Well-quasi-ordering Friedman ideals of finite trees Proof of Robertson's magic-tree conjecture

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Abstract

Applying a recent extension (2015) of a structure theorem of Robertson, Seymour and Thomas from 1993, in this paper we establish Robertson's magic-tree conjecture from 1997.

1 Introduction

We begin by explaining our motive for this work. A major motive for this paper is the Graph Minor Theorem (GMT). In [15], Robertson and Seymour establish Wagner's conjecture that finite graphs are 'well-quasi-ordered' (wqo) under the minor relation. This is indeed "among the deepest theorems that Mathematics has to offer" [1]. The depth and the scope of the theorem can be guessed by the mere fact that a series of twenty articles, Graph Minors I.-XX. had to appear in the span of two decades for the complete establishment of the conjecture. However, it is not known if the minor ideals (sets of finite graphs closed under taking minors) are wqo under the subset relation. The result in this paper is a preliminary to what we believe will be a proof of this open problem. Why is this interesting?

Due to Rado's example, [11], there exists a set Q, such that Q is wqo but the set LQ of ideals of Q (the set of all classes closed under $\leq$ in Q) is not wqo under $\subseteq$. However, if LQ is wqo under $\subseteq$, then Q is wqo under $\leq$. This is easy as shown by Lemma 3. Rado's example is very interesting since it shows that Higman's famous wqo extension lemma [3] for finite sequences is not generalizable to infinite sequences. However, if LQ is wqo under $\subseteq$, then by a similar argument as in Lemma 3 one can easily prove that the infinite sequences of Q are wqo under the embedding relation [2] (similar to Higman's embedding for finite sequences).

Furthermore, Kruskal has shown by an unpublished example (cited in [6]) that for each $n \geq 1$, there exists a set Q_n such that $L^n Q_n$ is wqo, but $L^{n+1} Q_n$ is not wqo. Also see [2] for a discussion and exercise on this particular problem. So for each n the wqo condition is increasingly stronger.

Motivated by Kruskal's examples Nash-Williams proved in [6] that the iterated sets of topological minor ideals of finite trees '$L^n\mathcal{F}$' is wqo under $\subseteq$, for every integer $n \geq 1$. Nash-Williams in fact expresses his opinion [6] that most 'natural' wqo sets have this extended wqo property, even if we iterate the ideals beyond the finite case. Iteration by transfinite induction is very interesting from the perspective of a theory called 'better-quasi-ordering' ('bqo theory'), introduced by Nash-Williams [7], but for simplicity we will not address this problem in this paper.

In parallel to Nash-Williams' motive, we are interested in proving that the iterated graph minor ideals are wqo under $\subseteq$. The result in this paper is a first step towards this goal. In Graph Minors IV, [14], it is shown that the Friedman gap-condition imposed on the 'linked tree-decomposition' of graphs plays a key role in verifying Wagner's conjecture for finite graphs of bounded tree-width. In fact, the authors prove that finite trees are wqo under a bit stronger form 'Robertson-Seymour gap-condition' (RS gap-condition) to achieve their goal. Without going into technical details we can state that the RS gap-condition in [14] requires strict equality of certain edge labels (in addition to the usual gap-condition). There is already a flavour of the RS gap-condition [14] concept in this paper. That is to say, in this paper, we allow the vertices of trees to be labelled by an arbitrary wqo set, while their edges are labelled by a well-ordered set. In [14, 15], the vertex labels correspond to what the authors call 'bags', and the edge labels correspond to the 'cut-size' at each vertex.

A pioneering work on finite structure is given by Robertson, Seymour and Thomas (RST). In [16], they show that every topological minor ideal $\mathcal{I}$ of finite trees has a finite 'structure-tree' $T_{\mathcal{I}}$ that contains all structural information about $\mathcal{I}$. In other words, the encoded information in $T_{\mathcal{I}}$ enables us to construct an arbitrary element of $\mathcal{I}$. This result was recently generalized for Friedman ideals in [8].

To prove that iterated graph minor ideals are wqo, we need to strengthen the finite structure theorem in [8], so that it is also valid for the stronger RS gap-condition. In [8], we have given an algorithm that computes the finite structure of a Friedman ideal $\mathcal{I}$, given the obstruction set of $\mathcal{I}$. Currently, we focus on Friedman ideals to avoid further complication.

Robertson conjectured that perhaps the structure-tree $T_{\mathcal{I}}$ has a more 'magical' use than merely holding instructions on how to construct the trees of $\mathcal{I}$. This additional and more dramatic benefit of $T_{\mathcal{I}}$ is known as the 'lifting property' of $T_{\mathcal{I}}$. Can an embedding relation between two finite trees 'lift' to imply that two infinite ideals are related by the subset relation, or is this wishful thinking? We will answer this question for Friedman ideals and call the tree a "magic-tree" for the purpose of dramatizing its surprising consequence. Prior to that we define the 'lifting' method formally in (1.1).

The main result of this paper is Theorem 6 which establishes the conjecture of Robertson discussed above. This is formally stated after some necessary definitions.

1.1 Basic definitions, a brief history and main results

A *quasi-order*, $\leq$, on a set Q is a reflexive and transitive relation. Let $I \subseteq Q$. If for every $y \in Q$ and $x \in \mathcal{I}$ such that $y \leq x$ we have $y \in \mathcal{I}$ then we say $\mathcal{I}$ is an *ideal of Q*. Let LQ denote the set of all ideals of Q quasi-ordered (qo, for short) under '$\subseteq$'. As usual, the notation $I \subset I'$ means strict inclusion, i.e. $I \subseteq I' \not\subseteq I$.

If $I \in LQ$ and Q any qo, then *an obstruction set of* I is a maximal set of incomparable minimal elements of $Q - I$. It is not hard to check that any two obstruction sets are the same up to equivalence of elements within Q. To avoid having to worry about this triviality, we fix for each ideal I a particular choice of obstruction set, to be denoted $\Omega(I)$. For $S \subseteq Q$, we define the ideal $Q/S = \{q \in Q | s \not\leq q \text{ for all } s \in S\}$.

For basic terminology and notations such as vertex set V, edge set E, a directed graph D and so forth, we follow standard conventions in [1]. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. For each $m \in \mathbb{N}$, we denote the set $\{0, 1, 2, \dots, m\}$ by $[[m]]$ and assume that both $\mathbb{N}$ and $[[m]]$ are well-ordered by the usual integer inequality. We also use, for a directed graph D, the following succinct notation inspired by [1]: A *directed path* from a vertex v to a vertex w in D is a sequence $(v_0, v_1, \dots, v_n)$ such that $v_0 = v, v_n = w$ and for any i with $1 \leq i \leq n$ there is an edge in D from v_{i-1} to v_i. If there is a unique such directed path, we shall denote it by vDw. For the special case where $V(D) = \{v_0, v_1, \dots, v_n\}$ and $E(D) = \{v_0v_1,\ v_1v_2,\ \dots, v_{n-1}v_n\}$, we write $D = [v_0, v_1, \dots, v_n]$ to be explicit.

1.1.1 Let $m \in \mathbb{N}$, and Q be a non-empty qo set. An (m, Q)-*tree* is a quadruple $T = (V, E, l, r)$, such that:

(i) V is a finite set, $r \in V$, and $E \subseteq V \times V$,
(ii) for every $v \in V$ there is a unique directed path rTv, and
(iii) $l : V \to [[m]] \times Q$ is a *labelling function*, where $l(v) = (l_E(v), l_V(v))$.

We denote the set of all (m, Q)-trees by $\mathcal{F}(m, Q)$.

Remark 1 *In this paper we work with (m, Q)-trees almost exclusively. In Section 3 we study ideals of unlabelled trees for simplicity. In this case we ignore all labels, by assuming $m = 0$ and $Q = \{0\}$, and write $\mathcal{F}$ instead of $\mathcal{F}(0, \{0\})$. However, we shall 'represent' ideals of unlabelled trees by trees in $\mathcal{F}(2, \mathbb{N})$. So, even in Section 3, it is the trees of $\mathcal{F}(2, \mathbb{N})$ that are interesting due to Theorem 11 and Theorem 15.*

As usual, elements of V are called *vertices*, and elements of E are called *edges* of T. We denote an edge $e = (u, v) \in V \times V$ by uv (for short). Also, if $V(T) = \{r\}$, then we denote the tree T by $[r]$. This avoids confusion with the notation $r \in V(T)$. So, by (1.1.1), $[r] = (\{r\}, \emptyset, l, r)$. Also note that $[r]$ is a special case of a directed path rTr as described above. We write $l(r) = (p, q), p \in [[m]], q \in Q$, when it is necessary to be explicit about the labels.

We often write T without specifying that $T \in \mathcal{F}(m, Q)$ when it is obvious that we are considering (m, Q)-trees. The same is true about the notation $[r]$. When the label set $[[m]] \times Q$ is obvious from the context, we use the term *tree* for short. The vertex r is called the *root* of T. When we need to be more explicit, we use the notation *root*(T) instead of r. We call $l_E(r)$, the *root-edge label*. On the other hand, if v is not a root, then there is a unique vertex u, such that the edge uv is directed into v (so v is an *out-neighbour of* u). Then, we can treat $l_E(v)$ as labelling the edge $e = uv$ and will sometimes refer to it as the *edge-label of* e. Where no misunderstanding occurs we will write $l(e)$ instead of $l_E(v)$, for clarity and emphasis. If $v \in V(T)$, then sometimes we write $v \in T$ (as this is widely used). The *out-degree* of a vertex $v \in T$ is the number of out-neighbours of v. A *leaf* of T is $v \in T$ with out-degree zero. If $v \in T$ is not a leaf, then it is an *interior* vertex. The *height* of T, *height*(T), is the maximum number of edges in a directed path in T. If $v \in T$, then T^v denotes the maximal subtree of T rooted at v. Thus, $T^r = T$, and $T^v = [v]$ if and only if v is a leaf of T. If uv is an edge of T, then T^v is *a child of* u. The set of all children of u is denoted by $Chil(u)$. If $X \subseteq \mathcal{F}(m, Q)$, then we define

$$Chil(X) = \bigcup_{T \in X} Chil(root(T)).$$

1.1.2 Let $T, T' \in \mathcal{F}(m, Q)$, with roots r, r', respectively. We say that T is a *topological minor of* T' (and write $T \preceq T'$) if there is an injective mapping $f : V(T) \to V(T')$, such that

(i) for every $u, v \in T$, $f(u \wedge v) = f(u) \wedge f(v)$, (where $u \wedge v$ means the last vertex that belongs to both rTv and rTu), and

(ii) $l_V(v) \leq_Q l_V(f(v))$ for each $v \in V(T)$.

Note that the relation $\preceq$ ignores the edge labels $l_E(v)$. Moreover, if $T \preceq T'$, then f maps every edge $uv = e$ of T to a unique directed path $f(e) = f(u)T'f(v)$ in T'. Thus, extending f so that it also maps $E(T)$ to a set of directed paths of T', we define the following:

1.1.3 Let $T, T' \in \mathcal{F}(m, Q)$, with roots r, r', respectively. Then, $T \preceq_F T'$ if

(i) $T \preceq T'$,

(ii) $l(e) \leq l'(e')$ for all $e' \in E(f(e))$ and $l(e) = l'(e'')$ for the last edge of $f(e)$, and

(iii) $l_E(r) \leq l_E(v)$, for every vertex v in $r'T'f(r)$.

The function f is said to satisfy the *Friedman gap-condition*, if (i), (ii) and (iii) of (1.1.3) hold. The set of rooted finite trees without edge labels (but with vertex labels) is denoted by $\mathcal{F}(Q)$.

Remark 2 *In (1.1.3)(ii) Robertson and Seymour further assume that equality holds for the first edge label of $f(e)$ with $l(e)$ (we write $T \preceq_{RS} T'$ when this additional condition is assumed).*

In $\mathcal{F}(m, Q)$, unless specified otherwise we always assume that trees are ordered by $\preceq_F$. A sequence $X = (q_1, q_2, \dots)$ of elements of Q is *good* if there exist indices i, j, such that $i < j$ and $q_i \leq q_j$. Otherwise X is *bad*. A qo set Q is *well-quasi-ordered* (*wqo*, for short) if every infinite sequence of Q is good.

Suppose that Q, Q' are qo sets such that Q' is wqo. Suppose also that we have a relation R from Q to Q' such that

1. for any x in Q there is some x' in Q' with xRx'.
2. for any x, y in Q and x', y' in Q' with xRx', yRy' and $x' \leq y'$ we also have $x \leq y$.

Then the wqo property of Q' is said to *lift to* Q. This is exactly how we are going to prove that Friedman ideals are wqo under the subset relation. We start with the following easy but fundamental lemma.

Lemma 3 *Let Q be a qo set. Then, the following are equivalent: (a) Q is wqo under $\leq$. (b) LQ is well founded. (c) Q is well founded and contains no infinite anti-chain. In particular, if LQ wqo under $\subseteq$, then Q is wqo under $\leq$.*

Proof. The equivalence of (a), (b), and (c) is easy and standard. Next, assume LQ is wqo and let $X = (q_1, q_2, \dots)$ be an infinite sequence of Q. Let $\downarrow q_i \downarrow = \{q \in Q \mid q \leq q_i\}$. Then consider the sequence $(\downarrow q_1 \downarrow, \downarrow q_2 \downarrow, \dots,)$ of ideals in LQ. For all i, i', $\downarrow q_i \downarrow \subseteq \downarrow q_{i'} \downarrow$ if and only if $q_i \leq q_{i'}$ and thus X is good if LQ is wqo. $\square$

The following theorem of Kruskal [5] is generalized by Friedman as Theorem 5 and by Robertson and Seymour as Theorem 5$'$.

Theorem 4 *(Kruskal, [5]) Let Q be wqo. Then $\mathcal{F}(Q)$ is wqo under $\preceq$.*

Theorem 5 *(Friedman, [13]) Let Q be wqo and $m \in \mathbb{N}$. Then $\mathcal{F}(m, Q)$ is wqo under $\preceq_F$.*

Theorem 5$'$ (*Robertson and Seymour, [14]*) *Let Q be wqo and $m \in \mathbb{N}$. Then $\mathcal{F}(m, Q)$ is wqo under $\preceq_{RS}$.*

In Section 2 we prove Theorem 15. In Section 3 we prove the following theorem which was a conjecture of Robertson, [12].

Theorem 6 *Let $m \in \mathbb{N}$ and let Q be a wqo set. Then there is an $m' \in \mathbb{N}$ and a qo set Q', so that for every $\mathcal{I} \in L\mathcal{F}(m, Q)$, there is a tree $T_\mathcal{I} \in \mathcal{F}(m, Q')$ with the property that $T_\mathcal{J} \preceq_F T_\mathcal{K}$ implies $\mathcal{J} \subseteq \mathcal{K}$, for all ideals $\mathcal{J}, \mathcal{K}$ of $\mathcal{F}(m, Q)$. Moreover, if LQ is wqo, then Q' can be chosen to be wqo.*

Remark 7 *In [9], we conjecture that no finite necessary and sufficient characterization of the subideal inclusion by finite tree inclusion exists.*

To explain Remark 7 in more detail we note that the construction we give later will not have the property that if $\mathcal{I} \subseteq \mathcal{J}$ then $T_\mathcal{I} \preceq_F T_\mathcal{J}$. We believe there is no way to improve the construction to gain this additional property.

2 Robertson-Seymour-Thomas (RST) Finite Structure

The three authors are the original architects of a finite structure for every topological ideal of finite trees, [16]. We owe it to the remarkable original work given by them that we are able to produce the results presented in this paper and in [8].

The labels on the edges of trees become a powerful tool when we impose the 'gap-condition', [13]. For simplicity, until Section 4 we assume trees have no labels and are ordered by the topological embedding $\preceq$. To make this paper self contained we introduce each concept assuming no prior knowledge. In Section 4 all the new concepts introduced here will be easily extended for $\mathcal{F}(m, Q)$.

2.1 What is a tree-sum?

The *null-graph*, Γ, is defined by $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, Γ is not a tree as it has no root. However, just as it is convenient to allow 0 as a natural number in order to have an additive identity, it is helpful to use Γ formally as the identity for the following 'tree-sum':

2.1.1 Let $T_1, T_2, \ldots, T_n, n \geq 0$, be pairwise vertex disjoint trees or Γ. A *tree-sum* of $T_1, T_2, \ldots, T_n$ is a new tree

$$T = Tree(T_1, T_2, \ldots, T_n),$$

where $V(T) = V(T_1) \cup \cdots \cup V(T_n) \cup \{r_0\}$, $r_0 = root(T)$ is a *new* vertex and $E(T) = E(T_1) \cup \cdots \cup E(T_n) \cup \{r_0 root(T_i); T_i \neq \Gamma, i = 1, 2, \ldots n\}$. So each T_i is a child of r_0.

Note that $Tree(\Gamma)$ is the vertex tree with no child.

2.2 What is a bit?

We now define a basic constructive tool called a 'bit', introduced in [16].

2.2.1 $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$ is said to be a *bit*, if

(i) $n \geq 0$ and for $1 \leq i \leq n+1$, $\mathcal{I}_i \in L\mathcal{F}$, and

(ii) $k \in \mathbb{N}$.

If $1 \leq i \leq n$, then $\mathcal{I}_i$ is called a *left component* of B and $\mathcal{I}_{n+1}$ is the *right component* of B, and k is the *width* of B. We do not distinguish between bits differing only by a permutation of their left components

2.3 What is $I(\mathcal{B})$?

We define a special ideal denoted by $I(\mathcal{B})$ where $\mathcal{B}$ is a set of bits. Three examples that can help grasp the definition of $I(\mathcal{B})$ easily are presented following (2.3.1).

2.3.1 Let $\mathcal{B}$ be a finite set of bits. Then, a set $S \subseteq \mathcal{F}$ is said to be $\mathcal{B}$-*closed* if the following two conditions hold:

1. If $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, then $\mathcal{I}_i \subseteq S$, for all $i = 1, 2, \ldots, n+1$, and

2. for every integer $s \geq 0$ and bit $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, every tree $T = Tree(T_1, T_2, \ldots, T_{n+k+s})$, which satisfies, (a)-(c) belongs to S, where

   (a) for $i = 1, 2, \ldots, n, T_i \in \mathcal{I}_i \cup \{\Gamma\}$,

   (b) for $i = 1, 2, \ldots, k, T_{n+i} \in S \cup \{\Gamma\}$,

   (c) for $i = 1, 2, \ldots, s, T_{n+k+i} \in \mathcal{I}_{n+1}$.

2.3.2 We define $I(\mathcal{B})$ to be the intersection of all $\mathcal{B}$-closed sets. We will show below that $I(\mathcal{B})$ is always an ideal.

Example 2 Let $\mathcal{P}$ denote the ideal of all finite directed paths. Consider the bit $B = (;1;\emptyset)$. Any $\{B\}$-closed set must contain the ideal $\{Tree(\Gamma), Tree(Tree(\Gamma)), \ldots\} = \mathcal{P}$, and $\mathcal{P}$ itself is $\{B\}$-closed. Hence $I(\{B\}) = \mathcal{P}$, is the *path* ideal.

Example 3 Let $\mathcal{S}$ denote the ideal of all trees of height at most 1. Consider the bit $B' = (;0;\{[v]\})$, where $\{[v]\}$ is the singleton vertex tree ideal. Any $\{B'\}$-closed ideal needs to contain the ideal $\{Tree([v_1], [v_2], \ldots, [v_s]) |$ for $i = 1, 2, \ldots, s, [v_i] \in \{[v]\}, s \in \mathbb{N}\} = \mathcal{S}$, and $\mathcal{S}$ itself is $\{B'\}$-closed. Hence $I(\{B'\}) = \mathcal{S}$, is the *star* ideal.

Example 4 Consider the set of bits $\{B, B'\}$, where $B = (;1;\emptyset)$ and $B' = (;0;\{[v]\})$. Let $\mathcal{I}$ be the set of all trees obtained by taking each $P \in \mathcal{P}$, each $S \in \mathcal{S}$, and gluing $root(S)$ to the leaf of P. Any $\{B, B'\}$-closed ideal needs to contain $\mathcal{I}$, and $\mathcal{I}$ itself is $\{B, B'\}$-closed. Hence, $I(\{B, B'\}) = \mathcal{I}$, is the *broom* ideal. As yet another example, let $\mathcal{I}' = \mathcal{P} \cup \mathcal{S}$, (the union of paths and stars), and let $B'' = (\mathcal{P}; 0; \emptyset)$. It is trivial to see that $I(\{B', B''\}) = \mathcal{I}'$.

2.4 What does conforming to a bit mean?

As we construct trees using bits, we need a terminology to relate a tree T and a bit B that is used to construct it.

2.4.1 Let T be a tree and $\mathcal{B}$ be a set of bits and $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$.

1. If $T \in \mathcal{I}_i$, for $1 \leq i \leq n+1$, then we say that T *conforms to* B *by base*, and

2. if $T = Tree(T_1, T_2, \ldots, T_{n+k+s})$ and it has three parts as listed in (2.3.1)(2), (a), (b) and (c), such that the trees in (b) are in $I(\mathcal{B}) \cup \{\Gamma\}$, then we say that T *conforms to* B *in* $I(\mathcal{B})$*, by tree-sum.*

We also say T *conforms to* B *in* $I(\mathcal{B})$, (when specifying "by base" or "by tree-sum" is not necessary).

Lemma 8 *[16] Let* $\mathcal{B}$ *be a set of bits. Let* T *be a tree. Then,* $T \in I(\mathcal{B})$ *if and only if* T *conforms to* B *in* $I(\mathcal{B})$*, for some* $B \in \mathcal{B}$.

Proof. First, suppose that T conforms to B in $I(\mathcal{B})$ for some $B \in \mathcal{B}$. For any $\mathcal{B}$-closed set X the set $I(\mathcal{B})$ is a subset of X and so T also conforms to B in X and so (since X is $\mathcal{B}$-closed) T is in X. Since X was arbitrary, it follows that $T \in I(\mathcal{B})$.

Conversely assume $T \in I(\mathcal{B})$ and define Y to be the set of all trees conforming to B in $I(\mathcal{B})$ for some $B \in \mathcal{B}$. Then any tree conforming to B in Y for some $B \in \mathcal{B}$ also conforms to B in $I(\mathcal{B})$, and so is in Y. Thus Y is $\mathcal{B}$-closed and so must be the whole of $I(\mathcal{B})$. Thus if T is in $I(\mathcal{B})$ then it is in Y and so it conforms to B in $I(\mathcal{B})$ for some $B \in \mathcal{B}$. $\square$

Lemma 9 *For any set of bits $\mathcal{B}$, $I(\mathcal{B})$ is an ideal.*

Proof. Suppose not for a contradiction, and let $T \in I(\mathcal{B})$ be a tree with $|T|$ minimal such that there is some tree $T' \preceq_F T$ and $T' \notin I(\mathcal{B})$. Let f be the map witnessing $T' \preceq_F T$. By Lemma 8 T conforms to B in $I(\mathcal{B})$, for some $B \in \mathcal{B}$, and it cannot do so by base since in that case so would T'. Thus T conforms to B by tree-sum: say $T = Tree(T_1, T_2, \ldots, T_{n+k+s})$ with three parts as listed in (2.3.1)(2) (a), (b) and (c) such that the trees in (b) are in $I(\mathcal{B}) \cup \{\Gamma\}$. By minimality of T, f must map the root r' of T' to the root r of T. For any i with $1 \leq i \leq n+k$ there is at most one out-neighbour r'_i of r' such that $f(r'_i)$ is in T_i; if there is such a neighbour then we set $T'_i := (T')^{r'_i}$ and otherwise we set $T'_i := \Gamma$. Let $r'_{n+k+1}, \ldots r'_{n+k+s'}$ be the remaining out-neighbours of r' and for $n+k+1 \leq i \leq n+k+s'$ set $T'_i := (T')^{r'_i}$. Say $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_n; k; \mathcal{I}_{n+1})$. Then for $1 \leq i \leq n$ we have $T'_i \in \mathcal{I}_i \cup \{\Gamma\}$ since $\mathcal{I}_i$ is an ideal, for $1 \leq i \leq k$ we have $T'_{n+i} \in I(\mathcal{B})$ by the minimality of T and for $1 \leq i \leq s'$ there is some j with $1 \leq j \leq s$ such that $f(r'_{n+k+i}) \in T_{n+k+j}$ and so $T'_i \in \mathcal{I}_{n+1}$ since $\mathcal{I}_{n+1}$ is an ideal. Thus T' conforms to B in $I(\mathcal{B})$ by tree-sum and so since $I(\mathcal{B})$ is $\mathcal{B}$-closed we have $T' \in I(\mathcal{B})$, contradicting our assumption. $\square$

2.5 RST proper description

The following tool called a 'proper description' of a tree ideal is a basic tool that lets us represent ideals by finite label trees. Recall that $X \subset Y$ means $X \subseteq Y \not\subseteq X$.

2.5.1 (A proper description) Let $\mathcal{I}$ be a topological tree ideal. Let $\mathcal{B}$ be a finite set of bits. Then, $\mathcal{B}$ is a *proper description* of $\mathcal{I}$ if

(N1) If $(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, then $\mathcal{I}_i \subset \mathcal{I}$, for all $i = 1, 2, \ldots, n+1$.
(N2) $I(\mathcal{B}) = \mathcal{I}$.

Example 5 Let $B_0 = (;0;\emptyset)$. Note that $\{B_0\}$ is the proper description for the singleton ideal $\{[v]\}$, as B_0 has nothing in its base, and $I(\{B_0\}) = \{Tree(\Gamma)\} = \{[v]\}$.

Theorem 10 *[16] Every $\mathcal{I} \in L\mathcal{F}$ has a proper description $\mathcal{B}$.*

2.6 Representing topological ideals by labelled trees

Now, using Theorem 10 we show that for every $\mathcal{I} \in L\mathcal{F}$ there is at least one tree $T \in \mathcal{F}(2, \mathbb{N}) \cup \{\Gamma\}$ that represents $\mathcal{I}$. We do not construct this tree. In a platonic sense, it already exists by Theorem 11 below. In this section we work only with proper ideals $\mathcal{I} \subset \mathcal{F}$.

2.6.1 Let $\mathcal{I} \in L\mathcal{F}$. If $\mathcal{I} = \emptyset$, then we represent $\mathcal{I}$ by Γ. Otherwise, a tree $T \in \mathcal{F}(2, \mathbb{N})$ represents $\mathcal{I} \in L\mathcal{F}$ if $T = Tree(T_1, T_2, \ldots, T_q)$ is a tree-sum of q trees with $q \geq 1$ such that

(i) T_i, for $i = 1, 2, \ldots, q$ represents a bit B_i where $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ is a proper description of $\mathcal{I}$, $l_E(root(T)) \in \{0, 1\}$, and

(ii) a tree $S \in \mathcal{F}(2, \mathbb{N})$ represents a bit $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$, if $S = Tree(S_1, S_2, \ldots, S_{n+1})$ is a tree-sum of $n+1$ trees representing $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n+1}$ in B, such that $l_E(root(S_{n+1})) = 1$ and $l(root(S)) = (2, k)$.

Any other label not mentioned here is taken to be 0.

Theorem 11 *For every ideal $\mathcal{I} \in L\mathcal{F}$, $\mathcal{I} \notin \{\emptyset, \mathcal{F}\}$ there exists a tree $T \in \mathcal{F}(2, \mathbb{N})$ representing $\mathcal{I}$.*

Proof. Suppose the contrary and by Lemma 3 and Theorem 4 choose $\mathcal{I}$ to be a subset minimal counterexample. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}, q \geq 1$ be a proper description of $\mathcal{I}$, where $i = 1, 2, \ldots, q$, and $B_i = (\mathcal{I}_1^i, \mathcal{I}_2^i, \ldots, \mathcal{I}_n^i; k_i; \mathcal{I}_{n_i+1}^i)$. By (N1), for all $l = 1, 2, \ldots, n_i + 1$, $\mathcal{I}_l^i \subset \mathcal{I}$ and hence by minimality of $\mathcal{I}$ we have a tree $T_l^i \in \mathcal{F}(2, \mathbb{N})$ representing $\mathcal{I}_l^i$. Therefore, for all $i = 1, 2, \ldots, q$ with $l(root(S^i)) = (2, k_i)$ the tree $S^i = Tree(T_1^i, T_2^i, \ldots, T_{n_i+1}^i) \in \mathcal{F}(2, \mathbb{N})$ represents B_i. Then with $l(root(T)) = (2, 0)$ the tree $T = Tree(S^1, S^2, \ldots, S^q) \in \mathcal{F}(2, \mathbb{N})$ represents $\mathcal{I}$, contrary to the assumption. The result follows. □

Remark 12 *For the rest of Section 2 we will use the notation "$T_{\mathcal{I}}$" for a tree representing $\mathcal{I}$ (instead of repeating the phrase "T is a tree representing $\mathcal{I}$"). Though it is not unique there is at least one by Theorem 11. The layers in this tree are of two alternating kinds. Starting with an ideal-layer, the odd level vertices are bit-layer vertices. For a bit-layer vertex u, $l_E(u) = 2$. For an ideal-layer v, $0 \leq l_E(v) \leq 1$ and so the following is easy.*

Lemma 13 *Let $\mathcal{I}, \mathcal{I}' \in L\mathcal{F}$ and $T_{\mathcal{I}}, T_{\mathcal{I}'} \in \mathcal{F}(2, \mathbb{N})$ be given. If f is a witness to $T_{\mathcal{I}} \preceq_F T_{\mathcal{I}'}$, then for any edge $e \in E(T_{\mathcal{I}})$, if $l(e) = 2$, then $f(e) = e'$ where $e' \in E(T_{\mathcal{I}'})$.*

Remark 14 *Thus, while normally $f(e)$ can be a long directed path, by Lemma 13, if $l(e) = 2$, then $f(e)$ is an edge. It also follows that f is layer preserving. That is, f sends an ideal- to an ideal- and a bit- to a bit-layer vertex.*

Theorem 15 *Let $\mathcal{I}, \mathcal{I}' \in L\mathcal{F}$ and $T_{\mathcal{I}}, T_{\mathcal{I}'} \in \mathcal{F}(2, \mathbb{N})$. If $T \preceq_F T,'$ then $\mathcal{I} \subseteq \mathcal{I}'$.*

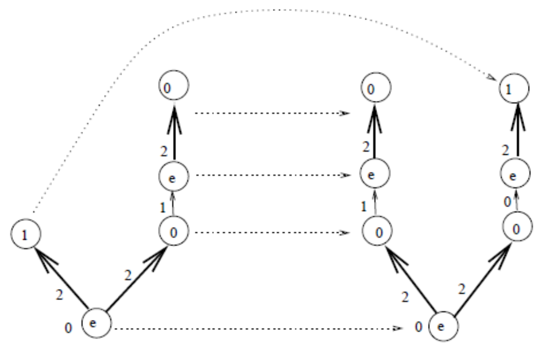

Figure 1: $T_{\mathcal{I}} \preceq T_{\mathcal{I}'}$ (see dotted lines) despite $\mathcal{I} \not\subseteq \mathcal{I}'$. Nevertheless $T_{\mathcal{I}} \not\preceq_F T_{\mathcal{I}'}$.

The edge labels play a key role here: if we used $\preceq$ rather than $\preceq_F$ then Theorem 15 would no longer hold, as the following example shows:

Example 6 *Let $\mathcal{I}, \mathcal{I}', B, B'$, and B'' be as defined in Example 4. Then, $T_{\mathcal{I}} = Tree(T_B, T_{B'}) \preceq T_{\mathcal{I}'} = Tree(T_{B''}, T_{B'})$, but $\mathcal{I} \not\subseteq \mathcal{I}'$. However, $T_{\mathcal{I}} \not\preceq_F T_{\mathcal{I}'}$ as the edge labelled by 2 is mapped to a long path violating (1.1.3)(ii) (see Figure 1).*

Proof. (**of Theorem 15**) Let $\mathcal{I}, \mathcal{I}' \in L\mathcal{F}$ be a pair and $T_{\mathcal{I}}, T_{\mathcal{I}'} \in \mathcal{F}(2, \mathbb{N})$. Suppose that $T_{\mathcal{I}} \preceq_F T_{\mathcal{I}'}$ by an embedding f. For a contradiction, assume that $\mathcal{I} \not\subseteq \mathcal{I}'$. By Lemma 3 and Theorem 4, we may take $\mathcal{I}$ to be a subset minimal counterexample. Let $\mathcal{B}$ and $\mathcal{B}'$ be the proper descriptions of $\mathcal{I}$ and $\mathcal{I}'$, respectively that correspond to $T_{\mathcal{I}}$ and $T_{\mathcal{I}'}$ according to (2.6.1).

Let $r_{\mathcal{I}} = root(T_{\mathcal{I}})$ and $r_{\mathcal{I}'} = root(T_{\mathcal{I}'})$. Since $\mathcal{I} \neq \emptyset$, we have $\mathcal{B} \neq \emptyset$. So $r_{\mathcal{I}}$ has at least one incident edge, say $r_{\mathcal{I}} r_B$, labelled 2, where $B \in \mathcal{B}$ and $root(T_B) = r_B$ and T_B represents B. By Lemma 13, $f(r_{\mathcal{I}})$ is an ideal layer vertex. We may assume $f(r_{\mathcal{I}}) = r_{\mathcal{I}'}$ by assuming $\mathcal{I}'$ subset minimal. Since $l(r_{\mathcal{I}} r_B) = 2$ by Lemma 13 we have $f(r_{\mathcal{I}} r_B) = r_{\mathcal{I}'} r_{B'}$ for some $B' \in \mathcal{B}'$. Say, $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$ and $B' = (\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_{n'+1})$. From $l_V(r_B) \leq l'_V(r_{B'})$, we have

$$k \leq k'. \tag{1}$$

Let $r_i = root(T_i)$ for $i = 1, 2, \ldots, n+1$ such that T_i represents $\mathcal{I}_i$. Similarly, let $r'_i = root(T'_i)$ for $i = 1, 2, \ldots, n'+1$ such that T'_i represents $\mathcal{I}'_i$. By (1.1.3), there is an injection $\iota : \{1, 2, \ldots, n+1\} \to \{1, 2, \ldots, n'+1\}$ such that

$$T_j \preceq_F T'_{\iota(j)}, \tag{2}$$

for $j = 1, 2, \ldots, n+1$. Since $l_V(r_{n+1}) = 1 > 0$, we have

$$\iota(n+1) = n'+1. \tag{3}$$

For each $j = 1, 2, \ldots, n+1$, by (N1), $\mathcal{I}_j \subset \mathcal{I}$. Then, by (2) and by minimality of $\mathcal{I}$,

$$\mathcal{I}_j \subseteq \mathcal{I}'_{\iota(j)} \quad for \quad j = 1, 2, \ldots, n+1. \tag{4}$$

Let $T \in \mathcal{I} - \mathcal{I}'$ be a tree of minimal height h, $h \geq 0$. We may assume that T conforms to B in $I(\mathcal{B})$, as B is arbitrary. By (4), T cannot conform to B by base as $\mathcal{I}' \supseteq \mathcal{I}'_{\iota(j)} \supseteq \mathcal{I}_j$ for all $j = 1, 2, \ldots, n+1$, and so has to conform by tree-sum. That is, (2.4.1)(2) holds so that $T = Tree(T_1, T_2, \ldots, T_{n+k+s})$, such that

(a) for $i = 1, 2, \ldots n, T_i \in \mathcal{I}_i \cup \{\Gamma\}$,

(b) for $i = 1, 2, \ldots, k, T_{n+i} \in \mathcal{I} \cup \{\Gamma\}$, and

(c) for $i = 1, 2, \ldots, s, T_{n+k+i} \in \mathcal{I}_{n+1}$.

In (a), by (4), we may replace $\mathcal{I}_i$ by $\mathcal{I}'_{\iota(i)}$; in (b), by minimality of h, $T_{n+i} \in \mathcal{I}' \cup \{\Gamma\}$, for all $i = 1, 2, \ldots, k$. Also, by (1), $k \leq k'$ and so, in (b) replace $\mathcal{I}$ by $\mathcal{I}'$; in (c) by (3) and by (4), replace $\mathcal{I}_{n+1}$ by $\mathcal{I}'_{n'+1}$. So, T conforms to B' in $I(\mathcal{B}')$. By Lemma 8 and by (N2), $T \in \mathcal{I}'$, contrary to the choice of T. The result follows. □

Corollary 16 *$L\mathcal{F}$ is wqo under '$\subseteq$'.*

3 From unlabelled to labelled tree ideals, tiny changes

In this section, we slightly modify each RST tool that we introduced in (2.1)-(2.5), by (3.1)-(3.5), respectively and generalize Theorem 15 by Theorem 6.

3.1 From tree-sum to labeled-tree-sum, a tiny change on (2.1)

The tree-sum we defined in (2.1) had nothing to do with labels. As in (2.1) the root r of a tree-sum is a new vertex, but we have to be clear about $l(r)$. The *null-graph*, Γ, with $V(\Gamma) = E(\Gamma) = \emptyset$ is exactly as defined in (2.1).

3.1.1 (Labeled-tree-sum) Let $p \in [[m]]$, $q \in Q$ and $T_1, T_2, \ldots, T_n$, be pairwise vertex disjoint elements of $\mathcal{F}(m, Q) \cup \{\Gamma\}$ where $n \geq 0$. Then, we say that T is a *labelled-tree-sum* of $T_1, T_2, \ldots, T_n$, if

$$T = Tree(T_1, T_2, \ldots, T_n)$$

such that (as in (2.1.1)), $V(T) = V(T_1) \cup \cdots \cup V(T_n) \cup \{r_0\}$, $r_0 = root(T)$ is a *new* vertex and $E(T) = E(T_1) \cup \cdots \cup E(T_n) \cup \{r_0 root(T_i), T_i \neq \Gamma, i = 1, 2, \ldots, n\}$. The labels of the children $T_1, T_2, \ldots, T_n$, remain as given, and $l(r_0) = (p, q)$. For the rest of the paper, (unless specified otherwise) by writing $T = Tree(T_1, T_2, \ldots, T_n)$, we mean that T is a labelled-tree-sum of $T_1, T_2, \ldots, T_n$. When we need to be specific about the root label, we will explicitly write down the root labels. Mostly, however, we can be less specific and only need to specify from which ideal the labels come. See (3.3.1)(2)(d), for example.

3.2 From a bit to a rooted-bit, a tiny change on (2.2)

When we work with $\mathcal{F}(m, Q)$ we introduce one more component on bits, called 'root component of a bit' besides the familiar left and right components.

3.2.1 $B = (\mathcal{I}_0; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$ is said to be a *rooted-bit*, if

(i) $n \geq 0$ and for $0 \leq i \leq n+1$, $\mathcal{I}_i \in L\mathcal{F}(m, Q)$,

(ii) $k \in \mathbb{N} \cup \{\infty\}$, and

(iii) every tree in $\mathcal{I}_0$ has height zero.

$\mathcal{I}_0$ is called the *root component* of B, and k is the *width* of B.

Remark 17 *In contrast to (2.2.1), here we allow $k = \infty$. In (2.2.1) allowing $k = \infty$ is not interesting as such a bit simply constructs every tree in $\mathcal{F}$. In $\mathcal{F}(m, Q)$, we may have a bit B of width ∞, and yet we cannot construct every tree of $\mathcal{F}(m, Q)$, unless the root component of B allows every possible labelling from $[[m]] \times Q$.*

From now on by a bit it is understood that we mean a rooted-bit. For emphasis, however, at times we use the word rooted-bit. We also continue to assume that bits are invariant under permutation of their left components.

3.3 Revising $I(\mathcal{B})$, a tiny change on (2.3)

We specify how we use a rooted-bit. In particular the use of a root-component, and a bit of width ∞ will be elaborated here as these are new concepts.

3.3.1 Let $\mathcal{B}$ be a finite set of rooted-bits. Then $\mathcal{I} \in L\mathcal{F}(m, Q)$ is said to be $\mathcal{B}$-*closed* if the following two conditions hold:

1. If $B = (\mathcal{I}_0; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, then $\mathcal{I}_i \subseteq \mathcal{I}$, for all $i = 1, \ldots, n+1$.

2. For every $s, \tilde{k} \in \mathbb{N}$ and $B = (\mathcal{I}_0; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, and every tree $T = Tree(T_1, T_2, \ldots, T_{n+\tilde{k}+s})$, which satisfies (a),(b),(c) and (d) belongs to $\mathcal{I}$, where $\tilde{k} \leq k$ and

   (a) *for* $i = 1, 2, \ldots, n, T_i \in \mathcal{I}_i \cup \{\Gamma\}$,

   (b) *for* $i = 1, 2, \ldots, \tilde{k}, T_{n+i} \in \mathcal{I} \cup \{\Gamma\}$,

   (c) *for* $i = 1, 2, \ldots, s, T_{n+\tilde{k}+i} \in \mathcal{I}_{n+1}$,

   (d) $[root(T)] \in \mathcal{I}_0$.

   We define $I(\mathcal{B})$ to be the intersection of all $\mathcal{B}$-closed sets.

Remark 18 *In (3.3.1)(2)(b), if $k = \infty$, then $\tilde{k} < k$ as $\tilde{k} \in \mathbb{N}$. This is necessary since we work with finite trees only. Note also that we have a new condition (d).*

3.4 Revising 'conforming to a bit', a tiny change on (2.4)

From (2.4) a couple of changes are added as follows:

3.4.1 Let $T \in \mathcal{F}(m, Q)$. Suppose $\mathcal{B}$ is a set of rooted-bits and $B = (\mathcal{I}_0; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$. Then,

1. if $T \in \mathcal{I}_i, 1 \leq i \leq n+1$, then we say that T *conforms to* B *by base*, and
2. if $T = Tree(T_1, T_2, \ldots, T_{n+\tilde{k}+s})$, $\tilde{k} \in \mathbb{N}$, $\tilde{k} \leq k$, has four parts as listed in (3.3.1)(2), (a), (b), (c), and (d), such that the trees in (b) are contained in $I(\mathcal{B})$, then we say that T *conforms to* B *by labelled-tree-sum in* $I(\mathcal{B})$.

We also say T *conforms to* B *in* $I(\mathcal{B})$, (when specifying "by base" or "by labelled-tree-sum" is not necessary). Observe that the new condition in (3.4.1) that is not in (2.4.1) is that we add condition (d) and also we have to choose $\tilde{k} \in \mathbb{N}$, since $k = \infty$ is a possibility now.

In (3.4.1)(1), we put $i \geq 1$ because if $T \in \mathcal{I}_0$, then $height(T) = 0$ and T conforms to B by labelled-tree-sum in $I(\mathcal{B})$ as T satisfies (3.4.1)(2) trivially. Explicitly, if $T = [root(T)]$, then (3.3.1)(a)-(c) are satisfied vacuously, whereas $[root(T)] \in \mathcal{I}_0$, thus also satisfying (d). Now we have analogs of Lemma 8 and Lemma 9 with similar proofs.

Lemma 19 *Let $\mathcal{B}$ be a set of rooted-bits. Let T be a tree. Then, $T \in I(\mathcal{B})$ if and only if T conforms to B in $I(\mathcal{B})$, for some $B \in \mathcal{B}$.*

Lemma 20 *For any set of rooted-bits $\mathcal{B}$, $I(\mathcal{B})$ is an ideal.*

For any nonempty set $X \subseteq \mathcal{F}(m, Q)$, let $\mu(X) = \max\{l_E(root(T)) \mid T \in X\}$, and let $\epsilon(X) = \min\{l_E(root(T)) \mid T \in X\}$. For $X = \emptyset$ let $\mu(X) = \epsilon(X) = 0$.

3.5 Proper description of (2.5) generalized

3.5.1 Let $\mathcal{I} \in L\mathcal{F}(m, Q)$ such that $\mu(\mathcal{I}) = m$. Let $\mathcal{B}$ be a finite set of rooted-bits. Then, B is a proper description of $\mathcal{I}$ if it satisfies the following:
(N1) If $(\mathcal{I}_0; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1}) \in \mathcal{B}$, then $\mathcal{I}_i \subset \mathcal{I}, \forall i, 0 < i \leq n+1$.
(N2) $I(\mathcal{B}) = \mathcal{I}$.

(3.5.1$'$) Let $\mathcal{I} \in L\mathcal{F}(m, Q)$ such that $\mu(\mathcal{I}) < m$. Then, we define $\{(\emptyset; 0; \mathcal{I})\}$ to be the proper description of $\mathcal{I}$.

3.6 A Structure Theorem

3.6.1 A structure theorem from [8] We will rely on the following structure theorem given in [8].

Theorem 21 *[8] For any ideal $\mathcal{I} \in L\mathcal{F}(m, Q)$ with $\mu(\mathcal{I}) = m$ there is a proper description $\mathcal{B}$ of $\mathcal{I}$ such that for any $T \in \mathcal{I}$ we have one of the following:*

(a) there exists a bit $B \in \mathcal{B} - \{tsb(\mathcal{I})\}$ where T conforms to B in $I(\mathcal{B})$ by a tree-sum, or

(b) $l_E(root(T)) < l_E(root(S))$ for some $S \in \Omega(\mathcal{I})$.

3.6.2 For every ideal $\mathcal{I}$ one canonical proper description $\mathcal{B}(\mathcal{I})$ From this stage onward, for every $\mathcal{I} \in L\mathcal{F}(m,Q)$ with $\mu(\mathcal{I}) = m$ we fix one canonical proper description as specified by Theorem 21 and use the notation $\mathcal{B}(\mathcal{I})$ to refer to it. If $\mu(\mathcal{I}) < m$, then it is easy to see that there is a unique $\mathcal{J} \in L\mathcal{F}(m,Q)$ such that $\mu(\mathcal{J}) = m$ and $\mathcal{I} = ||\mathcal{J}||_{\mu(\mathcal{I})+1}$. Hence we associate $\mathcal{I}$ the number $\mu(\mathcal{I})$ along with $\mathcal{B}(\mathcal{J})$. This is free from ambiguity as $\mathcal{B}(\mathcal{I})$ is derived from $\Omega(\mathcal{I})$. That is, for any $\mathcal{I} \in L\mathcal{F}(m,Q)$ as we have a fixed $\Omega(\mathcal{I})$ that unambiguously identifies $\mathcal{I}$ by obstruction (from without), so do we have a fixed proper description $\mathcal{B}(\mathcal{I})$ given by Theorem 21 that unambiguously identifies $\mathcal{I}$ by structure (from within). For the special case where $\mathcal{I} = \mathcal{F}(m,Q)$, we have the following.

Proposition 22 *$\mathcal{F}(m,Q)$ has a canonical proper description*

$$\mathcal{B}(\mathcal{F}(m,Q)) = \{(\mathcal{P}_0;\emptyset;\infty;\emptyset)\}\,,$$

where $\mathcal{P}_0$ is the ideal of all height zero trees of $\mathcal{F}(m,Q)$.

4 Refraction of ideals

Our construction will rely on an important operation, which we call refraction of ideals relative to other ideals. Before introducing this operation, we will need two simpler operations on ideals: truncation and refraction (but not relative to other ideals).

Definition 23 *Let $\mathcal{I} \in L\mathcal{F}(m,Q)$ be an ideal and let $0 \le i \le m$. Then we define the i^{th}* truncation *$||\mathcal{I}||_i$ of $\mathcal{I}$ to be the ideal containing all trees in $\mathcal{I}$ whose root edge label is less than i.*

We define the i^{th} refraction *$R_i(\mathcal{I})$ to be the ideal containing all trees T such that if the root edge label l of T is lowered to $\min(i,l)$ then the resulting tree is in $\mathcal{I}$.*

*If $\mathcal{I} = R_i(\mathcal{I})$ then we say that $\mathcal{I}$ is i-*refracted.

Remark 24 *If $\mathcal{I}$ is i-refracted then the root edge label of any element of $\Omega(\mathcal{I})$ is necessarily at most i. This allows us to prove the variant of Theorem 21 which we will need for our construction.*

Theorem 25 *Let $\mathcal{I}$ be any i-refracted ideal and let $T \in \mathcal{I}$. If $l_E(root(T)) = i$, then $\mu(\mathcal{I}) = m$ and there is a bit $B \in \mathcal{B}(\mathcal{I})$ such that T conforms to B in $\mathcal{I}$ by tree-sum.*

Proof. As $\mathcal{I}$ is i-refracted we have $i \ge \mu(\Omega(\mathcal{I}))$. So the tree obtained from T by replacing the root edge label with m is also in $\mathcal{I}$. We have $\mu(\mathcal{I}) = m$. So we may apply Theorem 21 to $\mathcal{I}$ and T. Alternative (b) cannot hold since for any $S \in \Omega(\mathcal{I})$ we have $l_E(root(T)) = i \ge l_E(root(S))$. So alternative (a) must hold, as required. □

We can now also define refraction of an ideal relative to another ideal.

Definition 26 *Let $\mathcal{J} \subseteq \mathcal{I}$ be ideals of $\mathcal{F}(m,Q)$. Then the i^{th}* refraction *$R_i(\mathcal{J},\mathcal{I})$ of $\mathcal{J}$ relative to $\mathcal{I}$ is*

$$R_i(\mathcal{J}) \cup ||\mathcal{I}||_i\,.$$

Note that $R_i(\mathcal{J},\mathcal{I})$ is an ideal as it is a union of two ideals. Clearly $R_i(\mathcal{J},\mathcal{I})$ is i-refracted.

Lemma 27 *Let $\mathcal{J} \subseteq \mathcal{I}$ be ideals of $\mathcal{F}(m,Q)$. If $i \leq \mu(\mathcal{J})$ then $\mu(R_i(\mathcal{J},\mathcal{I})) = m$.*

Proof. For brevity set $\mathcal{K} = R_i(\mathcal{J},\mathcal{I})$. By assumption $i \leq \mu(\mathcal{J})$, and so we can choose a tree $T \in \mathcal{J}$ with root-edge label i. By definition $\mathcal{J} \subsetneqq \mathcal{K}$ and so $T \in \mathcal{K}$. Next let T^* be the tree obtained from T by changing the root-edge label of T to m. Since by Remark 24 we know that $\mathcal{K}$ is i-refracted, we have $T^* \in \mathcal{K}$, that is $\mu(\mathcal{K}) = m$ as required. $\square$

Lemma 27 assures us that Theorem 21 will yield a proper description of the refraction $R_i(\mathcal{J},\mathcal{I})$ for all $i \in [[m]]$ and all $\mathcal{J} \subseteq \mathcal{I}$ of $\mathcal{F}(m,Q)$. There will be some values of i for which these refractions will be particularly useful in our later constructions.

Definition 28 *For ideals $\mathcal{J} \subseteq \mathcal{I}$ of $\mathcal{F}(m,Q)$ we define $\Delta(\mathcal{J},\mathcal{I})$ to be the set of $i \in [[\mu(\mathcal{J})]]$ for which there is at least one tree with root edge label i in $\mathcal{I} - \mathcal{J}$.*

For $i \in \Delta(\mathcal{J},\mathcal{I})$, there is a clear sense in which the refraction $R_i(\mathcal{J},\mathcal{I})$ is simpler than $\mathcal{I}$.

Definition 29 *For any ideal $\mathcal{I} \in L\mathcal{F}(m,Q)$ we define the* truncation sequence $ts(\mathcal{I})$ *of $\mathcal{I}$ to be the sequence $(||\mathcal{I}||_1, ||\mathcal{I}||_2, \ldots, ||\mathcal{I}||_{m+1})$. We then say that $\mathcal{I} \trianglelefteq \mathcal{I}'$ if $ts(\mathcal{I})$ is less than $ts(\mathcal{I}')$ in the lexicographic order defined with respect to the subset order on $L\mathcal{F}(m,Q)$.*

Remark 30 *The order $\trianglelefteq$ is wellfounded since the subset order on $L\mathcal{F}(m,Q)$ is. Further, by construction we have $R_i(\mathcal{J},\mathcal{I}) \triangleleft \mathcal{I}$ for any $i \in \Delta(\mathcal{J},\mathcal{I})$.*

4.1 Trees representing ideals, truncated ideals and bits

Our next aim is to explain the construction of trees representing ideals. However, the construction is a little involved, and proceeds by means of a definition of various different kinds of representation by a mutual nested recursion. We will define what it means for a tree T to represent an ideal, a refraction of an ideal or a bit. In the cases of an ideal or a bit, for technical reasons the representation will be given relative to another ideal.

More precisely, we will now explain when a tree counts as a representation of a bit B relative to an ideal $\mathcal{I}$. We will then explain when a tree counts as a representation of an ideal $\mathcal{J}$ relative to another ideal $\mathcal{I}$. Finally, we will explain when a tree counts as a refraction-representation of an ideal $\mathcal{I}$. The representations of ideals will be divided into two kinds: righty and lefty. All of these definitions will be given in terms of each other, which may at first appear circular. But in fact, to determine if a tree is a representation of one of these kinds, it will only be necessary to check whether children of that tree are representations of the other kinds. Thus by induction on the heights of the trees involved all the notions given below are well-defined.

Now we are ready for the main construction.

First of all, we say that a tree $T_{B,\mathcal{I}}$ with root r is a *representation* of a bit $B = (\mathcal{I}_0, \dots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$ relative to an ideal $\mathcal{I}$ if it is of the form $Tree(T_1, T_2, \dots, T_{n+1})$, where each T_i is a representation of $\mathcal{I}_i$ relative to $\mathcal{I}$, righty if $i = n+1$ and lefty otherwise, and we have:

$$l_V(r) = (k, \mathcal{I}_0) \text{ and } l_E(r) = m+1\,.$$

Secondly, for an ideal $\mathcal{J}$ we say that a tree $T_{\mathcal{J},\mathcal{I}}$ with root r is a representation of $\mathcal{J}$ relative to an ideal $\mathcal{I}$ if it is of the form $Tree(T_0, T_2, \dots T_m)$, where each T_i is an i-refraction representation of the ith refraction $R_i(\mathcal{J}, \mathcal{I})$ of $\mathcal{J}$ relative to $\mathcal{I}$ if $i \in \Delta(\mathcal{J}, \mathcal{I})$ and is Γ otherwise and we have

$$l_V(r) = (\mu(\mathcal{J}), \Delta(\mathcal{J}, \mathcal{I}), \delta) \text{ and } l_E(r) = m+1\,,$$

where $\delta \in \{0,1\}$. We say the representation is *lefty* if $\delta = 0$ and *righty* if $\delta = 1$.

When we talk of representations of an ideal without explicitly saying which other ideal they are relative to or whether they are righty or lefty, we always mean lefty representations relative to $\mathcal{F}(m, Q)$.

Finally for any $i \in [[m]]$ and for any i-refracted ideal $\mathcal{I}$ with $\mu(\mathcal{I}) = m$ and with proper description $\{B_1, B_2 \dots B_q\}$ we will say that a tree $T_{\mathcal{I},i}$ with root r is an i-refraction-representation of $\mathcal{I}$ if it is of the form $Tree(T_0, T_1, \dots T_q)$, where each T_j is a representation of B_j relative to $\mathcal{I}$ and we have

$$l_V(r) = l_E(r) = i\,.$$

It follows straightforwardly from the construction that any ideal has at most one representation, which if it exists we will call *the* representation of the ideal and will denote by $T_{\mathcal{I}}$. Our next aim is to show that such representations always exist.

Lemma 31 *For any $l \in [[m]]$ and any l-refracted ideal $\mathcal{I}$ in $\mathcal{F}(m, Q)$ with $\mu(\mathcal{I}) = m$ there is an l-refraction-representation $S_{\mathcal{I},l}$ of $\mathcal{I}$.*

Proof. Suppose not for a contradiction, and choose $\mathcal{I}$ to be $\trianglelefteq$-minimal such that this fails. Let $\mathcal{B} = \{B_1, B_2, \dots B_q\}$ be the canonical proper description of $\mathcal{I}$. For each $1 \leq i \leq q$ let $B_i = (\mathcal{I}^i_0, \dots, \mathcal{I}^i_{n_i}; k_i; \mathcal{I}^i_{n_i+1})$.

For any $1 \leq i \leq q$, $0 \leq j \leq n_i$ and $l \in \Delta(\mathcal{I}^i_j, \mathcal{I})$ we have $R_l(\mathcal{I}^i_j, \mathcal{I}) \triangleleft \mathcal{I}$ by Remark 30. Furthermore, since $\Delta(\mathcal{I}^i_j, \mathcal{I}) \subseteq [[\mu(\mathcal{I}^i_j)]]$ we have $l \leq \mu(\mathcal{I}^i_j)$. By Lemma 27 $\mu(R_l(\mathcal{I}^i_j, \mathcal{I})) = m$, and so by our minimal choice of $\mathcal{I}$ there is a tree $T_{i,j}$ l-refraction-representing $R_l(\mathcal{I}^i_j, \mathcal{I})$.

Now following the recipe above we may construct for each $1 \leq i \leq q$ and each $0 \leq j \leq n_i + 1$ a tree $T_{i,j}$ representing $\mathcal{I}^i_j$ relative to $\mathcal{I}$, righty if $j = n_i + 1$ and lefty otherwise. Following the recipe above again we may find for each $1 \leq i \leq q$ a tree T_i representing B_i relative to $\mathcal{I}$. Following the recipe one final time we may now construct a tree $S_{\mathcal{I},i}$ i-refraction-representing $\mathcal{I}$, giving the desired contradiction. □

Corollary 32 *For any ideal $\mathcal{I}$ of $\mathcal{F}(m, Q)$ there is a tree $T_{\mathcal{I}}$ representing $\mathcal{I}$.*

Proof. For any $i \in \Delta(\mathcal{I}, \mathcal{F}(m, Q)) \subseteq [[\mu(\mathcal{I})]]$, by Lemma 27 we have the equation $\mu(R_i(\mathcal{I}, \mathcal{F}(m, Q))) = m$, and so by Lemma 31 there is a tree $T_{i,j}$ l-refraction-representing $R_i(\mathcal{I}, \mathcal{F}(m, Q))$.

Now following the recipe above we may construct a tree T representing $\mathcal{I}$. □

4.2 Well-quasi-ordering Q_0 (the label set of trees representing $\mathcal{I}$)

For an integer $d \geq 1$ and a qo Q, let Q^d denote $Q \times Q \times \cdots \times Q$ (with d coordinates). For any $m \in \mathbb{N}$ define $]]m[[= \{0, 1, \ldots, m\}$ to be a set qo under *equality* so that for all $i, j \in]]m[[$, $i \leq j$ in $]]m[[$ if and only if $i = j$ in $\mathbb{N}$ so that $]]m[[$ is an anti-chain. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Recall that $P_0 = \{T \in \mathcal{F}(m, Q), height(T) = 0.\}$. If $Q_1, Q_2, \ldots, Q_r$ are sets such that each Q_i is qo under $\leq_{Q_i}$, then $Q_1 \times Q_2 \times \cdots \times Q_r = \{(x_1, x_2, \ldots, x_r) \mid x_i \in Q_i\}$ is qo by $(x_1, x_2, \ldots, x_r) \leq (x'_1, x'_2, \ldots, x'_r)$ if and only if $x_i \leq_{Q_i} x'_i$ for all $i = 1, 2, \ldots, r$. The following is an easy folklore lemma.

Lemma 33 *For $i = 1, 2, \ldots, r$ if each Q_i is wqo under $\leq_i$, then $Q_1 \times Q_2 \times \ldots Q_r$ is wqo under $\leq$.*

So, trivially LP_0 is wqo if and only if LQ is wqo, by Lemma 33 (by viewing LP_0 as a label ideal rather than as a tree ideal). The label sets Q_1, Q_2, Q_3 below (highlighted by bullets) correspond to labels defined in Subsection 4.1 such that Q_1 is a label set for bit-layer vertices, whereas Q_2 is for ideal-layer vertices, and Q_3 is for refraction-layer vertices. For Q_2 we use an arbitrary identification of the subsets of $[[m]]$ with the numbers from 0 to $2^{m+1} - 1$. Their quasi-order, '$\leq_0$', is defined below in (4.2.2).

4.2.1 Let $Q_0 = Q_1 \cup Q_2 \cup Q_3$ where

- $Q_1 = \bar{\mathbb{N}} \times LP_0$,
- $Q_2 =]]m[[\times]]2^{m+1} - 1[[\times]]1[[$, and
- $Q_3 =]]m[[$.

Let us define a quasi-order $\leq_0$ on Q_0 as follows.

4.2.2 Let $x, x' \in Q_0$. Then $x \leq_0 x'$ if and only if either

(a) $x, x' \in Q_1, x = (k, \mathcal{I}_0), x' = (k', \mathcal{I}'_0), k \leq k'$ and $\mathcal{I}_0 \subseteq \mathcal{I}'_0$, or

(b) $x, x' \in Q_2 \cup Q_3, x = x'$.

Lemma 34 *Q_0 is wqo under $\leq_0$ if and only if LQ is wqo under $\subseteq$.* □

4.3 Proof of the lifting

Our aim in this section is to show that that $\mathcal{I} \subseteq \mathcal{I}'$ follows from $T_{\mathcal{I}} \preceq_F T_{\mathcal{I}'}$ for ideals $\mathcal{I}$ and $\mathcal{I}'$. Before doing this, we must first establish a more technical result in a similar vein about refraction-representations. However, we begin with a small lemma which will be useful both for this technical interlude and for the final proof.

Lemma 35 *Let Δ be any finite totally ordered set and let $g\colon \Delta \to \Delta$ be any injective function with $g(i) \geq i$ for all $i \in \Delta$. Then g is the identity on Δ.*

Proof. For any $i \in \Delta$, let $\lfloor i \rfloor$ be $\{j \in \Delta\colon j \geq i\}$. Then $g|_{\lfloor i \rfloor}$ is an injective map from $\lfloor i \rfloor$ into itself and so must be a bijection. But the only element of $\lfloor i \rfloor$ which can possibly map to i is i, so we must have $g(i) = i$ as required. □

Theorem 36 *Let $0 \leq i \leq i' \leq m$. Let $\mathcal{I} \in L\mathcal{F}(m, Q)$ be an i-refracted ideal with $\mu(\mathcal{I}) = m$ and let $\mathcal{I}' \in L\mathcal{F}(m, Q)$ be an i'-refracted ideal with $\mu(\mathcal{I}') = m$. Let T and T' in $\mathcal{F}(m+1, Q_0)$ be trees i- and i'-refraction-representing $\mathcal{I}$ and $\mathcal{I}'$ respectively, and suppose that $T \preceq_F T'$. Let $T^* \in \mathcal{I}$ with $l_E(root(T^*)) = i$. Suppose further that for any tree $T^{**} \in \mathcal{I}$ of smaller height with $l_E(root(T^{**})) < i$ we have $T^{**} \in \mathcal{I}'$. Then we also have $T^* \in \mathcal{I}'$.*

Proof. Suppose not for a contradiction. Let $(\mathcal{I}, \mathcal{I}', T, T', T^*)$ be a five-tuple witnessing this with the height of T^* minimal and, subject to this, with the height of T' minimal. Let f be a map witnessing $T \preceq_F T'$. Let r and r' be the roots of T and T'.

Case 1: $f(r) = r'$: We have $i' = l_V(r') = l_V(r) = i$ and so $\mathcal{I}'$ is i-refracted.

Claim 1 *For any tree $T^{**} \in \mathcal{I}$ of height less than that of T^*, we have $T^{**} \in \mathcal{I}'$.*

Proof. This is true by assumption if $l_E(root(T^{**})) < i$. Otherwise, let $T^{***} \in \mathcal{I}$ be the tree obtained from T^{**} by reducing the root edge-label to i. By the minimality of the height of T^* we have $T^{***} \in \mathcal{I}'$, and so since $\mathcal{I}'$ is i-refracted we have $i \geq \mu(\Omega(\mathcal{I}'))$ and so $T^{**} \in \mathcal{I}'$. □

By Lemma 25 we know that there is a bit $B \in \mathcal{B}(\mathcal{I})$ such that T^* conforms to B in $\mathcal{I}$ by tree-sum. Let u be an out-neighbour of r such that T^u represents B relative to $\mathcal{I}$. By construction u has at least one out-neighbour v. For any such v the path obtained by concatenating $f(ru)$ with $f(uv)$ is a path beginning at r' in which all edges have label $m+1$, and so has length at least two. But any such path in T' has length at most two, so it must have length exactly two. This implies that $f(u)$ is an out-neighbour of r' and $f(v)$ is an out-neighbour of $f(u)$. In particular, $(T')^{f(u)}$ represents some bit $B' \in \mathcal{B}(\mathcal{I}')$ relative to $\mathcal{I}'$.

Say $B = (\mathcal{I}_0, \ldots, \mathcal{I}_n; k; \mathcal{I}_{n+1})$ and $B' = (\mathcal{I}'_0, \ldots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_{n'+1})$. Since $l_V(u) \leq l_V(u')$ we have $k \leq k'$ and $\mathcal{I}_0 \subseteq \mathcal{I}'_0$. By construction we can index the out-neighbours of u as $v_1 \ldots v_{n+1}$ where T^{v_i} represents the ideal $\mathcal{I}_i$ relative to $\mathcal{I}$ and is lefty for $i \leq n$ and righty for $i = n+1$. Similarly we can index the out-neighbours of u' as $v'_1 \ldots v'_{n'+1}$ where $(T')^{v'_i}$ represents the ideal $\mathcal{I}'{}_i$ relative to $\mathcal{I}'$ and is lefty for $i \leq n'$ and righty for $i = n'+1$.

Since $l_V(v_n) \leq l_V(f(v_n))$, it follows that $(T')^{f(v_n)}$ is righty and so $f(v_n) = v'_{n'}$. Since we consider bits to be invariant under permutation of their left components we may also assume without loss of generality that $f(v_i) = v'_i$ for any $i \leq n$.

Claim 2 *For $1 \leq i \leq n$, any tree $T^{**} \in \mathcal{I}_i$ with height strictly less than that of T^* is also in $\mathcal{I}'_i$.*

Proof. Since $l_V(v_i) \leq l_V(v'_i)$ we have $\mu(\mathcal{I}_i) = \mu(\mathcal{I}'_i)$ and $\Delta(\mathcal{I}_i, \mathcal{I}) = \Delta(\mathcal{I}'_i, \mathcal{I}')$, and we call these common values μ and Δ respectively.

For each $j \in \Delta$, let w_j be the out-neighbour of v_i with $l_E(w_j) = j$. By Lemma 35 the function g sending each j in Δ to the label of the first edge $v_i'w_j'$ of the path $f(v_iw_j)$ is the identity. That is, that label is simply j.

Now fix $j = l_E(root(T^{**}))$. We must have $j \leq \mu$.

If $j \in \Delta$ then note that for any tree $T^{***} \in R_j(\mathcal{I}_i, \mathcal{I})$ of height less than that of T^{**} with $l_E(root(T^{***})) < j$ we have $T^{***} \in \mathcal{I}$ and so by assumption $T^{***} \in \mathcal{I}'$. Hence, $T^{***} \in ||\mathcal{I}'||_j \subseteq R_j(\mathcal{I}_i', \mathcal{I}')$. By construction, T^{w_j} is a j-refraction representation of $R_j(\mathcal{I}_i, \mathcal{I})$ and $(T')^{w_j'}$ is a j-refraction representation of $R_j(\mathcal{I}_i', \mathcal{I}')$ and $f|_{T^{w_j}}$ witnesses that $T^{w_j} \preceq_F (T')^{w_j'}$. Since $T^{**} \in R_j(\mathcal{I}_i, \mathcal{I})$, by the minimality of the height of T^* in any five-tuple we have $T^{**} \in R_j(\mathcal{I}_i', \mathcal{I}')$. Since $l_E(root(T^{**})) = j$, we know that $T^{**} \notin ||\mathcal{I}'||_j$, Hence $T^{**} \in R_j(\mathcal{I}_i')$ by Definition 26. Next, by Definition 23 for $j = l_E(root(T^{**}))$ we have $T^{**} \in \mathcal{I}_i'$.

On the other hand, if $j \notin \Delta$ then since $T^{**} \in \mathcal{I}_i \subset \mathcal{I}$ we have $T^{**} \in \mathcal{I}'$ by minimality of $height(T^*)$. But then, by Definition 28, $j \notin \Delta$ and $l_E(root(T^{**})) = j$ implies that $T^{**} \in \mathcal{I}_i'$. $\square$

A similar argument shows that any tree $T^{**} \in \mathcal{I}_{n+1}$ with height strictly less than that of T^* is also in $\mathcal{I}'_{n'+1}$.

Since T^* conforms to B in $\mathcal{J}$ by tree-sum, it must be of the form

$$Tree(T_1, T_2, \ldots, T_{n+\tilde{k}+s})$$

with $\tilde{k} \in \mathbb{N}$ and $\tilde{k} \leq k \leq k'$ and such that:

(a) for $i = 1, 2, \ldots, n$, $T_i \in \mathcal{I}_i \cup \{\Gamma\}$,

(b) for $i = 1, 2, \ldots, \tilde{k}$, $T_{n+i} \in \mathcal{I} \cup \{\Gamma\}$,

(c) for $i = 1, 2, \ldots, s$, $T_{n+\tilde{k}+i} \in \mathcal{I}_{n+1}$,

(d) $[root(T^*)] \in \mathcal{I}_0$.

We now take T_i' to be T_i for $i \leq n$, to be Γ for $n < i \leq n'$, and to be $T_{i-(n'-n)}$ for $n' < i \leq n' + \tilde{k} + s$. Since the height of each T_i is strictly less than that of T^*, the argument above shows that (a) and (c) hold with the $\mathcal{I}_i$ and T_i replaced by the corresponding $\mathcal{I}_i'$ and T_i'. The minimality of the height of T^* implies that (b) holds with $\mathcal{I}$ replaced by $\mathcal{I}'$ and the T_i replaced by the corresponding T_i'. Finally, comparing the vertex labels of u and $f(u)$ implies that $\mathcal{I}_0 \subseteq \mathcal{I}_0'$, so that (d) holds with $\mathcal{I}_0$ replaced by $\mathcal{I}_0'$. Thus T^* also conforms to B' in $\mathcal{I}'$ by tree-sum, and it follows that $T^* \in \mathcal{I}'$, contradicting our choice of the tuple $(\mathcal{I}, \mathcal{I}', T, T', T^*)$.

Case 2: $f(r) \neq r'$**:** Since there is a path of length 2 beginning at r in which both edges have the label $m + 1$, the same must be true of $f(r)$. This implies that the path from r' to $f(r)$ in T' has length at least 3. Let the first four vertices on this path be r', u, v, w in that order. Then f also witnesses that $T \preceq_F (T')^w$. By construction there is an ideal $\mathcal{J} \subseteq \mathcal{I}'$ such that $(T')^v$ represents $\mathcal{J}$ relative to $\mathcal{I}'$ and there is some $j \in \Delta(\mathcal{J}, \mathcal{I}')$ such that $(T')^w$ j-refraction represents $R_j(\mathcal{J}, \mathcal{I}')$. Since vw lies on the path from r' to $f(r)$ we have $j \geq l_E(r) = i$.

For any tree $T^{**} \in \mathcal{I}$ of smaller height than T^* with $l_E(root(T^{**})) < i$ we have $T^{**} \in \mathcal{I}'$ by assumption and so $T^{**} \in R_j(\mathcal{J}, \mathcal{I}')$ since $j \geq i$ and $||\mathcal{I}'||_j \subseteq R_j(\mathcal{J}, \mathcal{I}')$. Thus the five-tuple $(\mathcal{I}, R_j(\mathcal{J}, \mathcal{I}'), T, (T')^w, T^*)$ fulfills the conditions of the statement, and so by the minimality of the height of T' we have $T^* \in R_j(\mathcal{J}, \mathcal{I}')$. Now since $i \leq j$, either $T^* \in ||\mathcal{I}'||_j$ and so $T^* \in \mathcal{I}'$ as required, or $T^* \in R_j(\mathcal{J})$. Then, $i = \min(i, j)$ and Definition 23 imply that $T^* \in \mathcal{J}$ as required since $\mathcal{J} \subseteq \mathcal{I}'$.

□

Theorem 37 *Let $\mathcal{I}$ and $\mathcal{I}'$ be ideals of $\mathcal{F}(m, Q)$. If $T_{\mathcal{I}} \preceq_F T_{\mathcal{I}'}$ then $\mathcal{I} \subseteq \mathcal{I}'$.*

Proof. Let f be a map witnessing that $T_{\mathcal{I}} \preceq T_{\mathcal{I}'}$. Let r and r' be the roots of T and T' respectively. Since $l_E(r) = m + 1$ but the label of any edge out of r' is at most m, we must have $f(r) = r'$. Comparing the labels of these vertices, we see that also $\mu(\mathcal{I}) = \mu(\mathcal{I}')$ and $\Delta(\mathcal{I}, \mathcal{F}(m, Q)) = \Delta(\mathcal{I}', \mathcal{F}(m, Q))$ and we call these common values μ and Δ respectively.

For each $i \in \Delta$, let u_i be the out-neighbour of r with $l_E(u_i) = i$. By Lemma 35 the function g sending each i in Δ to the label of the first edge of the path $f(ru_i)$ is the identity. That is, that label is simply i.

Now let $T^* \in \mathcal{I}$ and set $i := l_E(root(T^*))$. Thus $i \leq \mu$. If $i \notin \Delta$ then it follows that every tree of $\mathcal{F}(m, Q)$ with root edge label i is in $\mathcal{I}'$, and so in particular $T^* \in \mathcal{I}'$.

If instead $i \in \Delta$ then let $r'u$ be the first edge of the path $f(ru_i)$. Then T^{u_i} i-refraction-represents $R_i(\mathcal{I}, \mathcal{F}(m, Q))$ and $(T')^u$ i-refraction-represents $R_i(\mathcal{I}', \mathcal{F}(m, Q))$. The restriction of f to T^{u_i} witnesses that $T^{u_i} \preceq_F (T')^u$. For any tree $T^{**} \in R_i(\mathcal{I}, \mathcal{F}(m, Q))$ of smaller height than T^* with $l_E(root(T^{**})) < i$ we have $T^{**} \in R_i(\mathcal{I}', \mathcal{F}(m, Q))$ because $T^{**} \in \mathcal{F}(m, Q)$. By Theorem 36 follows $T^* \in R_i(\mathcal{I}', \mathcal{F}(m, Q))$ and so $T^* \in \mathcal{I}'$ in this case too, completing the proof. □

5 Conclusion

It is now easy to prove recursively for each $n \in \mathbb{N}$ that if $L^n Q$ is wqo, then $L^n \mathcal{F}(m, Q)$ is wqo, because we can represent every ideal $\mathcal{I} \in L\mathcal{F}(m, Q)$ by a finite tree $T \in \mathcal{F}(m + 2, Q_0)$ such that if LQ is wqo, then Q_0 wqo. So studying ideals of ideals of trees reduces to studying ideals of trees except for the fact that the labels are taken from a more complicated wqo set. This generalizes a theorem of Nash-Williams in [6]. It is however more interesting to prove that iterations that are not bounded by any $n \in \mathbb{N}$ are wqo. For this purpose we need to relax some of the strict quasi-orders we introduced in (3.8). See [2] for a deeper theory that involves a concept known as 'better-quasi-ordering', (bqo, for short). The concept of bqo was introduced by Nash Williams [7] in preparation for a proof that infinite trees are wqo under $\preceq$. However the proof becomes quite complicated. A goal of this paper is to pave the way for an easier finite tree representation of infinitely iterated ideals of finite trees so that we will have a better understanding of bqo theory.

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