Übung 1 zur Graphentheorie

Ihre Lösung zu der mit * markierten Aufgabe legen Sie bis zum 10. April um 12 Uhr in die mit GT1 beschrifteten Box im Schrank im Raum 227 ab. Ihre Lösungen zu den anderen Aufgaben bringen Sie bitte präsentierbereit zur Übung mit.

1. Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, V is the set of all 0–1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the *d-dimensional cube*. Determine the average degree, number of edges, diameter, girth and circumference of this graph.

   (Hint for the circumference: induction on d.)

2.$^-$ Show that the components of a graph partition its vertex set. (In other words, show that every vertex belongs to exactly one component.)

3. Let F, F' be forests on the same set of vertices, with $\|F\| < \|F'\|$. Show that F' has an edge $e \notin F$ such that $F + e$ is again a forest.

4. Determine $\kappa(G)$ and $\lambda(G)$ for G equal to the path P^m, the complete graph K^m, the complete bipartite graph $K_{m,n}$, the cycle C^n and the d-dimensional cube (Exercise 1) for all $m \geqslant 1$ and $d, n \geqslant 3$.

5.* Let $\mathcal{T}$ be a non-empty set of subtrees of a tree T, and let $k \geq 1$ be an integer.

   (i) Show that if the trees in $\mathcal{T}$ have pairwise non-empty intersection then their overall intersection $\bigcap \mathcal{T} = \bigcap_{T \in \mathcal{T}} T$ is non-empty.

   (ii) Show that either $\mathcal{T}$ contains k disjoint trees or there is a set of at most $k-1$ vertices of T meeting every tree in $\mathcal{T}$.

Hinweise

1. Average degree and edges: consider the vertex degrees. Diameter: how do you determine the distance between two vertices from the corresponding 0–1 sequences? Girth: does the graph have a cycle of length 3? Circumference: partition the d-dimensional cube into cubes of lower dimension and use induction.

2.$^-$ Assume the contrary, and derive a contradiction.

3. Corollary 1.5.3.

4. For each type of graph, the solution requires separate proofs of (coinciding) upper and a lower bounds. For the cube, use induction on n.

5.* The easiest solution is to apply induction on $|T|$. What kind of vertex of T will be best to delete in the induction step? Induction on $|\mathcal{T}|$ is also possible.