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Complex version of Catalan's problem

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Abstract. Catalan's problem consists of finding integer neighbors $j, j+1$ which are both powers of some - possibly different - integers ≥ 2. We will call such a pair a *Catalan pair*. The only pairs found so far are $(-1,0)$, $(0,1)$, $(8,9)$ and Eugène Catalań (Brügge, 1814 - Lüttich, 1894) conjectured 1844 that there are no other pairs. New research implies that if there is another pair at all, the corresponding powers must be very large. Details are given in the main text. The problem apparently lacks symmetry, in so far that a Catalan pair $(j, j+1)$ does not imply that $(-j-1, -j)$ is also a Catalan pair. If we look at $(8,9)$, then -9 is apparently not a square of a real number, but if we admit complex numbers, then $-9 = (3\,\mathrm{i})^2$ is a square of a complex integer. So by admitting complex integers, we obtain more symmetry. The definition of neighbors of complex integers will be introduced in a natural way. We not only gain symmetry, but we also find new pairs of complex integers showing that the real and complex problem of Catalan are not the same. There is also a small section on the density of the prime powers in the complex plane. We also find some complex integers, which are sandwiched between powers (Fermat numbers).

Subject Classifications: AMS: 00A08

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1. Introduction

Catalan's problem is well explained in an article by Roy & Mignotte [1997]. That paper contains new results and further hints to recent literature as well. It belongs to a series of papers on experimental mathematics collected by Levi [1997]. The problem consists of finding a pair of consecutive integers $(j, j+1)$, called a *Catalan pair*, such that both j and $j+1$ are (possibly different) powers of integers, i.e. we require $j = \ell^p, j+1 = m^q$, where $p, q \in \{2, 3, ...\}$ and $\ell, m \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. If $p \geq 2$ is not a prime, say $p = p_1 p_2$ where p_1 is a prime, then $j = \ell^p = \ell^{p_1 p_2} = (\ell^{p_2})^{p_1}$. Therefore, we may require, that the powers p, q are primes. Catalan's problem can be put into the form of an equation

$$m^q - \ell^p = 1, \quad \ell, m \in \mathbb{Z}, \quad p, q \text{ primes}, \tag{1.1}$$

where ℓ, m, p, q are unknowns. If we have found a quadruple (ℓ, m, p, q) which solves (1.1) we call the quadruple as well as the pair $(j, j+1) = (\ell^p, m^q)$ a *solution* of Catalan's problem. We shall call the three obvious solutions $(-1, 0)$, $(0, 1)$, $(8, 9)$ the *trivial solutions*.

According to Brindza, Győry & Tijdeman [1986], equation (1.1) has only finitely many solutions in any algebraic number field, provided m and ℓ are not roots of unity and $pq > 4$.

Theorem 1.1. (Roy & Mignotte [1997]) There is a (large, time dependent*) odd prime π_0 such that Catalan's problem (1.1) has no solution (ℓ, m, p, q) for all primes $p, q < \pi_0$ apart from the three trivial ones.

This means in particular that there is not much hope in finding another Catalan pair by using a computer. The aim here is to introduce some symmetry into the problem of Catalan, by admitting complex integers and finding properties of this new problem.

2. Complex Catalan pairs

By $\mathbb{G} = \mathbb{Z} + \mathrm{i}\mathbb{Z}$ we mean the set of all complex numbers with integer coordinates. We call these numbers *complex integers*. We also identify $\mathbb{G}$ with $\mathbb{Z}^2$ in an obvious manner. A *mesh* in $\mathbb{G}$ is a square with the corners $z, z+1, z+1+\mathrm{i}, z+\mathrm{i}$, where z is an arbitrary complex integer.

Definition 2.1. Two distinct complex integers j, k are called *neighbors* if they are vertices of the same mesh in $\mathbb{G}$. In other words, $j, k \in \mathbb{G}$ are neighbors if

$$|j - k| \in \{1, \sqrt{2}\} \quad \Leftrightarrow \quad \max\{|\Re(j-k)|, |\Im(j-k)|\} = 1, \tag{2.1}$$

where $\Re$, $\Im$ denote the real and the imaginary part of a complex number, respectively.

Definition 2.2. Let (j, k) form a pair of complex integers that are neighbors. If there are two complex integers $\ell, m \in \mathbb{G}$ and two primes p, q such that

$$j = \ell^p, \quad k = m^q, \tag{2.2}$$

* In the quoted paper $\pi_0 = 10651$, which is prime number 1299. However, Maurice Mignotte, in a message dated March 1997, says that it has been proven that π_0 is approximately 10^5 and he announces a proof that $\pi_0 \approx 2 \cdot 10^5$.

then the pair (j,k) is called a *Catalan pair of complex integers*, or if there is no confusion with the ordinary Catalan pairs it will be also called Catalan pair.

So, a Catalan pair is a set of two complex integers which are both neighbors and powers.

Theorem 2.3. Let (j,k) be a Catalan pair of complex numbers with $j=\ell^p, k=m^q, p,q$ prime and $\ell,m\in\mathbb{G}$. Then

a) $(-j,-k)$ is a Catalan pair,

b) $(\overline{j},\overline{k})$ is a Catalan pair, where the $\overline{}$ denotes complex conjugation.

c) If p,q are both odd, then $(\mathrm{i}j,\mathrm{i}k)$ is also a Catalan pair.

Proof: We have to prove, that the new pairs are formed out of neighbors, and that the two parts are again powers. First, it is immediately clear, that the new pairs are also formed out of neighbors. In order to show that they are powers, we give a proof only for j, the proof for k follows analogously. We have

$$-j=\begin{cases}(\mathrm{i}\ell)^2 & \text{if } p=2,\\ (-\ell)^p & \text{if } p \text{ is odd.}\end{cases}$$

That proves a). Now, $j=\ell^p$ implies $\overline{j}=\overline{\ell^p}=(\overline{\ell})^p$, which proves b). Let p be an odd prime. Then c) follows from

$$\mathrm{i}=\begin{cases}\mathrm{i}^p & \text{for } p \bmod 4=1,\\ (-\mathrm{i})^p & \text{for } p \bmod 4=3.\end{cases}$$ ∎

Using the already known trivial pairs, one can find new ones by symmetry, namely $(-9,-8)$, $(0,\mathrm{i})$, $(0,-\mathrm{i})$, $(\mathrm{i},1)$, $(-\mathrm{i},-1)$, $(-\mathrm{i},1)$, $(\mathrm{i},-1)$. Altogether we now have 10 trivial ones. Part c) of Theorem 2.3 gives particular information of Catalan pairs on the y-axis. It implies that pairs on the y-axis defined by odd prime powers would also define pairs on the x-axis and vice versa. Thus, the problems of finding pairs on the x-axis and finding pairs on the y-axis are the same if the corresponding prime powers are both odd. The case where one of the powers is two will be treated in the next section.

Corollary 2.4. Let (j,k) be a complex Catalan pair. Then $(-j,-k)$, $(\overline{j},\overline{k})$, $(-\overline{j},-\overline{k})$ are also Catalan pairs. If (j,k) is neither purely real nor purely imaginary, then the four pairs

$$(j,k),\quad(-j,-k),\quad(\overline{j},\overline{k}),\quad(-\overline{j},-\overline{k}) \tag{2.3}$$

are all different. In case (j,k) is purely real or purely imaginary, then

$$(j,k), \quad (-j,-k) \tag{2.4}$$

are different.

Proof: Follows immediately from a), b) of Theorem 2.3. ∎

To our surprise, we were able to find two such pairs (j,k), namely

$$j = 3+4\mathrm{i} = (2+\mathrm{i})^2, \quad k = 4+4\mathrm{i} = (-1-\mathrm{i})^5, \tag{2.5}$$

together with the three companions according to (2.3)

$$j = -3+4\mathrm{i} = (1+2\mathrm{i})^2, \quad k = -4+4\mathrm{i} = (1-\mathrm{i})^5, \tag{2.6}$$

$$j = -3-4\mathrm{i} = (-1+2\mathrm{i})^2, \quad k = -4-4\mathrm{i} = (1+\mathrm{i})^5, \tag{2.7}$$

$$j = 3-4\mathrm{i} = (2-\mathrm{i})^2, \quad k = 4-4\mathrm{i} = (-1+\mathrm{i})^5, \tag{2.8}$$

and

$$(\mathrm{i}, 2\mathrm{i}) = ((-\mathrm{i})^3, (1+\mathrm{i})^2), \tag{2.9}$$

and according to (2.4)

$$(-\mathrm{i}, -2\mathrm{i}) = (\mathrm{i}^3, (-1+\mathrm{i})^2). \tag{2.10}$$

Counting all pairs together, we have the ten trivial ones and the six listed in (2.5) to (2.10), see Figure 2.1. A simple computational check shows that these are the only ones in $[-100,100]^2$. The symmetry allows us to search for pairs (j,k) only in the first quadrant of $\mathbb{G}$. However, if we are searching for ℓ, m in the representation (2.2) with property (2.1), then the coordinates of ℓ and m are not restricted to a certain quadrant of $\mathbb{G}$, as we can see from the above given pairs.

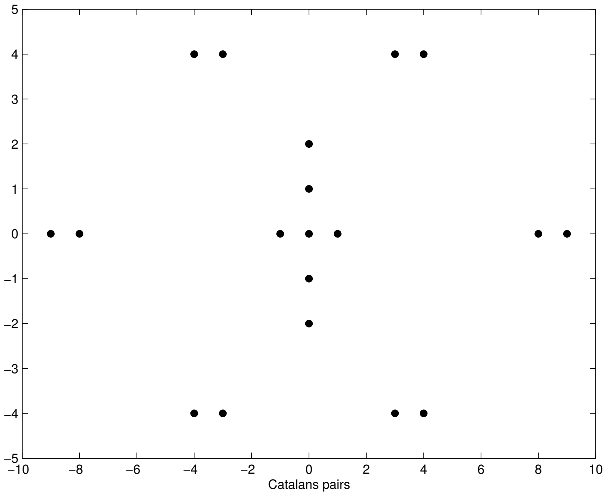

Figure 2.1. The 16 complex Catalan Pairs in $[-100, 100]^2$

3. An algorithm for detecting pairs

We describe a method here but cannot offer a fast solution. The method may also be stated independently of Catalan's problem.

Since we are looking for pairs (j, k) of complex integers with two properties, namely (a) being neighbors and (b) being powers, we may start with a set of complex integers that have already one of these properties and then make a search in this set for the other property. Though there is a slight difference. Being a power involves only one number, whereas the definition of neighborhood requires of course a pair of numbers.

In principle we could do the following.

Algorithm 1. Step 1: We sweep the first quadrant of $\mathbb{G}$ say in the form $j = j_1 + \mathrm{i}j_2$, $j_1, j_2 \geq 0$, $j_1 + j_2 = n$, $n = 0, 1, \ldots, N$ where N is a given (large) integer and put those j that are powers into a list, say $j^{(1)}, j^{(2)}, \ldots, j^{(M)}$.

Step 2: Find all neighbors within that list.

Algorithm 2. Step 1: Create all powers $j = j_1 + \mathrm{i}j_2 = (\ell_1 + \mathrm{i}\ell_2)^p$, $p = 2, 3, 5, 7, \ldots, P$, $-L \leq \ell_1, \ell_2 \leq L$, where P is a given (large) prime and L a given positive integer. There are some remarks on the size of these numbers in the next section. The result is again a list $j^{(1)}, j^{(2)}, \ldots, j^{(M)}$ of M powers distributed in $\mathbb{G}$. According to the symmetry given in Theorem 2.3 one can immediately delete those powers which are outside the first quadrant. So we assume that our list contains only numbers in the first quadrant of $\mathbb{G}$.

Step 2: Find all neighbors within that list.

In both algorithms, the second steps coincide. The first step of Algorithm 2 can be executed straightforwardly, whereas the first step of Algorithm 1 may create some unnecessary computational work. We also may assume, that the list does not contain multiple entries. This can be accomplished easily by the MATLAB command `unique` or by some other means involving an ordering process.

We will give some hints on how to solve Step 2. We assume that a (long) list of complex integers $j^{(1)}, j^{(2)}, \ldots, j^{(M)}$ is given, having no multiple entries. In principle one could compare list element $j^{(k)}$ with all consecutive list members, $k = 1, 2, \ldots, M-1$ and so find all neighboring pairs. This algorithm of comparing all pairs works correctly in principle. However, for larger M (say from $M = 10^5$ on), this consumes too much time.

Another idea is to eliminate all list elements which cannot possibly be a member of a pair. For this purpose one could project the list elements to the x- or y-axis and then eliminate those elements which do not have a neighbor on this projection. However, in our experience this only gives sufficient results for smaller M, say several hundred.

By using the algorithm of comparing all pairs in $[-N, N]^2$ with $N = 1000$ we found two new pairs, namely

$$j = 242\mathrm{i} = (11 + 11\mathrm{i})^2, \; k = 243\mathrm{i} = (3\mathrm{i})^5, \tag{3.1}$$

$$k = -243\mathrm{i} = (-3\mathrm{i})^5, \; j = -242\mathrm{i} = (11 - 11\mathrm{i})^2. \tag{3.2}$$

These are again Catalan pairs on the y-axis where one of the powers is two. Neighbors j, k on the y-axis obey the following equations:

$$j := (x+\mathrm{i}y)^p = \ell\mathrm{i}, \quad k := (u+\mathrm{i}v)^q = (\ell\pm1)\mathrm{i}, \quad x, y, u, v, \ell \in \mathbb{Z}, p, q \text{ primes}. \tag{3.3}$$

One can easily see, that $p = q = 2$ gives no solution. The special case $p = 2$, q odd reads

$$\begin{gathered} x^2 - y^2 = 0, \; 2xy = \ell; \\ u \sum_{r=\text{even}}^{q-1} \binom{q}{r} u^{q-r-1}(\mathrm{i}v)^r = 0, \; \sum_{r=\text{odd}}^{q} \binom{q}{r} u^{q-r}\mathrm{i}^{r-1}v^r = \ell \pm 1. \end{gathered} \tag{3.4}$$

The second set of equations has the special solution

$$u = 0,\ \mathrm{i}^{q-1} v^q = \ell \pm 1.$$

This results finally in finding integers $x \neq 1$, $v \neq 1$ with

$$|v^q - 2x^2| = 1, \quad q \text{ odd prime.} \tag{3.5}$$

The two solutions we found in the range $|l| \leq 10^{10}$, $3 \leq q \leq 67$ (and hence $2 \leq x \leq 71000$) are $q = 3:\ x = 78, v = 23$, $q = 5:\ x = 11, v = 3$. All pairs found are listed in Table 5.1. The case that both powers p, q are odd reduces essentially to the real case and has already been discussed in the previous section.

4. The density of powers

Since we are searching in the set of powers for Catalan pairs, it is interesting to know how many powers exist in a certain region. There are mainly two regions of interest: disks D_N with radius N, centered at the origin and squares $Q_N = [-N, N]^2$, where in both cases we assume that $N \in \mathbb{N}$.

We introduce some notation. Let $S \subset \mathbb{C}$ be a compact set and p a prime. We denote by

$$\mu_p(S) := \#\{z \in S\ :\ z = u^p,\ u \in \mathbb{G}\} \tag{4.1}$$

the number of p-powers in S, and by

$$\mu(S) := \#\{z \in S\ :\ z = u^p,\ p \text{ prime},\ u \in \mathbb{G}\} \tag{4.2}$$

the number of all prime powers in S. It is easy to find some estimates for these numbers in the case when $S = Q_N$.

Lemma 4.1. Let $N \in \mathbb{N}$ and p prime be given. Then

$$u^p \in Q_N,\ u \notin \{0, \pm 1, \pm \mathrm{i}\} \quad \Longrightarrow \quad p \leq 1 + \frac{2}{\log 2} \log N. \tag{4.3}$$

Proof: For any complex number z let $||z||_\infty := \max\{|\Re z|, |\Im z|\}$, $||z||_2 := \sqrt{|\Re z|^2 + |\Im z|^2}$. Let $u^p \in Q_N \Leftrightarrow ||u^p||_\infty \leq N \Rightarrow ||u^p||_2 = ||u||_2^p \leq \sqrt{2} N$. For $u = 1 + \mathrm{i}$ this gives $2^{p/2} \leq \sqrt{2} N$. If we multiply by $1/\sqrt{2}$ and take logarithms, (with respect to any basis) we obtain $(p-1)/2 \leq \log N / \log 2$, which is essentially the inequality given in (4.3). ∎

This means, given $N \in \mathbb{N}$, we need to consider only primes

$$p \le p_0(N) := \lfloor 1 + \frac{2}{\log 2} \log N \rfloor, \tag{4.4}$$

if we search for neighboring pairs in Q_N, where for $x \in \mathbb{R}$ the number $\lfloor x \rfloor$ is the largest integer $m \le x$. In particular, if $N = 10^n$, $n = 1, 2, \ldots, 16$ only the primes up to $p = 7, 13, 19, 23, 31, 37, 47, 53, 59, 67, 73, 79, 83, 89, 97, 107$, respectively, need to be considered.

Lemma 4.2. Let N, p be given as in the previous Lemma. Then

$$u^p \in Q_N \implies u \in Q_{(\sqrt{2}N)^{1/p}}. \tag{4.5}$$

Proof: : $u^p \in Q_N \Rightarrow u^p \in D_{\sqrt{2}N} \Rightarrow u \in D_{(\sqrt{2}N)^{1/p}} \Rightarrow u \in Q_{(\sqrt{2}N)^{1/p}}$. ∎

Corollary 4.3. The number $\mu_p(Q_N)$ defined in formula (4.1) can be estimated by

$$\mu_p(Q_N) \le (2\lfloor(\sqrt{2}N)^{1/p}\rfloor + 1)^2 = O(N^{2/p}), \tag{4.6}$$

Proof: The number of p-powers in Q_N is restricted by the number of complex integers in $Q_{(\sqrt{2}N)^{1/p}}$. The number of complex integers in $Q_{(\sqrt{2}N)^{1/p}}$ is given in the middle of (4.6). This is a consequence of Lemma 4.2. The given magnitude is obvious. ∎

Corollary 4.4. Let $N \in \mathbb{N}$ be given and $p_0(N)$ be defined in (4.4). Then for some constant $c > 0$ we have

(4.7)

$$\mu(Q_N) \le \sum_{p \text{ prime}}^{p_0(N)} \mu_p(Q_N) \le \sum_{p \text{ prime}}^{p_0(N)} (2\lfloor(\sqrt{2}N)^{1/p}\rfloor + 1)^2 \le cN + o(N^{2/3}\log N).$$

Proof: The first part follows from Corollary 4.3 together with Lemma 4.1. The remaining part needs some computations. We abbreviate $p_0(N)$ by p_0. We introduce $\nu(x)$ for real x, which is the number of primes not exceeding x. Since

$$(2\lfloor(\sqrt{2}N)^{1/p}\rfloor + 1)^2 \le (2\sqrt{2}N^{1/p} + 1)^2 = 8N^{2/p} + 4\sqrt{2}N^{1/p} + 1$$

the second sum in (4.7) is bounded from above by

$$\begin{aligned}
&\sum_{p \text{ prime}}^{p_0} 8N^{2/p} + \sum_{p \text{ prime}}^{p_0} 4\sqrt{2}N^{1/p} + \sum_{p \text{ prime}}^{p_0} 1 \\
&= 8N + \sum_{\substack{p \text{ prime} \\ p \ge 3}}^{p_0} 8N^{2/p} + \sum_{p \text{ prime}}^{p_0} 4\sqrt{2}N^{1/p} + \sum_{p \text{ prime}}^{p_0} 1 \\
&\le 8N + 8N^{2/3}(\nu(p_0) - 1) + 4\sqrt{2}N^{1/2}\nu(p_0) + \nu(p_0) \\
&\le cN + dN^{2/3}\nu(p_0)
\end{aligned}$$

with some constants $c, d > 0$. By standard arguments (see SCHOLZ & SCHOENEBERG [1961, p. 27]) from number theory $\nu(x) = O(x/\log x)$. Formula (4.4) implies $p_0 = O(\log N)$, and therefore

$$\nu(p_0) = O(\log N/\log\log N) = o(\log N). \qquad \blacksquare$$

We have proven the inequality (4.7) for $c = 8$. However, some explicit counting results suggest that it might even be possible to prove that (4.7) holds for $c \approx 2$. According to MÜLLER [1998], the best possible constant is $c = \log(3+2\sqrt{2}) = 1.762747\ldots$. The results are summarized in Table 4.1. It should be noted that the last row marked $\sum$ contains the number of all *different* powers. Thus, the entries in the last row (called $\sum$) are not just the algebraic sums of the corresponding entries in the columns. To mention a simple example, the powers $-\mathrm{i} = \mathrm{i}^3 = \mathrm{i}^7 = \mathrm{i}^{11} = \mathrm{i}^{19} = \mathrm{i}^{23} = \mathrm{i}^{31}$ are counted only once in the last row. We should remark that the given counting figures for $N = 10^6$ were obtained by using a Sun ultra2/1300 for about 45 h. Finding Catalan pairs in Q_N for this N would require sorting about $1.8 \cdot 10^6$ complex integers which is beyond the scope of most computers.

Table 4.1. Number of p-powers in squares Q_N

$p\backslash N$	1	10	10^2	10^3	10^4	10^5	10^6
2	3	19	183	1763	17637	176291	1762851
3	5	13	69	341	1577	7297	33901
5	5	9	21	49	129	341	829
7	5	9	9	21	45	89	169
11	5	5	9	9	21	25	37
13	5	5	9	9	13	21	25
17	5	5	5	9	9	9	21
19	5	5	5	9	9	9	13
23	5	5	5	5	9	9	9
29	5	5	5	5	5	9	9
31	5	5	5	5	5	9	9
37	5	5	5	5	5	5	9
$\sum$	5	35	271	2153	19357	183945	1797603

By looking at that table we see that the number of squares and the number of all powers in Q_N is proportional to N coinciding with formulae

(4.6) and (4.7). Similarly for the other powers. The figures are of the form $O(N^{2/p})$, see Table 4.2.

Table 4.2. Counting figures relative to $N^{2/p}$

$p \backslash N$	1	10	10^2	10^3	10^4	10^5	10^6
2	3	1.9000	1.8300	1.7630	1.7637	1.7629	1.7629
3	5	2.8008	3.2027	3.4100	3.3975	3.3870	3.3901
5	–	3.5830	3.3283	3.0917	3.2403	3.4100	3.3003
7	–	4.6615	2.4144	2.9179	3.2386	3.3176	3.2629
11	–	–	3.8959	2.5632	3.9350	3.0821	3.0012
13	–	–	4.4315	3.1096	3.1518	3.5726	2.9844
17	–	–	–	3.9930	3.0455	2.3228	4.1337
19	–	–	–	4.3496	3.4134	2.6787	3.0364
23	–	–	–	–	4.0403	3.3072	2.7071
29	–	–	–	–	–	4.0683	3.4710
31	–	–	–	–	–	4.2821	3.6910
37	–	–	–	–	–	–	4.2650

From the figures and the above computations we find that in a square Q_N there are about $2N$ prime powers, where the non-squares form a decreasing minority relative to N.

5. Summary

By searching in large squares we have found a total of 20 Catalan pairs.

Theorem 5.1. In Q_N with $N = 10^6$ there are exactly the following 20 pairs, which are listed in Table 5.1.

It should be observed, that all 20 Catalan pairs listed involve one square. However, if we do not count the trivial pairs and if we disregard the symmetries we have essentially found four new pairs. The search for pairs took about 6 1/2 hours on the machine mentioned.

Table 5.1. All 20 Catalan pairs in Q_N with $N = 10^6$

No	j	$=$	k	$=$
0.0	-9	$(3\mathrm{i})^2$	-8	$(-2)^3$
0.1	-1	$(-1)^3$	0	0^2
0.2	0	0^2	1	1^2
0.3	8	2^3	9	3^2
0.4	$-\mathrm{i}$	i^3	0	0^2
0.5	0	0^2	i	$(-\mathrm{i})^3$
0.6	$-\mathrm{i}$	i^3	1	1^2
0.7	1	1^2	i	$(-\mathrm{i})^3$
0.8	i	$(-\mathrm{i})^3$	-1	i^2
0.9	-1	i^2	$-\mathrm{i}$	i^3
1.0	$3+4\mathrm{i}$	$(2+\mathrm{i})^2$	$4+4\mathrm{i}$	$(-1-\mathrm{i})^5$
1.1	$-3+4\mathrm{i}$	$(1+2\mathrm{i})^2$	$-4+4\mathrm{i}$	$(1-\mathrm{i})^5$
1.2	$-3-4\mathrm{i}$	$(-1+2\mathrm{i})^2$	$-4-4\mathrm{i}$	$(1+\mathrm{i})^5$
1.3	$3-4\mathrm{i}$	$(2-\mathrm{i})^2$	$4-4\mathrm{i}$	$(-1+\mathrm{i})^5$
2.0	i	$(-\mathrm{i})^3$	$2\mathrm{i}$	$(1+\mathrm{i})^2$
2.1	$-\mathrm{i}$	i^3	$-2\mathrm{i}$	$(-1+\mathrm{i})^2$
3.0	$242\mathrm{i}$	$(11+11\mathrm{i})^2$	$243\mathrm{i}$	$(3\mathrm{i})^5$
3.1	$-242\mathrm{i}$	$(11-11\mathrm{i})^2$	$-243\mathrm{i}$	$(-3\mathrm{i})^5$
4.0	$12168\mathrm{i}$	$(78+78\mathrm{i})^2$	$12167\mathrm{i}$	$(-23\mathrm{i})^3$
4.1	$-12168\mathrm{i}$	$(78-78\mathrm{i})^2$	$-12167\mathrm{i}$	$(23\mathrm{i})^3$

6. A note on another problem by Fermat

"Fermat noticed that 26 is sandwiched between 25 and 27, one of which is a square ... and the other is a cube" (Singh [1997, p. 63]), and if we follow Singh, Fermat showed, that 26 is the only integer with this property. Some historical remarks on this problem can be found in the mentioned book by Singh [1997, p. 63–64]. It might be asked whether this is still true, if we admit complex integers. In slight generalization of the above problem, a complex integer f will be called a *Fermat number* if there are two prime powers a, b of complex integers such that $|a-b| = 2$ and $f = (a+b)/2$. If such numbers exist, then they will obey the symmetry rules expressed in Theorem 2.3. In particular, with f also $-f$, $\overline{f}$, $-\overline{f}$ are Fermat numbers. Since we had already computed all powers in the square $Q_N, N = 10^6$, it is natural to see if some Fermat numbers are included. The result (which took 57h computer time) is as follows:

Theorem 6.1. In the square $Q_{N-1}, N = 10^6$ there are the following Fermat numbers:

$$F_1 = 26, \quad F_2 = 1 + 2\mathrm{i}, \quad F_3 = 8 + 7\mathrm{i}. \tag{6.1}$$

The neighbors of F_2 are $2\mathrm{i} = (1+\mathrm{i})^2$, $2 + 2\mathrm{i} = (-1+\mathrm{i})^3$, the neighbors of F_3 are $8 + 6\mathrm{i} = (3+\mathrm{i})^2$, $8 + 8\mathrm{i} = (1-\mathrm{i})^7$. Apart from symmetries these are the only Fermat numbers in Q_{N-1}.

Again, there is one number, namely $F_2 = 1 + 2\mathrm{i}$ which is sandwiched between a square and a cube.

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