ON THE LAGRANGIAN EMBEDDING OF THE KLEIN BOTTLE

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Abstract. The notion of symplectic manifold takes its origin in mechanics. Trying to write down the canonical formalism in a coordinate-free way, one comes naturally to the definition of *symplectic structure* on manifolds. It is given by a closed non-degenerate 2-form ω, and such a symplectic manifold (X, ω) can be seen as a phase space of a Hamiltonian dynamical system. This point of view allows to pose global topological questions, for example, about special properties of (compact) manifolds admitting a symplectic structure, or what kind of non-isomorphic symplectic structures on a given manifold X one could have.

Symplectic manifolds admit locally special class of *Lagrangian submanifolds* $L \subset (X, \omega)$ which are especially interesting from the point of view of mechanics and symplectic geometry. They are characterised by the properties $\omega|_L \equiv 0$ and $\dim L = \frac{1}{2} \dim X$. For example, completely integrable systems can be viewed as foliations in Lagrangian tori.

In 1986 Givental' had shown that every compact embedded Lagrangian surface V in $\mathbb{R}^4$ satisfies the condition $\chi(V) = 0$ in the orientable case and $\chi(V) \equiv 0 \mathsf{mod} 2$ in the non-orientable case, and had constructed explicit examples of non-orientable Lagrangian surface $V \subset \mathbb{R}^4$ with $\chi(V) = -4k < 0$. The result was strengthened by M. Audin in 1990 who showed the condition $\chi(V) \equiv 0 \mathsf{mod} 4$ in the non-orientable case. However, the remaining question

Does there exists a Lagrangian embedding of the Klein bottle in $\mathbb{R}^4$*?*

was open until 2007.

In my talk I explain the ideas behind the proof of non-existence of Lagrangian embedding of the Klein bottle in $\mathbb{R}^4$ and in $\mathbb{CP}^2$.

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