The Max-Flow Min-Cut Theorem for Countable Networks

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Abstract. We prove a strong version of the the Max-Flow Min-Cut theorem for countable networks, namely that in every such network there exist a flow and a cut that are “orthogonal” to each other, in the sense that the flow saturates the cut and is zero on the reverse cut. If the network does not contain infinite trails then this flow can be chosen to be mundane, i.e. to be a sum of flows along finite paths. We show that in the presence of infinite trails there may be no orthogonal pair of a cut and a mundane flow. We finally show that for locally finite networks there is an orthogonal pair of a cut and a flow that satisfies Kirchhoff’s first law also for ends.

1. Introduction

Recently, the first two authors of this paper proved the following generalization of Menger’s theorem to the infinite case [5]:

**Theorem 1.1.** Given a possibly infinite digraph and two vertex sets $A$ and $B$ in it, there exists a set $P$ of vertex-disjoint $A$-$B$ paths and an $A$-$B$-separating set of vertices $S$, such that $S$ consists of a choice of precisely one vertex from each path in $P$.

In the finite case, the closely related edge version of Menger’s theorem can be viewed as the integral version of the Max-Flow Min-Cut (MFMC) theorem. In fact, the MFMC theorem can easily be reduced to Menger’s theorem, while the standard proofs of the MFMC theorem yield also its integral version, namely the edge version of Menger’s theorem.

Thus it is natural to ask also for a generalization of the MFMC theorem to the infinite case. Theorem 1.1, which was originally conjectured by Erdős, suggests a possible generalization. In the language of Linear Programming, the infinite version of Menger’s theorem is formulated in terms of the complementary slackness conditions, rather than of equality of the values of dual programs. Applying this to the MFMC theorem we are naturally led to the conjecture that in any network there exists an orthogonal pair of a flow and a cut, i.e. a flow and a cut related to each other by the complementary slackness conditions; these demand that every edge of the cut is saturated by the flow, and on each edge of the reverse cut the value of the flow is zero (see Section 2 for precise definitions).

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Conjecture 1.2. In any (possibly infinite) network there exists an orthogonal pair of a flow and a cut.

In this paper we prove this conjecture for countable networks (Section 6). For this proof we will use a generalization of the notion of waves, which was a central tool in the proof of Theorem 1.1.

This conjecture however does not generalize Theorem 1.1 even in its integral version, since the flow may contain infinite paths. This naturally raises the question whether Conjecture 1.2 is true for flows that do not allow any flow to escape to infinity; we will call such flows mundane (see Section 7 for the precise definition). As we shall see, this is not the case, since even in locally finite networks mundane flows do not necessarily attain the supremum of their values. We thus seek to relax this constraint that no flow escapes to infinity, and are led to two new types of flows: finite-cut-respecting flows, which are allowed to send flow to infinity but any amount flowing into an end of the graph must flow out of the same end, and cut-respecting flows, which are finite-cut-respecting flows with the additional constraint that if flow circumvents some cut $F$ by flowing through an end, then this circumvention does not exceed the amount that could in principle flow through $F$ (see Section 8 for precise definitions).

For each of these kinds of flows we show, at least for locally finite networks, that the infimum of the capacities of all relevant cuts equals the supremum of the values of the corresponding flows (Sections 7 and 8). Moreover, we decide when these infima and suprema are attained by some cut or flow (Sections 5, 7 and 8). Finally, we prove that in any locally finite network there is an orthogonal pair of a cut-respecting flow and a cut of minimum capacity (Section 8).

2. Definitions and Notation

As a general rule, we shall use the terminology of [1]. Deviations will be explicitly indicated.

The characteristic function of a subset $T$ of a set $S$ is denoted by $\chi_S(T)$, or simply $\chi(T)$ if the identity of $S$ is clear from the context.

For a directed edge $e = (u, v)$ we shall write $u = \text{init}(e)$, $v = \text{ter}(e)$. For a vertex $v$ in a digraph we denote by $\text{OUT}(v)$ the set of edges $e$ with $\text{init}(e) = v$, and by $\text{IN}(v)$ the set of edges $e$ with $\text{ter}(e) = v$.

Definition 2.1. A network $\Delta$ is a quadruple $(D, c, s, t)$, where $D = (V, E)$ is a digraph with no loops, $c$ is a function (called capacity) from $E$ to $\mathbb{R}_+$, and $s, t$ are vertices of $D$, called source and sink respectively. We shall assume that $\text{IN}(s) = \text{OUT}(t) = \emptyset$.

Let $\Delta = (D, c, s, t)$ where $D = (V, E)$ be a network fixed throughout this section.

For a function $g$ on $E$ and an edge $(u, v) \in E$ we abbreviate $g((u, v))$ to $g(u, v)$. If $F$ is a set of edges we write $g[F] = \sum \{ g(e) \mid e \in F \}$. We adopt the convention that if $\sum \{ g(e) \mid e \in F, g(e) > 0 \} = \infty$ and $\sum \{ g(e) \mid e \in F, g(e) < 0 \} = -\infty$ then $g[F] = 0$.

For a vertex $v \in V$ we write $d^-_g(v) = g[\text{IN}(v)]$, $d^+_g(v) = g[\text{OUT}(v)]$, and $d_g(v) = d^+_g(v) - d^-_g(v)$. Here, again, we adopt the convention $\infty - \infty = 0$. Given a function $f$ on the edge set of an undirected graph, the degree $d_f(v)$ of a vertex $v$ is the sum of $f(e)$ over all edges $e$ incident with $v$. 
**Definition 2.2.** Given a function \( f \) on \( E \), the set of vertices \( x \in V \) for which \( d^+_f(x) = 0 \) holds is denoted by \( SNK(f) \). The set of vertices \( x \) for which \( d^-_f(x) = 0 \) (and thus \( d^+_f(x) = d^-_f(x) \)) holds is denoted by \( KIR(f) \) (\( KIR \) standing for “Kirchhoff”).

**Definition 2.3.** A function \( f : E \to \mathbb{R}_+ \) is called a flow if:

- (Capacity constraint:) \( f(e) \leq c(e) \) for every \( e \in E \).
- (Flow conservation:) \( V \setminus \{s,t\} \subseteq KIR(f) \).

The set of edges \( e \) for which \( f(e) > 0 \) holds is denoted by \( supp(f) \). The value \( |f| \) of a flow \( f \) is defined by \( |f| := d^+_f(s) \). Note that in infinite networks this is not necessarily equal to \( d^-_f(t) \). If \( C \) is a directed in \( D \) then we say that \( C \) is a cycle of \( f \) if \( f(e) > 0 \) holds for every edge \( e \in E(C) \).

A cut is a set of edges of the form \( E(S, V \setminus S) \) for some \( S \subseteq V \), where \( E(X, Y) \) is the set of edges directed from \( X \) to \( Y \). An \( s-t \) cut is a cut \( E(S, V \setminus S) \) such that \( s \in S \) and \( t \not\in S \). A flow \( f \) is said to saturate an edge \( e \) if \( f(e) = c(e) \). It is said to saturate a set \( F \) of edges if it saturates all edges in \( F \). A flow \( f \) and an \( s-t \) cut \( E(S, V \setminus S) \) are orthogonal to each other if \( f \) saturates \( E(S, V \setminus S) \) and is zero on every edge in \( E(V \setminus S, S) \).

A 1-way infinite path in a graph \( G \) is called a ray. Two rays \( R, L \) in \( G \) are equivalent if no finite set of vertices separates them. The corresponding equivalence classes of rays are the ends of \( G \).

### 3. A vertex version

As already mentioned, in the finite case the edge version of Menger’s theorem is just the integral case of the MFMC theorem, namely the case in which the capacity function is identically 1 and the desired flow only takes the values 0 and 1. The vertex version and the edge version of Menger’s theorem are easily derivable from each other. To get the vertex version from the edge version, one splits each vertex into a “receiving” copy and an “emitting” copy, connected by an edge. The derivation of the edge version from the vertex version is done by a transformation that can be used also in the non-integral case, yielding an equivalent version of Conjecture 1.2. To state it we need the following definitions:

**Definition 3.1.** A weighted web \( \Gamma \) is a quadruple \((D, A, B, w)\), where \( D \) is a directed graph, \( A, B \subseteq V(D) \) and \( w \) is a function from \( V(D) \) to \( \mathbb{R}_+ \). Let \( V(\Gamma) = V(D) \) and \( E(\Gamma) = E(D) \).

Let \( \Gamma = (D, A, B, w) \) be a weighted web fixed throughout this section.

**Definition 3.2.** A current \( f \) in \( \Gamma \) is a function from \( E(D) \) to \( \mathbb{R}_+ \) such that

- (i) \( d^+_f(x) \leq w(x) \) and \( d^-_f(x) \leq w(x) \) for every vertex \( x \in V(D) \);
- (ii) \( d^+_f(x) \leq d^-_f(x) \) for every vertex \( x \in V(D) \setminus A \); and
- (iii) \( d^+_f(a) = 0 \) for every \( a \in A \) and \( d^-_f(b) = 0 \) for every \( b \in B \).

A vertex \( x \) is said to be saturated by \( f \) if \( x \in A \) or \( d^-_f(x) = w(x) \). The set of vertices that are saturated by \( f \) is denoted by \( SAT(f) \). The set \( SAT(f) \cap SNK(f) \) is denoted by \( TER(f) \) (standing for “terminal points”; recall that \( SNK(f) \) is the set of vertices \( x \) for which \( d^+_f(x) = 0 \)).
Definition 3.3. A current $f$ satisfying $KIR(f) \supseteq V(D) \setminus (A \cup B)$ is called a web-flow.

A set $S$ of vertices in $\Gamma$ is said to be $A$-$B$-separating (or simply separating) if every path from $A$ to $B$ meets $S$. Given a (not necessarily separating) subset $S$ of $V(D)$, a vertex $x \in S$ is said to be essential (for separation) in $S$ if it is not separated from $B$ by $S \setminus \{x\}$. The set of essential elements of $S$ is denoted by $E(S)$.

It is easy to show:

Lemma 3.4 ([5]). If $S$ is separating, then so is $E(S)$.

For a set $S$ of vertices in $\Gamma$ we denote by $RF(S) = RF_f(S)$ the set of vertices separated by $S$ from $B$. (The letters "RF" stand for "roofed", a term originating in the way the authors draw their weighted webs, with the "A" side at the bottom, and the "B" side on top.) In particular, $S \subseteq RF(S)$. We define $RF_0(S) := RF(S) \setminus E(S)$.

Given a current $f$, we write $RF(f) = RF(TER(f))$ and $RF_0(f) = RF_0(TER(f))$.

Definition 3.5. Let $f$ be a web flow and let $S$ be a separating set. We say that $S$ is orthogonal to $f$ if $S \subseteq SAT(f)$ and $f(u,v) = 0$ for every pair of vertices $u, v$ with $v \in RF(S)$ and $u \in V \setminus RF_0(S)$.

An equivalent conjecture to Conjecture 1.2 is:

Conjecture 3.6. In every weighted web there exists a web-flow $f$ and an $A$-$B$ separating set orthogonal to $f$.

The transformation used to deduce Conjecture 1.2 from Conjecture 3.6 is the following. Let $\Delta$ be a network, with notation as in Definition 2.1. Let $\Gamma = (D', A, B, w)$ be the web defined by $V(\Gamma) = E(\Delta)$, $E(\Gamma) = \{((x,y),(y,z)) \mid (x,y),(y,z) \in E(\Delta)\}$, $A = E(s,V(\Delta) \setminus \{s\})$, $B = E(V(\Delta) \setminus \{t\}, t)$, and $w(e) = e(e)$ for every $e \in V(\Gamma) = E(\Delta)$.

Easily, every $A$-$B$-separating set of vertices in $\Gamma$ is also an $s$-$t$-separating set of edges in $\Delta$. If $f$ is a web-flow in $\Gamma$, we can define a flow $g$ in $\Delta$ as $g(e) = \max(d_f^+(e), d_f^-(e))$ (recall, however, that $d_f^+(e) = d_f^-(e)$ if $e \notin A \cup B$). It is straightforward to check that $g$ is indeed a flow. Moreover, if $f$ is orthogonal to some $A$-$B$-separating set of vertices then $g$ is orthogonal to the corresponding set of edges in $\Delta$.

In the following sections we will prove Conjecture 3.6, and thus Conjecture 1.2, for the countable case (see Theorem 6.1).

4. Linkability in weighted webs, waves, and an equivalent conjecture

In this section we develop some tools that we will use for the proof of Conjecture 3.6 for countable weighted webs. These are generalizations of fundamental notions in the proof of Theorem 1.1 in [5], and they could turn out useful in proving the general case of Conjecture 3.6.

Let $\Gamma = (D, A, B, w)$ be a weighted web fixed throughout this section.

Definition 4.1. A web-flow $f$ in $\Gamma$ is called a linkage if $d_f^+(a) = w(a)$ for every $a \in A$. If a weighted web contains a linkage it is called linkable.

Definition 4.2. A current $f$ in $\Gamma$ is called a wave if $TER(f)$ is $A$-$B$-separating and $d_f^+(x) = 0$ for all $x \notin RF(f)$.
If \(f, g\) are waves, we write \(f \leq g\) if \(f(e) \leq g(e)\) for every edge \(e\).

**Lemma 4.3.** Let \(I\) be a totally ordered set, and let \((f_i \mid i \in I)\) be waves such that \(f_i \leq f_j\) whenever \(i \leq j\). Then \(f = \sup (f_i \mid i \in I)\) is a wave.

**Proof.** Let \(P\) be an \(A-B\) path. Easily, for \(j \geq i\) we have \(SNK(f_i) \supseteq SNK(f_j)\) and \(SAT(f_i) \subseteq SAT(f_j)\). As \(P\) is finite, this means that there exists an \(i\) such that for every \(j \geq i\) we have \(SNK(f_i) \cap V(P) = SNK(f_j) \cap V(P)\) and \(SAT(f_i) \cap V(P) = SAT(f_j) \cap V(P)\). Then, \(SNK(f) \cap V(P) = SNK(f_i) \cap V(P)\) and \(SAT(f) \cap V(P) \supseteq SAT(f_i) \cap V(P)\). Hence, \(TER(f) \cap V(P) \supseteq TER(f_i) \cap V(P)\), and since \(f_i\) is a wave, this implies that \(TER(f) \cap V(P) \neq \emptyset\). This proves that \(f\) is a wave.

By Zorn's lemma this implies:

**Corollary 4.4.** In every weighted web there exist a \((\leq)\)-maximal wave.

**Definition 4.5.** A wave \(f\) is called a hindrance if there exists a vertex \(a \in A \setminus \mathcal{E}(TER(f))\) such that \(d_+^f(a) < w(a)\). If \(0 < \varepsilon < w(a) - d_+^f(a)\) then \(f\) is said to be a \((> \varepsilon)\)-hindrance. A weighted web is called hindered (respectively \((> \varepsilon)\)-hindered) if it contains a hindrance (respectively a \((> \varepsilon)\)-hindrance). A weighted web is called loose if it contains no non-zero wave and the zero wave is not a hindrance.

The following is an easy consequence of the definitions.

**Observation 4.6.** Let \(\Gamma = (D, A, B, w)\) and \(\Gamma' = (D, A, B, w')\) be weighted webs such that \(w'(v) = w(v)\) for all \(v \in V \setminus A\) and \(w'(a) \leq w(a)\) for all \(a \in A\). Then every wave in \(\Gamma'\) is a wave in \(\Gamma\). Thus if \(\Gamma\) is loose then so is \(\Gamma'\).

**Definition 4.7.** Let \(f\) be a wave. A wave \(g\) is called a trimming of \(f\) if

(i) \(g \leq f\)

(ii) \(RF^o(f) \subseteq KIR(g) \cup A\) and:

(iii) \(TER(g) \setminus A = \mathcal{E}(TER(f)) \setminus A\).

A wave is called trimmed if it is a trimming of itself.

**Lemma 4.8.** Every wave has a trimming.

**Proof.** Let \(f\) be a wave that is not trimmed, let \(x \in RF^o(f) \setminus (KIR(f) \cup A)\) and let \(f_1\) be the wave obtained from \(f\) by decreasing the values on \(IN(x)\) so that \(d_+^{f_1}(x) = d_+^f(x)\). One can easily see that \(\mathcal{E}(TER(f_1)) = \mathcal{E}(TER(f))\), which means that \(f_1\) is indeed a wave. If \(f_1\) is trimmed, we are done. If not, we can find in a similar way a wave \(f_2 \leq f_1\) with \(\mathcal{E}(TER(f_2)) = \mathcal{E}(TER(f))\). Continuing this process, if necessary, transfinately, we obtain a trimmed wave \(f_\alpha\), which is then a trimming of \(f\). \(\Box\)

**Definition 4.9.** If \(\Gamma\) is a weighted web and \(f\) is a wave in \(\Gamma\), we write \(\Gamma/f\) for the web \(\Xi\) defined by \(A_\Xi = \mathcal{E}(TER(f)), B_\Xi = B, V_\Xi = V \setminus RF^o(f), D_\Xi = D[V_\Xi]\) (the subgraph of \(D\) induced on \(V_\Xi\)) and \(w_\Xi = w \restriction V_\Xi\).

Waves can be combined, as follows:

**Definition 4.10.** Let \(f\) be a wave and \(g\) be a current. We denote by \(f \prec g\) the function \(f + (g \restriction E(\Gamma/f))\).

It is easy to check that \(f \prec g\) is a current. In fact, if \(g\) is a wave, then \(g \restriction (\Gamma/f)\) is a wave in \(\Gamma/f\), and thus \(f \prec g\) is a wave, which follows from:
Lemma 4.11. If $g \mid (\Gamma/f)$ is a wave, then $TER(f \sim g) \supseteq \mathcal{E}(TER(f) \cup TER(g))$.

Proof. Let $x \in \mathcal{E}(TER(f) \cup TER(g))$. We wish to show that $x \in TER(f \sim g)$.

We first note that $x$ cannot lie in $RF^\circ(f)$ or $RF^\circ(g)$, so $x \in SNK(f) \cap SNK(g)$ hence $x \in SNK(f \sim g)$. It remains to show that $x \in SAT(f \sim g)$. Since $f \sim g \geq f$ we have $SAT(f \sim g) \supseteq SAT(f)$. Hence we are done in the case $x \in TER(f)$ and we may assume $x \in TER(g) \setminus TER(f)$.

Since $x \notin TER(f)$ and $x \notin RF^\circ(f)$, we have $x \notin RF(f)$ and thus, for an edge $e$ entering $x$ we have $e \in E(\Gamma/f)$ and thus $(f \sim g)(e) = g(e)$. Since $x \in SAT(g)$ this yields $x \in SAT(f \sim g)$, completing the proof. \hfill \Box

The following lemma is easy to prove:

Lemma 4.12. If $f$ and $g$ are waves in a bipartite web, then $TER(f \sim g) \cap B = (TER(f) \cup TER(g)) \cap B$.

We can now use our new machinery to reformulate Conjecture 3.6:

Conjecture 4.13. A loose weighted web is linkable.


Proof. Let $\Gamma$ be a weighted web, and let $f$ be a $(\leq)$-maximal wave in $\Gamma$. Let $T = \mathcal{E}(TER(f))$, and let $h$ be a trimming of $f$. Clearly, $\Gamma/f$ is loose. Assuming Conjecture 4.13, there exists a linkage $g$ in $\Gamma/f$. Then, $k = h + g$ is a web-flow, $T \subseteq SAT(k)$ and $T$ is $A-B$ separating. Since $\text{supp}(g) \subseteq V(\Gamma/f)$ we have $k(x, y) = 0$ for every pair of vertices $x, y$ with $x \in V \setminus RF^\circ(T)$ and $y \in RF(T)$, which proves that $T$ is orthogonal to $k$. \hfill \Box

5. Attainability of Flow Values in Infinite Networks

In this section we return to flows in networks, rather than web-flows. Our aim is to prove a result which will serve as a main ingredient in the proof of Conjecture 3.6 for countable weighted webs (Theorem 6.1), and which seems to be of independent interest:

Theorem 5.1. In a countable network $\Delta$ where $d_{\sim}^-(x) < \infty$ (i.e. the sum of the capacities of the edges pointing to $x$ is finite) for every vertex $x$, there exists a flow $f$ such that $|f| = \sup\{|g| : g$ is a flow in $\Delta\}$ and $d_{\sim}^-(x) \leq |f|$ for every vertex $x$. In particular, if the values of flows in $\Delta$ are unbounded, then there exists a flow of infinite value.

Definition 5.2. Let $f$ be a flow in a network $\Delta = (D, c, s, t)$ that contains no pair of edges with the same endvertices but opposite directions. The residual network $RES(\Delta, f)$ of $\Delta$ and $f$ is the network $(D', c_R, s, t)$ where $D'$ is the digraph obtained from $D$ by adding an edge $(u, v)$ for every edge $(v, u) \in E(D) \setminus (\text{OUT}(s) \cup \text{IN}(t))$, and where $c_R$ is defined by letting, for every edge $(x, y) \in E(D)$, $c_R(x, y) := c(x, y) - f(x, y)$ and $c_R(y, x) := f(x, y)$.

For a function $g$ on the edge-set of $RES(\Delta, f)$ let $f \oplus g$ denote the function $h$ on the edge-set of $\Delta$ defined by $h(x, y) = f(x, y) + g(x, y) - g(y, x)$. The following is a straightforward corollary of the definitions:

Lemma 5.3. Let $f$ be a flow in $\Delta$ and let $g$ be a flow in $RES(\Delta, f)$. Then $f \oplus g$ is a flow in $\Delta$, with $|f \oplus g| = |f| + |g|$.
The following result shows that it is possible to clean up a flow from cycles and current coming from infinity without reducing its value.

**Lemma 5.4.** If \( g \) is a flow in \( \Delta \) of finite value then there exists a flow \( h \leq g \) such that \( |h| = |g| \) and \( d^n_h(x) \leq |h| \) for every vertex \( x \).

**Proof.** We propagate the desired flow \( h \) from \( s \). We will define \( h \) recursively in infinitely many steps, in each step considering a vertex and adjusting its out-degree to its in-degree, and then removing any cycles we created in doing so. But as subsequent steps might change the in-degree of a vertex we already considered, we will have to return to each vertex infinitely often.

Formally, let \( v_1, v_2, \ldots \) be a sequence in which each vertex in \( V(\Delta) \setminus \{s, t\} \) appears infinitely often. We will recursively define sequences \( (h_i), (h^+_i) \) and \( (h^-_i) \) of functions on \( E(\Delta) \). Intuitively, \( h^+_i \) differs from \( h^-_{i-1} \) in that it makes the out-degree of \( v_i \) equal to its in-degree, while \( h^-_i \) is the sum of some unwanted cycles in \( h^-_{i-1} \). Subtracting the two we obtain the functions \( h_i \) that will converge to the desired flow \( h \).

While defining these sequences we will make sure that the following conditions are satisfied for all \( i \in \mathbb{N} \):

1. If \( i > 0 \), then \( h^+_i(e) \leq h^+_i(e) \) and \( h^-_{i-1}(e) \leq h^-_{i-1}(e) \) for every edge \( e \);
2. \( h^-_i(e) \leq h^+_i(e) \leq g(e) \) for every edge \( e \);
3. the support of \( h_i := h^+_i - h^-_i \) does not contain cycles;
4. \( d^+_h(v) \leq d^-_h(v) \) for every \( v \in V(\Delta) \setminus \{s, t\} \); and
5. \( d^+_h(v) = d^-_h(v) \) for every \( v \in V(\Delta) \setminus \{s, t\} \).

We start by defining \( h^+_0 = g \) on \( OUT(s) \) and \( h^-_0 = 0 \) on all other edges, and \( h^-_0 = 0 \). Clearly, conditions (i)–(v) are satisfied for \( i = 0 \). For \( i = 1, 2, \ldots \), assume that \( h^+_j \) and \( h^-_j \) have already been defined for every \( j < i \) and satisfy (i)–(v).

We define \( h^+_i \) first. If \( d^+_{h^-_{i-1}}(v_i) < d^-_{h^-_{i-1}}(v_i) \), then give each edge \( e \) in \( OUT(v_i) \) a value \( h^+_i(e) \) with \( h^+_i(e) \leq h^+_i(e) \) so that (having considered all edges in \( OUT(v_i) \)) \( d^+_{h^-_{i-1}}(v_i) = d^-_{h^-_{i-1}}(v_i) \) holds; this is possible since \( d^+_{h^-_i}(v_i) = d^-_{h^-_i}(v_i) \) and \( h^+_i(e) \leq g(e) \) for every \( e \in E(\Delta) \) by condition (ii). For every edge \( e \) in \( E(\Delta) \setminus OUT(v_i) \) let \( h^+_i(e) = h^-_{i-1}(e) \). Clearly, conditions (i)–(v) are not violated.

Next we define \( h^-_i \). The function \( h^-_i := h^+_i - h^-_{i-1} \) is non-negative since \( h^+_i \geq h^+_i \geq h^-_{i-1} \) by (i) and (ii). If \( supp(h^-_i) \) contains any cycles, then let \( C_1, C_2, \ldots \) be a (possibly infinite) enumeration of those cycles. (The \( C_j \) are not necessarily pairwise edge disjoint.) We are going to remove all cycles from \( supp(h^-_i) \) by performing infinitely many steps (within step \( i \)), in each step \( j \) eliminating the cycle \( C_j \) from \( supp(h^-_i) \). For every edge \( e \) we denote by \( h^-_i(e) \) the value that has to be subtracted from \( h^-_i(e) \) in order to eliminate the cycles \( C_1, \ldots, C_j \). To begin with, let \( h^-_0(e) = 0 \) for every \( e \in E(\Delta) \). For \( j = 1, 2, \ldots \), if \( C_j \) is a cycle in \( supp(h^-_i - h^-_{i-1}) \) then, for every edge \( e \in E(C_j) \), add to \( h^-_{i-1}(e) \) the value

\[
\min\{h^-_i(e) - h^-_{i-1}(e) \mid e \in E(C_j)\}
\]

to obtain \( h^-_i(e) \); let \( h^-_i(d) = h^-_{i-1}(d) \) for every other edge \( d \).

Having treated all cycles \( C_j \), define \( h^-_{i-1}(e) := h^-_{i-1}(e) + \lim_j h^-_i(e) \) for every \( e \in E(\Delta) \); this is well defined since \( h^-_i(e) \) is monotone increasing with \( j \) and bounded by \( h^-_i(e) \). It is not hard to see that conditions (i)–(v) are satisfied.
By (i) and (ii) the sequences \( h_i^+(e), h_i^-(e) \) and \( h_i(e) \) converge for every edge \( e \); we let \( h^+(e) := \lim_i h_i^+(e), h^-(e) := \lim_i h_i^-(e), \) and \( h(e) := \lim_i h_i(e) = h^+(e) - h^-(e) \). By (ii) we have \( h^+(e) \leq g(e) \). Since for every vertex \( v \in V(\Delta) \setminus \{s,t\} \) we have \( d_{h_i}^+(v) = d_{h_i}^-(v) \) for infinitely many \( i \), we have \( d_{h_i}^+(v) = d_{h_i}^-(v) \leq d_g^-(v) < \infty \).

By (i), (ii) and (v) we have \( d_{h_i}^-(v) = d_{h_i}^-(v) \leq d_{h_i}^+(v) \). Hence \( h \) is non-negative and \( d_{h_i}^+(v) = d_{h_i}^-(v) \), and therefore \( h \) is indeed a flow.

As \( IN(s) = \emptyset \), for every edge \( e \in OUT(s) \) no cycle considered in the construction of \( h^- \) contained \( e \), hence \( h^-(e) = 0 \) and \( h(e) = h^+(e) = h_0^+(e) = g(e) \). Therefore \( |h| = |g| \). Since \( h \leq h^+ \leq g \), all that remains to prove is that \( d_{h_i}^-(v) \leq |h| \) for every vertex \( v \).

Since \( |g| \) is finite, \( \sum_{e \in E} h_i(e) \) is finite for every \( i < \omega \) by the construction of the \( h_i \). Let \( x \in V \) and let \( X \) be the set of all vertices from which \( x \) is reachable via \( supp(h_i) \). Note that \( h_i(E(x, X-x)) = 0 \), since otherwise there would exist a cycle in \( supp(h_i) \). Thus, since \( d_{h_i}^-(y) \geq d_{h_i}^+(y) \) for every \( y \in V \setminus \{s\} \), we have

\[
\sum_{y \in X-x} (d_{h_i}^-(y) - d_{h_i}^+(y)) \geq -d_{h_i}^+(s) = -|h_i|.
\]

As \( \sum_{e \in E} h_i(e) \) is finite, we have

\[
\sum_{y \in X-x} (d_{h_i}^-(y) - d_{h_i}^+(y)) = h_i(E(V \setminus (X-x), X-x)) - h_i(E(X-x, V \setminus (X-x))).
\]

By the choice of \( X \), we have \( h_i(E(V \setminus (X-x), X-x)) = 0 \) and

\[
h_i(E(X-x, V \setminus (X-x))) \geq h_i(E(X-x, x)) = d_{h_i}^+(x).
\]

This yields \( -|h_i| \leq \sum_{y \in X-x} (d_{h_i}^-(y) - d_{h_i}^+(y)) \leq d_{h_i}^+(x) \) and hence \( d_{h_i}^+(x) \leq |h_i| = |h| \). Since \( h = \lim_i h_i \), we have \( d_{h}^+(x) \leq |h| \). Since \( x \) was chosen arbitrarily, this completes the proof. \( \square \)

**Proof of Theorem 5.1.** Easily, we may assume that \( OUT(s) \) consists of one edge only. Let \( \alpha = sup \{|g| : g \text{ is a flow in } \Delta\} \). If \( \alpha = \infty \), we may just choose flows \( f_i \) with \( |f_i| = 2^i \), and let \( f = \sum f_i \) which, then, is a flow with \( |f| = \infty \).

So assume that \( \alpha \) is finite. Define inductively flows \( f_i \) with \( |f_i| = (1 - (1/2)^i)\alpha \) as follows. Let \( f_0 \equiv 0 \). For every \( i > 0 \), let \( g_i \) be a flow in \( RES(\Delta, f_{i-1}) \) such that \( |g_i| = \frac{1}{2}(\alpha - |f_{i-1}|) = (1/2)^i \) (as \( |OUT(s)| = 1 \), every flow in \( \Delta \) of value \( |f_{i-1}| + \frac{1}{2}(\alpha - |f_{i-1}|) \) yields such a flow). By Lemma 5.4 there exists a flow \( k_i \leq g_i \) such that \( d_{k_i}(x) \leq |k_i| = |g_i| \) for every vertex \( x \). By Lemma 5.3, \( f_i := f_{i-1} \oplus k_i \) is a flow of the desired value.

By the choice of the flows \( k_i \), the values \( f_{i-1}(e) \) and \( f_i(e) \) differ by at most \( (1/2)^i \) for each edge \( e \). Hence the values \( f_i(e) \) converge for every \( e \); let \( f(e) = \lim_i f_i(e) \). It is easy to check that \( f \) is a flow. Further, every vertex \( x \) satisfies \( d_f^+(x) \leq \alpha \). Since \( |f| = \lim_i |f_i| = \alpha \), this proves the theorem. \( \square \)

We shall use Theorem 5.1 twice, and in both cases we shall use it with the roles of the source and the sink reversed. Still, we chose this formulation since it is more natural.
6. Orthogonal pairs in countable networks

The main result of this section is

**Theorem 6.1.** In any countable network there exists an orthogonal pair of a cut and a flow.

By Lemma 4.14, in order to prove Theorem 6.1 it suffices to show:

**Theorem 6.2.** A loose countable weighted web is linkable.

In order to prove Theorem 1.1 for the special case of digraphs containing no infinite paths [2], or no infinite outgoing paths [4], it was possible, and useful, to reduce the problem to the special case of bipartite digraphs. Here we are going to use a similar reduction in order to deduce Theorem 6.2 from its bipartite counterpart. This is done by the following transformation.

Let $\Delta = (D, A, B, w)$ be a weighted web. We define a bipartite weighted web $\Gamma = (D', A', B', w')$ in the following way. For each vertex $v \in V(D) \setminus A$ we introduce a new vertex $v_B$. For each vertex $v \in V(D) \setminus B$ we introduce a new vertex $v_A$. We set $A' = \{v_B \mid v \in V(D) \setminus B\}, B' = \{v_B \mid v \in V(D) \setminus A\}, V(D') = A' \cup B', E(D') = \{(u_A, v_B) \mid (u, v) \in E(D)\} \cup \{(v_A, v_B) \mid v \in V(D) \setminus (A \cup B)\}$, $w'(v_A) = w(v)$ for $v \in V(D) \setminus B$ and $w'(v_B) = w(v)$ for $v \in V(D) \setminus A$.

If $S$ is a separating set in $\Gamma$ and we write $A_S = \{v \mid v_A \in S\}, B_S = \{v \mid v_B \in S\}$ then it is straightforward to check that $S' = (A_S \cap B_S) \cup (A \cap A_S) \cup (B \cap B_S)$ is a separating set in $\Delta$. Moreover, waves in $\Gamma$ induce waves in $\Delta$. Indeed, given a wave $f$ in $\Gamma$ with $\text{TER}(f) = S$, define the function $f'$ on $E(D)$ by $f'(u, v) = f(u_A, v_B)$. We have:

**Lemma 6.3.** $f'$ is a wave in $\Delta$ with $\text{TER}(f') = S'$.

**Proof.** Let us first prove that $f'$ is a current. For this end, we only have to show that $d^+(v) \leq d^-(v)$ for every $v \in V \setminus A$. This is clearly true for $v \in B$, so we may assume $v \in V \setminus (A \cup B)$. By construction, we have $d^+(v) = d^+(v_B) - f(v_A, v_B)$ and $d^-(v) = d^-(v_B) - f(v_A, v_B)$. Hence we are done if $d^+(v_B) \leq d^-(v_B)$. So let us assume that $d^+(v_A) = d^+(v_B)$. Since $d^-(v_A) \leq w'(v_A) = w(v) = w'(v_B)$, we have $d^-(v_B) < w'(v_B)$ and hence $\text{VB} \notin \text{TER}(f)$. Likewise, we have $d^+(v_A) > d^+(v_B) \geq 0$ and hence $v_A \notin \text{TER}(f)$. Since $v_A$ and $v_B$ are connected by an edge, this contradicts the fact that $f$ is a wave.

Now let us prove $\text{TER}(f') = S'$. Clearly, we have $\text{TER}(f') \cap A = A \cap A_S$ and $\text{TER}(f') \cap B = B \cap B_S$. Thus, it remains to show that $\text{TER}(f') \setminus (A \cup B) = A_S \cap B_S$. Let $v \in A_S \cap B_S$. Since $v \in A_S$, we have $v_A \in \text{SNK}(f)$ and thus $v \in \text{SNK}(f')$. Finally, we have $v_B \in \text{SAT}(f)$, which yields $v \in \text{SAT}(f')$, since $f(v_A, v_B) = 0$. Hence $\text{TER}(f') \setminus (A \cup B) \supseteq A_S \cap B_S$. Now let $v \in \text{TER}(f') \setminus (A \cup B)$. Then, $w(v) = d^+(v) \leq d^+(v_B) \leq w'(v_B) = w(v)$, which means $v \in B_S$ and $d^+(v_B) = d^-(v_B)$. The latter yields $f(v_A, v_B) = 0$. Since $v \in \text{SNK}(f')$, we have $v \in A_S$.

Therefore, $\text{TER}(f') = S'$. Since $S'$ is an $A$-$B$-separator, $f'$ is a wave in $\Delta$. □

**Lemma 6.4.** If the zero wave in $\Gamma$ is a hindrance, then the zero wave in $\Delta$ is also a hindrance.

**Proof.** Suppose the zero wave $f_0$ in $\Gamma$ is a hindrance and let $v_A$ be a hindered vertex, that is, $w'(v_A) > 0 = d^+_n(v_A)$ and $v_A \notin \text{E}(\text{TER}(f_0))$. In other words, every
neighbour \( u_B \) of \( v_A \) lies in \( \text{TER}(f_0) \) and hence satisfies \( w'(u_B) = 0 \). If \( u_B \) existed, we would have \( w'(v_B) = w'(v_A) > 0 \). Hence \( v_B \) does not exist, which means that \( v \in A \). Further, every neighbour \( u \) of \( v \) in \( \Delta \) satisfies \( w(u) = w'(u_B) = 0 \), since \( u_B \) is a neighbour of \( v_A \) in \( \Gamma \). Therefore, \( v \in A \setminus \mathcal{E} (\text{TER}(f_0)) \) and \( w(v) = w'(v_A) = 0 \). Hence the zero wave \( f_0' \) in \( \Delta \) is a hindrance.

Our next aim is to prove:

**Theorem 6.5.** A countable loose bipartite weighted web is linkable.

Theorem 6.5 implies Theorem 6.2. Indeed, Lemmas 6.3 and 6.4 imply that if \( \Delta \) is loose then so is \( \Gamma \). On the other hand, if \( f \) is a linkage in \( \Gamma \), then the function \( f' \) defined above satisfies \( d_f^+(v) = w'(v_A) - f(v_A, v_B) \geq d_f^-(v_B) - f(v_A, v_B) = d_{f'}^-(v) \) for \( v \in V(D) \setminus B \) and \( d_f^+(a) = w(a) \) for \( a \in A \). Thus, applying ideas similar to those in the proof of Lemma 4.8, we can easily use \( f' \) to obtain a linkage of \( \Delta \).

The rest of this section will be devoted to the proof of Theorem 6.5. Henceforth \( \Gamma \) will denote a countable bipartite weighted web with sides \( A \) and \( B \) and weight function \( w \).

**Definition 6.6.** If \( f \) is a current in \( \Gamma \) we write \( \Gamma - f \) for the weighted web \((D, A, B, w - d_f)\).

**Lemma 6.7.** Let \( u \) be a non-negative function on \( B \) such that \( \varepsilon := \sum_{v \in B} u(v) \) is finite. Let \( w' \) be the weight function on \( V \) defined by \( w' \mid A = w \mid A \) and \( w' \mid B = (w \mid B) - u \). If \( \Xi = (D, A, B, w') \) is \((> \varepsilon)\)-hindered then \( \Gamma \) is hindered.

**Proof.** Let \( f \) be a \((> \varepsilon)\)-hindrance in \( \Xi \), and let \( a \in A \setminus \mathcal{E}(\text{TER}(f)) \) be a \((> \varepsilon)\)-hindered vertex for \( f \), that is \( w(a) - d_f(a) > \varepsilon \). We define a network \( \Psi \), as follows. The vertex set of \( \Psi \) is \( V(\Gamma) \cup \{t\} \), where \( t \) is a new vertex added (recovering, in fact, the sink vertex of the network from which the web \( \Gamma \) was obtained). The source vertex of \( \Psi \) is \( A \), and its sink vertex is \( t \). The edges of \( \Psi \) are all edges of \( \Gamma \), taken each in both directions, together with \( \{(y, t) \mid y \in B\} \). Its capacity function is defined by \( c_\Psi(x, y) = \max(w(x), w(y)) + 1 \), \( c_\Psi(y, x) = f(x, y) \) for all \( (x, y) \in E(\Gamma) \), and \( c_\Psi(y, t) = u(y) \) for all \( y \in B \). By Theorem 5.1 (with the roles of the source and the sink reversed) there exists in \( \Psi \) a flow \( j \) maximizing the in-degree of \( t \), and satisfying \( d_j^+(a) \leq d_j^-(t) \leq \varepsilon \). Note that for \( x \in \mathcal{E}(\text{TER}(f)) \cap A \), we have \( c_\Psi(e) = 0 \) for each \( e \in IN(x) \) and thus \( d_j^+(x) = d_j^-(x) = 0 \).

Call a vertex \( r \in V \) reachable (from \( a \)) if there exists a path \( P \) from \( a \) to \( r \) in \( \Psi \) such that \( c_\Psi(e) - j(e) > 0 \) for all \( e \in E(P) \). Note that \( c_\Psi(e) - j(e) > 0 \) for each \( A-B \) edge \( e \). Hence, if a vertex in \( A \) is reachable then so are all its neighbours in \( B \). Let \( g \) be the flow defined by letting \( g(e) = 0 \) if \( e \) has at least one unreachable endpoint and \( g(e) = (f \oplus j)(e) \) otherwise. We shall show that \( g \) is a wave in \( \Gamma \).

First note that \( g \) is a current since \( d_g(x) \leq d_{f \oplus j}(x) = d_f(x) + j(x, t) \leq w(x) \) for every \( x \in V(\Gamma) \). Suppose, for contradiction, that \( \text{TER}(g) \) is not \( A-B \) separating, in which case there exists an edge \( (x, y) \) such that neither \( x \) nor \( y \) are in \( \text{TER}(g) \). Since \( x \notin \text{TER}(g) \) it is reachable and so is \( y \); indeed, if \( x \) was unreachable, we would have \( x \in \text{SNK}(g) \) by definition of \( g \), and hence \( x \in \text{TER}(g) \). Thus, there exists a path \( P \) from \( a \) to \( y \) such that \( c_\Psi - j \) is positive on the edges of \( P \).

If \( x \in \text{TER}(f) \), we have \( d_g^-(x) = d_j^-(x) = 0 \). This yields \( d_g^+(x) = 0 \) and thus \( x \in \text{TER}(g) \), a contradiction. Thus, since \( f \) is a wave, \( y \in \text{TER}(f) \). Since \( y \notin \text{TER}(g) \), it is saturated by \( f \) but not by \( g \). This means that \( c_\Psi(y, t) - j(y, t) > 0 \).
Thus the flow $j$ in $\Psi$ can be augmented along $P$, by adding some small number $\zeta$ on all edges of $P$ and on $(y,t)$. This contradicts the maximality of $d_{j_{\gamma}}(t)$.

Therefore, $g$ is a wave in $\Gamma$. Since $d_{g_{\gamma}}(a) \leq d_{g_{\gamma}}(t) \leq \varepsilon$, we have $d_{g_{\gamma}}(a) < w_{\Gamma}(a)$. Thus $a$ witnesses the fact that $g$ is a hindrance in $\Gamma$, which proves the lemma. $\square$

Lemma 6.7 and Observation 4.6 imply:

**Corollary 6.8.** If $g$ is a current in $\Gamma$ with $\sum_{v \in B} g(v) = \varepsilon$, and if $\Gamma - g$ is ($>\varepsilon$)-hindered, then $\Gamma$ is hindered.

If $\Gamma = (D,A,B,w)$ is a weighted web and $g$ a real function on the vertices of $\Gamma$ such that $g(v) \leq w(v)$ for every $v \in V(D)$, we write $\Gamma - g$ for the weighted web $(D,A,B,w - g)$.

**Lemma 6.9.** Let $\Omega = (D,A,B,w)$ be a loose bipartite weighted web, and let $b$ be an element of $B$ with $w(b) > 0$. Then there exists $\varepsilon > 0$ such that $\Omega - \varepsilon\chi(\{b\})$ is unhindered.

Proof. Without loss of generality we may assume that $w(b) \geq 1$. This means that $\Omega - \frac{1}{n}\chi(\{b\})$ is defined for all positive integers $n$. Suppose, for contradiction, that $\Omega - \frac{1}{n}\chi(\{b\})$ contains a hindrance $g_n$ for every $n = 1, 2, 3, \ldots$. Clearly, $b \in TER(g_n)$, since otherwise $g_n$ would be a hindrance in $\Omega$. We define a wave $g_\omega$ in $\Omega - \chi(\{b\})$ as follows. First, for every $i$, let $\tilde{g}_i$ be a wave in $\Omega - \chi(\{b\})$ obtained from $g_i$ by reducing its value on some edges at $i$ as follows. First, for every $i$, let $\tilde{g}_i$ be a wave in $\Omega - \chi(\{b\})$ obtained from $g_i$ by reducing its value on some edges at $i$ so that $d_{g_{\gamma}}(b) = w(b) - 1$. Then, let $f_n = g_1 \sim g_2 \sim \ldots \sim g_n$ and let $g_\omega = \sup f_n$. By Lemma 4.3, $g_\omega$ is a wave in $\Omega - \chi(\{b\})$.

Now let $h_n = g_n \sim g_\omega$. It is easy to check that $h_n$ is a wave in $\Omega - \frac{1}{n}\chi(\{b\})$, even though $g_\omega$ is not: $g_\omega = ((\Omega - \frac{1}{n}\chi(\{b\}))/g_n)$ is a wave in $\Omega - \frac{1}{n}\chi(\{b\})/g_n$ and hence $h_n$ is a wave, by Lemma 4.11. Let $T = TER(g_n) \cap B$ and let $S = A \setminus RF(T)$. Then, by Lemma 4.12, $T \supset TER(g_n) \cap B$ for all $n$, and hence $T = TER(h_n) \cap B$ for all $n$. The waves $h_n$ all play in the same arena - the web induced on $(A \setminus S) \times T$.

Similarly with Lemma 6.7, we can define a network $\Psi$ with sink $b$ and source $s$, where $s$ is a new vertex added, joined to all vertices in $A \setminus S$. In $\Psi$, we can apply Theorem 5.1 to the flows $h_2 - h_1, h_3 - h_1, \ldots$, to deduce that there exists a current $k$ in $\Psi$ of value 1. Then, $h_1 \oplus k$ is a current in $\Omega$ saturating all vertices in $T$, and is thus a non-zero wave in $\Omega$, contradicting the fact that $\Omega$ is loose. $\square$

We shall use Lemma 6.9 for our next lemma:

**Lemma 6.10.** Let $\Omega = (D,A,B,w)$ be a loose bipartite weighted web, and let $a$ be any element of $A$. Then, there exists a current $f$ such that $d_{f}(a) = w(a)$ and $\Omega - f$ is loose.

Proof. We may assume that $w(a) > 0$ since otherwise we could choose $f \equiv 0$. We choose recursively vertices $y_0 \in B$, flows $f_0$ and networks $\Omega_0$, for countable ordinals $\theta$, as follows. Since $w(a) = 0$ and since $\Omega$ is unhindered, there exists an edge $(a, y_0) \in OUT(a)$, such that $w(y_0) > 0$. By Lemma 6.9 and Observation 4.6 we can find $\delta_0 > 0$ such that $\Omega - \delta_0\chi(\{a, y_0\})$ is unhindered. Let $k_0$ be a maximal wave in $\Omega - \delta_0\chi(\{a, y_0\})$. Define $f_0 = \delta_0\chi(\{a, y_0\}) + k_0$. Since $k_0$ is maximal, $\Omega - f_0$ is loose; for if $\Omega - f_0$ contained a wave $g$ which is non-zero or a hindrance, $k_0 + g$ would be a wave in $\Omega - f_0$, contradicting either the maximality of $k_0$ or the fact that $\Omega - f_0$ is unhindered. Let $\Omega_1 = \Omega - f_0$. If $w_{\Omega_1}(a)$, i.e. the capacity of $a$ in $\Omega_1$, is greater than 0, then there exists $(a, y_1) \in OUT(a)$ with $w_{\Omega_1}(y_1) > 0$. Thus we
can find $\varepsilon_1 > 0$ such that $\Omega_1 - \varepsilon_1 \chi(\{a, y_1\})$ is unhindered. Taking a maximal flow $k_1$ in $\Omega_1 - \varepsilon_1 \chi(\{a, y_1\})$ and defining $f_1 = \varepsilon_1 \chi(\{a, y_1\}) + k_1$, the weighted web $\Omega_1 - f_1 = \Omega - f_0 - f_1$ is then loose.

We continue this way transfinitely until either the capacity of $a$ has been reduced to 0 or we have obtained a hindered weighted web. For each ordinal $\alpha$ write $f_\alpha = \sum_{\theta < \alpha} f_\theta$. For successor ordinals, the currents $f_\alpha$ and the weighted webs $\Omega_\alpha$ are defined as exemplified above. For limit ordinals $\alpha$ define $\Omega_\alpha = \Omega - f_\alpha$.

We wish to show that $\Omega_\alpha$ is unhindered for every $\alpha$. By the construction, this is automatically true for successor ordinals $\alpha$. Thus we only have to show:

**Assertion 6.11.** $\Omega_\alpha$ is unhindered for all limit countable ordinals $\alpha$.

The proof is by induction on $\alpha$. Let $\alpha$ be a limit ordinal, and assume that $\Omega - f_\nu$ is unhindered for all limit ordinals $\nu < \alpha$. Clearly, hindrances cannot appear at non-limit ordinals, and thus we may assume that $\Omega - f_\nu$ is unhindered for all $\nu < \alpha$. Assume, for contradiction, that there exists a hindrance $h$ in $\Omega_\alpha$. Let $z \in A$ be a hindered vertex and let $\delta = w_{\Omega_\alpha}(z) - d_h(z)$. Since $\sum_{\nu < \alpha} d_{f_\nu}(a)$ is bounded (by $w(a)$ for instance), there is some ordinal $\nu$ such that $\sum_{\nu < \theta < \alpha} d_{f_\theta}(a) < \delta$. In particular, $\sum_{\nu < \theta < \alpha} \varepsilon_\theta < \delta$. Since $f_\alpha = f_\nu + \sum_{\nu < \theta < \alpha} \varepsilon_\theta \chi(a, y_\theta) + \sum_{\nu < \theta < \alpha} k_\theta$, the current $\sum_{\nu < \theta < \alpha} k_\theta + h$ is a ($\geq \delta$)-hindrance in $\Omega - f_\nu - \sum_{\nu < \theta < \alpha} \varepsilon_\theta \chi(a, y_\theta)$. But since $\sum_{\nu < \theta < \alpha} \varepsilon_\theta < \delta$, this contradicts the fact that $\Omega - f_\nu$ is unhindered by Corollary 6.8. This proves the assertion.

Since $w_{\Omega_{\alpha+1}}(a) < w_{\Omega_\alpha}(a)$ for every $\theta$, the process must stop at some countable ordinal $\alpha$. But this can only happen when $w_{\Omega_{\alpha}}(a) = 0$. Taking $f = f_\alpha$ for $\alpha$ satisfying this condition yields the lemma. \hfill \Box

Applying this lemma recursively, we can now achieve our aim:

**Proof of Theorem 6.5.** Enumerate the vertices in $A$ as $a_1, a_2, \ldots$. Applying Lemma 6.10 to $\Delta$ with $a = a_1$ we get a current $f_1$ in $\Delta$ saturating $a_1$, and having the property that $\Delta - f_1$ is loose. Using the same lemma again, we get a current $f_2$ in $\Delta - f_1$ saturating $a_2$ in this weighted web, and such that $\Delta - f_1 - f_2$ is loose. Continuing this way, we find a sequence $f_i$ of currents, where $f_i$ saturates $a_i$ in $\Delta - \sum_{j<i} f_j$. The current $\sum f_i$ is then the desired linkage of $\Delta$. \hfill \Box

As already mentioned, Theorem 6.5 implies Theorem 6.2, which in turn implies Theorem 6.1.

### 7. Mundane Flows and Attainability

As mentioned in the introduction, Theorem 6.1 does not generalize Theorem 1.1, since the flow is allowed to contain infinite paths. One could try to generalize Theorem 1.1 by only considering flows that do not contain infinite paths:

**Definition 7.1.** A flow $f$ is mundane if (seen as a vector in $\mathbb{R}_E^I$) it can be written as $f = \sum_{i \in I} \theta_i \chi_E(E(P_i))$, where $\theta_i$ is a positive real number and $P_i$ is an $s$-$t$ path.

**Problem 7.2.** Does there exist an orthogonal pair of a cut and a mundane flow for every infinite network?

The results proved so far answer this question for certain networks. A trail in a network is a directed walk in which no edge appears more than once.
Corollary 7.3. In every countable network $\Delta = (D, s, t, c)$ that contains no infinite trail, there is an orthogonal pair of a cut and a mundane flow.

Proof. By the transformation of Section 3, $\Delta$ yields a weighted web $\Gamma = (D', A, B, w)$. Recall that $V(D') = E(D)$. By Theorem 6.2 and Lemma 4.14 there is an orthogonal pair of a separating set $S$ and a web-flow $f$ in $\Gamma$. We may assume $S$ to be essential. We claim that there is a mundane web-flow $f' \leq f$ that is also orthogonal to $S$, where mundane web-flows are defined analogously to mundane flows.

Since $\Delta$ contains no infinite trails, there are no infinite paths in $\Gamma$; we will use this fact to construct $f'$. Inductively for countable ordinals $i$ we will choose $A$-$B$ paths $P_i$ and positive real numbers $\theta_i$ so that the function $f_i := \sum_{j<i} \theta_j \chi(E(P_j))$ is a mundane web-flow with $f_i(e) \leq f(e)$ for each edge $e$. Let $i$ be a countable ordinal and assume that $P_j$ and $\theta_j$ have been defined for all $j < i$. Then, since each $f_j$ satisfies $f_j \leq f$ by assumption, the function $f_{<i} := \sum_{j<i} \theta_j \chi(E(P_j))$ is a mundane web-flow with $f_{<i} \leq f$. If $f_{<i}(e) = f(e)$ for every $e \in E(A, V(D') \setminus A)$, we terminate the construction and put $f' := f_{<i}$. Otherwise, since $\Gamma$ contains no infinite paths, the support of the web-flow $f - f_{<i}$ contains an $A$-$B$ path $P_i$; let $\theta_i := \min\{f(e) - f_{<i}(e) \mid e \in E(P_i)\}$. Clearly, $f_i$ is a mundane web-flow with $f_i \leq f$. Since $\text{supp}(f_i) \subseteq \text{supp}(f_{<i})$ and $\Gamma$ is countable, the construction terminates after countably many steps.

We thus have a mundane web-flow $f'$ that coincides with $f$ on $E(A, V(D') \setminus A)$. We have to show that $f'$ is orthogonal to $S$. Since $f' \leq f$ and $f$ is orthogonal to $S$, it suffices to show that $S \subset \text{SAT}(f')$. If $d_{f'-f'}(s) < w(s)$ for a vertex $s \in S \setminus A$, then $d_{f'-f'}^{-1}(s) > 0$. Since $f - f'$ is a web-flow, no vertex in the digraph $\hat{D} = (V(D'), \text{supp}(f - f'))$ that does not lie in $A \cup B$ has degree 1. Hence $s$ lies on an $A$-$B$ path in $\hat{D}$, or on an infinite path, or on a cycle. By the choice of $f'$, there are no $A$-$B$ paths in $\hat{D}$, and $\hat{D}$ does not contain infinite paths since $D'$ does not. So $s$ lies on a cycle $C$ in $\hat{D}$, which is clearly also a cycle in $\text{supp}(f)$. Since $S$ is essential we have $s \in RF(S) \setminus RF^c(S)$, and hence $C$ contains an edge $e$ from $V(D') \setminus RF^c(S)$ to $RF(S)$. But then $e \in \text{supp}(f)$ and thus $f(e) > 0$, contradicting the fact that $f$ is orthogonal to $S$.

We have shown that there is an orthogonal pair of a separating set $S$ and a mundane web-flow $f' = \sum_{i \in I} \theta_i \chi(E(P_i))$ in $\Gamma$. These pair can easily be translated into an orthogonal pair of a cut $F$ and a mundane flow $g$ in $\Delta$: The vertex set $S$ in $D'$ is an edge set in $D$ and it is $s$-$t$ separating in $D$ since it is $A$-$B$ separating in $D'$; hence it contains a cut $F$ in $D$. Every $A$-$B$ path $P_i$ in $D'$ corresponds to an $s$-$t$ trail $P_i'$ in $D$; let $g'$ be the function on $E(D)$ defined by $g' := \sum_{i \in I} \theta_i \chi(E(P_i'))$. It is easy to see that $g'$ is a flow in $\Delta$ orthogonal to $S$ and hence also to $F$. Therefore, each $P_i'$ meets $F$ in precisely one edge. Every $P_i'$ contains an $s$-$t$ path $Q_i$; let $g := \sum_{i \in I} \theta_i \chi(E(Q_i))$. Then $Q_i$ meets $F$ at the same edge as $P_i'$ does, and hence $g$ is a mundane flow in $\Delta$ orthogonal to the cut $F$. \qed

In the remainder of this section we show that the infimum $\sigma$ of the capacities of the $s$-$t$ cuts in a network equals the supremum $\tau_m$ of the values of the mundane flows. Moreover, we show that $\sigma$ is attained by some cut but $\tau_m$ need not be attained by any mundane flow.
**Definition 7.4.** Given a countable network $\Delta = (D, c, s, t)$, let
\[
\sigma := \inf\{c[F] : F \text{ is an } s-t \text{ cut}\},
\]
and
\[
\tau_m := \sup\{|f| : f \text{ is a mundane flow in } \Delta\}.
\]

**Theorem 7.5.** Let $\Delta = (D, c, s, t)$ be a countable network. The following statements hold:

(i) $\Delta$ has an $s-t$ cut $F$ of minimal capacity $\sigma$, and
(ii) $\tau_m = \sigma$.

**Proof.** For every positive integer $i$, let $c_i$ be the function obtained from $c$ by cutting off everything behind the $i$th decimal; formally, $c_i(e) = \lfloor 10^i c(e) \rfloor / 10^i$. In the network $\Delta_i = (D, c_i, s, t)$, all capacities are multiples of $10^{-i}$, hence we can use Theorem 1.1 to find an orthogonal pair of a mundane flow $f_i$ and a cut $F_i$ in $\Delta_i$. Since $c_i \leq c$, $f_i$ is also a flow in $\Delta$. This yields $c_i[F_i] = |f_i| \leq \tau_m$. We will use the cuts $F_i$ to construct a cut $F$ with capacity $\tau_m$.

First, enumerate all edges in $E(D)$ as $e_1, e_2, \ldots$. Then, inductively for every positive integer $i$, if there is an integer $m$ such that $m > j_i$ for all $l < i$ and the set $\{e_j, \ldots, e_{i-1}, m\}$ is contained in infinitely many of the cuts $F_1, F_2, \ldots$, then let $j_i$ be the smallest such integer $m$. If no such $m$ exists, then stop.

If $j_i$ exists for all $i$, we end up with a set $F' = \{e_{j_1}, e_{j_2}, \ldots\}$ of edges. Now choose a subsequence of $F_1, F_2, \ldots$ as follows: For every positive integer $i$, let $k_i$ be the smallest integer such that $k_i > k_l$ for all $l < i$ and the set $\{e_{j_1}, \ldots, e_{j_i}\}$ is contained in $F_{k_i}$.

If for some $i$ there is no $j_i$ as desired, we end up with a finite set $F' = \{e_{j_1}, \ldots, e_{j_{i-1}}\}$ and we choose $F_{k_1}, F_{k_2}, \ldots$ to be the subsequence of $F_1, F_2, \ldots$ consisting of all cuts that contain $F'$.

In both cases, every edge $e_l$ that is contained in infinitely many of the cuts $F_{k_1}, F_{k_2}, \ldots$ is contained in $F'$, since it must have been chosen as $e_j$ at some step $i$. We claim that $c[F'] \leq \tau_m$. Indeed, for every $\varepsilon > 0$, there is a finite subset $F''$ of $F'$ with $c[F''] \geq c[F'] - \frac{1}{2} \varepsilon$. For sufficiently large $i$, we have $c_i[F''] \geq c[F''] - \frac{1}{2} \varepsilon$ and thus $c[F'] \leq c[F''] + \varepsilon \leq \tau_m + \varepsilon$. With $\varepsilon \to 0$, this yields $c[F'] \leq \tau_m$. We further claim that $F'$ separates $s$ from $t$. Indeed, let $P$ be an $s-t$ path. Since $P$ is finite and $F_{k_1}, F_{k_2}, \ldots, \infty$, $P$ contains an edge that is contained in infinitely many of the cuts $F_{k_1}, F_{k_2}, \ldots$, and is thus contained in $F'$, so $F'$ meets every $s-t$ path. Therefore, $F'$ contains a cut $F$ which, then, satisfies $c[F] \leq \tau_m$.

This shows that $\sigma \leq c[F] \leq \tau_m$. Combining with the trivial inequality $\tau_m \leq \sigma$ we obtain the required result.

The remaining question is whether there is always a mundane flow of value $\tau_m$.

The following example shows that this is not the case, providing a negative answer to Problem 7.2.

**Example 7.6.** We construct a locally finite network in which there is no mundane flow of maximal value. We start with a disjoint union of (directed) paths $Q_i = x_0^i, x_1^i, x_2^i, x_3^i, i = 1, 2, \ldots$. For every positive integer $k$, let each edge $e$ on any path $Q_i$ with $2^{k-1} \leq i < 2^k - 1$ have capacity $c(e) = 1/2^k$. Further, for each such $k$ and $i$, we attach the paths $Q_{2i}$ and $Q_{2i+1}$ to $Q_i$ by adding the edges $(x_0^i, x_0^{2i})$, $(x_3^i, x_2^i)$ (to attach $Q_{2i}$), $(x_1^i, x_0^{2i+1})$, and $(x_3^{2i+1}, x_3^i)$ (to attach $Q_{2i+1}$). Let each such edge $e$ have capacity $c(e) = 1/2^k$. We denote the resulting digraph by $D$. The definition
of the network $\Delta = (D, c, s, t)$ is completed by choosing $s = x_0^1$ and $t = x_3^1$ (see Figure 7.1).

![Figure 7.1. A locally finite network with no mundane flow of maximal value](image)

Clearly, $D$ is locally finite (in fact it has maximum degree 3). For every positive integer $k$, there exists a mundane flow of value $1 - 1/2^k$: It is easy to see that for each positive integer $i$, there is exactly one $s-t$ path that contains $Q_i$; denote it by $P_i$. Then $f_k := \sum_{i=1}^{2^k-1} \frac{1}{2^k} P_i$ is a mundane flow of value $1 - 1/2^k$. This shows that $\tau_m \geq 1$.

We claim that there is no mundane flow in $\Delta$ that has value 1. Indeed, suppose for contradiction that $f$ is a mundane flow with $|f| = 1$. Let $e := (x_1^1, x_2^1)$ and $d := (x_1^2, x_3^0)$. Applying Kirchhoff’s first law to $x_1^1$, we obtain $f(d) \leq 1/2 - f(e)$. However, since $F = \{d, (x_2^1, x_3^1)\}$ is an $s-t$ cut with $c[F] = 1$, $f$ must saturate $F$ and thus $f(d) = 1/2$ whence $f(e) = 0$ holds. Similarly, we can prove that $f(g) = 0$ holds for every edge $g$ of the form $(x_1^i, x_2^i)$. Since these edges form an $s-t$ cut we obtain a contradiction to the fact that $f$ is mundane.

8. Flowing through an end

In this section we consider constraints on flows that are weaker than being mundane, in order to allow for flows to flow, in a sense, through ends of the digraph. As an example look at the flows in Figure 8.2 and Figure 8.3. The definition of a mundane flow does not distinguish between the two and rejects both. However, there is an important difference: The flow in Figure 8.2 disappears in the left end of the graph and comes back from the right one, while the flow in Figure 8.3 just flows through the left end. In this section we study flows of the second kind. In order to distinguish them formally from other flows we need an analog of Kirchhoff’s first law for ends. In the case of Figure 8.3 it is possible to say how much flow arrives at the left end and how much flow leaves it, but in general this is not possible: look for
example at the network in Figure 7.1. The flows $f_k$ used there have a limit flow $g$. Now for every ray $R$ in this network the values of $g$ along $R$ converge to 0, however there is some flow running to infinity and coming back. Similarly to the examples in Figure 8.2 and Figure 8.3, it is possible to construct flows like $g$ where the flow does flow out of the same ends it flows in (like in Figure 8.3 and Figure 7.1) or it does not (like in Figure 8.2). For flows like $g$ it is not clear how to make precise the assertion than the ends satisfy Kirchhoff’s first law. The following definition accomplishes this task in an elegant way:

\[
\text{Figure 8.2.} \quad \text{A network and a flow. Thick edges carry a flow of value 1; thin edges carry no flow. This flow flows into the left end of the graph and returns through the right one.}
\]

\[
\text{Figure 8.3.} \quad \text{A flow flowing through the left end of the graph.}
\]

**Definition 8.1.** We will call a flow in a network $\Delta = (D, c, s, t)$ **finite-cut-respecting** if for every cut $E(S, T)$ in $D$ (where $T = V(D) \setminus S$) with $s \in S$ that consists of finitely many edges we have

\[
(1) \quad f[E(S, T)] = \begin{cases} 
  f[E(T, S)] & \text{if } t \in S, \\
  f[E(T, S)] + |f| & \text{if } t \in T.
\end{cases}
\]

Let $\tau_w := \sup \{|f| : f \text{ is a finite-cut-respecting flow}\}$, and let $\sigma_w$ be the infimum of the capacities of all $s$–$t$ cuts consisting of finitely many edges.

To see why this definition can be thought of as an analog of Kirchhoff’s first law for ends note that in a locally finite network a cut consisting of finitely many edges cannot separate two rays in the same end. It is easy to check that $g$ as well as the flow in Figure 8.3 is finite-cut-respecting while the flow in Figure 8.2 is not.

**Theorem 8.2.** In every locally finite network $\Delta = (D, c, s, t)$, $\sigma_w = \tau_w$ holds. Moreover, there is a finite-cut-respecting flow $f$ such that $|f| = \tau_w$.

**Proof.** For every edge $e$ in $D$ let $I_e$ be the real interval $[0, c(e)]$, and define the topological space $X := \prod_{e \in E(D)} I_e$. By Tychonoff’s theorem $X$ is compact.

Pick an $s$–$t$ path $P$ in $D$, and for every $i \in \mathbb{N}$, let $\Delta_i = (D_i, c_i, s, t)$ be the finite network obtained from $\Delta$ by contracting each component $C$ of $D - \{x \in V(D) \mid d(x, P) \leq n\}$ to a vertex $v_C$, and letting $c_i(e) = c(e)$ for every edge in this network.
By the MFMC theorem (for finite networks) there is a flow $f_i$ in $\Delta_i$ such that $|f_i| = \sigma_i$, where $\sigma_i$ denotes the minimum capacity of an $s$–$t$ cut in $\Delta_i$. For every $n$, $f_i$ corresponds to a point $x_i$ in $X$: the point that has value $f_i(e)$ at the coordinate $I_e$ of $X$ for every $e \in E(D_i)$ and value 0 at every other coordinate. Since $X$ is compact, the sequence $x_1, x_2, \ldots$ has an accumulation point $x$, which determines a function $f : E(D) \rightarrow \mathbb{R}$.

We claim that $f$ is a finite-cut-respecting flow in $\Delta$; indeed, if (1) is violated by $f$ for some finite cut $B$, in particular if Kirchhoff’s law is violated at some vertex, then there is a basic open neighbourhood $O \ni x$ in $X$, chosen by taking a small enough interval of $I_e$ around $f(e)$ for every $e \in B$, such that every function in $O$ also violates (1) at $B$. But this cannot be the case since any such $O$ contains some $x_i$ where $i$ is large enough so that $B$ is a cut in $D_i$.

Similarly, it is not hard to check that $|f|$ is an accumulation point of the sequence $\{|f_i|\}_{i \in \mathbb{N}}$. Since any cut in some $D_i$ is also a cut in $D$, we have $\sigma_i \geq \sigma_w$, and since $|f_i| = \sigma_i$, we obtain $|f| \geq \sigma_w$. But $|f| \leq \tau_w \leq \sigma_w$ by (1), thus $|f| = \tau_w = \sigma_w$. \hfill \Box

Thus the value $\tau_w$ is always attained by some finite-cut-respecting flow. However, $\sigma_w$ does not have to be attained by some finite cut, as shown by the following example.

**Example 8.3.** Starting with the network of Example 7.6, we modify the capacities of its edges as follows. For every edge $e$ that is the middle edge $(x_i^1, x_i^2)$ of some path $Q_i$, let $c'(e) = 0$; for every other edge $f$, if $c(f) = 1/2^k$ then let $c'(f) = 1/4^k$. Now the resulting network $\Delta' = (D, c', s, t)$ has $\sigma_w = 0$ but the only cut of capacity 0 is the infinite cut consisting of all the middle edges of the $Q_i$.

Although the definition of a finite-cut-respecting flow allows flows through ends and forbids flows like the one in Figure 8.2, there are also instances of finite-cut-respecting flows that may seem unnatural. Look for example at Figure 8.4: it shows a finite-cut-respecting flow of value 1 from $s$ to $t$, in a network that contains no finite directed $s$–$t$ path. The following definition bans such flows.

**Figure 8.4.** A non-zero finite-cut-respecting flow in a network with no finite directed $s$–$t$ path.

**Definition 8.4.** We will call a flow $f$ in a network $\Delta = (D, c, s, t)$ cut-respecting if it is finite-cut-respecting and moreover for every $s$–$t$ cut $E(S, T)$ in $D$ we have

$$ |f| + f[E(T, S)] \leq c[E(S, T)], \quad \text{and} \quad f[E(S, T)] \leq c[E(T, S)] + |f|. $$
Intuitively, the first condition demands that if some flow circumvents an infinite $s$--$t$ cut $E(S,T)$, then this circumvention does not exceed the amount that could flow through $E(S,T)$ given its capacity $c[E(S,T)]$, taking into account that the flow through $E(S,T)$ should also compensate for any flow $f[E(T,S)]$ in the inverse direction. The second condition demands that if some $s$--$t$ cut carries more flow than $|f|$, then the excess is not greater than the amount than could go back through the inverse cut.

Let $\tau_s := \sup\{|f| : f \text{ is a cut-respecting flow}\}$.

**Theorem 8.5.** In every locally finite network $\Delta = (D, c, s, t)$ we have $\sigma = \tau_s$. Moreover, there is a cut-respecting flow $f$ such that $|f| = \tau_s$ and an $s$--$t$ cut $F$ with $c[F] = \sigma$ orthogonal to $f$.

**Proof.** Since, clearly, every mundane flow is cut-respecting, we have $\tau_s \geq \tau_m$, and thus, by Theorem 7.5 and condition (2), $\sigma = \tau_s$. Let $f_1, f_2, \ldots$ be a sequence of mundane flows in $\Delta$ whose values converge to $\tau_m = \tau_s$. As in the proof of Theorem 8.2, for every edge $e$ in $D$ let $I_e$ be the real interval $[0, c(e)]$, and define the topological space $X := \Pi_{e \in E(D)} I_e$. Every $f_i$ corresponds to a point $x_i$ in $X$: the point that has value $f_i(e)$ at the coordinate $I_e$ of $X$ for every $e \in E(D)$. Since $X$ is compact, the sequence $x_1, x_2, \ldots$ has an accumulation point $x$, which determines a function $f : E(D) \to \mathbb{R}$. Similarly with the proof of Theorem 8.2, it is straightforward to check that $f$ is a cut-respecting flow since every $f_i$ is, and that $|f| = \tau_s$.

Let $F$ be an $s$--$t$ cut with $|F| = \sigma$, which exists by Theorem 7.5. We claim that $f$ saturates $F$. Suppose for contradiction that there is an edge $e \in F$ such that $f(e) < c(e) - \epsilon$ for some $\epsilon > 0$. Then, there is an infinite subsequence $(f_i)$ of $(f_i)$ with $f'_i(e) < c(e) - \epsilon$. But this means that the $f'_i$ are mundane flows in the network $\Delta'$ obtained from $\Delta$ by reducing $c(e)$ by $\epsilon$. Thus, $\lim |f'_i| \leq \tau_m - \epsilon$ by Theorem 7.5 since $F$ is a cut of capacity $\sigma - \epsilon$ in that network. This contradicts the choice of $(f_i)$, so $f$ saturates $F$ as claimed. Similarly, it is easy to show that for every $T$--$S$ edge $e$ we have $f(e) = 0$, which proves that $f$ and $F$ form an orthogonal pair.

It is possible to consider networks where the source $s$ or sink $t$ or both are ends of the underlying digraph $D$ instead of vertices. An $s$--$t$ flow of value $m$ is, then, a function $f$ on $E(D)$ such that $KIR(f) = V(D)$ and moreover, for every finite cut $E(S,T)$ such that $s$ lives in $S$ we have $f(E(S,T)) = m$ unless $t$ also lives in $S$, in which case we have $f(E(S,T)) = 0$. Here, we say that an end lives in $S$ if one of its rays, and thus, since $E(S,T)$ is finite, a subray of any of its rays, is contained in $S$; we also say that the vertices of $S$ live in $S$. The interested reader will be able to confirm that the results of this section carry over to such networks and flows.

**References**


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