. Now, by definition of definable, we have $a = a'$. □

The next lemma will be used frequently. It roughly says that most of mathematics can be modeled in $H(\theta)$ for big enough ordinals θ.

Lemma 5.16 ([7, p. 110])**.** *If κ is a regular uncountable cardinal, then $H(\theta)$ is a model of $ZFC - P$.* □

Here are some properties for elementary submodels:

Lemma 5.17 ([7, p. 230]). *Suppose that θ is an uncountable cardinal and that M is a countable elementary submodel of $H(\theta)$. Then*

(1) $\omega \cup \{\omega, \omega_1\} \subseteq M$.

(2) *$M \cap \omega_1$ is a countable limit ordinal.*

(3) *If $f \in M$ is a function and $x \in \text{dom}(f) \cap M$, then $f(x) \in M$.*

(4) *If $x \in M$ and $|x| \subseteq M$, then $x \subseteq M$.*

(5) *If $\langle P, \leq \rangle \in M$ is a well ordering of type ω_1, then $M \cap P$ is an initial segment of P and $\min_{\leq}(P \setminus M)$ is a $\leq$-limit.*

Proof. For (1) let us show that ordinals smaller than ω as well as ω and ω_1 are definable. The set $\emptyset$ is definable witnessed by the formula $\forall y[y \notin x]$. If n is a positive integer, then n is definable by the formula $x = S^n(\emptyset)$. The ordinal ω is definable by the formula 'x is the smallest limit ordinal' and ω_1 is definable by 'x is the smallest uncountable ordinal'.

(2) can be seen as follows: Since $H(\theta) \models \forall x \in \omega_1 \, \forall y \in x \, \exists z[z = y]$ and $M \preceq H(\theta)$ we have that $M \cap \omega_1$ is an initial segment of ω_1; in particular $M \cap \omega_1$ is an ordinal. Moreover, $H(\theta) \models \forall x \, \exists y[y = S(x)]$. Again by $M \preceq H(\theta)$ we have that $\xi + 1$ is contained in M whenever $\xi \in M$. It follows that $M \cap \omega_1$ is a limit ordinal.

For (3) suppose that $f \in M$ and $x \in \text{dom}(f) \cap M$. Then $H(\theta) \models \exists y[\langle x, y \rangle \in f]$ and by $M \preceq H(\theta)$, there is some $y \in M$ such that $f(x) = y$.

For (4) let $x \in M$ and $\lambda := |x| \subseteq M$. Then $H(\theta) \models$ 'there is a bijective function f with $\text{dom}(f) = \lambda$ and $\text{ran}(f) = x$'. By $M \preceq H(\theta)$, we find a bijection $f \in M$ with $\text{dom}(f) = \lambda$ and $\text{ran}(f) = x$. Now, (3) implies x is included in M.

Finally, let us check (5). We have that $H(\theta) \models$ 'there is an order preserving bijection ϕ, with $\text{dom}(\phi) = \omega_1$ and $\text{ran}(\phi) = P$'. By $M \preceq H(\theta)$, we find such a function ϕ in M. Now, (3) implies that the intersection $M \cap P$ is an initial segment of P as well and $\min_{\leq}\{p \in P : p \notin M\}$ is a $\leq$-limit because the intersection $M \cap \omega_1$ is a limit by (2). $\square$

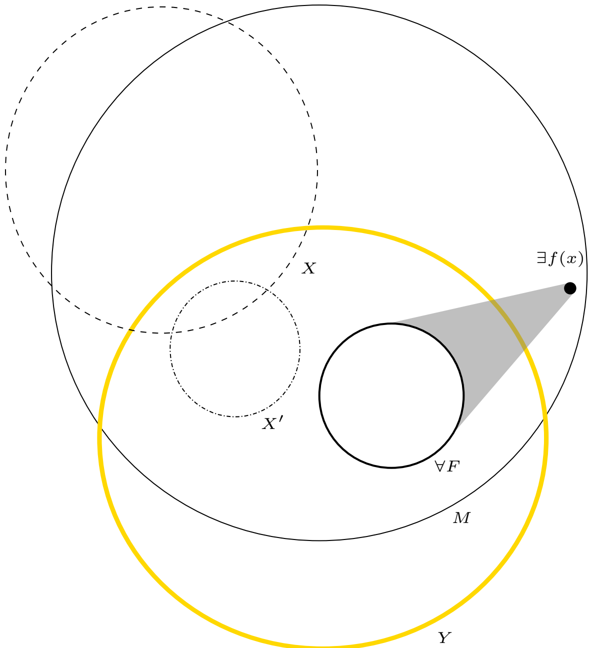
Lemma 5.18. *Let θ be an uncountable cardinal and M a countable elementary submodel of $H(\theta)$. Moreover, let $G \in M$ be a graph and $Y \in M$ a set of vertices in G that satisfies $N_G[F]$ is uncountable for every finite $F \subseteq Y$. If $X \in M$ is a countable set of vertices, then $N_{G-X}[F] \cap M$ is infinite for every finite $F \subseteq (Y \cap M) \setminus X$.*

Proof. First, recall that $M \preceq H(\theta)$ implies $M \subseteq H(\theta)$ and thus G, Y, and X lie in $H(\theta)$. Let us fix finite sets $F \subseteq (Y \cap M) \setminus X$ and $X' \subseteq Y \cap M$. Note that X' and F lie in M (since X' and F are finite and $X', F \subseteq M$). We show that $N_{G[M]-X}[F] \setminus X'$ is non-empty. Consider the formulas

- $\varphi_1(f', F', V') :=$ f' is a function, $\mathrm{dom}(f') = F'$, and $\mathrm{ran}(f') = V'$.
- $\varphi_2(f', E') := \forall \langle x, y\rangle \in f'[xy \in E']$.
- $\varphi_3(f') := \exists y \in \mathrm{ran}(f')\, \forall x \in \mathrm{dom}(f')[\langle x, y\rangle \in f']$.

Since $N_G[F]$ is uncountable, we have that

$$H(\theta) \models \exists f[\varphi_1(f, F, V(G)\backslash(X \cup X')) \wedge \varphi_2(f, E(G)) \wedge \varphi_3(f)].$$



Now, by $M \preceq H(\theta)$, we find such a function $f \in M$ and by (3) of Lemma 5.17, it follows that $f(x) \in M$. □

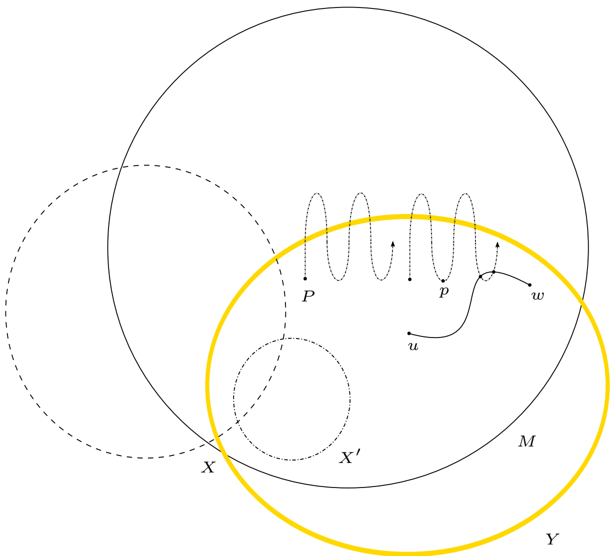
Lemma 5.19. *Let θ be an uncountable cardinal and M a countable elementary submodel of $H(\theta)$. Moreover, let $\langle P, \preceq_P\rangle \in M$ be a generalized path, $G \in M$ a graph, $X \in M$ a countable set of vertices, and $Y \in M$ a $<\aleph_1$-inseparable set of vertices in G. If $p \in P \cap M$, then $(M \cap Y)\backslash(X \cup p\!\downarrow)$ is $<\aleph_0$-inseparable in the graph $G[M] - (X \cup p\!\downarrow)$.*

Proof. Note that $\langle P, \preceq_P\rangle$, G, p, and X lie in $H(\theta)$ (since $M \preceq H(\theta)$). Let us fix vertices $u, w \in (M \cap Y)\backslash(X \cup p\!\downarrow)$ and let $X' \subseteq Y \cap M$ be finite. Since X' is finite and $X' \subseteq M$ it follows that X' is contained in M. We show that there is an u–w-path in the graph $G[M] - (X \cup p\!\downarrow \cup X')$. Consider the formulas:

- $\varphi_1(f', u', w', m', G') :=$ f' is a finite u'–w'-path[5] of length m'.
- $\varphi_2(f', \preceq', p') := \forall y \in \text{ran}(f')\, [y \not\preceq', p']$.

Since Y is $<\aleph_1$-inseparable in G, we have that

$$H(\theta) \models \exists f\, \exists m \in \omega[\varphi_1(f, m, u, w, G - (X \cup X')) \wedge \varphi_2(f, \preceq_P, p)].$$



By $M \preceq H(\theta)$, we can find such sets m and f in M and by (3) of Lemma 5.17, it follows that $f(k)$ is contained in M for every $k < m$. □

Lemma 5.20. *Let θ be a regular uncountable cardinal and M an elementary submodel of $H(\theta)$. Moreover, suppose that $G \in M$ is a graph, $\langle P, \preceq_P \rangle \in M$ is a generalized path of G, and that $A \in M$ is a countable set. If $X \in M$ is an uncountable set of vertices with X that is contained in P, then there is a vertex $x \in X \setminus A$ such that $x + \omega$ lies in M.*

Proof. Note that G, $\langle P, \preceq_P \rangle$, and A are contained in $H(\theta)$ (by $M \preceq H(\theta)$). Since X is uncountable, the set $X' := \{x + \omega \colon x \in X\}$ is uncountable as well (otherwise, the set $\bigcup\{[x, x + \omega) \colon x + \omega \in X'\}$ would be countable and contain the uncountable set X as a subset). Using that A is countable, we obtain that $X' \setminus A$ is uncountable. Hence we find $x \in X$ with $x + \omega \in X' \setminus A$. It follows that

$$H(\theta) \models \exists x'[\exists x \in X \setminus A[x' \text{ is the smallest } \preceq_P\text{-limit above x}]].$$

Since $M \preceq H(\theta)$, we can find such a vertex x' in the submodel M. □

[5]Here, we identify a finite path with an injective function f' that has $\text{dom}(f') = m'$, $\text{ran}(f') \subseteq V(G)$ and satisfies that $f'(k)f'(f+1)$ is an edge of G' for $k < m' - 1$.

Lemma 5.21. *Let θ be a regular uncountable cardinal and M an elementary submodel of $H(\theta)$. Moreover, suppose that $G \in M$ is a graph, $\langle P, \preceq_P \rangle \in M$ is a generalized path of order type ω_1 in G, and that $A \in M$ is a countable set. Then $(P \cap M) \backslash A$ is non-empty.*

Proof. Note that G, $\langle P, \preceq_P \rangle$, and A are contained in $H(\theta)$ (by $M \preceq H(\theta)$). Consider the formula $\varphi(P', A') := \exists v[v \in P' \backslash A']$. Since P' is uncountable and A' is countable we find a vertex v in $P \backslash A$ and such a vertex is contained in $H(\theta)$ (using that $G \in H(\theta)$). Hence $H(\theta) \models \varphi(P, A)$ and by elementarity it follows that $M \models \varphi(P, A)$. □

Definition 5.22 ([7, p. 238])**.** Let θ be an uncountable regular cardinal. The sequence $\langle M_\alpha \colon \alpha < \omega_1 \rangle$ is a *nice chain* (of countable elementary submodels for $H(\theta)$) if the following holds:

(1) $M_0 = \emptyset$ and $M_\alpha \preceq H(\theta)$ for $0 < \alpha < \omega_1$.

(2) M_α is countable for $\alpha < \omega_1$.

(3) The sequence is increasing, i.e., $M_\alpha \subseteq M_\beta$ for ordinals $\alpha \leq \beta < \omega_1$.

(4) $M_\alpha \in M_\beta$ for ordinals $\alpha < \beta < \omega_1$.

(5) $M_\beta = \bigcup\{M_\alpha \colon \alpha < \beta\}$ for every limit ordinal $\beta < \omega_1$.

Lemma 5.23. *Let θ be an uncountable regular cardinal, $\mathcal{M} = \langle M_\alpha \colon \alpha < \omega_1 \rangle$ a nice chain for $H(\theta)$ and C a club of ω_1 with $0 \in C$. Moreover, let ϕ be the unique order preserving bijection with* $\mathrm{dom}(f) = \omega_1$ *and* $\mathrm{ran}(f) = C$ *where C inherits the well order on ω_1. Then $\langle M'_\alpha \colon \alpha < \omega_1 \rangle$ is a nice chain for $H(\theta)$ where $M'_\alpha := M_{\phi(\alpha)}$.*

Proof. Let us check conditions (1)-(5) from the definition of nice chain. The first part of (1) holds since 0 is contained in C. The second part of (1) and statement (2) follow right from the definition of M'_α. Conditions (3) and (4) hold since ϕ is order preserving. For (5) let $\beta < \omega_1$ be a limit ordinal. Then

$$\begin{aligned} M'_\beta &= M_{\phi(\beta)} \\ &\overset{\text{(a)}}{=} \bigcup\{M_\alpha \colon \alpha \in \phi(\beta)\} \\ &\overset{\text{(b)}}{=} \bigcup\{M_\alpha \colon \alpha \in \phi(\beta) \cap C\} \\ &= \bigcup\{M_{\phi(\alpha)} \colon \phi(\alpha) < \phi(\beta)\} \\ &\overset{\text{(c)}}{=} \bigcup\{M_{\phi(\alpha)} \colon \alpha < \beta\} \\ &= \bigcup\{M'_\alpha \colon \alpha < \beta\}. \end{aligned}$$

This needs some further explanation: (a) holds by condition (5) of the definition of nice chain. For the non-trivial inclusion of (b), we claim that $\sup(\phi(\beta) \cap C) = \phi(\beta)$. Indeed, since C is closed, it follows that $\sup(\phi(\beta) \cap C)$ lies in C. By assumption, the bijection ϕ is order preserving, which implies that $\phi(\beta)$ is a limit in C. Now, if $\sup(\phi(\beta) \cap C) < \phi(\beta)$, then $\phi(\beta)$ would be a successor in C; a contradiction. Finally, (c) holds, since ϕ is order preserving. □

The next lemma is a characterisation of elementarity and is very useful in proofs where elementary submodels are constructed directly.

Lemma 5.24 (**Tarski-Vaught criterion**, [7, p. 88])**.** *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be structures for some lexicon* $\mathcal{L}$*. Then the following are equivalent:*

(1) $\mathfrak{A} \preceq \mathfrak{B}$.

(2) *If* $\mathfrak{B} \models \exists y \psi(\vec{a}, y]$ *for a formula* $\psi(\vec{x})$ *and* $\vec{a} \in A$*, then there exists* $b \in A$ *such that* $\mathfrak{B} \models \psi(\vec{a}, b)$*.*

□

Lemma 5.25 ([7, p. 238])**.** *Let* θ *be a regular uncountable cardinal and* $x \in H(\theta)$*. Then there is a nice chain* $\langle M_\alpha \colon \alpha < \omega_1 \rangle$ *of elementary submodels for* $H(\theta)$ *with* $x \in M_1$*.*

Proof. We define a nice chain $\langle M_\alpha \colon \alpha < \omega_1 \rangle$ of elementary submodels for $H(\theta)$ with $x \in M_1$ by recursion on α. The recursion base starts by letting $M_0 := \emptyset$. Now assume that M_α has already been defined for $\alpha < \beta$. If $\beta = \alpha + 1$ choose M_β as a countable elementary submodel of $H(\theta)$ with $M_\alpha \cup \{M_\alpha\} \cup \{x\} \subseteq M_\beta$ (possible by Theorem 5.13).

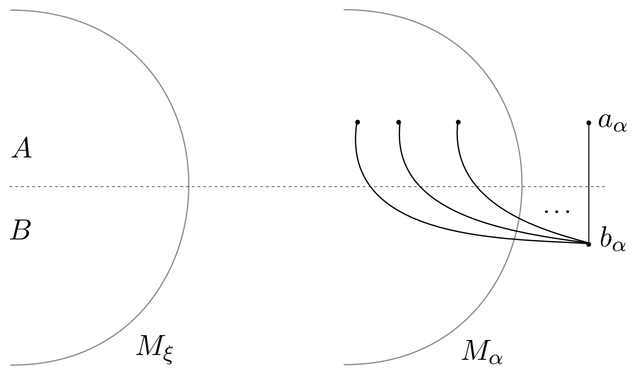
Otherwise, if β is a limit ordinal let $M_\beta := \bigcup \{M_\alpha \colon \alpha < \beta\}$. We need to check that M_β is an elementary submodel of $H(\theta)$. By Lemma 5.24, it suffices to show that given an existential formula $\varphi(\vec{x}) = \exists y \psi(\vec{x}, y)$ and $\vec{a} \in M_\beta$ with $H(\theta) \models \varphi(\vec{a})$ we find $b \in M_\beta$ with $H(\theta) \models \psi[\vec{a}, b]$. This is seen as follows: the sets that define $\vec{a}$ are contained in M_α for some large enough ordinal α. Hence we can find $b \in M_\alpha$, also using that M_α is an elementary submodel of $H(\theta)$. Finally, b also lies in M_β, since $M_\alpha \subseteq M_\beta$ (by condition (3) of the definition of nice chain). □

Definition 5.26. Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph on ω_1 with bipartition classes A,B and let $X \in [A]^{\aleph_1}$. We say that $\langle A, B, X\rangle$ is a *trail* iff there is a nice chain $\langle M_\alpha\colon \alpha < \omega_1\rangle$ of countable elementary submodels for $H(\aleph_2)$ satisfying the following:

(1) For every $\alpha < \omega_1$ there are vertices $a_\alpha \in A\backslash(M_\alpha \cup X)$ and $b_\alpha \in B\backslash M_\alpha$ with $a_\alpha b_\alpha \in E$ and

$$N_G(b_\alpha) \cap (M_\alpha\backslash M_\xi) \cap (A\backslash X)$$

is infinite for every $\xi < \alpha < \omega_1$.



(2) $X \cap (M_{\alpha+1}\backslash M_\alpha)$ consists of exactly one vertex.

(3) $G, A \in M_1$.

Lemma 5.27. *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph on ω_1, with bipartition classes A, B and $X \subseteq A$. Moreover, suppose that $\langle A, B, X\rangle$ is a trail. Then there is a set $X' \subseteq X$ such that $\langle A, B, X'\rangle$ is a trail and a nice chain $\langle M'_\alpha\colon \alpha < \omega_1\rangle$ witnessing that $\langle A, B, X\rangle$ is a trail with the additional property that a'_α and b'_α lie in $M'_{\alpha+1}$.*

Proof. Let $\mathcal{M} := \langle M_\alpha\colon \alpha < \omega_1\rangle$ be a nice chain of countable elementary submodels witnessing that $\langle A, B, X\rangle$ is a trail. Moreover, let a_α and b_α be vertices for $\alpha < \omega_1$ witnessing that $\langle A, B, X\rangle$ is a trail and appending to $\mathcal{M}$. By recursion on α we define a sequence $\langle \xi_\alpha\colon \alpha < \omega_1\rangle$ of ordinals satisfying the following:

(1) The sequence is increasing, i.e., $\xi_\alpha < \xi_\beta$ whenever $\alpha < \beta$.

(2) If $\sup(C_\beta \cap \zeta) = \zeta$ for some $\zeta \le \xi_\beta$, then ζ lies in $C_\beta := \{\xi_\alpha\colon \alpha \le \beta\}$.

(3) $a_{\xi_\beta}, b_{\xi_\beta} \in M_{\xi_{\beta+1}}$.

(4) $N_G(b_{\xi_\beta}) \cap (M_{\xi_\beta}\backslash M_{\xi_\alpha}) \cap (A \backslash X)$ is infinite whenever $\alpha < \beta$.

We begin our recursion by letting $\xi_0 := 0$. Now suppose that ξ_α has already been defined for $\alpha < \beta$. If β is a limit ordinal, then let $\xi_\beta := \bigcup\{\xi_\alpha\colon \alpha < \beta\}$. Condition (1) follows right from the definition of ξ_β. For (2) let $\zeta \le \xi_\beta$ with $\sup(C_\beta \cap \zeta) = \zeta$. If $\zeta \le \xi_\alpha$

for some $\alpha < \beta$, then ζ is contained in C_α, which implies together with $C_\alpha \subseteq C_\beta$ that ζ is contained in C_β. Otherwise, by choice of ξ_β, we have that ζ is the ordinal ξ_β and thus ζ is contained in C_β. Condition (3) only needs to be checked in the case that β is a successor and condition (4) holds by the definition of trail and (1).

Now suppose that $\beta = \alpha + 1$. Choose $\xi_\beta > \xi_\alpha$ minimal with the property that $a_{\xi_\alpha}, b_{\xi_\alpha} \in M_{\xi_\beta}$. Then (1),(3), and (4) hold. For (2) fix $\zeta \leq \xi_\beta$ with $\sup(C_\beta \cap \zeta) = \zeta$. If $\zeta \leq \xi_\alpha$, then $\zeta \in C_\alpha$ and $C_\alpha \subseteq C_\beta$ imply that ζ is contained in C_β. Otherwise, ζ is the ordinal ξ_β and thus ζ is contained in C_β. This completes our construction.

Let $a'_\alpha := a_{\xi_\alpha}$, $b'_\alpha := b_{\xi_\alpha}$ for $\alpha < \omega_1$, and $C := \{\xi_\alpha \colon \alpha < \omega_1\}$. We want to show that C is a club, that is, C is closed and unbounded. Unbounded follows from (1), so let us check that C is closed. If ζ is a countable ordinal with $\sup(C \cap \zeta) = \zeta$, then fix some ordinal β satisfying $\zeta \leq \xi_\beta$. It then holds that

$$\sup(C_\beta \cap \zeta) = \sup(C \cap \zeta) = \zeta.$$

Now, by (2) we have that ζ is contained in C_β, which implies that ζ is contained in C as well.

Let ϕ be the unique order preserving bijection with $\mathrm{dom}(\phi) = \omega_1$ and $\mathrm{ran}(\phi) = C$. An application of Lemma 5.23 to the sequence $\langle M'_\alpha \colon \alpha < \omega_1 \rangle$ shows that $\langle A, B, X' \rangle$ is a trail, where $M'_\alpha := M_{\phi(\alpha)}$ and X' is a suitable subset of X. The conditions (3) and (4) guarantee that a'_α and b'_α have the desired property. $\square$

5.3 Proof of the Main Lemmas

We begin this subsection with the proof of the first main lemma (Lemma 5.10). It uses a nice chain of elementary submodels in order to rebuilt the given generalized path so that it has all the properties we want. The rest of this subsection is aimed at proving the second main lemma (Lemma 5.11). For this purpose, we show how to constructed generalized paths of uncountable order type (that have some additional properties) using nice chains of elementary submodels. Such generalized paths are the starting point of the proof of Lemma 5.11. Covers of bipartition classes by disjoint monochromatic graphs of different colors are carried out by induction on the number of colors.

Lemma 5.10 (First Main Lemma)**.** *Let G be a bipartite graph with bipartition classes A, B both of size $\aleph_1$ and Y a $<\!\aleph_1$-inseparable set of vertices. Moreover, suppose that A is included in Y. If G has a generalized path that is strong on A, then it also has a generalized path that is strong on A and covers Y. (Cf. Figure 14.)*

Proof. We may assume that G has vertex set ω_1 (then $H(\aleph_2)$ contains the graph G). Let X be an uncountable set of vertices witnessing that the generalized path P is strong on A and let us fix a nice chain $\langle M_\alpha \colon \alpha < \omega_1 \rangle$ of countable elementary submodels for $H(\aleph_2)$ satisfying that G, $\langle P, \preceq_P \rangle$, and X lie in M_1 (possible by Lemma 5.25). Let us write $V_\alpha := M_\alpha \cap V(G)$ and choose the vertex p_α to be the $\preceq_P$-minimal vertex in $P \setminus V_\alpha$ for $\alpha < \omega_1$. Then, by (5) of Lemma 5.17, the vertex p_α is a $\preceq_P$-limit and V_α is an initial segment of P. By recursion on α, we define sequences $\langle Q_\alpha \colon \alpha < \omega_1 \rangle$, $\langle \preceq_{Q_\alpha} \colon \alpha < \omega_1 \rangle$ and $\langle x_{\alpha+1} \colon \alpha < \omega_1 \rangle$ satisfying the following:

(1) $\langle Q_\alpha, \preceq_{Q_\alpha}\rangle$ is a generalized path.

(2) If α is not 0, then the order type of Q_α is a limit ordinal.

(3) $Y_\alpha := Y \cap M_\alpha \subseteq Q_\alpha \subseteq M_\alpha$.

(4) $Q_\alpha^\frown p_\alpha$ is a generalized path concentrated on $A \backslash X_\alpha$, where $X_\alpha := \{x_{\xi+1} \colon \xi < \alpha\}$.

(5) Q_β extends $Q_\alpha^\frown p_\alpha$, whenever $\alpha \leq \beta < \omega_1$.

(6) X_α is scattered on Q_α and $x_{\alpha+1} \in (Q_{\alpha+1} \backslash Q_\alpha) \cap X$.

(7) Q_α is locally X_α-robust.

The union of all the generalized paths Q_α will be our desired generalized path and the set of all the vertices $x_{\alpha+1}$ will witness that this generalized path is strong on A. We begin our recursion by letting $Q_0 = \preceq_{Q_0} = \emptyset$. Then suppose that Q_α, $\preceq_{Q_\alpha}$, and $x_{\alpha+1}$ have been defined for ordinals $\alpha < \beta$. We consider the usual cases:

Case 1. If β is a limit ordinal, then let $Q_\beta := \bigcup\{Q_\alpha \colon \alpha < \beta\}$ and choose the set $\preceq_{Q_\beta} := \bigcup\{\preceq_{Q_\alpha} \colon \alpha < \beta\}$ as partial order of Q_β. We first show that Q_α is an initial segment of Q_β for $\alpha < \beta$. For this purpose, consider vertices $q_1 \in Q_\beta$ and $q_2 \in Q_\alpha$ satisfying that $q_1 \preceq_{Q_\beta} q_2$. We find an ordinal $\alpha \leq \zeta < \beta$ such that q_1 is contained in Q_ζ. By condition (5), the ordered set Q_α is an initial segment of Q_ζ and thus q_1 is contained in Q_α.

Next, we show that condition (1) holds for β. The binary relation $\preceq_{Q_\beta}$ is by construction a total order. In order to show that it is well-founded let us fix a non-empty set $U \subseteq Q_\beta$. We find an ordinal α such that the intersection $Q_\alpha \cap U$ is non-empty and using condition (1) for α, it follows that $Q_\alpha \cap U$ has a $\preceq_{Q_\alpha}$-minimal element u_0. Since Q_α is an initial segment of Q_β, this vertex u_0 is also $\preceq_{Q_\beta}$-minimal in the intersection $Q_\beta \cap U$.

To complete the proof that $\langle Q_\beta, \preceq_{Q_\beta}\rangle$ is a generalized path, consider a vertex $q \in Q_\beta$. We find an ordinal $\alpha < \beta$ satisfying that Q_α contains q and that the set $q{\downarrow}$ of $\preceq_{Q_\alpha}$-down-neighbors is cofinal in $q{\downarrow}$. Since Q_α is an initial segment of Q_β, we also have that $q{\downarrow} \cap N_G(q)$ is cofinal in $q{\downarrow}$ with respect to the well order $\langle Q_\beta, \preceq_{Q_\beta}\rangle$.

The first inclusion of condition (3) holds, since every vertex in Y_β lies in some Q_α with $\alpha < \beta$ and condition (3) for this Q_α. The second inclusion holds by $M_\beta = \bigcup\{M_\alpha \colon \alpha < \beta\}$, the definition of Q_β, and condition (3) for ordinals less than β.

For condition (4), consider two vertices $q_1, q_2 \in Q_\beta^\frown p_\beta$ with $q_1 \prec_{Q_\beta^\frown p_\beta} q_2$ and suppose that q_2 is a $\preceq_{Q_\beta^\frown p_\beta}$-limit. If q_2 is not the vertex p_β, then we find an ordinal $\alpha < \beta$ satisfying that q_1 and q_2 are contained in Q_α. Since Q_α is concentrated on $A \backslash X_\alpha$, there is a vertex q that is contained in

$$(A \backslash X_\alpha) \cap Q_\alpha \restriction [q_1, q_2) \cap N_G(q_2).$$

By condition (5) and the second part of condition (6) for α, we also have that q is contained in

$$(A \backslash X_\beta) \cap Q_\beta \restriction [q_1, q_2) \cap N_G(q_2).$$

Now suppose that q_2 and p_β coincide. We find an ordinal $\alpha < \beta$ satisfying that q_1 is contained in Q_α. By Lemma 5.21, we have that the intersection $P \cap (M_\alpha \backslash M_{\alpha+1})$ is non-empty and it follows that p_α is contained in $M_{\alpha+1}$ (also using the choice of p_α). As a consequence we have that $p_\alpha \prec_{Q_\beta^\frown p_\beta} p_\beta$. Since the generalized path P is concentrated on $A \backslash X$, we find a vertex q in

$$(A \backslash X) \cap P \restriction [p_\alpha, p_\beta) \cap N_G(p_\beta).$$

Note that X_β is included in X by the second part of condition (6). Furthermore, note that the intersection $A \cap P \restriction [p_\alpha, p_\beta)$ coincides with $A \cap (Q_\beta^\frown p_\beta) \restriction [p_\alpha, p_\beta)$ by condition (3) and using that A is included in Y. It follows that q is contained in

$$(A \backslash X_\beta) \cap (Q_\beta^\frown p_\beta) \restriction [q_1, q_2) \cap N_G(q_2)$$

as well.

Condition (5) is right from the definitions. For condition (6) let us fix a standard interval S of Q_β. We find an ordinal $\alpha < \beta$ such that S is included in Q_α (also using condition (2)). Since Q_α is an initial segment of Q_β, we have that S is also a standard interval of Q_α. Hence $X_\alpha \cap A$ consists of at most one vertex and if a vertex x is contained in $X_\alpha \cap S$, then x is not the first vertex on S. By the second part of condition (6) for α, we also have that the intersection $X_\alpha \cap Q_\alpha$ coincides with the intersection $X_\beta \cap Q_\alpha$ and it follows that $X_\beta \cap S = X_\alpha \cap S$.

For condition (7) let us fix a vertex $x \in X_\beta$ and a standard interval S of Q_β that contains x. We find an ordinal $\alpha < \beta$ satisfying that S is included in Q_α. By the second part of condition (6) for ordinals less than β, we have that x is contained in X_α. Now, condition (7) for α implies that S is x-robust.

Case 2. Suppose that $\beta = \alpha + 1$ is a successor ordinal. By Lemma 5.20 applied to the elementary submodel M_β of $H(\aleph_2)$, the countable set M_α, the generalized path P, and the uncountable set X we find a vertex $x_\beta \in X \backslash M_\alpha$ satisfying that $x_\beta + \omega$ is contained in M_β (cf. Figure 16). Let y be the $\preceq_P$-minimal vertex in y above $x_\beta + \omega$ and let $Q_{[p_\alpha, y)}$ be the restricted generalized path $Q_\alpha^\frown P \restriction [p_\alpha, y)$.

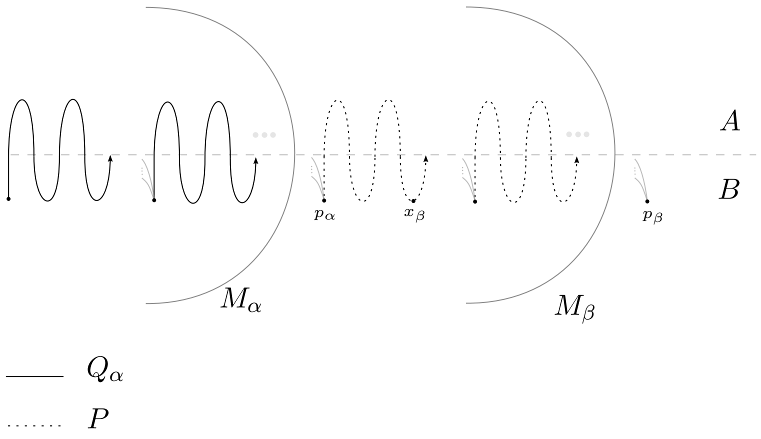


Figure 16: Construction Q_β when β is a successor ordinal.

Let us fix an enumeration $y_0, y_1, \ldots$ of the countably infinite set $Y_\beta \backslash Q_{[p_\alpha, y)}$ with $y_0 = y$. Note that $Y_\beta \backslash Q_{[p_\alpha, y)}$ is indeed infinite: the intersection $A \cap P \restriction [y, y+\omega)$ has size $\aleph_0$ and is included in $Y_\beta \backslash Q_{[p_\alpha, y)}$. Moreover, A is included in Y.

By recursion on n, we define an increasing sequence $\langle S_n \colon n < \omega \rangle$ of finite generalized paths satisfying that S_n is a subset of $V_\beta \backslash Q_{[p_\alpha, y)}$, that S_n contains the vertex y_n and that the last vertex of S_n is contained in Y. Let $S_0 := y$ and suppose that S_n has already been defined. If y_{n+1} is contained in S_n, then just let $S_{n+1} := S_n$. Otherwise, let $\tilde{y}_n$ denote the last vertex on S_n. By Lemma 5.19 applied to the elementary submodel M_β, the generalized path P, the countable set V_α, and $y = p$ we find a finite $\tilde{y}_n$–y_n-path T in $V_\beta \backslash (V_\alpha \cup y{\downarrow} \cup S_n)$. Letting $S_{n+1} := S_n ^\frown T$ completes our construction of the S_n.

We define Q_β as the generalized path $Q_{[p_\alpha, y)} ^\frown S$, where $S := \bigcup \{S_n \colon n < \omega\}$. Then (1)-(3) and (5) follow straight from the construction (using (1)-(3) and (5) for α). For condition (4) consider vertices $q_1, q_2 \in Q_\beta ^\frown p_\beta$ with $q_1 \prec_{Q_\beta ^\frown p_\beta} q_2$ and suppose that q_2 is a $\preceq_{Q_\beta ^\frown p_\beta}$-limit. We need to find a vertex q that is contained in the set

$$(A \backslash X_\beta) \cap (Q_\beta ^\frown p_\beta) \restriction [q_1, q_2) \cap N_G(q_2).$$

If q_2 is part of Q_α, then we find q using condition (4) and the second part of condition (6) for α. Otherwise, if q_2 lies on the generalized path $Q_{[p_\alpha, y)}$, then we let $\tilde{q}_1$ be the vertex $\max_{\preceq_{Q_{[p_\alpha, y)}}} \{q_1, p_\alpha\}$ and we find q in

$$(A \backslash X) \cap P \restriction [\tilde{q}_1, q_2) \cap N_G(q_2)$$

since P is concentrated on $A \backslash X$. Finally, if q_2 is the vertex p_β, then let $\tilde{q}_1$ be the vertex $\max_{\preceq_{Q_\beta}} \{q_1, y\}$ and find q in the set

$$(A \backslash X) \cap (S \backslash S \restriction [y, \tilde{q}_1)) \cap N_G(q_2)$$

again using that P is concentrated on $A \backslash X$.

For (6) consider a standard interval S' of Q_β. If S' is included in Q_α, then S' is also a standard interval of Q_α and x_β is not contained in S'. Hence (6) for α shows that $Q_\alpha \cap X_\beta$ consists of at most one vertex and if x is such a vertex that besides is contained in $S' \cap X_\beta$, then x is not the first vertex on S'. On the other hand, if S' is not included in Q_α, then $S' \cap X_\beta$ is either empty or consists only of the vertex x_β. In the second case S' is also a standard interval of P and thus x_β is not the first vertex on S' since X is scattered on P.

For (7) let $x \in X_\beta$ and let S' be a standard interval of Q_β that contains x. If S' is included in Q_α, then x is contained in X_α and S' is x-robust by (7) for α. Otherwise, x is the vertex x_β and S' is a standard interval of P. It follows that S' is x-robust using that P is locally X-robust and x_β is contained in X.

Now, let $Q := \bigcup\{Q_\alpha \colon \alpha < \omega_1\}$, $\preceq_Q := \bigcup\{\preceq_{Q_\alpha} \colon \alpha < \omega_1\}$, and $X' := \bigcup\{X_\alpha \colon \alpha < \omega_1\}$. We show that $\langle Q, \preceq_Q\rangle$ is a generalized path of order type ω_1 that is strong on A (witnessed by X') and covers Y.

The pair $\langle Q, \preceq_Q\rangle$ is a generalized path by (1) and (5) and Q covers Y by condition (3). The set $A \backslash X'$ is concentrated on Q by (4) and by the second part of (6). To see that X' is scattered on Q fix a standard interval S of Q. Then there is some ordinal $\alpha < \omega_1$ such that S meets Q_α. By condition (2), we have that S is also a standard interval of Q_α. Now by (6), it follows that $Q_\alpha \cap X_\alpha = Q \cap X'$ and that X_α is scattered on Q_α. Thus $|S \cap X'| = |S \cap X_\alpha| = 1$ and if $x \in S \cap X'$, then x is not the first vertex on S.

Finally, let us show that Q is locally X'-robust. Fix a vertex $x \in X'$ and a standard interval S of Q that contains x. As above we find an ordinal α such that S is also a standard interval of Q_α. By condition (6), we have that $Q_\alpha \cap X_\alpha = Q_\alpha \cap X'$ and thus condition (7) shows that S is x-robust. □

Lemma 5.28. *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph with bipartition classes A, B both of size $\aleph_0$. Moreover, suppose that $N_G[F]$ is infinite for every finite $F \subseteq A$ and that ab is an edge of G with $a \in A$ and $b \in B$. If $x \in A \backslash \{a\}$ is a vertex, then there is an x-robust generalized path S of order type ω that covers A and starts with $b^\frown a$.*

Proof. Let us fix an edge ab of G with $a \in A$ and $b \in B$. Moreover, let $A = \{a_n \colon n < \omega\}$ be an enumeration of A satisfying that $a_0 = a$, $a_1 = x$ and let $b_0 := b$. By recursion on n, we find vertices $b_1, b_2 \ldots$ such that $b_n \in B \backslash \{b_0, \ldots, b_{n-1}\}$ and $b_{n+1} \in N_G[\{a_n, a_{n+1}, a_{n+2}\}]$ for every positive integer n (using that $N_G[\{a_n, a_{n+1}, a_{n+2}\}]$ is infinite for every $n < \omega$).

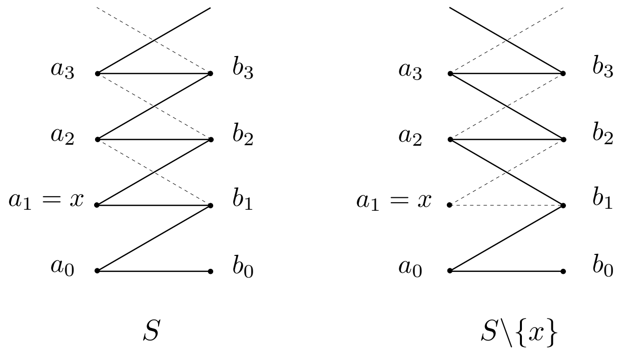


Figure 17: The well-orderings of S and $S\backslash\{x\}$ respectively.

Let $S := A \cup B'$ where $B' := \{b_n : n < \omega\}$ and assign S with the well order $\preceq_S$ that is induced by the enumeration

$$b_0, a_0, b_1, a_1, b_2, a_2, \ldots.$$

Then $\langle S, \preceq_S \rangle$ is a generalized path of order type ω and S is x-robust for $x := a_1$. Indeed, we can assign $S\backslash\{x\}$ with the well ordering $\preceq_{S\backslash\{x\}}$ that is induced by the enumeration

$$b_0, a_0, b_1, a_2, b_2, a_3, b_3, \ldots$$

(cf. Figure 17). □

Lemma 5.29. *Suppose that $G = \langle A \sqcup B, E \rangle$ is a bipartite graph with bipartition classes A,B both of size $\aleph_1$. If $\langle A, B, X \rangle$ is a trail and $N_G[F]$ is uncountable for every finite $F \subseteq A$, then there is a generalized path P of order type ω_1 that is strong on A (witnessed by a set $X' \subseteq X$) and covers A.*

Proof. We may assume that G is a graph on ω_1 (then G is contained in $H(\aleph_2)$). By Lemma 5.27, we find a set $X' \subseteq X$, a nice chain $\mathcal{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels for $H(\aleph_2)$, and vertices $a_\alpha, b_\alpha \in M_{\alpha+1}$ for $\alpha < \omega_1$ witnessing that $\langle A, B, X' \rangle$ is a trail. Let us write $V_\alpha := V(G) \cap M_\alpha$.

By recursion on α, we define sequences $\langle P_\alpha : \alpha < \omega_1 \rangle$, $\langle \preceq_{P_\alpha} : \alpha < \omega_1 \rangle$ and $\langle x_{\alpha+1} : \alpha < \omega_1 \rangle$ satisfying the following:

(1) $\langle P_\alpha, \preceq_{P_\alpha}\rangle$ is a generalized path.

(2) If α is not 0, then the order type of P_α is a limit ordinal.

(3) $A_\alpha := A \cap M_\alpha \subseteq P_\alpha \subseteq M_\alpha$.

(4) $P_\alpha^\frown b_\alpha$ is a generalized path concentrated on $A \backslash X_\alpha$ where $X_\alpha := \{x_{\xi+1} : \xi < \alpha\}$.

(5) P_β extends $P_\alpha^\frown b_\alpha$, whenever $\alpha \leq \beta < \omega_1$.

(6) X_α is scattered on P_α and $x_{\alpha+1} \in (P_{\alpha+1} \backslash P_\alpha) \cap X'$.

(7) P_α is locally X_α-robust.

Let $P_0 = \preceq_{P_0} = \emptyset$ and then suppose that P_α, $\preceq_{P_\alpha}$, and $x_{\alpha+1}$ have been defined for $\alpha < \beta$. We consider the usual cases:

Case 1. If β is a limit ordinal, then we let P_β be the union $P_\beta := \bigcup\{P_\alpha : \alpha < \beta\}$ and similar $\preceq_{P_\beta} := \bigcup\{\preceq_{P_\alpha} : \alpha < \beta\}$. Let us check condition (4) (all the other conditions can be seen as in the proof of Lemma 5.10). Consider vertices $p_1, p_2 \in P_\beta^\frown b_\beta$ and suppose that p_2 is a $\preceq_{P_\beta^\frown b_\beta}$-limit. If p_2 is not the vertex b_β, then we find an ordinal $\alpha < \beta$ satisfying that p_1 and p_2 both lie in P_α. Since P_α is concentrated on $A \backslash X_\alpha$, there is a vertex p that is contained in the set

$$(A \backslash X_\alpha) \cap P_\alpha \restriction [p_1, p_2) \cap N_G(p_2).$$

By condition (5) and the second part of condition (6) for α, we also have that p is contained in

$$(A \backslash X_\beta) \cap P_\beta \restriction [p_1, p_2) \cap N_G(p_2).$$

Now, suppose that p_2 and b_β coincide. We find an ordinal $\alpha < \beta$ satisfying that p_1 is contained in P_α. By the choice of b_β, we have that

$$N_G(b_\beta) \cap (M_\beta \backslash M_\alpha) \cap (A \backslash X')$$

is infinite and thus contains a vertex p that is not part of the finite set $P_\beta \restriction [b_\alpha, \tilde{p}_1)$, where $\tilde{p}_1 := \max_{\preceq_{P_\beta^\frown b_\beta}}\{p_1, b_\alpha\}$. Using condition (3), one finds out that p is also contained in

$$(A \backslash X_\beta) \cap P_\beta \restriction [p_1, p_2) \cap N_G(p_2).$$

Case 2. Suppose that $\beta = \alpha + 1$. Let x_β be the unique vertex in the intersection $X' \cap (M_\beta \backslash M_\alpha)$. Note x_β is distinct from the vertex a_α that is not contained in X' (by the definition of trail). By Lemma 5.18 applied to G, the countable elementary submodel M_β of $H(\aleph_2)$, the countable set V_α, and $Y = A$ we have that $N_{G-V_\alpha}[F] \cap M_\beta$ is infinite for every finite $F \subseteq A_\beta \backslash V_\alpha$. Hence Lemma 5.28 applied to $G[M_\beta] - V_\alpha$, the vertex x_β and the edge $b_\beta a_\beta$ yields an x_β-robust generalized path S that starts with $b_\alpha^\frown a_\alpha$ and covers $A_\beta \backslash V_\alpha$. Let us define P_β as the generalized path $P_\beta := P_\alpha^\frown S$. Then (1)-(3) and (5) hold right from the definitions.

For condition (4), consider vertices $p_1, p_2 \in P_\beta^\frown b_\beta$ with $p_1 \prec_{P_\beta^\frown b_\beta} p_2$ and suppose that p_2 is a $\preceq_{P_\beta^\frown b_\beta}$-limit. We need to find a vertex p that is contained in the set

$$(A \backslash X_\beta) \cap (P_\beta^\frown b_\beta) \restriction [p_1, p_2) \cap N_G(p_2).$$

If p_2 is part of P_α, then we find p using condition (4) and the second part of condition (6) for α. Otherwise, p_2 is the vertex b_β. We can pick the vertex p in the infinite set

$$N_G(b_\beta) \cap (M_\beta \backslash M_\alpha) \cap (A \backslash X')$$

and outside of $P_\beta \restriction [\tilde{p}_1, b_\alpha)$ where $\tilde{p}_1 := \max_{\preceq_{P_\beta^\frown b_\beta}}\{p_1, b_\alpha\}$.

For condition (6), consider a standard interval S' of P_β. If S' is included in P_α, then S' is also a standard interval of P_α and x_β is not contained in S'. Hence condition (6) for P_α shows that the intersection $P_\alpha \cap X_\beta$ consists of at most one vertex and if $x \in S' \cap X_\beta$ is a vertex, then x is not the first vertex on S'. On the other hand, if S' is not included in P_α, then S' coincides with S. Hence the intersection $S' \cap X_\beta$ consists exactly of the vertex x_β and x_β is not the first vertex on S'.

For condition (7) let $x \in X_\beta$ and S' be a standard interval of P_β that contains x. If S' is included in P_α, then x is contained in X_α and S' is x-robust by condition (7) for α. Otherwise, x is the vertex x_β and $S' = S$. It follows that S' is x-robust.

Now, let $P := \bigcup\{P_\alpha \colon \alpha < \omega_1\}$, $\preceq_P := \bigcup\{\preceq_{P_\alpha} \colon \alpha < \omega_1\}$. The set X' witnesses that $\langle P, \preceq_{P_\alpha}\rangle$ is a generalized path that is strong on A and covers A (cf. the proof of Lemma 5.10). □

Lemma 5.30 (cf. [11])**.** *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph with bipartition classes A, B both of size $\aleph_1$ and suppose that A is $\aleph_1$-complete in B. Moreover, let $c\colon E \to \{0, 1\}$ be an edge-coloring of G. If $\langle A, B, X\rangle$ is not a trail in color 0 for any $X \subseteq A$, that is, $\langle A, B, X\rangle$ is not a trail in the graph G_0, then we can find a set $A' \subseteq A$ and a copy of H_{ω_1,ω_1} in color 1 with main class A'.*

Proof. We may assume that G has vertex set ω_1 (then G is contained in $H(\aleph_2)$). Let $\langle M_\alpha \colon \alpha < \omega_1\rangle$ be a nice chain of countable elementary submodels for $H(\aleph_2)$ with $G, A \in M_1$ and let $V_\alpha := V(G) \cap M_\alpha$ for $\alpha < \omega_1$. Fix a set $X \subseteq A$ with $|X \cap (M_{\alpha+1} \backslash M_\alpha)| = 1$ for every $\alpha < \omega_1$. Moreover, let W consist of exactly those ordinals $\beta < \omega_1$ such that there are vertices $a \in A \backslash (V_\beta \cup X)$ and $b \in B \backslash V_\beta$ with $ab \in E(G_0)$ and

$$(N_G(b, 0) \cap V_\beta) \backslash (V_\alpha \cup X)$$

is infinite for every $\alpha < \beta$. We claim that $S := \omega_1 \backslash W$ is stationary. Indeed suppose for a contradiction that C is a club of ω_1 that fails to meet S. Note that C is included in W. Let ϕ be the unique order preserving bijection with $\mathrm{dom}(\phi) = \omega_1$ and $\mathrm{ran}(\phi) = C$. By Lemma 5.23, the sequence $\langle M'_\alpha \colon \alpha < \omega_1\rangle$ is a nice chain of elementary submodels for $H(\aleph_2)$, where $M'_\alpha := M_{\phi(\alpha)}$. Let X' be a subset of X with $|X' \cap (M'_{\alpha+1} \backslash M_\alpha)| = 1$. Then $\langle A, B, X'\rangle$ is a trail in color 0. Indeed let $\beta < \omega_1$. By $C \subseteq W$, the choice of W, and the definition of M'_β we find vertices $a \in A\backslash(M'_\beta \cup X')$ and $b \in B\backslash V_{\phi(\beta)}$ such that ab is an edge of G_0 and the set

$$(N_G(b, 0) \cap M'_\beta)\backslash(M'_\alpha \cup X')$$

is infinite for every $\alpha < \beta$. We now consider two cases:

Case 1. Suppose that there is an ordinal $\alpha \in S$ such that every vertex $a \in A\backslash(V_\alpha \cup X)$ is adjacent to only countably many vertices in G_0. Let us fix such an ordinal $\alpha < \omega_1$. Then A' is $\aleph_1$-complete in B in color 1, for $A' := A\backslash(V_\alpha \cup X)$. Moreover, A' has size $\aleph_1$ (using that A has size $\aleph_1$, that V_α is countable and that $V_{\alpha+1}\backslash(V_\alpha \cup X)$ is non-empty for every $\alpha < \omega_1$ by elementarity). Hence Lemma 5.4 yields a copy of H_{ω_1,ω_1} in color 1 with main class A'.

Case 2. Suppose that for every ordinal $\alpha \in S$ there is a vertex $a \in A\backslash(V_\alpha \cup X)$ satisfying that $N_G(a, 0)$ is uncountable. For $\alpha < \omega_1$ let us fix vertices $a_\alpha \in A\backslash(V_\alpha \cup X)$ and $b_\alpha \in B\backslash V_\alpha$, pairwise distinct and satisfying that $a_\alpha b_\alpha$ is an edge of color 0. By choice of S we find for every ordinal β an ordinal $\alpha = \alpha(\beta)$ with $\alpha < \beta$ and $N_G(b_\beta, 0)\backslash(V_\alpha \cup X)$ is finite. If β is a limit ordinal, then we can even achieve that $N_G(b_\beta, 0)\backslash(V_\alpha \cup X)$ is empty. Fix such ordinals $\alpha(\beta)$ for every limit ordinal $\beta \in S$.

Apply Fodor's Pressing Down Lemma (see [7, p. 220]) to the stationary set

$$S \cap \{\xi < \omega_1 \colon \xi \text{ limit}\}$$

in order to obtain a stationary set T and an ordinal $\tilde{\alpha}$ such that $N_G(b_\beta, 0)\backslash(V_{\tilde{\alpha}} \cup X) = \emptyset$ for every ordinal $\beta \in T$. In other words, every vertex $a \in A\backslash V_{\tilde{\alpha}}$ is linked to every vertex b_β with $\beta \in T$ by a 1-colored edge. We can now apply Lemma 5.4 to the graph $G_1[A', B']$ where $A' := A\backslash(V_{\tilde{\alpha}} \cup X)$ and $B' := \{b_\beta \colon \beta \in T\}$ in order to obtain a copy of H_{ω_1,ω_1} in color 1 with main class A'. □

Lemma 5.31 (cf. [11])**.** *Let $\langle H, \phi\rangle$ be a copy of H_{ω_1,ω_1} with main class A. Then $\langle V(H), \preceq_H\rangle$ contains a generalized path P of order type ω_1 that is strong on A and covers A.*

We give a short proof using elementary submodels and some of our preliminary results. However, one can prove Lemma 5.31 by hand as well.

Proof. We may assume that H has vertex set ω_1 (then H is contained in $H(\aleph_2)$. For each ordinal $\alpha < \omega_1$ let $a_\alpha := \phi(\langle\alpha, 0\rangle)$ and $b_\alpha := \phi(\langle\alpha, 1\rangle)$. We show that $N_H[F]$ has size $\aleph_1$ for every finite $F \subseteq A$ and that there is an uncountable set $X \subseteq A$ such that $\langle A, B, X\rangle$ is a trail, where $B := V(H)\backslash A$. Then Lemma 5.29 applied to H yields the desired generalized path. So suppose that $F = \{a_{\alpha_1}, \ldots a_{\alpha_n}\} \subseteq A$ is finite and let $\alpha^* := \max\{\alpha_k \colon k < n\}$. Then $N_G[F]$ has the uncountable subset $\{b_\beta \colon \alpha^* \leq \beta\}$, which shows the first statement.

For the second statement let $\langle M_\alpha \colon \alpha < \omega_1\rangle$ be a nice chain of countable elementary submodels for $H(\aleph_2)$ with $G, \langle V(H), \preceq_H\rangle \in M_1$. Let us fix a set $X \subseteq A$ satisfying that $X \cap (M_{\alpha+1}\backslash M_\alpha)$ consists of exactly one vertex for every $\alpha < \omega_1$. For each $\beta < \omega_1$ let ξ_β be the smallest ordinal such that b_{ξ_β} is not contained in M_β, i.e.,

$$\xi_\beta := \min\{\alpha < \omega_1 \colon b_\alpha \notin M_\beta\}.$$

By (5) of Lemma 5.17, the vertex b_{ξ_β} is a $\preceq_H$-limit and $M_\beta \cap V(H)$ is a $\preceq_H$-initial segment of $V(H)$ for every $\beta < \omega_1$. Hence the set

$$N_H(b_{\xi_\beta}) \cap (M_\beta \backslash M_\alpha) \cap (A \backslash X)$$

is infinite whenever $\alpha < \beta < \omega_1$ (using that $V(H) \cap (M_\beta \backslash (M_\alpha \cup X))$ is infinite by elementarity). So the triple $\langle A, B, X \rangle$ is indeed a trail. □

Lemma 5.32. *Let $G = \langle A \sqcup B, E \rangle$ be a bipartite graph with bipartition classes A, B both of size $\aleph_1$ and suppose that A a is $\aleph_1$-complete in B. Moreover, let $c\colon E \to \{0, \dots, r-1\}$ be an edge-coloring. Then there is a color j, an uncountable subset $A' \subseteq A$ and disjoint sets $B', B'' \subseteq B$ of vertices satisfying the following:*

- *There is a generalized path P of $G_j[A', B']$ that is strong on A', has order type ω_1, and satisfies that $A' \cap P$ is $<\aleph_1$-inseparable in $G_j[A', B']$.*
- *$G_j[A', B'']$ has a perfect matching.*

Proof. The proof is via induction on r. If the edges of G are colored with only one color, then Lemma 3.10 yields disjoint uncountable subsets $B', B'' \subseteq B$ such that $G[A, B'']$ has a perfect matching and A is $\aleph_1$ complete in B' (for the second statement we only need that B' is uncountable, which is clearly satisfied by the lemma). By Lemma 5.4, we find a copy H of H_{ω_1,ω_1} with main class A in $G[A, B']$ and by Lemma 5.31, this graph H together with the H_{ω_1,ω_1}-ordering is a generalized path that is strong on A and A is $<\aleph_1$-inseparable. Letting $A' := A$ completes the induction base.

Now assume that the lemma holds for edge-colorings with less than r colors. By Lemma 3.6, we find a set $A' \in [A]^{\aleph_1}$ and a color k such that $N_G[F, k]$ is uncountable for every finite $F \subseteq A'$. We may assume that k is the color 0. By Lemma 3.10 we find disjoint sets $B', B'' \subseteq B$ such that $N_G[F, 0] \cap B'$ is uncountable for every finite $F \subseteq A'$ and $G_0[A', B'']$ has a perfect matching. Let $c'\colon E \to \{0, 1\}$ be the coloring that assigns an edge e with color 0 if it has color 0 with respect to c and color 1 else. We consider the following two cases: If there is a 1-colored copy H of H_{ω_1,ω_1} in $G[A', B']$ (with respect to c') whose main class is included in A', then we can find everything we need in H using the induction hypothesis.

Otherwise, by Lemma 5.30 applied to $G[A', B']$ and c', there is a set $X \subseteq A$ such that the triple $\langle A', B', X \rangle$ is a trail in $G_0[A', B']$. Applying Lemma 5.29 to $G_0[A', B']$ yields the desired generalized path. □

Lemma 5.11 (Second Main Lemma). *Let $G = \langle A \sqcup B, E\rangle$ be a bipartite graph with bipartition classes A, B, both of size $\aleph_1$ and suppose that A is $\aleph_1$-complete in B. Moreover, let $c\colon E \to \{0, \dots, r-1\}$ be an edge-coloring of G. Then there is a partition $\{A_i : i < r\}$ of A, disjoint sets $B_0, \dots, B_{r-1} \subseteq B$, well-orders $\preceq_{A_0 \sqcup B_0}, \dots, \preceq_{A_{r-1} \sqcup B_{r-1}}$, and distinct colors $i_0, \dots, i_{r-1}$ satisfying the following:*

- *A_0 is $<\aleph_1$-inseparable in $G_{i_0}[A_0, B_0]$.*
- *$\langle A_0 \sqcup B_0, \preceq_{A_0 \sqcup B_0}\rangle$ is a generalized path of order type ω_1 in the graph G_{i_0}, that is strong on A.*
- *$\langle A_j \sqcup B_j, \preceq_{A_j \sqcup B_j}\rangle$ is a generalized path in the graph G_{i_j} for $j = 1, \dots, r-1$.*

  (Cf. Figure 15.)

Proof. The proof is via induction on r. For the induction base suppose that the edges of G are colored with only one color. By Lemma 5.4, we find a copy H of H_{ω_1,ω_1} with main class A and by Lemma 5.31 this H together with the H_{ω_1,ω_1}-ordering is a generalized path of order type ω_1 that is strong on A and A is $<\aleph_1$-inseparable.

Now, assume that the lemma holds for edge-colorings of G with less than r colors. Let us fix a color j, sets A', B', B'' of vertices and a generalized path P_0' as in Lemma 5.32. By symmetry, we may assume that $j = 0$. Let $Y' := P_0' \cap A$ and let Y consist of all the vertices in A that lie in Y' or send uncountably many 0-colored edges to B'', i.e.,

$$Y := Y' \cup \{a \in A\colon |N_G(a, 0) \cap B''| = \aleph_1\}.$$

We consider the following two cases:

Case 1. Suppose that Y and Y' coincide. Then the set $A \backslash Y$ is $\aleph_1$-complete in B with respect to the graph $G_{\neq 0}[A\backslash Y, B'']$. By the induction hypothesis if $A\backslash Y$ is uncountable and by Corollary 4.4 else, we find disjoint monochromatic generalized paths $P_1, \dots, P_{r-1}$ of different colors in the graph $G_{\neq 0}[A\backslash Y, B'']$ covering $A\backslash Y$. Letting $P_0 := P_0'$, $A_i := P_i \cap A$, $B_i := P_i \cap B$, and $\preceq_{A_i \sqcup B_i} := \preceq_{P_i}$ for $i < r$ completes the proof.

Case 2. Suppose that Y' is a proper subset of Y. Let us fix a sequence $\langle y_\alpha : \alpha < \omega_1\rangle$ of vertices in $Y\backslash Y'$ satisfying that every vertex in $Y\backslash Y'$ occurs uncountably often. By recursion on α, we find distinct vertices $b^i_\alpha \in B''$ for $i = 0, 1$ and $\alpha < \omega_1$ such that $b^0_\alpha y_\alpha$ is an edge of G_0 (the vertices b^1_α will serve as a buffer). Indeed if the b^i_α have already been defined for $\alpha < \beta$, then let $B_{<\beta} := \{b^i_\alpha : \alpha < \beta, i = 0, 1\}$. Since $B_{<\beta}$ is countable and since y_α sends uncountably many 0-colored edges to B'', we can pick the vertex b^0_β inside $B''\backslash B_{<\beta}$. The vertex b^1_β can be chosen arbitrarily in $B''\backslash B_{<\beta} \cup \{b^0_\beta\}$ (again using that $B_{<\beta}$ is countable).

Let B''' be the set $\{b^0_\alpha : \alpha < \omega_1\}$. By Lemma 3.7 and Lemma 3.9, we have that the set Y is $<\aleph_1$-inseparable in $G_0[Y, B' \cup B''']$. Hence Lemma 5.10 yields a generalized path P_0 of order type ω_1 that is strong on Y and covers Y with respect to the graph $G_0[Y, B' \cup B''']$. Note that $A\backslash Y$ is $\aleph_1$-complete in $B''\backslash B'''$ with respect to the graph $G_{\neq 0}$. By the induction hypothesis if $A\backslash Y$ is uncountable and by Corollary 4.4 else, we can find disjoint monochromatic generalized paths $P_1, \dots, P_{r-1}$ of different colors in

$G_{\neq 0}[A\backslash Y, B'']$ covering $A\backslash Y$. Letting $A_i := P_i \cap A$, $B_i := P_i \cap B$ and $\preceq_{A_i \sqcup B_i} := \preceq_{P_i}$ for $i < r$ completes the proof. $\square$

6 Open Problems

It remains an open problem whether Theorem 1.6 can be generalized to complete bipartite graphs with bipartition classes of arbitrary uncountable size:

Conjecture 6.1 ([11]). *For every positive integer r, every infinite cardinal κ and every r-edge-coloring of the complete bipartite graph with bipartition classes of size κ, there is a partition of the vertex set into $2r-1$ monochromatic generalized paths.*

In [11] Daniel Soukup discusses ω-edge-colorings of complete infinite graphs and it turns out, that there are ω-edge-colorings of the complete graph on ω_1 such that there is no monochromatic uncountable generalized path. In particular, Theorem 1.4 can not be generalized to ω-edge-colorings. However, considering *locally finite* edge-colorings might yield some interesting problems:

Definition 6.2. Let $G = \langle V, E\rangle$ be a graph. An edge-coloring $c\colon E \to \kappa$ is *locally finite* iff the image of $E(v)$ under c is finite for every vertex $v \in V$.

Problem 6.3. Let $G = \langle V, E\rangle$ be an infinite complete graph and let $c\colon E \to \omega$ be a locally finite edge-coloring of G. Is it true that G has a partition $\mathcal{P}$ of its vertex set into monochromatic generalized paths of different colors? Moreover, if $|G|$ has cofinality greater than $\aleph_0$, can then $\mathcal{P}$ be chosen finite?

This would be best possible because if $|G|$ has cofinality $\aleph_0$, then there is a locally finite edge coloring $c\colon E \to \omega$ such that every partition $\mathcal{P}$ of V into monochromatic generalized paths of different colors needs to be infinite:

Example 6.4. Let κ be an infinite cardinal, let $G = \langle V, E\rangle$ be a complete graph of order κ, and suppose that κ has cofinality $\aleph_0$. Moreover, let $\langle \kappa_n \colon n < \omega\rangle$ be an increasing sequence of cardinals which exhaust κ, that is, $\bigcup\{\kappa_n \colon n < \omega\} = \kappa$. Since G has order κ, we find a partition $\{V_n \colon n < \omega\}$ of V such that V_n has size κ_n.

Consider the locally finite edge-coloring $c\colon E \to \omega$ of G defined as follows. Let $e = vw$ be an edge. Additionally, suppose that $v \in V_m$, $w \in V_n$ and that $m \le n$ are positive integers. If m and n coincide, then c assigns the edge e with color 0; else with color $m+1$.

We claim that there is no partition of V into finitely many monochromatic generalized paths of different colors. Indeed if $\mathcal{P}$ is such a partition, then there is a generalized path P contained in $\mathcal{P}$ that has size κ. However, this is not possible: The graph G_0 has no set of κ many vertices that is connected and each of the graphs G_n for positive integers $n \neq 0$ can be considered as bipartite graph with bipartition classes having size κ_n (such graphs can only have generalized path of size at most κ_n).

The 'moreover' part of Problem 6.3 is motivated by the following observation:

Example 6.5. Let $G = \langle V, E \rangle$ be an infinite complete graph and let $c\colon E \to \omega$ be a locally finite edge-coloring. Moreover, suppose that $|G|$ has cofinality greater than $\aleph_0$. Then there is a bipartition $\{V_1, V_2\}$ of V with bipartition classes both of size $|G|$ and a positive integer N, satisfying that all the edges in $E(V_1) \cup E(V_1, V_2)$ have color less than N. Indeed, let $n < \omega$ the set $\tilde{V}_{\leq n}$ consist of those vertices of G that are only incident with edges of color at most n. Consider the partition $\{\tilde{V}_n \colon n < \omega\}$ of V, where $\tilde{V}_n := \tilde{V}_{\leq n} \setminus \bigcup\{\tilde{V}_m \colon m < n\}$. Since $|G|$ has cofinality greater than $\aleph_0$, we find a positive integer N such that $\tilde{V}_N$ has size $|G|$. Every bipartition $\{V_1, V_2\}$ of V with bipartition classes both of size $|G|$ and satisfying that V_1 is included in $\tilde{V}_N$ can be taken.

Maybe one could slightly modify the proof of Theorem 1.4 given in [11], such that the above example leads to a proof that confirms the 'moreover' part of Problem 6.3.

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