Übung 2 zur Graphentheorie

Ihre Lösung zu der mit * markierten Aufgabe geben Sie bitte am Anfang Ihrer jeweiligen Übungsgruppe ab. Ihre Lösungen zu den anderen Aufgaben bringen Sie bitte präsentierbereit zur Übung mit.

1. Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

2. Let α, β be two graph invariants with positive integer values. Formalize the two statements below, and show that each implies the other:

   (i) β is bounded above by a function of α;

   (ii) α can be forced up by making β large enough.

   Show that the statement

   (iii) α is bounded below by a function of β

   is not equivalent to (i) and (ii). Which small change will make it so?

3. Show that every graph that is k-edge-connected but loses this property whenever we delete an edge has a vertex of degree k.

4. Find two very short proofs, one by induction and another without, that every tree has more leaves than vertices of degree at least 3.

5.* Prove Theorem 1.5.1.

Die folgende Aufgabe ist eine Bonusaufgabe, die also nicht zu der zu erreichende Gesamtpunktzahl hinzuzählt.

6.+ Show for every $k \in \mathbb{N}$ that every graph of minimum degree $2k$ has a $(k+1)$-edge-connected subgraph.

Hinweise

1. Consider how the path intersects C. Where can you see cycles, and can they all be short?
2. Rephrase (i) and (ii) as statements about the existence of two $\mathbb{N} \to \mathbb{N}$ functions. To show the equivalence, express each of these functions in terms of the other. Show that (iii) may hold even if (i) and (ii) do not, and strengthen (iii) to remedy this.
3. Chase that vertex by considering separations with one side as small as possible.
4. Average degree.

5.* Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) from the definitions of the relevant concepts.

6.+ If G itself is not $(k+1)$-edge-connected, it has a vertex partition into sets A, B such that A sends at most k edges to B. If $G[A]$ is not $(k+1)$-edge-connected, it has a similar partition. Continuing in this way, we obtain vertex sets $A_0 \supsetneq A_1 \supsetneq \ldots$. At most how many edges do they send to the entire rest of G if we choose them well? Is the number of these edges bounded by a constant depending only on k?