Bachelor Thesis
Describing highly cohesive structures in the plane via tangles

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1 Introduction

In their famous proof of the graph minor theorem [7] Robertson and Seymour defined tangles as a new tool. These tangles turned out to be a great way to find regions of high cohesion in a graph. The idea was to no longer identify a region of high cohesion by looking at vertices which are contained inside it but by considering the separations of low order which point towards most of it. This allows these tangles to describe regions which have a certain fuzziness: Where it is possible to cut out small parts and hence difficult to describe a precise boundary. An example for this is a large square grid: If we want to separate a single vertex from the rest of the grid it suffices to delete the four neighboring vertices but if we wanted to separate a grid into large parts then we would need to delete many vertices. Since then, the idea of tangles was generalized many times and in this thesis we are going to use the, as of today, most generalized version of tangles, as described by R. Diestel in [2]. In this generalization we now longer care what elements these separations separate but instead consider just the separations. This generalization allows us to apply the idea of tangles not just to graphs but to everything which can be separated in some sense, since the theory works just with the separations.

Suppose we have a shape in the plane which is homeomorphic to the closed unit disk. In this thesis we will try to split apart this shape, from this point on simply called a disk, into highly connected regions. For an example, consider the union of two large circles which slightly overlap. Intuitively one would see two highly connected regions, namely these two circles. What we will do in this thesis is to apply the tangles framework on this disk in the hopes of finding these same highly connected regions which we intuitively indentified.

This would be very interesting for multiple reasons. First of all we would be able to analyze these shapes in a new way where we are even able to identify the number of highly cohesive regions the disk contains. Secondly seeing tangles work in this relatively simple context could valdiate the use of tangles in more complicated situations.

To define tangles we are going to approximate them using tangles on graphs. Specifically we are going to use increasingly fine triangulations, which we define in section 2, to approximate the shape of the disk. Then we can define tangles in section 3 on these triangulations, which are simple going to be plane graphs. This has the advantage that tangles on graphs are well understood and furthermore that there are only finitely many separations, which makes many arguments simpler.

One of the strengths of the tangle framework from [2] are two fundamental theorems about tangles. These theorems are the cornerstone for working with tangles as they describe a very interesting structure about tangles. The first of these is the tangle-tree theorem which states that tangles form a tree structure which is further witnessed by nested separations. This is very important as this theorem gives us nested cuts along the disk which separate all of the regions of high cohesion from each other. The other fundamental theorem is the tangle-tree duality theorem. It is not obvious that tangles actually exist in many setups since they are defined using large sets of separations, but using the tangle-tree duality theorem one is able to prove that a tangle exists in a given set of separations. The tangle-tree duality theorem states that either a tangle exists or a tree of forbidden stars exist which means that either there obviously can not exist a tangle or if we can show that no such tree of forbidden stars exist we have already shown that there must be a tangle.

We will first prove these theorems for the tangles on graphs in section 4. Afterwards we are going to use an inverse limit construction to define tangles for the entire disk. This inverse limit construction has the benefit of allowing us to answer questions about tangle of the disk by considering these questions on all of the triangulations. This will allow us to prove the tree-of-tangles theorem and the tangle-tree duality on the tangles of the disk. The inverse limit will avoid many problems which could arise due to infinity. In the case of the tree-of-tangles theorem we will prove a theorem that tangles of the disk form a tree structure which is witnessed in a triangulation by a set of nested separations. In the tangle-tree duality theorem we will show that if we do not have a tangle this is in fact witnessed by a tree of forbidden stars in one of the triangulations.

In section 7 after we have laid the groundwork for using tangles to find structures of high cohesion in disks we will revisit the question from the beginning. We will show that if we have a shape which contains an arbitrary number of overlapping circles which can pairwise be separated from each other then each of these circles corresponds to a distinct tangle. To do this we will need some geometric consideration which we will do in section 6. The geometric considerations will show that circles have to much cohesion to be cut apart in the context of tangles.

2 Triangulations

This section will assume familiarity with basic concepts of topology and the chapter about plane graphs in [1].

Definition 2.1. *A disk D is a bounded subset of $\mathbb{R}^2$ that is topologically homeomorphic to the closed unit disk.*

The goal in this thesis will be to look at the structure of D, to do this we will not be looking at D itself but instead at triangulations of D that we hope will capture the structure of D. We will model these triangulations by using plane graphs. For plane graphs we will use the following definition taken from [1].

Definition 2.2. *A plane graph G is a pair (V, E) of vertices V and edges E such that:*

(i) $V \subseteq \mathbb{R}^2$

(ii) every edge is an arc between two vertices, where an arc is a finite union of straight line segments that are topologically homeomorphic to the unit interval $[0, 1]$

(iii) different edges have different sets of endpoints

(iv) the interior of an edge contains no vertex and no point of any other edge

A concept we need to introduce to define triangulations is the concept of regions and faces.

Definition 2.3. *A region is an arc-connected component of a subset of $\mathbb{R}^2$. The frontier of a region X is every point for which every neighbourhood meets both X and $\mathbb{R}^2 \setminus X$. In a plane graph $G = (V, E)$ we call the regions of $(\mathbb{R}^2 \setminus V) \setminus \bigcup E$ the faces of G. Since G is bounded there exists one unbounded face which we call the outer face and every other face we call inner faces.*

Definition 2.4. *A triangulation G of a disk D is a connected plane graph $G = (V, E)$ with vertices $V \subseteq D$ and edges $E \subseteq D$ where each of its inner faces is bounded by a triangle and its outer face by a cycle. Furthermore if v lies on the boundary of the outer face then v lies in the topological boundary of D. The inner faces of G excluding the unbounded outer face are noted as $F(G)$.*

We will later use bipartitions of the faces to define separations so we'll go over some more basic terminology.

Definition 2.5. *If we have a set $A \subset F(G)$ then let $\partial A \subset E(G)$ be the set of all edges that are contained in the frontier of a face from A and a face from $F(G) \setminus A$. We call ∂A the boundary of A.*

Important to note is that edges that lie on the frontier of the outer face are not part of ∂A. This makes sense as we will use ∂A to define the order of separations and we only care about the inner faces, as they are the ones that describe D.

Definition 2.6. *A set $A \subset F(G)$ is called connected if for every two faces $a, b \in A$ there exists a sequence of faces $f_1, ..., f_n \in A$ such that $a = f_1$, $b = f_n$ and the faces f_i and f_{i+1} have a common edge on their frontier.*

So in other words: A is connected if it is connected in the dual graph. Importantly a set $A \subset F(G)$ being connected topologically in the sense of $\bigcup A$ being connected does not imply A being connected as we do not count connections that consist of a single vertex.
A problem we might run into when we look at a triangulation is that it might not cover the entire detail of the shape and so we want to look at a more detailed triangulation.

Definition 2.7. *A triangulation G_1 is finer than a trianglulation G_2 if $V(G_2) \subset V(G_1)$ and $\bigcup E(G_2) \subset \bigcup E(G_1)$. In that case we say that G_2 is coarser than G_1.*

In order to work with triangulations we might run into a situation where we have two triangulations, that each cover different parts of D well, where we wish to have one triangulation which is finer than both of them and thus captures the detail of both triangulations. Luckily this is indeed possible to find by combining the points of both triangulations, adding points at the intersections and adding some edges.

Lemma 2.1. *If we have two triangulations G_1 and G_2 we can find a triangulation G which is finer than both G_1 and G_2.*

Proof. We construct G in the following way: First we define a plane multigraph H where not every face is a triangle but which we will then use to define G. We start by defining $V(e_1, e_2)$ for $e_1 \in G_1$ and $e_2 \in G_2$ as the set of points which lie on the topological boundary of $e_1 \cap e_2$ in the topology of $E(G_1) \cup E(G_2)$. With that we can define the vertex set of H as

$$V(H) := V(G_1) \cup V(G_2) \cup \Big(\bigcup_{e_1 \in G_1, e_2 \in G_2} V(e_1, e_2) \Big)$$

The set $V(H)$ is finite, since $V(G_1)$ and $V(G_2)$ are finite and since e_1 and e_2 are finite unions of straight line segments the set $V(e_1, e_2)$ is also finite. For the edges $E(H)$ of H we take the closure of the connected components of $(\bigcup E(G_1) \cup \bigcup E(G_2)) \setminus V(H)$.
We now show that H is in fact a plane graph. Since $V(G_1), V(G_2) \subseteq \mathbb{R}^2$ and $\bigcup E(G_1), \bigcup E(G_2) \subseteq \mathbb{R}^2$ this also means that $V(H) \subseteq \mathbb{R}^2$.
Next we need to show that every edge in $E(H)$ is an arc between two vertices. The set $\bigcup E(G_1) \cup \bigcup E(G_2)$ is a finite union of straight line segments since every edge is a finite union of straight line segments. If we delete a point of a straight line segment and then look at the closure of the connected components we have a (finite) union of two straight line segments. This means that every element of $E(H)$ is also a finite union of straight line segments. Now we need to show that every element of $E(H)$ is also topologically homeomorphic to the closed unit inverval $[0, 1]$. This follows from the fact that every element of $E(H)$ is contained either in an edge from $E(G_1)$ or $E(G_2)$. Every closed connected subset of a set that is topologically homeomorphic to $[0, 1]$ that is not a set containing just one point is also topologically homeomorphic to $[0, 1]$. The elements of $E(H)$ have a non empty interior and thus contain more than one point. Every element of $E(H)$ is also bound by two vertices of $V(H)$ by definition and thus we know that every edge in $E(H)$ is an arc between two vertices.
Finally we show that the interior of an edge in $E(H)$ contains no vertex from $V(H)$ and no point of any other edge. Because of the definition of $E(H)$ we can see that the interior of an edge can not contain any point from another edge since the interior of an edge is one of the components of $(\bigcup E(G_1) \cup \bigcup E(G_2)) \setminus V(H)$. Hence the components also can not contain a vertex from $V(H)$. Thus we have shown that H is in fact a plane multigraph and by definition we have $V(G_1) \subseteq V(H)$, $V(G_2) \subseteq V(H)$, $\bigcup E(G_1) \subseteq \bigcup E(H)$ and $\bigcup E(G_2) \subseteq \bigcup E(H)$.
Now we construct G in three steps. First we subdivide every edge in H by adding one vertex and thus splitting the edge into two edges, so that we get a graph H' since this removes all multiple edges. Next if there is a vertex v in H' which lies on the boundary of H' but not on the boundary of the outer face C then there exists a path $v_0 v_1 ... v_n$ containing v in one of the inner vertices, where v_i for $i \in \{0, ..., n\}$ all lie on the boundary of H' and where only v_0 and v_n lie on the boundary of C. Then we add an edge connecting v_0 to v_n and thus v no longer lies on the boundary of H', we repeat until all vertices which lie on the boundary of H' also lie on the boundary of C . Finally we add edges to the bounded faces of H until we cannot add any more edges. Adding edges can only be done finitely many times as plane graphs with a fixed number of vertices can only have a

bounded number of edges. After this process every inner face is a triangle (see Proposition 4.2.8 in [1]) and thus G is a triangulation with $V(H) \subseteq V(G)$ and $\bigcup E(H) \subseteq E(G)$. Hence we know that $V(G_1) \subseteq V(G)$, $V(G_2) \subseteq V(G)$, $\bigcup E(G_1) \subseteq E(G)$ and $\bigcup E(G_2) \subseteq E(G)$. This means that G is finer than both G_1 and G_2. $\square$

Definition 2.8. *A weight function of a triangulation G is a function $w : E(G) \to \mathbb{R}_{>0}$ which maps every edge to its weight.*

In this thesis we will always use $w(E)$ where E is a set of edges to note the sum of the weights of all the edges contained in E.
We will later need a way to measure the area of subsets of D, hence we will define a measure now.

Definition 2.9. *Let Δ be a measure that is finite on every bounded measurable subset of $\mathbb{R}^2$.*

3 Separations and Tangles

We will now look at the basic terminology of separations and tangles that will be used in this thesis. For the readability of this thesis, we collect most of the important definitions from [2] in this section.

Definition 3.1. *A separation system* $(\vec{S}, \leq, ^*)$ *is partially ordered set* $\vec{S}$ *with an order-reversing involution* *. *Its elements are called oriented separations.*

We will write an oriented separation as $\vec{s}$ and its inverse $\vec{s}^*$ we will write as $\overleftarrow{s}$. Separations themselves will be noted by $s = \{\overleftarrow{s}, \vec{s}\}$. In this thesis the separations we look at are bipartitions that divide some set into two parts and the oriented separations orient the separation towards one of the sides.

Definition 3.2. *A universe is a separation system with two binary operators* $\vee$ *and* $\wedge$ *that make it into a lattice. A universe is submodular if it has a submodular order function, a function* $|\cdot| : \vec{S} \to \mathbb{R}_{\geq 0}$ *that satisfies* $|\vec{s}| = |\overleftarrow{s}|$ *and* $|\vec{r} \vee \vec{s}| + |\vec{r} \wedge \vec{s}| \leq |\vec{r}| + |\vec{s}|$.

We call two separations $s, r \in S$ nested if each has an orientation $\vec{s} \in s$ and $\vec{r} \in r$ so that $\vec{s}$ and $\vec{r}$ are comparable. If two separations are not nested we call them crossing. Separations themselves are sets $\{\overleftarrow{s}, \vec{s}\}$ and the set of all separations is called S, without the arrow. A separation $\vec{r} \in \vec{S}$ is trivial if there exists a separation $\vec{s} \in \vec{S}$ such that $\vec{r} < \vec{s}$ and $\vec{r} < \overleftarrow{s}$. A separation $\vec{r}$ such that $\vec{r} < \overleftarrow{r}$ is called small. Every trivial separation is small but not every small separation is trivial. If $\vec{r}$ is trivial then $\overleftarrow{r}$ is called co-trivial.

We define the separations as bipartitions of the faces of our triangulation.

Definition 3.3. *An ordered separation of a triangulation G is an ordered pair (A, B) with $A \cup B = F(G)$ and $A \cap B = \emptyset$. We denote the set of all ordered seperations as $\vec{S}(G)$ or $\vec{S}$ if the triangulation is clear from the context.*

Note that for some later theorems we will still use $\vec{S}$ to mean a general system of separations and not necessarily the separations of a triangulation.

The ordered separations of a set together with the partial-order $(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \Leftrightarrow B \geq D$ form a partially-ordered set. We also have an order-reversing involution * by defining $(A, B)^* = (B, A)$. With this partial order there can only exist one trivial element, that being $(\emptyset, F(G))$[1], hence the only co-trivial element is $(F(G), \emptyset)$. Next we define the supremum and infimum of the separations in the way it is normally defined for bipartitions of sets.

Definition 3.4. *The supremum of two separations (A, B) and (C, D) is defined as $(A, B) \vee (C, D) := (A \cup C, B \cap D)$. The infimum of two separations (A, B) and (C, D) is defined as $(A, B) \wedge (C, D) := (A \cap C, B \cup D)$.*

We can see that the supremum and infimum are indeed contained in the set of separations and that this makes $\vec{S}$ a lattice.

Now that we have defined supremum and infimum we can define an order function for $\vec{S}$.

Definition 3.5. *The order of a separation (A, B) of a triangulation G with weight function w is $|(A, B)| = w(\partial A)$.*

This makes our set of separations into a submodular universe.

[1]assuming the separation system is not empty, as every separation (A, B) witnesses $(\emptyset, F(G))$ being trivial.

Lemma 3.1. *The above order function is well-defined and submodular.*

Proof. To show that the order function is well-defined we need to prove that $|(A, B)| = |(B, A)|$ since $|(A, B)| = w(\partial A)$ and $|(B, A)| = w(\partial B)$ it suffices to prove that $\partial A = \partial B$. If $e \in \partial A$ this means that e is contained in the frontier of a face from A as well as a face from $F(G) \setminus A = B$. This is equivalent to e being contained in the frontier of a face from B and a face from $F(G) \setminus B = A$. Hence we know that $\partial A = \partial B$.
Now we just need to show that $|\cdot|$ is submodular for that we look at two separations (A, B) and (C, D). We need to show that

$$|(A, B) \vee (C, D)| + |(A, B) \wedge (C, D)| \leq |(A, B)| + |(C, D)|.$$

By the definition of the infimum and supremum this means that

$$|(A \cup C, B \cap D)| + |(A \cap C, B \cup D)| \leq |(A, B)| + |(C, D)|.$$

We can prove this by showing that $\partial(A \cup C) \cup \partial(A \cap C) \subseteq \partial A \cup \partial C$ and $\partial(A \cup C) \cap \partial(A \cap C) = \partial(A) \cap \partial(C)$. We start by showing $\partial(A \cup C) \subseteq \partial A \cup \partial C$. If $e \in \partial(A \cup C)$ then e lies on the frontier of A or C and on the frontier of $F(G) \setminus (A \cup C)$. This means that $e \in \partial A$ or $e \in \partial C$. If on the other hand $e \in \partial(A \cap C)$ then e lies on the frontier of both A and C but it is also on the frontier of a face that is not in A or not in C. This means that again $e \in \partial A$ or $e \in \partial C$.
All that is left to show is that $\partial(A \cup C) \cap \partial(A \cap C) = \partial(A) \cap \partial(C)$. Let $e \in \partial(A \cup C) \cap \partial(A \cap C)$, that means that e is on the frontier of a face of $A \cap C$ and on the frontier of a face of $F(G) \setminus (A \cup C)$. Since any edge can only be on the frontier of two faces we know that e lies on the frontier of a face of A, on the frontier of a face of C as well as the frontier of a face of $F(G) \setminus (A \cup C)$. That is now equivalent to being in the boundary of A and in the boundary of C which shows $c \in \partial(A) \cap \partial(C)$. Hence we have shown the order function to be submodular.

□

We now define what it means for all of the separations of a set $S' \subset S(G)$ to be oriented towards some region.

Definition 3.6. *An orientation is a subset $O \subset \overrightarrow{S'}$ so that for each $s \in S'$ the set O contains either $\overrightarrow{s}$ or $\overleftarrow{s}$. An orientation is consistent if there are no distinct $r, s \in S'$ with $\overrightarrow{r} < \overrightarrow{s}$ such that $\overleftarrow{r}, \overrightarrow{s} \in O$.*

For an orientation to make sense when thinking about the separations as pointing at some region we need to avoid one case - where one separation points at one region of the triangulation and the other in a completely different direction. That motivates the definition of consistency. A consistent orientation is simply an orientation which does not have separations pointing away from each other.
We are now at the point where we wonder what the consistent orientations of the entire set $S(G)$ look like, sadly there is not much structure there. The problem is that the set of consistent orientations of $S(G)$ just depend on the number of elements in $F(G)$ and not on the structure of G. To deal with this we make the following two changes. We do not look at every separation but only at a certain subset of separations that respect the shape of D.

Definition 3.7. *For every $k \in \mathbb{R}_+$ let $\overrightarrow{S_k} := \{\overrightarrow{s} \in \overrightarrow{S} : |\overrightarrow{s}| < k\}$*

There is one more problem though. We could have an orientation which only points towards one small triangle and thus does not point towards a region of high cohesion as we would like. To deal with this we require an orientation to avoid certain forbidden structures. These forbidden structures should witness our orientation not pointing towards a region of high cohesion. The standard approach is to use stars.

Definition 3.8. *A star is a set σ of oriented separations whose elements point towards each other: $\vec{r} \le \overleftarrow{s}$ for all distinct $\vec{r}, \vec{s} \in \sigma$.*

If we have a structure with high cohesion then we expect this structure to stay intact, that is to say we expect there not to be any small stars with small area contained in an orientation that orients the separations towards such a structure. More formally if $\sigma = \{(A_i, B_i)\}_{i\in I}$ then the area of the intersection of the right sides of σ denoted as $\cap\sigma := \bigcap_{i\in I} B_i$ should be larger than some small number, ε say. This motivates the following definition.

Definition 3.9. *For every $k \in \mathbb{R}_+$ and $\varepsilon > 0$ let*

$$\mathcal{F}_k^\varepsilon := \{\sigma\, star : |\sigma| \le 3 \wedge \Delta(\cap\sigma) < \varepsilon\} \cap 2^{\vec{S_k}}$$

This finally allows us to define tangles of triangulations.

Definition 3.10. *Let G be a triangulation of a disk D then an $\mathcal{F}_k^\varepsilon$-tangle or simply a tangle of G is a consistent orientation of $\vec{S}_k(G)$ that does not have a subset in $\mathcal{F}_k^\varepsilon$.*

3.1 Simple separations

One kind of separation that we wish to look at more is one that is generated by the cut of a path P. To generate a separation we need this path to cut the graph into two components so we expect this path to either be a path with ends contained in the outer cycle C or a closed path, that is a cycle, specifically a cycle that does not meet C in more than one vertex. We call paths that fall into one of these categories a cutting path. Our intuition is that a separation (A, B) is simple if it is in some sense generated by a single cutting path P. What we mean by that is that the boundary ∂A is equal to the edge set of P. Simple separations (A, B) have the property that both sides A and B are connected so let us take this as the definition as it is easier to work with. We will later see that our first intuition already follows from that definition.

Definition 3.11. *A simple separation (A, B) is a separation where A and B are connected.*

Note that in general simple separations do not form a lattice, hence why we chose all bipartitions for our separation system.
To show that in general every simple separation is generated by a cutting path we first need to make precise what it means for a path to generate a separation.

Lemma 3.2. *A separation (A, B) is simple if and only if there exists a cutting path P such that $E(P) = \partial A$.*

Proof. Let us say that P is a good cycle if it intersects the boundary of the outer cycle C in at most one vertex, or is a path that connects to C only with both ends. We start by showing that if P is a good cycle that there is a simple separation with P as a boundary. If P is a cycle this can be done by letting A be the inner faces and B the outer faces. If P is instead a C path then P together with one of the two parts of the outer cycle it encloses gives us a cycle and we get the simple separation in the same way.
For the other direction we start by showing that if a separation (A, B) is simple then there exists such a cutting path P with $E(P) = \partial A$.
Fist we show that the graph $P := \partial A$ does not have a vertex v of degree greater than two. Suppose there was a vertex $v \in V(P)$ with $d(v) > 2$ then we consider a neighbourhood N of v where the only vertex of $V(P)$ contained within it is v. This neighbourhood N is divided into $d(v)$ parts $N_1, ..., N_{d(v)}$ by P. Without loss of generality these parts are ordered in a clockwise fashion and $N_1 \subseteq \bigcup A$. We know that $N_2 \subseteq \bigcup B$ as the edge separating N_1 from N_2 in the neighbourhood of

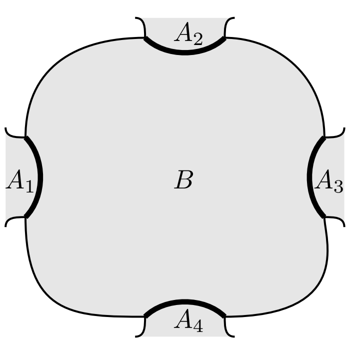
v has to lie on both the boundary of $\bigcup A$ and $\bigcup B$, analogously we can show that $N_3 \subseteq \bigcup A$. In that case B is not connected as a path from N_1 to N_3 separates N_2 from the rest of the parts.
Now we need to show that if P has vertices of degree 1 then these vertices lie in C. This can be easily seen from the fact that otherwise these leaves would not lie on a boundary of ∂A since they would not be separating (Lemma 4.1.3 from [1]).
Since P is a graph where every vertex has degree at most 2 every component of P has to be a cutting path or a cycle intersecting C in more than one vertex or a path which intersects C in at least 3 vertices, but the last two can not happen since that would mean that either A or B are not connected since no edge of C can lie in ∂A and a cycle intersecting C would thus generate three components.
The last thing we need to prove is that P only contains one component. We prove this by considering, that if we had multiple components $P_1, ..., P_n$ of P then each of these components would be a cutting path and thus by the other direction of this theorem each would generate a simple separation. Since these cutting paths P_i do not intersect either A or B contains more than one component which is a contradiction to the connectedness of A and B. □

This is where we use that a disk is homeomorphic to the unit circle. If our disk has a hole then this lemma does not hold. The problem arises in the part where we show that a vertex of degree 1 has to lie on C, if we had a hole in D then v could also lie on the boundary of this hole.

Lemma 3.3. $|(A, B)| = \sum_{i=1}^{n} |(A_i, F(G) \setminus A_i)|$ *if* $A_1, ..., A_n$ *are the components of* A.

Proof. To prove this we look at the set ∂A. It follows instantly that $\partial A \subseteq \partial A_1 \cup ...\partial A_n$, since every edge that lies on ∂A also has to lie on the boundary of one of the components ∂A_i. To show $\partial A_i \subseteq A$ we look at the definition of A_i. If there is an edge e on the boundary of A_i then it also has to be on the boundary of another face. This face has to be a face in B as the A_i are components of A and as such are not connected to other components. Since e is on the boundary of A_i and B it is also on the boundary of B and thus we have shown $\partial A_i \subseteq A$.
The last thing we need to show to prove this lemma is that the union $\partial A_1 \cup ...\partial A_n$ is disjoint. This can again be seen from the fact that if there is an edge e that is $e \in \partial A_i$ and $e \in \partial A_j$. Then $\partial A_i \cup \partial A_j$ are connected and thus A_i and A_j would not be components. Hence we have shown this lemma. □

Every tangle of all of the separations S_k induces a tangle on only the simple separations in S_k but is the converse also true? Does every tangle of the simple separations also extend to a tangle on all of S_k? For that to be the case it has to be the case that if we have a forbidden star σ in a tangle there must also be a forbidden star made of simple elements contained inside of the same tangle. This is not generally the case as the following diagram shows.



Let $|(A_i, F(G) \setminus A_i)| < \frac{k}{2}$ for $i \in \{1, 2, 3, 4\}$ then if $\Delta B < \varepsilon$ we can get a forbidden star made out of two separations $(A_1 \cup A_3, B \cup A_2 \cup A_4)$ and $(A_2 \cup A_4, B \cup A_1 \cup A_3)$. But to make a forbidden star out of simple separations we would need the four separations $(A_i, F(G) \setminus A_i)$ for $i \in \{1, 2, 3, 4\}$.

4 Tree-of-tangles and Duality

There are two fundamental theorems about tangles. The first one is the tree of tangles theorem and the other one is tangle-tree duality. The former gives us a tree structure on all of the tangles, even with different order k, while the latter gives us a criterium with which we can find out if we have tangles or find a certificate that proves it does not.

4.1 Tree-of-tangles Theorem

Let us first start with the tree-of-tangles theorem. To formulate the theorem we first need some more definitions as the theorem does not look at tangles but instead at profiles. The definitions of this section and the main theorem are taken from [3].

Definition 4.1. *A profile P of S is a consistent orientation of S that satisfies the profile-property: For all $\vec{r}, \vec{s} \in P$ the separation $\overleftarrow{r} \wedge \overleftarrow{s}$ is not in P.*

To use the tree-of-tangles theorem for the $\mathcal{F}_k^\varepsilon$-tangles we have to show that they already fulfill the profile-property.

Lemma 4.1. *An $\mathcal{F}_k^\varepsilon$-tangle is a profile.*

Proof. Suppose there is an $\mathcal{F}_k^\varepsilon$-tangle τ that is not a profile. In that case there are separations $\vec{r}, \vec{s} \in \tau$ and $\overleftarrow{t} = \overleftarrow{r} \wedge \overleftarrow{s} = (\vec{r} \vee \vec{s})^* \in \tau$. But that would mean that $\vec{t} \leq \vec{r}$ while $\overleftarrow{t}, \vec{r} \in \tau$ which is a contradiction to the consistency of τ. $\square$

Sets of profiles do not always form a tree structure but robust profiles do.

Definition 4.2. *A profile P of S is robust if it is n-robust for every n, and it is n-robust if for every $\vec{r} \in P$ and every $s \in S_n$ the following holds: If $\overleftarrow{r} \wedge \vec{s}$ and $\overleftarrow{r} \wedge \overleftarrow{s}$ both have order $< |r|$, they do not both lie in P.*

This is to exclude one special case which does not allow the tree structure we want. In our case this again cannot happen.

Lemma 4.2. *An $\mathcal{F}_k^\varepsilon$-tangle is a robust profile.*

Proof. Suppose there is an $\mathcal{F}_k^\varepsilon$-tangle τ that is not a robust profile. Then there exists an n so that τ is not n-robust. This means that there exists an $\vec{r} \in \tau$ and an $s \in S_n$ where $\overleftarrow{r} \wedge \vec{s}$ and $\overleftarrow{r} \wedge \overleftarrow{s}$ both have order $< |r|$ but they both lie in τ. We now define $\sigma = \{\vec{r}, \overleftarrow{r} \wedge \vec{s}\} \subseteq \tau$. Since σ is a star and $\cap\sigma = \emptyset$ and thus $\Delta(\cap\sigma) = 0$ we have $\sigma \in \mathcal{F}_k^\varepsilon$ which is a contradiction. $\square$

We say that two profiles P, P' are distinguished by a separation $s \in S$ if $\vec{s} \in P$ and $\overleftarrow{s} \in P'$, in that case we call them distinguishable. If the separation s that separates them has order $|s| < k$ we say that they are k-distinguishable. The smallest k for which P and P' are k-distinguishable is called $\kappa(P, P')$ and if s distinguishes P and P' and $|s| = \kappa(P, P')$ we say that s distinguishes P and P' efficiently. A set $T \subseteq U$ distinguishes a set $\mathcal{P}$ of profiles efficiently if for any two profiles in $\mathcal{P}$ there is an element of T which distinguishes them efficiently. If a set of profiles $\mathcal{P}$ is pairwise distinguishable and every element is robust we call $\mathcal{P}$ robust. After this preparation we can now formulate the tangle-tree theorem for separation universes (taken from [3]).

Theorem 4.1. *Let $\overrightarrow{U} = (\overrightarrow{U}, \leq, {}^*, \vee, \wedge, ||)$ be a submodular universe of separations. Then for every robust set $\mathcal{P}$ of profiles in $\overrightarrow{U}$ there is a nested set $S_{\mathcal{P}} \subseteq U$ of separations such that:*

(i) every two profiles in $\mathcal{P}$ are efficiently distinguished by some separation in $S_{\mathcal{P}}$;

(ii) every separation in $S_{\mathcal{P}}$ efficiently distinguishes a pair of profiles in $\mathcal{P}$;

This theorem tells us a lot about the structure of the $\mathcal{F}_k^\varepsilon$-tangles. First of all we can use it to find a tree structure for a set of $\mathcal{F}_k^\varepsilon$-tangles T.

Corollary 4.1. *There is a nested set $S_T \subseteq \overrightarrow{S}_k$ of separations such that:*

(i) every two $\mathcal{F}_k^\varepsilon$-tangles are efficiently distinguished by some separation in S_T;

(ii) every separation in S_T efficiently distinguishes a pair of $\mathcal{F}_k^\varepsilon$-tangles;

But we can also find a tree-structure for bigger sets of tangles containing tangles with different orders. For that we first have to show that tangles of higher order can also be seen as tangles of lower order. More precisely

Lemma 4.3. *Let $k > \ell$ then every $\mathcal{F}_k^\varepsilon$-tangle τ_k induces an $\mathcal{F}_\ell^\varepsilon$-tangle $\tau_\ell := \tau_k \cap 2^{\overrightarrow{S}_\ell}$ that can not be distinguished from τ_k.*

Proof. The $\mathcal{F}_\ell^\varepsilon$-tangle τ_ℓ is a consistent orientation since τ_k already had to consistently orient $\overrightarrow{S}_\ell$ to consistently orient $\overrightarrow{S}_k$. If there is a star $\sigma \in \mathcal{F}_\ell^\varepsilon$ in τ_ℓ then this star must have also been in τ_k since $\mathcal{F}_\ell^\varepsilon \subseteq \mathcal{F}_k^\varepsilon$. This proves that τ_ℓ is indeed an $\mathcal{F}_\ell^\varepsilon$-tangle. The only thing left to show is that τ_ℓ can not be distinguished from τ_k. This can easily be seen from the fact that if $\overleftarrow{s} \in \tau_\ell$ and $\overrightarrow{s} \in \tau_k$ then $\overrightarrow{s}$ must also be in τ_ℓ since $|s| < \ell$ which is a contradiction. □

Now we can define the tangles that are pairwise distinguishable which we need for the tree-of-tangles theorem.

Definition 4.3. *Let τ be an $\mathcal{F}_k^\varepsilon$-tangle that is not induced by an $\mathcal{F}_\ell^\varepsilon$-tangle for an $\ell > k$ then we call τ maximal.*

These tangles are indeed distinguishable.

Lemma 4.4. *Maximal tangles are pairwise distinguishable.*

Proof. Let τ_1 and τ_2 be two distinct maximal tangles. If the order of τ_1 and τ_2 is equal then since they are distinct there must be a separation $s \in S$ so that $\overrightarrow{s} \in \tau_1$ and $\overleftarrow{s} \in \tau_2$. If the order of τ_1 and τ_2 is not equal then without loss of generality we can assume the order ℓ of τ_1 to be smaller than the order k of τ_2. Since τ_1 is not induced by τ_2 there must be an oriented separation $\overrightarrow{s} \in \overrightarrow{S}_\ell$ so that $\overrightarrow{s} \in \tau_1$ but $\overrightarrow{s} \notin \tau_2$. Since $s \in S_\ell$ this also means that $s \in S_k$ but that means that τ_2 also has to orient s. That implies $\overleftarrow{s} \in \tau_2$ and so τ_1 and τ_2 are distinguished by s. □

Finally we get a tree structure for all of our maximal tangles.

Corollary 4.2. *For a set of maximal tangles T there exists a nested set $S_T \subseteq \overrightarrow{S}$ of separations such that:*

(i) every two maximal tangles of T are efficiently distinguished by some separation in S_T;

(ii) every separation in S_T efficiently distinguishes a pair of maximal tangles of T;

We can even make this theorem a little stronger by only including simple separations. For that it suffices to show that the separations we get from the tree-of-tangles theorem are simple due to efficiently distinguishing the tangles.

Lemma 4.5. *If two tangles $\tau_1, \tau_2 \in T$ are efficiently distinguished by a separation r then r is a simple separation.*

Proof. Let $\overrightarrow{r} = (A, B)$ and $\overrightarrow{r} \in \tau_1, \overleftarrow{r} \in \tau_2$. The set A has components $A_1, ..., A_n$ and B has components $B_1, ..., B_m$.

We want to show that there is an i and a j such that $\overrightarrow{s_1} := (F(G) \setminus B_j, B_j) \in \tau_1$ and $\overleftarrow{s_2} := (F(G) \setminus A_i, A_i) \in \tau_2$. For that let us look at a sequence of separations $\overrightarrow{r_i} := (A \cup B_1 \cup ... \cup B_i, B_{i+1} \cup ... \cup B_m)$. If $\overrightarrow{r_m} \in \tau_1$ then $\overrightarrow{s_1} = \overrightarrow{r_m}$ otherwise there is a smallest i such that $\overleftarrow{r_i} \in \tau_1$ but $\overrightarrow{r_{i-1}} \in \tau_1$ if we can show that $|r_i| \leq |r|$. This can be seen by calculating

$$|r_i| = \sum_{j=i+1}^{m} |(F(G) \setminus B_j, B_j)| \leq \sum_{j=1}^{m} |(F(G) \setminus B_j, B_j)| = |r|.$$

Now that we have shown that there is a smallest i such that $\overleftarrow{r_i} \in \tau_1$ but $\overrightarrow{r_{i-1}} \in \tau_1$, suppose $\overrightarrow{s_1} := (F(G) \setminus B_i, B_i) \notin \tau_1$ then $(B_i, F(G) \setminus B_i) \in \tau_1$. This would mean that $\sigma := \{\overleftarrow{s_1}, \overleftarrow{r_i}, \overrightarrow{r_{i-1}}\} \subseteq \tau_1$. Now we can calculate that

$$\cap \sigma = (B_i \cup ... \cup B_m) \cap (A \cup B_1 \cup ... \cup B_i) \cap (F(G) \setminus B_i) = \emptyset.$$

Since that would mean that σ is a forbidden star it must follow that $\overrightarrow{s_1} \in \tau_1$. Showing $\overleftarrow{s_2} \in \tau_2$ works analogously.
Both s_1 and s_2 distinguish τ_1 and τ_2 and hence since r distinguished τ_1 and τ_2 efficiently it must follow that $|s_1| = |s_2| = |r|$. Since $|r| = \sum_{i=1}^{n} |(F(G) \setminus A_i, A_i)| = \sum_{j=1}^{m} |(F(G) \setminus B_j, B_j)|$, this can only be the case if $B = B_j$ and $A = A_i$ as the order of every nontrivial separation is greater than zero and thus we know that r is in fact simple. □

Corollary 4.3. *There is a nested set $S_T \subseteq \overrightarrow{S}$ of simple separations such that:*

(i) every two maximal tangles are efficiently distinguished by some simple separation in S_T;

(ii) every separation in S_T efficiently distinguishes a pair of maximal tangles;

4.2 S_T-trees

Let us look some more at these nested sets of separations and what these have to do with trees. This will help us further understand the tree-of-tangles theorem. So suppose we have been given a nested set of separations S_T as in the tree-of-tangles theorem. The definition of an S_T-tree is taken from [6].

Definition 4.4. *An S_T-tree is a pair (H, α) of a tree H and a bijective map $\alpha : \overrightarrow{E(H)} \to \overrightarrow{S_T}$ such that*

- *α is order-preserving, where the edges of the tree are ordered such that $(e_1, v_1, w_1) \leq (e_2, v_2, w_2)$ if the unique path in H connecting v_1 to w_2 passes through w_1 and v_2,*
- *$\alpha(e, v, w)^* = \alpha(e, w, v)$.*

We are now going to define such a tree H and a map α for the S_T we got from the tree-of-tangles theorem.

Definition 4.5. *For the nested set S_T let H have vertex set $V(H)$ defined as the consistent orientations of S_T and let the edge set $E(H)$ be sets consistent orientations which only differ in one separation. Let $\alpha : \overrightarrow{E} \to \overrightarrow{S_T}$ map (e, τ_1, τ_2) to the distinguishing separation $\vec{s}$ if $\vec{s} \in \tau_1$ and otherwise to $\overleftarrow{s}$.*

Lemma 4.6. *(H, α) is an S_T-tree.*

Proof. We first need to show that H is a tree. By definition H is a graph. What we need to show is that H is connected and circle free. We first show that we are circle free, for that consider a circle made of the edges $e_1, ..., e_n$. Then let us consider the edge $e_1 = v_1v_2$ and the path $P = v_2e_2v_3...e_nv_1$ connecting v_2 to v_1 in the circle avoiding e_1. Since v_1 and v_2 differ in the separation s. There must exist v_i and v_{i+1} which only differ in s, since otherwise the orientation of s would not be changed in any edge. This is not possible since $\vec{s}$ forces every separation smaller than $\vec{s}$ to be in v_1 and $\overleftarrow{s}$ forces every separation smaller than $\overleftarrow{s}$, hence greater than $\vec{s}$ to lie in v_2 hence also in v_1. This means that s already completely determines v_1 and v_2.
Now we need to show that H is connected. For that let us consider orientations τ_1 and τ_2 which differ in a minimal number of separation while still not being connected. Then we choose the largest separation $\vec{s}$ in τ_1 which distinguishes τ_1 and τ_2. We define $\tau_3 := \tau_1 \cup \{\overleftarrow{s}\} \setminus \vec{s}$. Since $\overleftarrow{s} \in \tau_2$ we know that $\overleftarrow{s}$ is consistent in τ_3 with all of the separations which do not differ between τ_1 and τ_2 and since we chose $\vec{s}$ maximal, we are also consistent with the separations which distinguish τ_1 and τ_2. Hence τ_3 is consistent and we have a contradiction.
Next we need to show that α is bijective. First we show that α is injective. This can be easily seen from the fact that α maps different orientations of the same edge to different separations and α must map different edges to different separations since we have already shown that each edge has a unique separation distinguishing the two orientations in the proof that H is a tree. Next we need to show that α is also surjective. For that suppose we want to find an edge which is mapped by α to a separation s. Then s must distinguish two orientations τ_1, τ_2 of S_T by definition of S_T but since H is a tree τ_1 and τ_2 are connected. Hence since τ_1 and τ_2 disagree on s at some edge in the path connecting τ_1 and τ_2 there must exist two orientations which disagree only on s. The edge between them is mapped to s and so we have shown that α is bijective.
Next we need to show that α is order-preserving. For that it suffices to show that given vertices u, v, w with $uv \in E(H)$ an $vw \in E(H)$ it follows that $\vec{s} := \alpha(uv, u, v) \leq \alpha(vw, v, w) =: \vec{r}$. We know that $\vec{s} \in v$ and $\overleftarrow{r} \in v$, $\overleftarrow{s} \in u$ and $\overleftarrow{r} \in u$ and finally $\vec{s} \in w$ and $\vec{r} \in w$. We also know that s and r are nested this just leaves the option that $\vec{s} \leq \vec{r}$ since otherwise r and s would be pointing away from each other in one of the orientations.
Finally $\alpha(e, v, w)^* = \alpha(e, w, v)$ can be seen from the definition of α. □

With this tree in mind we can see that every tangle τ can be identified with a vertex v_τ of H. This is because a tangle $\tau \in T$ orients every separation in S it also orients every separation in S_T. Since every orientation of edges of a tree has a sink, a vertex which every edge points to, we define v_τ as this sink. The edges $\overrightarrow{E}(V(H), v_\tau)$ are mapped by α to a set of separations which we define as σ_τ.

Lemma 4.7. *For every $\tau \in T$ the set σ_τ is a star of cardinality $d(v)$ that is contained in τ.*

Proof. The only thing we have to show is that σ_τ is a star. For that let us consider two separations $\vec{r}, \vec{s} \in \sigma_\tau$. Then there exist oriented edges with $\alpha(e_r, v_r, v_\tau) = \vec{r}$ and $\alpha(e_s, v_s, v_\tau) = \vec{s}$. But since $(e_r, v_r, v_\tau) \leq (e_s, v_\tau, v_s)$ it follows that $\alpha(e_r, v_r, v_\tau) \leq \alpha(e_s, v_\tau, v_s)$ hence $\vec{r} \leq \overleftarrow{s}$. □

Note that σ_τ cannot be contained in $\mathcal{F}_k^\varepsilon$ since τ is a tangle.

Lemma 4.8. *For two non-equal tangles $\tau, \rho \in T$ the sets $\cap\sigma_\tau$ and $\cap\sigma_\rho$ are disjoint.*

Proof. Since τ and ρ are distinct there exists a separation $s \in S_T$ which distinguishes them, without loss of generality let $\vec{s} \in \tau$ and $\overleftarrow{s} \in \rho$. Then, by definition of σ_τ there exists an $\vec{s_\tau} \in \sigma_\tau$ with $\vec{s_\tau} \geq \vec{s}$. Similarly there exists an $\vec{s_\rho} \in \sigma_\rho$ with $\vec{s_\rho} \geq \overleftarrow{s}$. Suppose $\vec{s} = (A, B)$, then by the previous argument we know that $\cap\sigma_\tau \subseteq B$ and $\cap\sigma_\rho \subseteq A$. Since A and B are disjoint the sets $\cap\sigma_\tau$ and $\cap\sigma_\rho$ are also disjoint. $\square$

We will use these lemmas later to look more at the structure of the tangles.
We have seen that every tangle can be mapped to a vertex in H but is the other direction also true? Can every vertex in H be mapped to a tangle in S_T? In general this is not true, but if we assume our nested set from the tree-of-tangle theorem to be of minimal cardinality, then it does become true.

Lemma 4.9. *Let $S_T \subseteq S$ be a smallest nested set of separation such that every pair of tangles in T is efficiently distinguished by some separation in S_T and each separation efficiently distinguishes a pair of tangles of T. If (H, α) is an S_T-tree then for every vertex v of H there exists a tangle $\tau \in T$ with $v = v_\tau$.*

Proof. Suppose there is a vertex $v \in V(H)$ such that there is no tangle τ for which $v = v_\tau$. If we delete the vertex v then H is separated into $d(v)$ many components $H_1, ..., H_{d(v)}$. Each component H_i is connected with v by an edge e_i that is mapped by α to a separation s_i. Let $j \in \{1, ..., d(v)\}$ be a natural number such that $|\alpha(e_j)| \geq |\alpha(e_i)|$ for all $i \in \{1, ..., d(v)\}$. We hope to prove that s_j can in fact be removed. This can be shown in the following way: Let $\tau, \rho \in T$ be two tangles that were efficiently distinguished by s_j. Without loss of generality $v_\tau \in H_j$ as otherwise s_j would not be distinguishing τ and ρ and $v_\rho \in H_k$ with $k \neq j$ as v_ρ can not be v. This now shows that s_k also distinguishes τ and ρ and since $|s_k| \leq |s_j|$ it also does so efficiently. This proves that after deleting s_j we can still distinguish every pair of tangles efficiently and every separation still efficiently distinguishes two tangles. This is a contradiction to the minimality of the nested set. $\square$

4.3 Tangle-tree Duality

In this section we will find a way to determine whether we find a tangle and if we do not find any tangles in a triangulation we will find a tree structure which witnesses this. Most of the definitions and the main theorem of this section are taken from [6].

Definition 4.6. *Let $\vec{S}$ be a separation system, $\vec{r} \in \vec{S}$ nontrivial and $\vec{s_0} \in \vec{S}$ with $\vec{r} \leq \vec{s_0}$.*

$$S_{\geq \vec{r}} := \{s \in S | \vec{s} \geq \vec{r} \, or \overleftarrow{s} \geq \vec{r}\}$$

Then if $\vec{s} \in \vec{S}_{\geq \vec{r}}$ and $\vec{s} \geq \vec{r}$ we define $f_{\vec{s_0}}^{\vec{r}} : \vec{S}_{\geq \vec{r}} \to \vec{U}$ as $f_{\vec{s_0}}^{\vec{r}}(\vec{s}) := \vec{s} \vee \vec{s_0}$ and $f_{\vec{s_0}}^{\vec{r}}(\overleftarrow{s}) := (\vec{s} \vee \vec{s_0})^*$

We call $f_{\vec{s_0}}^{\vec{r}}$ the shifting map. We imagine it to shift the separations over to $\vec{s_0}$.

Definition 4.7. *Let $\mathcal{F}$ be a set of stars then $\vec{s_0} \in \vec{S}$ emulates $\vec{r} \in \vec{U}$ in $\vec{S}$ for $\mathcal{F}$ if*

- *$\vec{s_0} \geq \vec{r}$;*
- *every $\vec{s} \in \vec{S}$ with $\vec{s} \geq \vec{r}$ satisfies $\vec{s} \vee \vec{s_0} \in \vec{S}$;*
- *every star in $\mathcal{F}$ that has an element $\vec{s} \geq \vec{r}$ satisfies $f_{\vec{s_0}}^{\vec{r}}(\sigma) \in \mathcal{F}$.*

A separation $\vec{r} \in \vec{S}$ is said to be forced by $\mathcal{F}$ if $\{\overleftarrow{r}\} \in \mathcal{F}$.

Definition 4.8. *A separation system $\vec{S}$ is $\mathcal{F}$-separable if for every two nontrivial $\vec{r}, \overleftarrow{r'} \in \vec{S}$ that are not forced by $\mathcal{F}$ and satisfy $\vec{r} \leq \vec{r'}$ there exists an $s_0 \in S$ with an orientation $\vec{s_0}$ that emulates $\vec{r}$ in $\vec{S}$ for $\mathcal{F}$ and such that $\overleftarrow{s_0}$ emulates $\overleftarrow{r'}$ in $\vec{S}$ for $\mathcal{F}$.*

Lemma 4.10. *For a triangulation G the separation system $\vec{S_k}(G)$ is $\mathcal{F}_k^\varepsilon$-separable.*

Proof. Let $\vec{r}, \overleftarrow{r'} \in \vec{S_k}$ with $\vec{r} \leq \vec{r'}$. We now define $s_0 \in S_k$ as a separation which fulfills $\vec{s_0} \geq \vec{r}$ and $\overleftarrow{s_0} \geq \overleftarrow{r'}$, we say a good separation, with minimal order. Such a good separation exists since $\vec{s_0} = \vec{r}$ is good. Now we want to show that for all $\vec{s} \in \vec{S_k}$ with $\vec{s} \geq \vec{r}$ it follows that $\vec{s} \vee \vec{s_0} \in \vec{S_k}$. Suppose $\vec{s} \vee \vec{s_0} \notin \vec{S_k}$, in that case we define $\vec{t} := \vec{s} \wedge \vec{s_0}$. Since $\vec{r} \leq \vec{s}$ and $\vec{r} \leq \vec{s_0}$ it follows that $\vec{r} \leq \vec{t}$. It also follows from $\vec{s_0} \leq \vec{r'}$ that $\vec{t} \leq \vec{r'}$. Hence $\vec{t}$ is a good separation. Since $\vec{s} \vee \vec{s_0} \notin \vec{S_k}$ it follows that $|\vec{s} \vee \vec{s_0}| \geq k$.

$$|\vec{t}| + |\vec{s} \vee \vec{s_0}| = |\vec{s} \wedge \vec{s_0}| + |\vec{s} \vee \vec{s_0}| \leq |\vec{s}| + |\vec{s_0}| < k + |\vec{s_0}|$$

This gives us the inequality $|\vec{t}| < k + |\vec{s_0}| - |\vec{s} \vee \vec{s_0}| \leq |\vec{s_0}|$. This is a contradiction as we chose $\vec{s_0}$ with minimal order.
Now we show that $\vec{s_0}$ emulates $\vec{r}$ in $\vec{S_k}$ for $\mathcal{F}_k^\varepsilon$. By definition $\vec{s_0} \geq \vec{r}$ and every $\vec{s} \in \vec{S_k}$ with $\vec{s} \geq \vec{r}$ satisfies $\vec{s} \vee \vec{s_0} \in \vec{S_k}$. Now let $\sigma \in \mathcal{F}_k^\varepsilon$ be a star that has an element $\vec{s} \geq \vec{r}$. We want to show $f_{\vec{s_0}}^{\vec{r}}(\sigma) \in \mathcal{F}_k^\varepsilon$. Let us first look at the case where $|\sigma| = 1$. In that case $\sigma = \{\vec{u}\}$. Since $\vec{r'}$ is not forced by $\mathcal{F}_k^\varepsilon$ we know that $\vec{u} \geq \vec{s_0}$ and so $f_{\vec{s_0}}^{\vec{r}}(\sigma) = \sigma \in \mathcal{F}_k^\varepsilon$. Now we will look at the case $|\sigma| \in \{2, 3\}$. In this case there is an element $\vec{t_1} \in \sigma$ with $\vec{t_1} \geq \vec{r}$ and elements $\vec{t_2}, \vec{t_3} \in \sigma$ with $\overleftarrow{t_2} \geq \vec{r}$ and $\overleftarrow{t_3} \geq \vec{r}$ such that $\sigma = \{\vec{t_1}, \vec{t_2}, \vec{t_3}\}$. Let $\vec{t_j} = (A_j, B_j)$ and $\vec{s_0} = (C, D)$.

$$f_{\vec{s_0}}^{\vec{r}}(\vec{t_1}) = (\vec{t_1} \vee \vec{s_0}) = (A_1 \cup C, B_1 \cap D) := \vec{t_1'}$$

$$f_{\vec{s_0}}^{\vec{r}}(\vec{t_2}) = (\overleftarrow{t_2} \vee \vec{s_0})^* = (A_2 \cap D, B_2 \cup C) := \vec{t_2'}$$

$$f_{\vec{s_0}}^{\vec{r}}(\vec{t_3}) = (\overleftarrow{t_3} \vee \vec{s_0})^* = (A_3 \cap D, B_3 \cup C) := \vec{t_3'}$$

$\sigma' := \{\vec{t_1'}, \vec{t_2'}, \vec{t_3'}\}$ is still a star and $\cap\sigma' = (B_1 \cap D) \cap (B_2 \cup C) \cap (B_3 \cup C) \subseteq B_1 \cap B_2 \cap B_3 = \cap\sigma$. Hence we know that $\sigma' \in \mathcal{F}_k^\varepsilon$. With that we have shown that $\vec{s_0}$ emulates $\vec{r}$ in $\vec{S_k}$ for $\mathcal{F}_k^\varepsilon$.
The proof that $\overleftarrow{s_0}$ emulates $\overleftarrow{r'}$ in $\vec{S_k}$ for $\mathcal{F}_k^\varepsilon$ follows in the same way. □

Definition 4.9. *The set of stars $\mathcal{F}$ is standard for $\vec{S}$ if it contains the stars $\{\overleftarrow{r}\}$ for all trivial $\vec{r} \in \vec{S}$.*

Lemma 4.11. *$\mathcal{F}_k^\varepsilon$ is standard for $\vec{S_k}(G)$.*

Proof. Let $\vec{r} = (A, B)$ be a trivial element of $\vec{S_k}(G)$. Then there must be a $(C, D) \in \vec{S_k}$ such that $(A, B) < (C, D)$ and $(A, B) < (D, C)$, this implies $A = \emptyset$ and $B = F(G)$. That means that $\cap\{\overleftarrow{r}\} = \cap\{(B, A)\} = \emptyset$ and thus $\{\overleftarrow{r}\} \in \mathcal{F}_k^\varepsilon$. □

This allows us to show the following tangle-tree duality theorem, proven in [6], for our setup.

Theorem 4.2. *Let $(\vec{U}, \leq, ^*, \vee, \wedge)$ be a universe of separations containing a finite separation system $(\vec{S}, \leq, ^*)$. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$. If $\vec{S}$ is $\mathcal{F}$-separable, exactly one of the following assertions holds:*

(i) There exists an $\mathcal{F}$-tangle of S.

(ii) There exists an S-tree over $\mathcal{F}$.

We can now use this theorem for the $\mathcal{F}_k^\varepsilon$-tangles.

Corollary 4.4. *For $k \in \mathbb{N}$ either there exists an $\mathcal{F}_k^\varepsilon$-tangle or there exists an $\vec{S_k}$-tree over $\mathcal{F}_k^\varepsilon$.*

This tells us that if we do not have a tangle in G for a certain k it is because of this $\vec{S_k}$-tree and if we can show that we can not have this $\vec{S_k}$-tree, we know that there must be a tangle.

5 Tangles of the disk

We will now use the definition of tangles on triangulations to define tangles on the disk D. The intuition behind this is that a tangle of the disk D would orient any separation of low order. We will not use every separation of the disk though as there could be infinitely many, instead for an arbitrary triangulation G of D every separation of $S_k(G)$ would be oriented by one of those tangles. This motivates us to define a tangle as a map τ that maps every triangulation G of D to a tangle on G. First we need to look at the set of triangulations of G. For that we define

Definition 5.1. $\varepsilon(G) := \varepsilon - \Delta(D \setminus G)$.

Definition 5.2. *A system of triangulations D^Δ is a set of pairs (G, w_G), sometimes abbreviated as G, of a triangulation G and a weight function w_G such that*

- *For two triangulations G and G' where G' is finer than G the weight function have to fulfill for every $e \in E(G)$ that $w_G(e) = w_{G'}(E')$ where $E' := \{e' \in E(G') | e' \subseteq e\}$. We call this property consistency.*
- *For two pairs $(G_1, w_{G_1}), (G_2, w_{G_2}) \in D^\Delta$ there exists a pair $(G', w_{G'}) \in D^\Delta$ with G' being finer than both G_1 and G_2.*
- *For any $\varepsilon' > 0$ there exists a $(G, w_G) \in D^\Delta$ such that $\Delta(D \setminus G) < \varepsilon'$.*
- *$\varepsilon(G) > 0$ for all $G \in D^\Delta$.*

We say that for $G_1, G_2 \in D^\Delta$ if G_2 is finer than G_1, that $G_1 \leq G_2$.

Lemma 5.1. *A system of triangulations D^Δ is a directed partially ordered set, ordered by $\leq$.*

Proof. The set being directed can instantly be seen from the definition hence it suffices to show that $\leq$ is a partial order. First we show that $\leq$ is reflexive. Let $(G, w_G) \in G^\Delta$, since $V(G) = V(G)$ and $E(G) = E(G)$ we know that G is finer than G and since the edges of $E(G)$ are disjoint we know that $E' = \{e' \in E(G) | e' \subseteq e\} = \{e\}$ and thus $w_G(e) = w_G(E')$. Next we show that $\leq$ is antisymmetric. So let $(G_1, w_{G_1}), (G_2, w_{G_2}) \in D^\Delta$ be two triangulations and $G_1 \leq G_2$ and $G_2 \leq G_1$. This means that $V(G_1) = V(G_2)$ and $\bigcup(E(G_1)) = \bigcup(E(G_2))$. If $E(G_1)$ and $E(G_2)$ were not equal, that would mean that the end of one edge would lie inside the interior of another edge which is forbidden since edges only intersect the vertex set at their ends. As before we see that $w_{G_1}(e) = w_{G_2}(E') = w_{G_2}(e)$ and thus the weight functions are equal. Now we just need to show the transitivity. Let $(G_1, w_{G_1}), (G_2, w_{G_2}), (G_3, w_{G_3}), \in D^\Delta$ and $G_1 \leq G_2$ as well as $G_2 \leq G_3$. It follows from $V(G_1) \subseteq V(G_2)$, $V(G_2) \subseteq V(G_3)$, $\bigcup(E(G_1)) \subseteq \bigcup(E(G_2))$ and $\bigcup(E(G_2)) \subseteq \bigcup(E(G_3))$ that $V(G_1) \subseteq V(G_3)$ and $\bigcup(E(G_1)) \subseteq \bigcup(E(G_3))$. Hence G_3 is finer than G_1. Thus we have shown that $\leq$ is indeed a partial order. $\square$

What we need to do is given two triangulations $G, G' \in D^\Delta$ where G' is finer than G and given a separation (A, B) of $\vec{S}(G)$ we want to find a separation (X, Y) of $\vec{S}(G')$ which in some way has the same properties. For that we need to assign a side for every face of G'. Some faces of G' are contained inside of a face of G so we can simply use the side assigned to that face in G but some faces of G' lie outside of G. To deal with this case we first consider a face f of G which is bounded by an edge e_f which lies on the boundary of G. Then there exists a component, which we will from now on call $C(f)$, of $D \setminus G$ which also contains e_f in its boundary. If a face f does not have an edge which lies on the boundary then we define $C(f) := \emptyset$. It is not hard to see that the $C(f)$ are pairwise disjoint for $f \in F(G)$ and that every interior point of D is contained in either a face of G or a $C(f)$ for a face $f \in F(G)$. This means that every face of G' lies completely in one of the generalized faces $f \cup C(f)$ for an $f \in F(G)$.

Definition 5.3. *For triangulations $G, G' \in D^{\Delta}$ of a system of triangulations with G' finer than G we define $\iota_G^{G'} : \vec{S}(G) \to \vec{S}(G')$ which maps a separation (A, B) to the separation (X, Y) where $X := \{f \in F(G') | \exists a \in A : f \subseteq a \cup C(a)\}$ and $Y := \{f \in F(G') | \exists b \in B : f \subseteq b \cup C(b)\}$.*

Lemma 5.2. *$\iota_G^{G'}$ fulfills the following properties:*

(i) $\iota_G^{G'}$ is well defined.

(ii) $\iota_G^{G'}(\overleftarrow{s}) = \iota_G^{G'}(\overrightarrow{s})^*$. Implying that if $s \in S(G)$ then $\iota_G^{G'}(s) \in S(G')$.

(iii) $|\iota_G^{G'}(s)| = |s|$.

(iv) $\iota_G^{G'}(\overrightarrow{s}) \leq \iota_G^{G'}(\overrightarrow{t}) \Leftrightarrow \overrightarrow{s} \leq \overrightarrow{t}$.

(v) $\iota_G^{G'}$ is injective.

(vi) If we have G'' with $G'' \geq G'$ then we have $\iota_G^{G''} = \iota_{G'}^{G''} \circ \iota_G^{G'}$.

Proof. The points (ii), (iv), (v) and (vi) can be easily seen from the definition.

(i) We have to show, that (X, Y) is indeed a separation. For that we first need to show that $X \cap Y = \emptyset$. If that were not the case there would be a $f \in F(G')$ such that there is an $a \in A$ with $f \subseteq a \cup C(a)$ and $b \in B$ with $f \subseteq b \cup C(b)$. That would mean that $a \cap b \neq \emptyset$ since $b \cap C(a)$, $a \cap C(b)$ and $C(a) \cap C(b)$ are empty by definition and since a and b are faces in $F(G)$ this means that $a = b$ contradicting $A \cap B = \emptyset$. To show that $X \cup Y = F(G')$ it suffices to see that every face of $F(G')$ is a subset of a generalized face $f \cup C(f)$ of $F(G)$.

(iii) We have to show that $w_{G'}(\partial X) = w_G(\partial A)$. For that if suffices by the definition of systems of triangulations to show that $\bigcup \partial X = \bigcup \partial A$. This can be seen since a point $p \in \mathbb{R}^2$ lies on the boundary of A and B if it contains a point of a face in A and a point of a face of B in every neighbourhood, but that means that it also contains points of faces of X and Y in every neighbourhood. For the other direction, if we have a point p on an edge which lies both on the boundary of X and Y then there must be faces $x \in X$ and $y \in Y$ on whose boundary p lies but x and y both must lie in faces of G. This can be seen from the fact that if y lied in $C(b)$ for a $b \in B$ then x would need to lie in b since that is the only face which shares an edge on its boundary with $C(b)$ but then x would already be in b. Similarly we can see that x also cannot lie inside of a $C(a)$ for an $a \in A$. This shows that p already lies in $\bigcup \partial A$.

□

Definition 5.4. *Let G and G' be triangulations of D with G' finer than G then an $\mathcal{F}_k^{\varepsilon}$-tangle τ' of G' induces an $\mathcal{F}_k^{\varepsilon}$-tangle τ of G if for every separation $\overrightarrow{s} \in \tau$ the separation $\iota_G^{G'}(\overrightarrow{s}) \in \tau'$.*

Lemma 5.3. *Let $(G, w_G), (H, w_H) \in D^{\Delta}$ be two triangulations with $G \leq H$ then every $\mathcal{F}_k^{\varepsilon(H)}$-tangle τ of H induces a $\mathcal{F}_k^{\varepsilon(G)}$-tangle on G.*

Proof. We prove this by defining the tangle ρ of G in the following way. If we have a separation $s \in S_k(G)$ then we have a separation $s' := \iota_G^H(s) \in S(H)$. This separation s' has the same order as s and thus s' is oriented by the $\mathcal{F}_k^{\varepsilon}$-tangle τ towards the orientation $\overrightarrow{s'}$. Now this means that there is a separation $\overrightarrow{s} \in s$ such that $\iota_G^H(\overrightarrow{s}) = \overrightarrow{s'}$. We define $\overrightarrow{s} \in \rho$. Now we need to show that ρ is indeed a $\mathcal{F}_k^{\varepsilon}$-tangle. First we show that ρ is an orientation. This can be seen from the fact that from every separation we have chosen one orientation to put into ρ. Next we show that ρ is consistent. Suppose ρ were not consistent. Then there are distinct $r, s \in \rho$ with $\overrightarrow{r} < \overrightarrow{s}$ such that $\overleftarrow{r}, \overrightarrow{s} \in \rho$. But that would mean that $\iota_G^H(\overrightarrow{r}) < \iota_G^H(\overrightarrow{s})$ while $\iota_G^H(\overrightarrow{r})^*, \iota_G^H(\overrightarrow{s}) \in \tau$. That is a contradiction as this would mean that τ is not consistent. Now we just need to show, that there is no forbidden star in ρ. For that is suffices to show, that if we had a forbidden star $\sigma' \subseteq \rho$ we have a star $\sigma \subseteq \tau$ such that $\cap\sigma' \setminus (H \setminus G) \subseteq \cap\sigma$ as this implies

$$\Delta(\cap\sigma') \leq \Delta(\cap\sigma) - (\Delta(H) - \Delta(G)) \leq \varepsilon - \Delta(D \setminus G) - (\Delta(H) - \Delta(G))$$

$$\leq \varepsilon - \Delta(D \setminus H) = \varepsilon(H)$$

which would be a contradiction. We define $\sigma := \iota_G^H(\sigma')$. Since if $\iota_G^H(A, B) = (X, Y)$ if follows that $\bigcup X \cap G = \bigcup A$ and $\bigcup Y \cap G = \bigcup B$ we get that $\cap\sigma' \setminus (H \setminus G) \subseteq \cap\sigma$. $\square$

Now we have everything that we need to define tangles on D given a system of triangulations D^Δ.

Definition 5.5. *Let D be a disk then an $\mathcal{F}_k^\varepsilon$-tangle on D is a map τ from the triangulations $G \in D^\Delta$ of D to a tangle $\tau(G) = \tau_G$ where τ_G is a $\mathcal{F}_k^{\varepsilon(G)}$-tangle on G. Where if G_1 and G_2 are triangulations of D where G_1 is coarser than G_2 then $\tau(G_2)$ must induce $\tau(G_1)$.*

What follows from this is that if we have two triangulations G_1 and G_2 then there is a triangulation G_3 that is finer than both G_1 and G_2 so that $\tau(G_3)$ induces both $\tau(G_1)$ and $\tau(G_2)$. This is in line with the intuition that this is supposed to be one tangle that orients the separations of every triangulation.

5.1 Tree-of-tangles Theorem

We now apply the tree of tangle theorem to tangles of D and use the insights from that to find out more about the structure of the tangles.
We can easily see that a $\mathcal{F}_k^\varepsilon$-tangle of D induces a $\mathcal{F}_\ell^\varepsilon$-tangle of D for $k \geq \ell$ as they are induced in every triangulation. This allows us to define a set of maximal tangles T as a set of tangles which are maximal in T which means that each tangle in T is not induced by a different tangle in T.

Theorem 5.1. *Let T be a finite set of maximal tangles of D then there is a nested set S_T of separations of a triangulation G such that*

(i) any two tangles in T are distinguished by some separation in S_T;

(ii) every separation in S_T distinguishes a pair of tangles in T.

We say that G witnesses the tangles T. Two tangles τ_1 and τ_2 in T are distinguished by a separation $s \in S_T$ if s distinguishes $\tau_1(G)$ and $\tau_2(G)$.

Proof. We start by constructing the triangulation G that contains enough separations to allow us to apply the tree of tangle theorem. For the theorem to work we need to be able to distinguish any two tangles in T with separations in G. We start by noting that for any two distinct tangles $\tau_1, \tau_2 \in T$ there exists a triangulation G_{τ_1,τ_2} such that $\tau_1(G_{\tau_1,\tau_2})$ and $\tau_2(G_{\tau_1,\tau_2})$ are different. In that triangulation G_{τ_1,τ_2} there must be a separation $s \in S(G_{\tau_1,\tau_2})$ that distinguishes $\tau_1(G_{\tau_1,\tau_2})$ and $\tau_2(G_{\tau_1,\tau_2})$. There must a triangulation G that is finer than all of these finitely many G_{τ_1,τ_2} and since there exists a separation $s \in S(G_{\tau_1,\tau_2})$ that separates $\tau_1(G_{\tau_1,\tau_2})$ from $\tau_2(G_{\tau_1,\tau_2})$ in $S(G)$ the separation $\iota^G_{G_{\tau_1,\tau_2}}(s)$ distinguishes $\tau_1(G)$ from $\tau_2(G)$.
Now the rest of the proof follows instantly. The tangles $T(G)$ are pairwise distinct $\mathcal{F}_k^{\varepsilon(G)}$-tangles of G. Thus there exists a nested set S_T of separations such that every two tangles of $T(G)$ are efficiently distinguished by some separation in S_T and every separation in S_T efficiently distinguishes a pair of tangles in $T(G)$. This is exactly what we wanted to show. $\square$

Note that from the proof follows that the separations that distinguish the tangles in G are in fact efficient in $S(G)$.

Our next goal is to show that there can only be finitely many tangles. This is important to show as it would be very counterintuitive if in a finite space there would be infinitely many regions of high cohesion which each are not contained within a small area in some sense since they do not contain a forbidden star. That would in fact lead us to question the definitions we have made in this thesis. For that goal let us first proof a basic result about trees to later apply it to the tree of tangles.

Lemma 5.4. *In a tree H with k edges there are at least $\frac{1}{2}k$ vertices with degree at most 2.*

Proof. We know that the sum of all edge degrees is

$$\sum_{v \in V(H)} d(v) = 2k.$$

Now let m be the number of vertices with degree ≤ 2 then

$$2k = \sum_{v \in V(H)} d(v) \geq m \cdot 1 + (k + 1 - m) \cdot 3 = 3k + 3 - 2m \Leftrightarrow$$

$$2k \geq 3k + 3 - 2m \Leftrightarrow 2m \geq k + 2 \Leftrightarrow m \geq \frac{1}{2}k.$$

□

With this preparation we can prove that there can only be a finite number $\mathcal{F}_k^\varepsilon$-tangles of D, in fact we can prove an upper bound for the number of tangles.

Lemma 5.5. *For every finite set T of maximal tangles of D it is valid that $|T| \leq \frac{4\Delta D}{\varepsilon}$.*

Proof. Let G witness the tangles of T. Without loss of generality we can assume that $\varepsilon(G) > \frac{\varepsilon}{2}$ since otherwise we could find a triangulation G' with $\varepsilon(G') > \frac{\varepsilon}{2}$ and thus find a triangulation G'' which is both finer than G and G' which means that it both witnesses the tangles T as well as having $\varepsilon(G'') > \frac{\varepsilon}{2}$.
Let S_T be the nested set of separation from $S_k(G)$ that we get from the tree-of-tangle theorem and let (H, α) be the S_T-tree. Now let us look at all of the vertices $V' \subseteq V(H)$ of H that have degree ≤ 2. For each of these vertices $v \in V'$ there exists a tangle $\tau \in T$ such that $v = v_\tau$. Let us call the set of these tangles $T' \subseteq T$. Since $d(v) \leq 2$ we know that $|\sigma_\tau| \leq 2$ and thus since $\sigma_\tau \subseteq \tau$ we know that $\Delta(\cap\sigma_\tau) \geq \varepsilon(G) \geq \frac{\varepsilon}{2}$. Since the sets $\cap\sigma_\tau$ are disjoint for all $\tau \in T'$ we know by

$$\Delta F(G) \geq \sum_{\tau \in T'} \Delta(\cap\sigma_\tau) \geq |T'|\frac{\varepsilon}{2} \Leftrightarrow |T'| \leq \frac{2\Delta F(G)}{\varepsilon}.$$

and $|T'| \leq |T| \leq |V(H)| \leq 2|T'|$ that $|T| \leq \frac{4\Delta F(G)}{\varepsilon}$. Hence since $F(G) \subseteq D$ it follows that $\Delta F(G) \leq \Delta D$. Thus $|T| \leq \frac{4\Delta D}{\varepsilon}$. □

This allows us to formulate the tree-of-tangles theorem for tangles of D.

Theorem 5.2. *Let T be a set of maximal tangles of a system of triangulations D^Δ then there is a nested set S_T of simple separations of a triangulation $G \in D^\Delta$ such that*

(i) any two tangles in T are distinguished by some separation in S_T;

(ii) every separation in S_T distinguishes a pair of tangles in T.

Proof. We just need to show that T is finite. Suppose T was infinite then we can find an arbitrarily large finite subset of T. Suppose we chose a subset T_{fin} of size $|T_{fin}| \geq \frac{4\Delta D}{\varepsilon}$. Then this set T_{fin} would contradict Lemma 5.5. Hence we have proven this theorem. □

5.2 Tangle-tree Duality

In this section we find a result similar to the tangle-tree duality for the tangles of D. To understand the consequences of an $\mathcal{F}_k^{\varepsilon}$-tangle not existing for a certain k we instead first look at what is necessary for it to exist.

Lemma 5.6. *If we have a system of triangulations D^{Δ} and for every triangulation $G \in D^{\Delta}$ there exists an $\mathcal{F}_k^{\varepsilon(G)}$-tangle then there exists an $\mathcal{F}_k^{\varepsilon}$-tangle of D^{Δ}.*

Proof. We use the generalized infinity-lemma to show this lemma. Let X_G be the set of all $\mathcal{F}_k^{\varepsilon(G)}$-tangles on $G \in D^{\Delta}$ then we have a family of non-empty finite sets $(X_G)_{G \in D^{\Delta}}$. The system of triangulations is a directed poset. Now we define maps $f_{GH} : X_G \to X_H$ for all $G \geq H$ where we map a tangle of X_G to its induced tangle in X_H. This map is compatible as $f_{GH} \circ f_{HI} = f_{GI}$. Then we get from the generalized infinity-lemma a tangle T_G for every $G \in D^{\Delta}$ such that $f_{GH}(T_G) = T_H$ for $G \geq H$, or in other words: T_H is induced by T_G. $\square$

This means that if we do not have an $\mathcal{F}_k^{\varepsilon}$-tangle of D^{Δ} that there exists a triangulation $G \in D^{\Delta}$ that does not have a tangle. By tangle-tree duality this means that there is an $\vec{S_k}(G)$-tree over $\mathcal{F}_k^{\varepsilon(G)}$.

Theorem 5.3. *Let D^{Δ} be a system of triangulations then for every $\varepsilon > 0$ and $k \in \mathbb{N}$ exactly one of the following assertions holds:*

(i) There exists an $\mathcal{F}_k^{\varepsilon}$-tangle of D^{Δ}.

(ii) There exists a triangulation $G \in D^{\Delta}$ with an $\vec{S_k}(G)$-tree over $\mathcal{F}_k^{\varepsilon(G)}$.

6 Geometry of a circle

In this section we prepare for an important theorem in the next section. What we will want to show is that a circle is difficult to cut apart in some sense. More specifically we want to show that given a length $k > 0$ and a circle of radius $R := \frac{k}{2} + \delta$ with $\delta > 0$ then this circle can not be "covered easily". What we mean by this is that for every $\delta > 0$ there exists a $\varepsilon > 0$ such that the left sides of three separations of arbitrary triangulations are not able to cover the entire area of the circle of radius R except for an area of size ε, instead they cover too little. To show this we will instead show a similar but slightly stronger geometric result. To do this we will reimagine the components of the left sides of separations as mere sets of points with geometric boundaries and prove the result in this more general framework. In this section let $X := \{x \in \mathbb{R}^2 || |x|| \leq R\}$ be a closed ball, centered in the origin of $\mathbb{R}^2$, of radius $R > \frac{k}{2}$ and let C_X be the boundary of X in $\mathbb{R}^2$.

Definition 6.1. $\delta := R - \frac{k}{2}$.

Definition 6.2. *Let $\psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2, (r, \varphi) \mapsto re^{i\varphi}$ be the parametrisation of the plane in polar coordinates.*

Definition 6.3. *Two sets of points A and B are close if there exist $a \in A$ and $b \in B$ such that $d(a, b) < \frac{\delta}{4}$. We then also say that A is close to B or B is close to A.*

Definition 6.4. *For a set $A \subseteq X$ we define the boundary ∂A as the set of all of the points $x \in X$ which contain in every neighbourhood an element of A as well as an element of $X \setminus A$.*

Definition 6.5. *A path P is a function $P : [0, 1] \to X$ whose image is a one-dimensional measurable manifold. A point a is said to be the starting point of P if $a = P(0)$ and b is said to be the end point of P if $b = P(1)$. P is then also said to connect a to b. A path with a starting point in a set A and an end point in a set B but which is otherwise disjoint with A and B is called an A-B-path or if $A = B$ simply an A-path. If a path connects one point to the same point we call it a cycle-path. A path with starting point a and end point b is said to connect a to b and such a path can also be written as aPb. If Q is another path connecting a point b to c then the path $aPbQc$ is defined as the concatenation of P and Q. The image of P will be written as $\overline{P} := Im(P)$.*

Definition 6.6. *Let K be an open connected subset of X not close to C_X and for which there exists a cycle-path P such that $\partial K = \overline{P}$, then we call K an inner component of X.*

Definition 6.7. *Let K be an open connected subset of X close to C_X, for which there exists a C_X-path or cycle-path P such that $\partial K = \overline{P}$, we then call K an outer component of X. If the distance between K and C_X is zero then we call K a strong outer component. Any inner or outer component which is not a strong outer component is called a weak inner component.*

The order or length of a component K, written as $|K|$, is defined as the lebesgue measure λ^1 of the boundary. Let L be a set of components then $|L| = \sum_{K \in L} |K|$.

Lemma 6.1. *If we have a weak inner component K with order $|K|$ then its area is $\lambda^2(K) \leq \frac{|K|^2}{4\pi}$. Hence if we have a set L of weak inner components each of length smaller than k then*

$$\lambda^2(L) \leq \frac{k}{4\pi}|L| \text{ and } \lambda^2(L) \leq \frac{|L|^2}{4\pi}$$

Proof. The first equation is simply the isoperimetric inequality [8].
For the other inequalities we calculate

$$\sum_{K \in L} \lambda^2(K) \leq \sum_{K \in L} \frac{|K|^2}{4\pi} \leq \sum_{K \in L} \frac{k|K|}{4\pi} = \frac{k}{4\pi} \sum_{K \in L} |K| \text{ and}$$

$$\sum_{K\in L} \lambda^2(K) \le \sum_{K\in L} \frac{|K|^2}{4\pi} \le \frac{(\sum_{K\in L}|K|)^2}{4\pi} = \frac{|L|^2}{4\pi}.$$

$\square$

Definition 6.8. *Let $A \subseteq X$ be a set of points then $r(A) := \inf_{a\in A} ||a||$ is the radius of A.*

Definition 6.9. *Let $\overline{L}$ be a set of inner and outer components, then we call $\overline{L}$ a cut of length k and margin ε or simply a cut for short if,*

i) The components in $\overline{L}$ are disjoint and cover X except for a set of size less than ε, more formally $\lambda^2(X \setminus \bigcup \overline{L}) < \varepsilon$,

ii) $\sum_{K\in\overline{L}} |K| \le 3k$ and $|K| < k$ for all $K \in \overline{L}$.

iii) $\lambda^2(K) < \frac{\pi R^2}{2}$ for every $K \in \overline{L}$.

Definition 6.10. *Let K_A be the union of a set of outer components $C := \{K_A^1, ..., K_A^n\} \subseteq \overline{L}$ of a cut $\overline{L}$ such that*

i) for every outer component $K \in \overline{L}$ which is not contained in C it follows that $r(K) \ge r(K')$ for every $K' \in C$

ii) for every $i \in \{1, ..., n\}$ the set K_A^i is close to an K_A^j for a $j < i$

iii) if two points A_1 and A_2 lie in $K_A \cup \partial K_A$ then they do not lie opposite each other, which means that there does not exist a $\lambda < 0$ such that $A_1 = \lambda A_2$.

If a set K_A fulfills these properties then we call K_A a cluster.

Lemma 6.2. *Let K_A be a cluster then there exists a straight line going trough the origin $t = \mathbb{R}v$ for a vector $v \in \mathbb{R}^2$ such that K_A lies completely in one of the components of $X \setminus t$.*

Proof. If K_A is empty this is trivially true. Hence assume that K_A is not empty and let $A \in K_A$ be an arbitrary point of K_A. A may not lie in the origin as it would otherwise be opposite itself which contradicts property iii). Now let us rotate X such that $A = (-a, 0)$ for an $a > 0$. This implies that there is no element of K_A which lies close to the positive section of the x-axis $\{(x, 0) \in D | x > 0\}$. Let

$$\varphi_1 := \inf\{\varphi' \in (0, 2\pi) : \exists r \in \mathbb{R}_+, \psi(r, \varphi') \in K_A\}$$

Then there exists an $r_1 \in \mathbb{R}_+$ such that $\psi(r_1, \varphi_1) \in \overline{\partial K_A}$. Since $A = (-a, 0) \in K_A$ we know that $\varphi_1 < \pi$. Let $\varphi_2 := \varphi_1 + \pi \in (\pi, 2\pi)$ then we know that there is no element of K_A close to $t_+ := \psi(\mathbb{R}_+ \times \{\varphi_2\})$. Let us define $t_- := \psi(\mathbb{R}_+ \times \{\varphi_1\})$ and $t = t_- \cup t_+$. Then t is a straight line going through the origin. The only thing left to show is that K_A lies completely in one of the components of $X \setminus t$, note that in particular this means that K_A and t are disjoint. Let $T_1 := \varphi(\mathbb{R}_+ \times (\varphi_1, \varphi_2)) \cap (X \setminus t)$ and $T_2 := X \setminus t \setminus T_1$. Since there is an element of K_A in T_1 we have to show that $K_A \cap T_2 = \emptyset$. Suppose there exists an element in $K_A \cap T_2$, then there must also exist an element $B \in K_A \cap T_2$ which is close to an element from $K_A \cap T_1$ since $K_A \cap t = \emptyset$. This further implies that B must be close to t. But by definition B cannot be close to t_+, hence B must be close to t_- but not close to the origin. Furthermore by definition of φ_1 the point B must thus lie in the sector $T_2' := \psi((\frac{\delta}{4}, R) \times (\varphi_2, 2\pi))$. Let $B = (b_x, b_y)$ then it suffices to show that $b_y \le -\frac{\delta}{4}$ as this is a contradiction to B being close to t_- since every element of t_- has a positive y-coordinate. If $b_x > 0$ and $b_y > -\frac{\delta}{4}$ then B is close with the positive section of the x-axis. If $b_x < 0$ and $b_y > -\frac{\delta}{4}$, then $\varphi_2 < \frac{3\pi}{2}$ and thus $\varphi_1 < \frac{\pi}{2}$ which means that $(b_x, \tan(\varphi_1)b_x) \in t_+$ and $\tan(\varphi_1)bx < 0$. This means that the distance d between t_+ and B is $d < |b_y - \tan(\varphi_1)bx| < \frac{\delta}{4}$ because of $-\frac{\delta}{4} < b_y < \tan(\varphi_1)bx < 0$ where the second inequality follows from $B \in T_2'$. $\square$

Lemma 6.3. *Let $\overline{L}$ be a cut and K_A a cluster with radius a' and let $b' > a'$ such that there exists an outer component $K_B \in \overline{L}$ with $r(K_B) = b'$ and every outer component of $\overline{L}$ not contained in K_A has radius $\geq b'$. Then for $\varepsilon < \frac{\delta k}{4\pi}$ the length $|\overline{L}| > 2k - 2a' - 2b' + \frac{2\pi^2}{k}a'^2 + \frac{2\pi^2}{k}b'^2$.*

Proof. Let t be the straight line going through the origin such that K_A lies completely in one of the components of $X \setminus t$. Let C_A be this component and let C_B be the other component. Let $M_A \subset C_A$ be every element not contained in an outer component and let $M_B \subset C_B$ be defined the same way. Then $B_A := B_{a'} \cap C_A \subset M_A$ and $B_B := B_{b'} \cap C_B \subset M_B$, where $B_r := \{x \in \mathbb{R}^2 : ||x||^2 < r\}$, since in C_A every outer separation has radius greater than a' and in C_B every outer separation has radius greater than b'. Then the area of $B_A \cup B_B$ must, except for an area of size ε, be contained in inner components. Let L' be the set of all inner components of $\overline{L}$ which intersect $B_A \cup B_B$. Then $\lambda^2(L') > \lambda^2(B_A \cup B_B) - \varepsilon = \frac{\pi}{2}a'^2 + \frac{\pi}{2}b'^2 - \varepsilon$. Since by Lemma 6.1 $\lambda^2(L') \leq \frac{k}{4\pi}|L'|$ this means that $|L'| > \frac{2\pi^2}{k}a'^2 + \frac{2\pi^2}{k}b'^2 - \frac{4\pi}{k}\varepsilon$. Since $\varepsilon < \frac{\delta k}{4\pi}$ this means that $|L'| > \frac{2\pi^2}{k}a'^2 + \frac{2\pi^2}{k}b'^2 - \delta$. We now also need to consider the length of outer components. The length $|K_A| > k + \frac{\delta}{2} - 2a'$ and $|K_B| > k + \frac{\delta}{2} - 2b'$. This is because the boundary of the component K_A contains two disjoint paths connecting a point with distance a' to the origin to a point with distance $k + \frac{\delta}{2}$ from the origin, for K_B we can argue analogously. Adding both of these together we get

$$|\overline{L}| > 2k - 2a' - 2b' + \tfrac{2\pi^2}{k}a'^2 + \tfrac{2\pi^2}{k}b'^2.$$

□

Lemma 6.4. *Let $\overline{L}$ be a cut, K_A a cluster and $\varepsilon < \frac{\delta k}{4\pi}$, then there exists an outer component K_B which is disjoint from K_A.*

Proof. Suppose not, then we can do the same calculation of the previous lemma, but without K_B and with $B_B = B_{\frac{k}{2}}$ and we get

$$|\overline{L}| > k - 2a' + \frac{2\pi^2}{k}a'^2 + \frac{2\pi^2}{k}\left(\frac{k}{2}\right)^2 > \frac{2\pi^2}{k}\left(\frac{k}{2}\right)^2 = \frac{2\pi^2}{k}\frac{k^2}{4} > 3k$$

which is a contradiction. □

This means that suppose we have been given a cut $\overline{L}$ containing a cluster K_A with radius a' then there exists another disjoint outer component K_B with minimal radius b'.

Lemma 6.5. *Let $\varepsilon < \frac{\delta k}{4\pi}$, K_A be a cluster and K_B be the outer component of smallest radius which is disjoint with K_A and $a' := r(K_A)$, $b' := r(K_B)$ then $2a' + 2b' \leq k$.*

Proof. Suppose $2a' + 2b' > k$ hence $2b' > k - 2a'$. This means that

$$|\overline{L}| > 2k - 2a' - 2b' + \frac{2\pi^2}{k}a'^2 + \frac{2\pi^2}{k}(k - 2a')^2 > 2\pi^2 k - (2 + 8\pi^2)a' + \frac{10\pi^2}{k}a'^2$$

For the second inequality we used that $2k - 2b' > 0$. This is a parabola with respect to a' and thus we can calculate the lowest value this parabola takes.

$$2\pi^2 k - (2 + 8\pi^2)a' + \frac{10\pi^2}{k}a'^2 > 2\pi^2 k - \frac{(2 + 8\pi^2)^2}{4\frac{10\pi^2}{k}} = (\frac{2}{5}\pi^2 - \frac{1}{10\pi^2} - \frac{4}{5})k$$

One can calculate that

$$(\frac{2}{5}\pi^2 - \frac{1}{10\pi^2} - \frac{4}{5})k > 3k$$

which is a contradiction since $3k > |\overline{L}|$. □

Definition 6.11. *Given an outer component K let $S_L(K)$ be the supremum of all $y \in [-R, R]$ such that $(x, y) \in K$ for an $x \in [-R, 0]$ and let $I_L(K)$ be the infimum of the same set. Analogously let $S_R(K)$ and $I_R(K)$ be defined but with $x \in [0, R]$ instead.*

Definition 6.12. *An upper-component $K \in \overline{L}$ is an outer component for which there exists a $y > 0$ such that $(0, y) \in \partial K$. A lower-component $K \in \overline{L}$ is an outer component for which there exists a $y < 0$ such that $(0, y) \in \partial K$.*

Lemma 6.6. *We prove a few simple properties about upper- and lower components. Let K be an outer component of a cut $\overline{L}$.*

i) Two points close to ∂K cannot lie opposite each other hence a component can not be an upper- and a lower component.

ii) Let K be a lower component, then $|K| > k + \frac{3}{2}\delta + 2\min(S_L(K), S_R(K))$. If K is also a strong outer component then $|K| > 2R + 2\min(S_L(K), S_R(K))$. Let K' be an upper component, then $|K'| > k + \frac{3}{2}\delta - 2\max(I_L(K'), I_R(K'))$. If K' is also a strong outer component then $|K'| > 2R - 2\max(I_L(K'), I_R(K'))$.

iii) Any outer component K_B is an upper component or it is disjoint to one of the two upper quadrants $X_1 := \psi([0, R] \times (0, \frac{\pi}{2}])$ and $X_2 := \psi([0, R] \times [\frac{\pi}{2}, \pi)$.

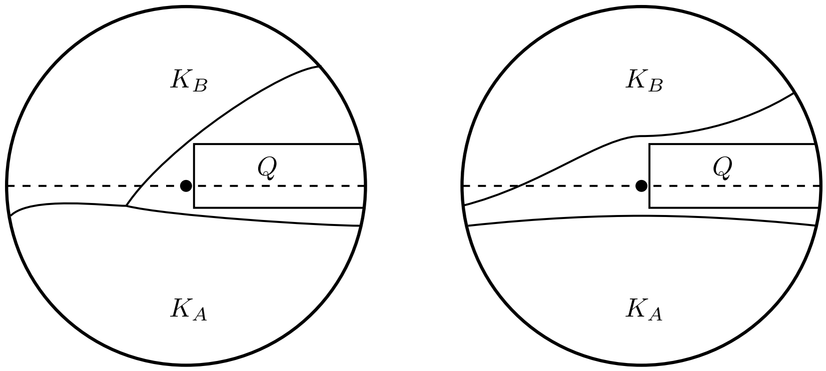
Proof. i) Suppose there are A_1 and A_2 close to ∂K with $\lambda < 0$ such that $A_2 = \lambda A_1$. Then there exists a path P which lies in the boundary of K and which starts at a point C_1 close to the boundary of X, goes through A_1, then A_2 and then goes close to C_2 which is again close to the boundary of X. If we replace P with a (shorter) path P' which follows P but connects A_1 to A_2 in a straight line then P' is a path which starts close to the boundary of X goes close to the origin and then goes back to a point close to the boundary of X. This path P' would then have a length longer then k which is a contradiction since $k > |\partial K| > |P| > |P'|$.

ii) We will just show the first inequality as the second can be generated from the first by reflecting X along the x-axis. Since K is a lower component there exist points C_1, C_2 close to the boundary of X and $A = (-x_a, S_L(K)) \in \overline{\partial K}$, $B = (x_b, S_R(K)) \in \overline{\partial K}$ with $x_a, x_b > 0$ such that there exists a C_1ABC_2-path P. Let us now replace the AB-section of P with a straight line and name the new path P'. This path P' is shorter than P and since K is a lower component it may not go above the origin as can be seen from i). Hence $S_L(K)$ or $S_R(K)$ must be smaller than zero. This path P' thus has distance less than $-\min(S_L(K), S_R(K))$ to the origin. Which means that its length $|P'| > 2((\frac{k}{2} + \frac{3\delta}{4} - (-\min(S_L(K), S_R(K)))) = k + \frac{3}{2}\delta + 2\min(S_L(K), S_R(K))$. In the case that K is a strong outer component we can do the same argument but the path P now connects two points on C_X which means that the length of the path is $2R + 2\min(S_L(K), S_R(K))$.

iii) If an outer component is not an upper component but still intersects both X_1 and X_2 then there must be a path P with $\overline{P} \subset \partial K_B$ connecting X_1 to X_2, this path must contain a point $(0, -p)$ with $p > 0$ which means that K_B is a lower component. Since $S_L(K_B) > 0$ and $S_R(K_B) > 0$ this means by ii) that $|K_B| > k$ which is a contradiction.

$\square$

Now that we have done some preparation let us sketch the rest of the section. We will want to show that for small ε no cuts of X exist (Theorem 6.1), to do this we will do a proof by contradiction. We will consider two cases which are sketched blow. We have the cluster K_A and the outer component K_B. Either K_A and K_B are close or they are not. In both cases we are going to approximate the length of K_A and K_B as well as the length of boundaries of components which lie close to K_A and K_B. But this is not enough to prove the theorem, so we also have to look at the length of paths contained inside of the rectangle Q which is not close to K_A or K_B hence these lengths were not yet counted. The lemmas we have just shown will help us prove that such a Q indeed exists.



Let us now do this formally.

Lemma 6.7. *Let S be an open set, let $\overline{L}$ be a set of components and let $M \subseteq [0, R]$ be a measurable where for every element $m \in M$ there exist n distinct values $\varphi_1, ..., \varphi_n \in [0, 2\pi)$ such that $\psi(m, \varphi_i) \in S \cap \partial\overline{L}$ for all $i \in \{1, ..., n\}$. Then $|\overline{L} \cap S| \geq n\lambda^1(M)$.*

Proof. We first recognize that for every $p \in S \cap \partial\overline{L}$ there exists a $w(p) > 0$ such that for every ball $B_w(p)$ of radius $w < w(p)$ it holds that $|B \cap S \cap \partial\overline{L}| \geq 2w$. This can be seen from S being open as well as the definition of the boundary of components. Next we can define for every $m \in M$ a $d(m) := \min_{i \neq j} |\varphi_i - \varphi_j|$ and a $w(m)$ which is defined as $w(m) := \min(d(m), \min_{i \in \{1,...,n\}}(w(\psi(m, \varphi_i))))$. This means that for every $w < w(m)$ it follows that

$$|\psi((m - w, m + w, [0, 2\pi)) \cap S \cap \partial\overline{L}| \geq$$
$$\sum_{i \in \{1,...,n\}} |B_w(\psi(m, \varphi_i)) \cap S \cap \partial\overline{L}| \geq n \cdot 2w$$

Where the first inequality follows from the fact that the balls $B_w(\psi(m, \varphi_i))$ do not intersect.
Let us now define a sequence $M_1, M_2, ...$ of closed subsets of M for which $\lambda^1(M_i) \to \lambda^1(M)$ for $i \to \infty$. We now want to show that the theorem holds for M_i. Since $|\overline{L} \cap S| \geq \lambda^1(M_i)$ for every $i \in \mathbb{N}$ we have $|\overline{L} \cap S| \geq \lambda^1(M)$.
Since M_i is bounded by $[0, R]$ and closed it is also compact. This means that there exists a $w_i > 0$ such that $w(m) > w_i$ for all $m \in M_i$. Now we can define for every $m \in M_i$ an interval $I_m := (m - w_i, m + w_i)$. These sets I_m cover the entirety of M_i hence there is a finite subset of these intervals which also cover M_i. Let I' be the finite subset of these intervals. We now wish to construct a set I of disjoint intervals with centers in M_i and length less than $2w_i$ that still cover M_i except for finitely many points. To do that it suffices to check that we can untangle two intersecting intervals. Since I' only contains finitely many intervals we then can construct I in finitely many steps. Suppose now that there exist two intervals $I_1 = (a, b)$ and $I_2 = (c, d)$ without loss of generality fulfilling $d > b > c > a$. Then let d' be the maximal element of M_i contained in $(b, \frac{b+d}{2})$ then $I'_1 = (a, b), I'_2 = (b, b + 2(d' - b))$ are disjoint, their centers are contained in M_i and they just cover one point less than I_1 and I_2.
Now we can finish the proof by showing that $|\overline{L} \cap S| \geq \lambda^1(M_i)$. For that we consider

$$|\overline{L} \cap S| \geq |\psi(M, [0, 2\pi)) \cap \overline{L} \cap S| \geq$$
$$\sum_{i \in I} |\psi(i, [0, 2\pi)]) \cap \overline{L} \cap S| \geq \sum_{i \in I} n|i| \geq n\lambda^1(M_i).$$

$\square$

Lemma 6.8. *Let S be an open set with two orthogonal and normalized vectors $v, w \in \mathbb{R}^2$ such that there exists a measurable set $M \subseteq \mathbb{R}$ where for every element $m \in M$ there exist n distinct values $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $mv + \lambda_i w \in S \cap \partial \overline{L}$ for all $i \in \{1, ..., n\}$. Let $\overline{L}$ be a set of components. Then $|\overline{L} \cap S| > n\lambda^1(M)$.*

Proof. The proof of this lemma follows analogously to the proof of the previous lemma. □

Lemma 6.9. *If outer components K_A and K_B are not close and $\varepsilon \leq \frac{\delta^3}{16R}$ then if S is the set of points which are close to $K_A \cup K_B$ we have $|\overline{L} \cap S| \geq 2k - 2a' - 2b' + |K_A| + |K_B| + \delta$.*

Proof. Let $S_1 \subset X$ be the set of all points which have a distance of $(0, \frac{\delta}{8})$ to K_A and let $S_2 \subset X$ be the set of all points which have distance $(0, \frac{\delta}{8})$ to K_B. These sets have the property that $S_1 \cap S_2 = \emptyset$ as well as $S_1 \cup S_2 \subseteq S$. To show the lemma it suffices to prove $|\partial\overline{L} \cap S_1| \geq k - 2a' + \frac{\delta}{2}$ and $|\partial\overline{L} \cap S_2| \geq k - 2b' + \frac{\delta}{2}$. To prove $|\partial\overline{L} \cap S_1| \geq k - 2a' + \frac{\delta}{2}$ we start by looking at the structure of S_1. Let without loss of generality X be rotated such that every element of K_A has a negative y-coordinate. Then for every $r \in (a', R - \frac{\delta}{4})$ let $\Phi_r \subset (\pi, 2\pi)$ be the set of all angles φ such that $\psi(r, \varphi) \in K_A$. Let $\varphi_r^+ := \sup \Phi_r$ and $\varphi_r^- = \inf \Phi_r$ then we can define $I_r^+ := \psi(r, [\varphi_r^+, \varphi_r^+ + \frac{\delta}{8R}))$ and $I_r^- := (r, (\varphi_r^- - \frac{\delta}{8R}, \varphi_r^-])$, these intervals have the property that $I_r^+ \subset S_1$ and $I_r^- \subset S_1$ as well as $I_r^- \cap I_r^+ = \emptyset$ since $\frac{\delta}{8R} < \frac{1}{8} < \frac{\pi}{2}$. Let us now define M_+ as the set of all $r \in (a', R - \frac{\delta}{4})$ such that there exists a φ_1 with $\psi(r, \varphi_1) \in \partial\overline{L} \cap I_r^+$ and let us define M_- as the set of all $r \in (a', R - \frac{\delta}{4})$ such that there exists a φ_2 with $\psi(r, \varphi_2) \in \partial\overline{L} \cap I_r^-$. Both sets M_+ and M_- are measurable. This can be seen by considering $I^+ := \bigcup_{r \in (a', R - \frac{\delta}{4})} I_r^+$ and $I^- := \bigcup_{r \in (a', R - \frac{\delta}{4})} I_r^-$. Since I^+,I^- and $\partial\overline{L}$ are measurable and we can view M_+ and M_- as projections of $\partial\overline{L} \cap I^+$ and $\partial\overline{L} \cap I^-$ by the measurable map $\pi_r : \mathbb{R}^2 \to \mathbb{R}^+, x \mapsto ||x||$ onto the radial component, this means that M_+ and M_- are also measurable. Hence $M := M_+ \cap M_-$ is also measurable. For every value $r \in M$ there exist distinct angles $\varphi_1, \varphi_2 \in [0, 2\pi)$ such that $\psi(r, \varphi_1) \in \partial\overline{L} \cap S_1$ and $\psi(r, \varphi_2) \in \partial\overline{L} \cap S_1$. This means that $|\partial\overline{L} \cap S| \geq 2\lambda^1(M)$. To prove $|\partial\overline{L} \cap S_1| \geq k - 2a' + \frac{\delta}{2}$ it thus suffices to show that $\lambda^1(M) \geq \frac{k}{2} - a' + \frac{\delta}{4}$. Suppose $\lambda^1(M) < \frac{k}{2} - a' + \frac{\delta}{4}$ then there exists a set $N := (a', R - \frac{\delta}{4}) \setminus M$ of length $\lambda^1(N) = (R - \frac{\delta}{4} - a') - (\frac{k}{2} - a' + \frac{\delta}{4}) > \frac{\delta}{2}$. For every $r \in N$ there exists an interval $I_r \in \{I_r^+, I_r^-\}$ such that $\psi(r, I_r)$ does not contain an element of $\partial\overline{L}$. This would mean that $\psi(r, I_r)$ also does not contain an element of $\overline{L}$ since $\psi(r, I_r)$ is connected to K_A and thus if $\psi(r, I_r)$ contained an element x of $K' \in \overline{L}$ there would then exist an x-K_A-path P inside of $\psi(r, I_r)$ which means that there must exist a $y \in P$ such that $y \in \partial K'$ since $K' \cap K_A = \emptyset$. This means that we can define $N' := \bigcup_{n \in N} \psi(n, I_n)$. We know that $\overline{L} \cap N' = \emptyset$. Now we just need to estimate the area of N' and show that it is larger than ε.

$$\lambda^2 N' = \int_N \lambda^1(I_r) \mathrm{d}r \geq \int_N r\frac{\delta}{8R} \mathrm{d}r \geq a' \frac{\delta}{8R} \int_N 1 \mathrm{d}r \geq \frac{\delta^2}{8R} \lambda^1(N) \geq \frac{\delta^3}{16R} \geq \varepsilon$$

This proves $|\partial\overline{L} \cap S_1| \geq k - 2a'$. The proof of $|\partial\overline{L} \cap S_2| \geq k - 2b'$ follows in the same way. □

Lemma 6.10. *Let $Q := (\frac{\delta}{4}, R] \times (-a + \frac{\delta}{4}, b - \frac{\delta}{4})$ be a rectangle where $a < b'$ and $b < b'$ then if Q is not close to $K_A \cup K_B$ and $\varepsilon \leq \frac{\delta^2}{8}$ then $|\partial\overline{L} \cap Q| > 2(a + b) - 2\delta$.*

Proof. Let us start by defining $I_y^- := (\frac{\delta}{4}, \frac{\delta}{2}) \times \{y\}$, $I_y^+ := (\sqrt{R^2 - y^2} - \frac{\delta}{4}, \sqrt{R^2 - y^2}) \times \{y\}$, $I^+ := \bigcup_{y \in (-a + \frac{\delta}{4}, b - \frac{\delta}{4})} I_y^+$ and $I^- := \bigcup_{y \in (-a + \frac{\delta}{4}, b - \frac{\delta}{4})} = [\frac{\delta}{4}, \frac{\delta}{2}] \times [-a + \frac{\delta}{4}, b - \frac{\delta}{4}] \subset B_{b'}$. To show $I^- \subset B_{b'}$ let $(x, y) \in I^-$ be an arbitrary point, then

$$|(x, y)| = \sqrt{x^2 + y^2} \leq \sqrt{\frac{\delta^2}{4} + (b' - \frac{\delta}{4})^2} = \sqrt{b'^2 - \frac{b\delta}{2} + \frac{\delta^2}{2}} \leq \sqrt{b'^2} = b',$$

since $\delta < b'$. This proves $I^- \subset B_{b'}$.
Let $\pi : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto y$ be the projection onto the y-coordinate. Let us now define $M_1 := \pi(\overline{L} \cap I^-)$ and $M_2 := \pi(\overline{L} \cap I^+)$. Then by definition M_1 and M_2 are measurable. Hence $M := M_1 \cap M_2$ is also measurable.
We now show that $\lambda^1(M) > a + b - \delta$, to do this suppose that $\lambda^1(M) < a + b - \delta$ and let $N := (-a + \frac{\delta}{4}, b - \frac{\delta}{4}) \setminus M$ then $\lambda^1(N) > (b - \frac{\delta}{4}) - (-a + \frac{\delta}{4}) - (a + b - \delta) = \frac{\delta}{2}$. But for every $n \in N$ there exists an $I_n \in \{I_n^+, I_n^-\}$ such that $I_n \cap \overline{L} = \emptyset$. Let us now define the set $N' := \bigcup_{n \in N} I_n$. For N' it also follows that $N' \cap \overline{L} = \emptyset$. Let us now show that $\lambda^2 N' > \varepsilon$ to get to a contradiction.

$$\lambda^2 N' = \int_N \lambda^1(I_y)\mathrm{d}y \geq \int_N \frac{\delta}{4}\mathrm{d}y = \lambda^1(N)\frac{\delta}{4} \geq \frac{\delta^2}{8} \geq \varepsilon$$

This means that $\lambda^1(M) \geq a + b - \delta$. Next we want to show that for every $m \in M$ there exist $x_1, x_2 \in (\frac{\delta}{4}, R]$ such that $(x_1, m) \in \partial\overline{L}$ and $(x_2, m) \in \partial\overline{L}$. For that let us consider $z_1, z_2 \in (\frac{\delta}{4}, R]$ with $(z_1, m) \in \overline{L} \cap I_m^-$ and $(z_2, m) \in \overline{L} \cap I_m^+$. These must exist by definition. Let $K_1 \in \overline{L}$ be the component containing (z_1, m) and let $K_2 \in \overline{L}$ be the component containing (z_2, m). Then $K_1 \neq K_2$ since K_1 must be an inner component since $|(z_1, m)| < b'$ and every outer component with radius $\leq b'$ is accounted for in K_A and K_B which are both not close to Q. On the other hand K_2 must be an outer component since (z_2, m) is close to ∂X. Now we can consider the path P which is a straight line connecting (z_1, m) and (z_2, m). On this path there must be x_1, x_2 such that $(x_1, m) \in \partial K_1$ and $(x_2, m) \in \partial K_2$.
Now we can apply Lemma 6.8. Let S be the set of inner points of Q then we have vectors $v = (0, 1)$ and $w = (1, 0)$ which are orthogonal and normalized. Let $L_1 \subset \overline{L}$ be the set of all inner components and let $L_2 \subset \overline{L}$ be the set of all outer components. Then for every $m \in M$ we have values x_1, x_2 such that $mv + x_i w \in S \cap \partial L_i$ for $i \in \{1, 2\}$. Hence $L_i \cap Q > \lambda^1(M) = (a + b) - \delta$. Since $\overline{L} \cap Q = (L_1 \cap Q) \cup (L_2 \cap Q)$ and L_1 and L_2 are disjoint this means $\overline{L} \cap Q > 2(a + b) - 2\delta$. □

Lemma 6.11. *Let K_A be a cluster and K_B a different outer component not contained in K_A, let K_A and K_B be close and let $K_A \cup K_B$ not be a cluster, then inside of the set S of points which are close to $K_A \cup K_B$ there are paths in $\overline{L}$ of length $l \geq k + |K_A| + |K_B|$ if $\varepsilon < \frac{\delta^2}{8}$.*

Proof. Let S' be the set of all points which have a distance to $K_A \cup K_B$ contained in $(0, \frac{\delta}{4})$, that is to say all of the points which are close to $K_A \cup K_B$ but not contained within K_A, K_B or their boundary. We want to show $|\overline{L} \cap S'| \geq k$ which suffices to prove this lemma.
Since $K_A \cup K_B$ is not a cluster this means that there exist points A, B such that $A \in K_A \cup \partial K_A$ and $B \in K_B \cup \partial K_B$ and such that A and B lie opposite each other. This means that we can rotate X such that $A = (0, -a')$ and $B = (0, b')$ for $a, b > 0$. We now choose a, b minimal with the property that $(0, -a) \in K_A \cup \partial K_A$ and $(0, b) \in K_B \cup \partial K_B$. These a, b must exist since $K_A \cup \partial K_A$ and $K_B \cup \partial K_B$ are closed. By definition $(0, -a) \in \partial K_A$ and $(0, b) \in \partial K_B$. Let C further be a point which is close to both K_A and K_B then let us reflect X such that C has negative x-coordinate. This means that there exists an $r' \in (0, R)$ and an $\alpha' \in (\frac{\pi}{2}, \frac{3\pi}{2})$ such that $C = \psi(r', \alpha')$. Then for $\alpha := \alpha' - \pi$ we know that $\psi((0, R) \times \{\alpha\})$ is not close to $K_A \cup K_B$.
We now consider the intervals $Y_1 := (-\frac{k}{2} - \frac{\delta}{2}, -a)$, $Y_2 := (-a, b)$ and $Y_3 := (b, \frac{k}{2} + \frac{\delta}{2})$. For $y \in Y_1$ let $I_y^1 := \psi([-y] \times (\varphi_y^1, \varphi_y^1 + \frac{\delta}{8y}))$ where

$$\varphi_y^1 := \sup\{\varphi' \in [\alpha, \alpha + 2\pi) : \psi(-y, \varphi') \in K_A \cup K_B\}.$$

For $y \in Y_2$ let $I_y^2 := (x_y, x_y + \frac{\delta}{8}) \times [y]$ where

$$x_y := \sup\{x \in [-R, 0] : (x, y) \in K_A \cup K_B\}.$$

For $y \in Y_3$ let $I_y^3 := \psi([y] \times (\varphi_y^3 - \frac{\delta}{8y}, \varphi_y^+))$ where

$$\varphi_y^3 := \inf\{\varphi' \in [\alpha, \alpha + 2\pi) : \psi(y, \varphi') \in K_A \cup K_B\}.$$

And finally let $I^1 := \bigcup_{y \in Y_1} I_y^1$, $I^2 := \bigcup_{y \in Y_2} I_y^2$ and $I^3 := \bigcup_{y \in Y_3} I_y^3$.
Let $U^j(\gamma) := \bigcup_{i \in I^j} B_\gamma(i) \cap S'$ for $j \in \{1, 2, 3\}$. Since $I^j \subseteq S'$ this implies that $I^j \subseteq U^j$. Let us now show that there exists a $\gamma > 0$ such that the $U^j(\gamma)$ are disjoint subsets of S'. Suppose there is not such a γ then this would mean that there exist distinct $j_1, j_2 \in \{1, 2, 3\}$ such that $\overline{U^{j_1}} \cap \overline{U^{j_2}} \neq \emptyset$. We first look at the case $j_1 \in \{1, 3\}$ and $j_2 \in \{1, 3\}$. In this case there exists a $y_1 \in Y_1$ and a $y_3 \in Y_3$ such that $\overline{I_{y_1}^1} \cap \overline{I_{y_3}^3} \neq \emptyset$. This implies $y_3 = -y_1$. Let $\psi(y_3, \varphi) \in \overline{I_{y_1}^1} \cap \overline{I_{y_3}^3}$. Since $\psi(y_3, \varphi) \in \overline{I_{y_1}^1}$ this means by definition that $\psi(\{y_3\} \times (\varphi, \alpha + 2\pi]) \cap (K_A \cup K_B) = \emptyset$. Since $\psi(y_3, \varphi) \in \overline{I_{y_3}^3}$ this on the other hand means that $\psi(\{y_3\} \times [\alpha, \varphi)) \cap (K_A \cup K_B) = \emptyset$. This means that if we consider the circle $C = \partial B_{y_3}$ then for every $y \in C \setminus \{\psi(y_3, \varphi)\}$ it follows that $y \notin K_A \cup K_B$. But since $C \setminus \{\psi(y_3, \varphi)\}$ intersects every neighbourhood of $\psi(y_3, \varphi)$ and since $K_A \cup K_B$ is open, this means that $\psi(y_3, \varphi)$ is also not contained in $K_A \cup K_B$ which means that $C \cap (K_A \cup K_B) = \emptyset$. Which is a contradiction since there exists a point $A_1 \in K_A$ which has radius less than y_3 which then must be connected by a path P in K_A to a point $A_2 \in K_A$ close to the boundary of X, since K_A is connected and an outer component. Because $r(A_1) < y_3$ and $r(A_2) > y_3$ the path P must intersect C which is a contradiction.
Without loss of generality we can thus assume that $j_1 = 2$ and $j_2 = 3$ since the case $j_1 = 2$ and $j_2 = 1$ can be proven analogously. Now there exists a $y_2 \in Y_2$ and a $y_3 \in Y_3$ such that $\overline{I_{y_2}^2} \cap \overline{I_{y_3}^3} \neq \emptyset$. Let $Z = (z_x, y_2) = \psi(y_3, \varphi_z)$ be an element of the intersection. By definition of $\overline{I_{y_2}^2}$ and $\overline{I_{y_3}^3}$ it follows that $C_1 := ((z_x, 0] \times \{y_2\}) \cap (K_A \cup K_B) = \emptyset$ and $C_2 := \psi(\{y_3\} \times [\alpha, \varphi_z)) \cap (K_A \cup K_B) = \emptyset$. With the same argument as before we can see that $Z \notin (K_A \cup K_B)$. Additionally we know that $C_3 := \{0\} \times [\min(0, y_2), \max(0, y_2)] \cap (K_A \cup K_B) = \emptyset$ and also that $C_4 := \psi([0, y_3] \times \{\alpha\}) \cap (K_A \cup K_B)$. But now this topological cycle $C_1 \cup C_2 \cup C_3 \cup C_4 \cup Z$ separates small neighbourhoods of $(0, -b)$ from the boundary of X similarly to the previous section. Contradiction.
Let us now define $\overline{L'}$ which is $\overline{L}$ where we delete every component of K_A and K_B and we define $M_j := \{y \in Y_j : I_y^j \cap \overline{L'} \neq \emptyset\}$ for $j \in \{1, 2, 3\}$. Then for every $y \in M_j$ it follows that $\overline{I_y^j} \cap \partial \overline{L'} \neq \emptyset$. This means using the lemmas 6.7 and 6.8 that $|\overline{L'} \cap S| \geq \sum_{j \in \{1,2,3\}} \lambda^1(M_j)$. Let us define $N := \bigcup_{j \in \{1,2,3\}} Y_j \setminus M_j$ then

$$\lambda^1(N) = \sum_{j \in \{1,2,3\}} \lambda^1(Y_j) - \lambda^1(M_j) = k + \delta - \sum_{j \in \{1,2,3\}} \lambda^1(M_j)$$

$$\Leftrightarrow |\overline{L'} \cap S| \geq \sum_{j \in \{1,2,3\}} \lambda^1(M_j) = k + \delta - \lambda^1(N)$$

Which means that showing $\lambda^1(N) < \delta$ finishes the proof. For that let us consider $N' = \bigcup_{j \in \{1,2,3\}} \bigcup_{y \in Y_j \setminus M_j} I_y^j$. We know that $N' \cap \overline{L} = \emptyset$ and hence $\lambda^2 N' < \varepsilon$. But since $\lambda^2 N' \geq \frac{\delta}{8} \lambda^1(N)$, see proof of Lemma 6.9 and 6.10, this means that $\lambda^1(N) \leq \frac{\delta}{8} \lambda^2 N' \leq \frac{8}{\delta} \varepsilon < \delta$. □

Lemma 6.12. *Let K_A be a cluster and let K_B be an outer component of $\overline{L}$ disjoint from K_A with minimal radius. If K_A and K_B are not close then for $\varepsilon < \min(\frac{\delta k}{4\pi}, \frac{\delta^3}{16R})$ the set $\overline{L}$ is not a cut.*

Proof. First of all we show that we can apply all of the previous lemmas by showing that $\frac{\delta^3}{16R} < \frac{\delta^2}{8}$. This inequality is equivalent to the inequality $\frac{\delta}{R} < 2$ which is true since $\delta < R$.
For this proof we need to consider two cases: The first being that K_A is a cluster which is the union of more than one element and the other is K_A being a cluster containing exactly one outer component.

Let us first consider the case that K_A is the union of more than one outer component. For that let us first rotate X such that the line t restricting K_A lies on the x-axis and K_A lies in the lower component. Let $a := 0$ and let K_A be the union of $\{K_A^1, ..., K_A^n\}$ then if we consider $K_A' := \bigcup\{K_A^1, ..., K_A^{n-1}\}$ and $K_B' = K_A^n$ then from Lemma 6.5 it follows that $2r(K_A')+2r(K_B') \le k$ which means that

$$|K_A| = |K_A'| + |K_B'| \ge k + 2\delta - 2r(K_A') + k + 2\delta - 2r(K_B') \ge k - 2a + \tfrac{\delta}{2}$$

Let us now consider the case of K_A itself being an outer component.
Let A_1 and A_2 be two non linear dependent points of K_A, which exist since K_A is open. Now let us show that it is possible for X to be transformed by an orthogonal matrix such that in the image we have $S_L(K_A) = S_R(K_A)$ and also such that the images of A_1 and A_2 lie below the x-axis. For that let us rotate X such that A_1 lies on the left side of the x-axis then either A_2 lies below the x-axis or we can reflect X along the x-axis and afterwards A_2 lies below the x-axis. This means that there exist $r_1, r_2 \in (R - \frac{\delta}{4}, R)$ and $0 < \alpha < \pi$ such that $A_1 = \psi(r_1, \pi)$ and $A_2 = \psi(r_2, 2\pi - \alpha)$. Let R_φ be the map rotating $\mathbb{R}^2$ by the angle φ, then we can see that in $R_\varphi(X)$ for $\varphi \in (0, \alpha)$ both A_1 and A_2 lie below the x-axis while in $R_0(X)$ the point A_1 lies on the x-axis on the left side and A_2 lies below the x-axis and in $R_\alpha(X)$ the point A_1 lies below the x-axis and A_2 lies on the x-axis on the right side. Let $S_L(\varphi)$ be the value of $S_L(K_A)$ in $R_\varphi(X)$ and let $S_R(\varphi)$ be the value of $S_R(K_A)$ in $R_\varphi(X)$. Then we know that $S_L(0) \ge 0$ since the y-coordinate of A_1 is 0 but since $k > |K| > k + \frac{\delta}{2} + 2\min(S_L(0), S_R(0), 0)$ this implies $S_R(0) < -\frac{\delta}{2}$. Analogously we can show that $S_R(\alpha) \ge 0$ and $S_L(\alpha) < \frac{\delta}{2}$. Since $S_R(\varphi)$ and $S_L(\varphi)$ are continuous this means that there must exist an angle $\varphi' \in (0, \alpha)$ such that $S_L(\varphi') = S_R(\varphi') =: -a$. Since $k > |K| > k + \frac{\delta}{2} + 2\min(-a, 0)$ this implies $a > \frac{\delta}{2}$.
After having defined a, in both cases, such that $S_L(K_A) \le -a$ and $S_R(K_A) \le -a$ while still $|K_A| \ge k - 2a + \frac{\delta}{2}$, we now consider the component K_B.
If K_B is a lower-component or neither a lower- nor an upper-component then K_B is disjoint with one of the two upper quadrants of X. Without loss of generality we can assume that it is disjoint from $X_1 := \psi((0, R) \times (0, \frac{\pi}{2}))$ and then we can define $b := b'$ which implies $|K_B| > k - 2b + \frac{\delta}{2}$ as well as $Q := [\frac{\delta}{4}, R] \times [-a + \frac{\delta}{4}, b - \frac{\delta}{4}]$ not being close to K_A or K_B. On the other hand if K_B is an upper-component then we define $b > \frac{\delta}{2}$ such that without loss of generality, after possibly reflecting X along the y-axis, $b = I_R(K_B) > I_L(K_B)$. This again implies $|K_B| > k - 2b + \frac{\delta}{2}$ as well as $Q := [\frac{\delta}{4}, R] \times [-a + \frac{\delta}{4}, b - \frac{\delta}{4}]$ not being close to K_A or K_B.
After having defined a and b we can now show that $|K_A| > k - 2a + \frac{\delta}{2}$ and $|K_B| > k - 2b + \frac{\delta}{2}$ which follows directly from the definition of a and b.
We can now use the Q defined above to finish this lemma. Using Lemma 6.9 and 6.10 we can find a lower bound for the length of the entire cut $\overline{L}$.

$$|\overline{L}| > |K_A| + |K_B| + |\partial\overline{L} \cap Q| + 2k - 2a' - 2b' > 4k - 2a' - 2b'$$

Since $3k > \overline{L}$ this implies $2a' + 2b' > k$ which is a contradiction hence we have proven this lemma. $\square$

Lemma 6.13. *Let K_A be a cluster and let K_B be an outer component of $\overline{L}$ disjoint from K_A with minimal radius. If K_A and K_B are close then for $\varepsilon < \min(\frac{\delta k}{4\pi}, \frac{\delta^2}{8})$ $\overline{L}$ is not a cut.*

Proof. In this proof we will apply the Lemmas 6.5, 6.10 and 6.11 which explains the bound for ε. Let n be the number of components contained in K_A then $|K_A| \ge n|K_B|$ which means that $|\overline{L}| \ge (n+1)|K_B|$. Since $2r(K_A) + 2r(K_B) < k$ this means that $r(K_B) < \frac{k}{4}$, hence $|K_B| > \frac{k}{2}$. This implies together with $3k > \overline{L}$ that $3k > (n+1)\frac{k}{2}$ which implies $n < 5$.
Now since K_A and K_B are close and K_A cannot consist of arbitrarily many components we can

see that we can choose the cluster K_A maximal (by inclusion) such that K_B, the outer component of $\overline{L}$ with the smallest radius which is not contained in K_A, is not close to K_A.
We now need to consider two possible cases: Either $K_A \cup K_B$ is a cluster, or $K_A \cup K_B$ is not a cluster. Let us first consider the case that $K_A \cup K_B$ is a cluster. In this case define $K'_A := K_A \cup K_B$ and K'_B as the outer component of $\overline{L}$ which has the smallest radius while being disjoint with K'_A. Then we can see that K'_A and K'_B must be close since we have chosen K_A maximal, which means that Lemma 6.12 implies that $\overline{L}$ is not a cut.
Let us now look at the case that $K_A \cup K_B$ is not a cluster which means that there exist points A' close to K_A and B' close to K_B such that $A' = -\lambda B'$ for $\lambda > 0$. Let C be the closest point to the origin such that C is close to both K_A as well as K_B. Let X be rotated such $C = (-c, 0)$ for $c > 0$. Then both K_A and K_B can not be close to the right side of the x-axis, if either one of them are close to a point C' on the right side of the x-axis then C' is opposite to C. Let us define two quadrants $X_1 := \psi((0, R) \times (0, \frac{\pi}{2}))$ and $X_2 := \psi((0, R) \times (\frac{3\pi}{2}, 2\pi))$. If K_A or K_B lie in X_i then they can not lie in X_{2-i}. Since $K \in \{K_A, K_B\}$ intersected X_1 and X_2 then there would exist angles φ_1, φ_2 and distances r_1, r_2 such that $\psi(r_1, \varphi_1) \in K \cap X_1$ and $\psi(r_2, \varphi_2) \in K \cap X_2$, but then for every r it follows that $\psi(r, \varphi_1 + \pi)$ and $\psi(r, \varphi_2 - \pi)$ are disjoint and not close to K, hence $T := \bigcup_{r \in (0,R)} \psi(r, \varphi_1 + \pi) \cup \psi(r, \varphi_2 - \pi)$ is disjoint and not close to K. But any path connecting C to a point in X_1 or X_2 must intersect T since T separates the circle into two components T_1, T_2, hence K would not be connected and there would be sets $K^1 := T_1 \cap K$ and $K^2 := T_2 \cap K$ and K must thus be a cluster, but in a cluster K_1 and K_2 would be close which would imply that K_1 and K_2 must also be close to T. This is a contradiction hence if K_A or K_B lie in X_i then they can not lie in X_{2-i}.
If K_A does not intersect X_2 then we define $a := \min(I_R(K_A), b')$ which implies that $Q_A := (\frac{\delta}{2}, R] \times (-b' + \frac{\delta}{2}, a - \frac{\delta}{2})$ is not close to K_A on the other hand if K_A does not intersect with X_1 then we define $a := \min(-S_R(K_A), b')$ and $Q_A := (\frac{\delta}{2}, R] \times (-a + \frac{\delta}{2}, b' - \frac{\delta}{2})$ is not close to K_A.
If K_B does not intersect X_2 then we define $b := \min(I_R(K_B), b')$ which implies that $Q_B := (\frac{\delta}{2}, R] \times (-b' + \frac{\delta}{2}, b - \frac{\delta}{2})$ is not close to K_B on the other hand if K_B does not intersect with X_1 then we define $b := \min(-S_R(K_B), b')$ and $Q_B := (\frac{\delta}{2}, R] \times (-b + \frac{\delta}{2}, b' - \frac{\delta}{2})$ is not close to K_B.
Now we define $Q' := Q_A \cap Q_B$ which is not close to either K_A or K_B. Then $Q' \subseteq (\frac{\delta}{2}, R] \times (-a + \frac{\delta}{2}, b - \frac{\delta}{2})$ or $Q' \subseteq (\frac{\delta}{2}, R] \times (-b + \frac{\delta}{2}, a - \frac{\delta}{2})$ we define Q as the one of the two rectangles such that $Q' \subseteq Q$. Note that by definition $|K_A| > k + \delta - 2a$ and $|K_B| > k + \delta - 2b$.
We know that $|\overline{L} \cap Q| > 2a + 2b - 2\delta$ and if S is the set of points close to $K_A \cup K_B$ then $|\overline{L} \cap S| > k + |K_A| + |K_B|$. Together this means, since Q and S are disjoint, that

$$3k > |\overline{L}| \geq |\overline{L} \cap Q| + |\overline{L} \cap S| \geq k + k + \delta - 2a + k + \delta - 2b + 2a + 2a - 2\delta = 3k$$

Which is a contradiction. □

Theorem 6.1. *For every circle X of radius $R > \frac{k}{2}$ and for $\varepsilon < \min(\frac{\delta k}{4\pi}, \frac{\delta^2}{8})$ there is no cut $\overline{L}$ of X with length k and margin of ε.*

Proof. Suppose there exists a cut $\overline{L}$ with length k and margin of ε. We use Lemma 6.4 on the cluster $\emptyset$ to show that there exists an outer component in $\overline{L}$. Hence there also exists an outer component of minimal radius K_A. We again use Lemma 6.4, this time using the cluster K_A, to show that there exists another outer component. We choose K_B with minimal radius from the outer components of $\overline{L} \setminus \{K_A\}$. Now K_A and K_B are either close or not close. In both cases either lemma 6.12 or lemma 6.13 imply that $\overline{L}$ is not a cut. Contradiction. □

To conclude this chapter we will prove a small lemma which we will need for the next section.

Lemma 6.14. *If we have a set L of strong outer components such that $|L| \leq k$ then $\lambda^2(L) \leq \lambda^2(M_{\frac{|L|}{2}})$ where $M_h := \{(x,y) \in X | y < h - R\}$ for $h < R$. Furthermore*

$$\lambda^2(M_h) = R^2 \mathit{arccos}\frac{R-h}{R} - (R-h)\sqrt{h(2R-h)}.$$

Proof. We start by showing the formula for $\lambda^2(M_h)$. For that consider

$$\begin{aligned}
\lambda^2(M_h) &= \int_{-R}^{h-R} 2\sqrt{R^2-y^2}\mathrm{d}y = \left[R^2\arctan\frac{y}{\sqrt{R^2-y^2}} + y\sqrt{R^2-y^2}\right]_{-R}^{h-R} \\
&= R^2\arctan\frac{h-R}{\sqrt{h(2R-h)}} - (R-h)\sqrt{h(2R-h)} + R^2\frac{\pi}{2} \\
&= R^2\arctan\frac{\sqrt{h(2R-h)}}{R-h} - (R-h)\sqrt{h(2R-h)} \\
&= R^2\arccos\frac{R-h}{R} - (R-h)\sqrt{h(2R-h)}
\end{aligned}$$

Next we show that $\lambda^2(L) \leq \lambda^2(G_{\frac{|L|}{2}})$ holds for sets L containing a single outer separation K. To see this let us rotate X in the same way as in Lemma 6.12 such that $S_L(K) = S_R(K) =: h - R$ with $0 < h < R$. From Lemma 6.6 ii) we get $|K| \geq 2R + 2(h - R)$ which means $h \leq \frac{|K|}{2}$. Since $S_L(K) = S_R(K) = h - R$ we have $H \subseteq M_h \subseteq M_{\frac{|K|}{2}}$ and thus $\lambda^2(K) \leq \lambda^2(M_{\frac{|K|}{2}})$.
Finally we only have to show that this also works if L contains more than one element since for $L = \emptyset$ the lemma is trivially true. For that we first calculate that

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 h}(\lambda^2(M_h)) = \frac{2(R-h)}{\sqrt{h(2R-h)}}$$

which is greater than 0 for $h \in (0, R)$. Since $\lambda^2(M_0) = 0$ this means that for any $a, b > 0$ with $a + b < R$ we get $\lambda^2(M_a) + \lambda^2(M_b) \leq \lambda^2(M_{a+b})$, which means that by induction and limit

$$\lambda^2(L) = \sum_{K\in L} \lambda^2(M_{\frac{|K|}{2}}) \leq \lambda^2(M_{\sum \frac{|K|}{2}}) = \lambda^2(M_{\frac{|L|}{2}})$$

□

Lemma 6.15. *For every circle X of radius $R > \frac{k}{2}$ and for any set $\overline{L}$ of disjoint inner and outer components contained in X which fulfill the following properties:*

- *The components in $\overline{L}$ are disjoint,*
- $\sum_{K \in \overline{L}} |K| \leq k$*,*
- $\lambda^2(K) < \frac{\pi R^2}{2}$*,*

it follows that $\lambda^2(\overline{L}) < \frac{\pi R^2}{2}$.

Proof. Let us consider a disjoint partition $\overline{L} = I \cup O$ into the weak inner components and the strong outer components of $\overline{L}$ and suppose $\lambda^2(\overline{L}) \geq \frac{\pi R^2}{2}$. Let $a := |I|$ then $b := |O| \leq k - a$. Let us define

$$N_h := \{(x, y) \in X | x < 0 \text{ and } x^2 + y^2 > (R - h)^2\}$$

then clearly $M_h \subseteq N_h$. Further $\lambda^2(N_h) = \frac{\pi R^2}{2} - \frac{\pi (R-h)^2}{2}$. This means that

$$\lambda^2(I) \geq \frac{\pi R^2}{2} - \lambda^2(O) \geq \frac{\pi R^2}{2} - \lambda^2(M_{\frac{a}{2}}) \geq \frac{\pi R^2}{2} - \lambda^2(N_{\frac{a}{2}}) = \frac{\pi (R - \frac{a}{2})^2}{2}.$$

But then $\lambda^2(I) \geq \frac{\pi b^2}{8}$ which means that using Lemma 6.1 we get that

$$\frac{\pi b^2}{8} \leq \lambda^2(I) \leq \frac{b^2}{4\pi} \Leftrightarrow \frac{\pi}{2} \leq \frac{1}{\pi}$$

Which is a contradiction. $\square$

7 Tangles in the euclidean plane

We will now apply the results of the previous sections to the euclidean space and look at one sensible way to define tangles.

The weight function in the general case could represent a kind of density of the shape at certain points but in the euclidean plane we will just be looking at the geometric length of the polygonal arc connecting the endpoints of the edge as its length. More precisely if a graph contains an edge e which is connected by a polygonal arc going through the points $a = a_0, a_1, ..., a_n = b$. Then we define

$$d(e) := \sum_{i=0}^{n-1} d'(a_i, a_{i+1})$$

where d' is the euclidean distance in $\mathbb{R}^2$.

For looking at areas we will use the Lebesgue measure λ^2.
We are going to use d as the weight function for our separations. This means that the order of a separation is the length of the boundary. Specifically for a simple separation it is the length of the cutting path. Now we want to define the triangulation system D^Δ that we want to use for our tangles.

$$D^\Delta := \{(G, d) | G \text{ is a triangulation of } D \text{ with } \varepsilon(G) > 0\}$$

Lemma 7.1. *D^Δ is a system of triangulations.*

Proof. First we need to show that the weight functions are consistent. To do this let us consider $(G, w_G), (H, w_H) \in D^\Delta$ where H is finer than G. Let $e \in E(G)$ be an edge of G and let $E := \{e' \in E(H) | e' \subseteq e\}$. Then by definition of $d(e)$ it is not hard to see that $d(E) = d(e)$ since except for finitely many points the union of E is equal to e and thus by definition of $d(e)$ as the geometric length, both sets have the same length.
Next we need to show that if there are two triangulations $(G_1, w_{G_1}), (G_2, w_{G_2}) \in D^\Delta$ that there exists a $(G', w_{G'}) \in D^\Delta$ where G' is finer than both G_1 and G_2. But this follows instantly from lemma 2.1.
Lastly we only need to show that for any $\varepsilon' < \varepsilon$ there exists a $(G, w_G) \in D^\Delta$ such that $\varepsilon(G) > \varepsilon'$. From measure theory we know that since $D \setminus \partial D$ is an open set we can approximate it by using closed sets. Hence there exists a closed set $Y \subseteq D \setminus \partial D$ such that $\lambda^2(D \setminus Y) > \varepsilon - \varepsilon'$. We can now find a triangulation G which contains the entirety of Y inside of it by choosing three points along the boundary of D and connecting them by polygonal arcs which do not intersect Y. Hence

$$\varepsilon(G) = \varepsilon - \lambda^2(D \setminus G) \geq \varepsilon - \lambda^2(D \setminus Y) = \varepsilon - \varepsilon + \varepsilon' = \varepsilon'$$

$\square$

Now that we have found the system of triangulations that we wish to use we can look at the structure of the tangles we get. We can directly apply the tree-of-tangles theorem and the tangle-tree duality theorem.

Theorem 7.1. *Let T be a set of tangles of D^Δ then there is a nested set S_T of simple separations of a triangulation $G \in D^\Delta$ such that*

(i) any two tangles in T are distinguished by some separation in S_T;

(ii) every separation in S_T distinguishes a pair of tangles in T.

Theorem 7.2. *For every $\varepsilon > 0$ and $k \in \mathbb{N}$ exactly one of the following assertions holds:*

(i) There exists an $\mathcal{F}_k^\varepsilon$-tangle of D^Δ.

(ii) There exists a triangulation $G \in D^\Delta$ with an $\overrightarrow{S_k}(G)$-tree over $\mathcal{F}_k^\varepsilon$.

7.1 Deciders

In the previous section we have looked at the structure of tangles of D. In this section we find a method with which we can prove that a tangle exists.

Definition 7.1. *A decider of a tangle τ is a set $X \subseteq \mathbb{R}^2$ such that for all $(A, B) \in \tau$ it follows that $\lambda^2(\bigcup A \cap X) < \lambda^2(\bigcup B \cap X)$.*

We can use arbitrary sets X to orient the separations of a separation system S by letting for all $s = \{(A, B), (B, A)\} \in S$ the set $X(S)$ be the orientation of a subset of S that contains (A, B) iff $\lambda^2(\bigcup A \cap X) < \lambda^2(\bigcup B \cap X)$. The set $X(S)$ orients every separation for which $\lambda^2(\bigcup A \cap X) \neq \lambda^2(\bigcup B \cap X)$.

Lemma 7.2. *For any set $X \subseteq \mathbb{R}^2$ and separation system S the $X(S)$ is consistent. Note that $X(S)$ does not orient separations for which $\lambda^2(\bigcup A \cap X) = \lambda^2(\bigcup B \cap X)$.*

Proof. Let $(A, B) \in X(S)$ and let $(C, D) \in S$ with $(C, D) \leq (A, B)$. Then

$$\lambda^2(\bigcup C \cap X) \leq \lambda^2(\bigcup A \cap X) < \lambda^2(\bigcup B \cap X) \leq \lambda^2(\bigcup D \cap X).$$

Hence $(C, D) \in X(S)$ and thus $X(S)$ is consistent. □

Obviously if X is a decider of a $\mathcal{F}_k^\varepsilon$-tangle τ then it follows that $\tau = X(S_k)$. On the other hand, does every tangle have a decider? It is speculated that indeed every tangle as defined in this chapter has a decider.

Conjecture 7.1. *For every $\mathcal{F}_k^\varepsilon$ tangle τ of a disk D with respect to the triangulation D^Δ, there exists a set $X \subseteq D$ such that $\tau(G) = X(S_k(G))$.*

Even if in general not every tangle has a decider, there are certain important cases where we do have a decider. Circles of certain size induce tangles. This can most easily be seen with large circles where each separation has $\frac{3}{4}$ of the area of the circle on its right side. In that case every star must have $\frac{1}{4}$ of the area of the circle in the intersection of the right sides and thus such a circle is in fact a decider. Interestingly enough we do not need large circles but any size greater than k suffices as the following theorem shows.

Theorem 7.3. *Let D be a disk that contains a closed circle X of radius $R > \frac{k}{2}$ then τ_X is an $\mathcal{F}_k^\varepsilon$-tangle for every $\varepsilon < \min(\frac{\delta k}{4\pi}, \frac{\delta^2}{8})$ where we define τ_X in the following way: For every $G \in D^\Delta$ we let $\tau_X(G) = X(S_k(G))$.*

Proof. We know that $\tau_X(G)$ is a consistent orientation since $S_k(G)$ are not able to cut X in half, so what we need to show is that $\tau_X(G)$ does not contain a forbidden star. Suppose there was a $G \in D^\Delta$ and an $\varepsilon < \min(\frac{\delta k}{4\pi}, \frac{\delta^2}{8})$ that contained a forbidden star $\sigma \in \mathcal{F}_k^{\varepsilon(G)}$.
Let σ be a star containing three nested separations (A, A_0), (B, B_0) and (C, C_0). Since σ is a star we know that A, B and C are disjoint. Hence there are disjoint components (using our definition of connectedness of sets of faces) $A^1, ..., A^{n_a}$ of A, $B^1, ..., B^{n_b}$ of B and $C^1, ..., C^{n_c}$ of C. We write

$$L := \{A^1, ..., A^{n_a}, B^1, ..., B^{n_b}, C^1, ..., C^{n_c}\}$$

for the set of all of these components and $\overline{L'} := \{\bigcup K \cap X | K \in L\}$ for the point sets contained in these components, where in $\bigcup K$ we also include edges which lie on the boundary of two faces in K and vertices which only lie on boundaries of edges which we included.
First we need to show that these elements $K \in \overline{L'}$ are open. For that let us take a point $x \in K$. If x lies inside of a face then we can find a neighbourhood which lies completely in that face. If x lies

on an edge, then the edge lies on the frontier of two faces, hence every small enough neighbourhood only contains points from the two faces and the edge. If x on the other hand is a vertex then all of the neighbouring edges are contained in K and thus also all of the faces neighbouring x. This means that a small enough neighbourhood of x contains only the neighbouring edges and faces and itself.

Next we want to show that K is topologically connected. For that we first notice that it suffices to show that any two points x, y which lie inside of faces are path-connected since any point not in a face can be connected to a face of K. Suppose x lies in a face f_x and y lies in a face f_y then there exists a sequence of faces in K $f_x = f_1, ..., f_n = f_y$ where f_i and f_{i+1} have a common edge e_i. We can now connect x by a path to the edge e_1 then we can connect e_i through f_{i+1} to the edge e_{i+1} for all $i \in \{1, ..., n-1\}$ which means that we can connect x to f_y where we can then find a connection to y. To define $\overline{L}$ we for every element $K \in \overline{L'}$ the components $\mathcal{C}$ of $X \setminus \overline{K}$. If there exists a component $C' \in \mathcal{C}$ with area greater than half of $\lambda^2(X)$ then we can replace K by $X \setminus \overline{C'}$. Otherwise all elements of $\mathcal{C}$ have to be inner and outer components which cover more than half of X, but by Lemma 6.15 this is impossible. We define $\overline{L}$ as the set we are left with after replacing every element of K with multiple components in $X \setminus \overline{K}$.

We now show that the elements of $\overline{L}$ are disjoint inner or outer components. This can now be seen from the fact that since $X \setminus \overline{K}$ only contains one component for all $K \in \overline{L}$ we can find a path P such that $Im(P) = \partial K$, similarly to the simple separations in chapter 3.

This would mean that $\overline{L}$ is a cut as X can be covered except for an area of size less than $\varepsilon(G) + \lambda^2(X \setminus G) < \varepsilon$ by inner and outer components which have boundary less than k by definition of the separations and which have area less than $\frac{\pi R^2}{2}$ by virtue of being separations contained in $\tau_X(G)$. This is a contradiction. $\square$

This theorem could have been shown a lot easier by using the following conjecture.

Conjecture 7.2. *For every k exists an $\varepsilon > 0$ such that every consistent orientation of simple separations that avoids $\mathcal{F}_k^\varepsilon$ is induced by a consistent orientation of S_k that avoids $\mathcal{F}_k^\varepsilon$.*

We have already seen that this does not hold with arbitrary order functions but if we define the order functions as in this chapter by using the euclidean distance of the endpoints of each edge it is conjectured to hold. This would have made the proof of the theorem much simpler as we would not need to consider as many cases.

To conclude this thesis, we will prove the result promised in the introduction. Suppose we have a shape containing different circles which can be pairwise separated, then we want to show that each of these circles corresponds to a different tangle. To do this we first define what it means for sets to be separated.

Definition 7.2. *Let $X_1, X_2 \subseteq \mathbb{R}^2$ then we say that X_1 and X_2 can be k-separated if there exists a triangulation $G(X_1, X_2) \in D^\Delta$ with a separation $(A, B) < k$ such that $\lambda^2(\bigcup B \cap X_2) > \frac{1}{2}\lambda^2(X_2)$ and $\lambda^2(\bigcup A \cap X_1) > \frac{1}{2}\lambda^2(X_1)$*

Corollary 7.1. *Let $k > 0$ and D be a disk containing closed circles $X_i := \{x \in \mathbb{R}^2 || |x - o_i|| \leq r_i\}$ with midpoints $o_i \in \mathbb{R}^2$ and radii $r_i > \frac{k}{2}$ which can be pairwise k-separated from each other, then τ_{X_i} are distinct $\mathcal{F}_k^\varepsilon$-tangles of the disk where $\varepsilon \leq \min_{i \in I}\{\frac{\delta_i k}{4\pi}, \frac{\delta_i^2}{8}\}$ with $\delta_i := r_i - \frac{k}{2}$.*

Proof. From Theorem 7.3 we know that the τ_{X_i} are $\mathcal{F}_k^\varepsilon$-tangles. The only thing left to show is that two tangles τ_{X_i}, τ_{X_j} are distinct for $i \neq j$. For that consider $G(X_i, X_j)$ and the separation $(A, B) \in \overrightarrow{S}_k(G(X_i, G_j))$ which fulfills $\lambda^2(\bigcup B \cap X_j) > \frac{1}{2}\lambda^2(X_j)$ and $\lambda^2(\bigcup A \cap X_i) > \frac{1}{2}\lambda^2(X_i)$. Then because of $\lambda^2(\bigcup B \cap X_j) + \lambda^2(\bigcup A \cap X_j) \leq \lambda^2(X_j)$ we have $\lambda^2(\bigcup A \cap X_j) < \frac{1}{2}\lambda^2(X_j) < \lambda^2(\bigcup B \cap X_j)$ which means that $(A, B) \in X_j(S_k(G)) = \tau_{X_j}(G)$. We can analogously show that $(B, A) \in X_i(S_k(G)) = \tau_{X_i}(G)$. This proves the corollary. $\square$

8 Outlook

In this thesis we have used tangles to analyse the structure of shapes in the plane $\mathbb{R}^2$. We have restricted ourselves by only looking at those shapes that are homeomorphic to the unit disk since shapes that have holes are intuitively less likely to have the tree structure that the tangles framework is built to detect. It might be fruitful though to look at more general subsets of the plane or even of other more general surfaces. It could very well be the case that similar theorems to the one we have proven in this thesis hold for these other cases and this could have interesting implications for the topological study of surfaces.

This thesis is another one that uses bipartitions for the tangle framework. In recent years there have been many interesting results found using this method and we hope that this thesis inspires others to also look at these possibilities. For people interested we recommend looking at [5] for an application in the social sciences.

Another example for using bipartitions to apply the tangle framework would be [4] were they use a similiar method to the one used in this paper to analyse images. They use bipartitions on the pixels which we can imagine in the language of this thesis as the faces of a triangulation. Thus this thesis is also applicable to that context and it could be interesting to see if the results from this thesis could help in that context as well.

One thing that we did not look at in this thesis was classifying all of the tangles that we find, not even the ones in the case of the euclidean plane. If one showed that every tangle is induced by an orienting subset that would be a very impressive result and could be applied to many other examples of tangles that would greatly benefit from understanding what a tangle ”looks like” and thus could help in applying the tangle framework for other purposes.

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