Continuity of the discrete curve evolution

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ABSTRACT

Recently Latecki and Lakämper (*Computer Vision and Image Understanding* 73:3, March 1999) reported a novel process for a discrete curve evolution. This process has various application possibilities, in particular, for noise removal and shape simplification of boundary curves in digital images.

In this paper we prove that the process of the discrete curve evolution is continuous: if polygon Q is close to polygon P, then the polygons obtained by their evolution remain close. This result follows directly from the fact that the evolution of Q corresponds to the evolution of P if Q approximates P. This intuitively means that first all vertices of Q are deleted that are not close to any vertex of P, and then, whenever a vertex of P is deleted, then a vertex of Q that is close to it is deleted in the corresponding evolution step of Q.

Keywords: discrete curve evolution, polygon evolution, shape simplification, curve approximation

1. INTRODUCTION

Latecki and R. Lakämper in Ref. 1 presented a shape similarity measure for object contours. An application of this measure to retrieval of similar objects in a database of object contours is demonstrated in Figure 1, where the user query is given by a graphical sketch.

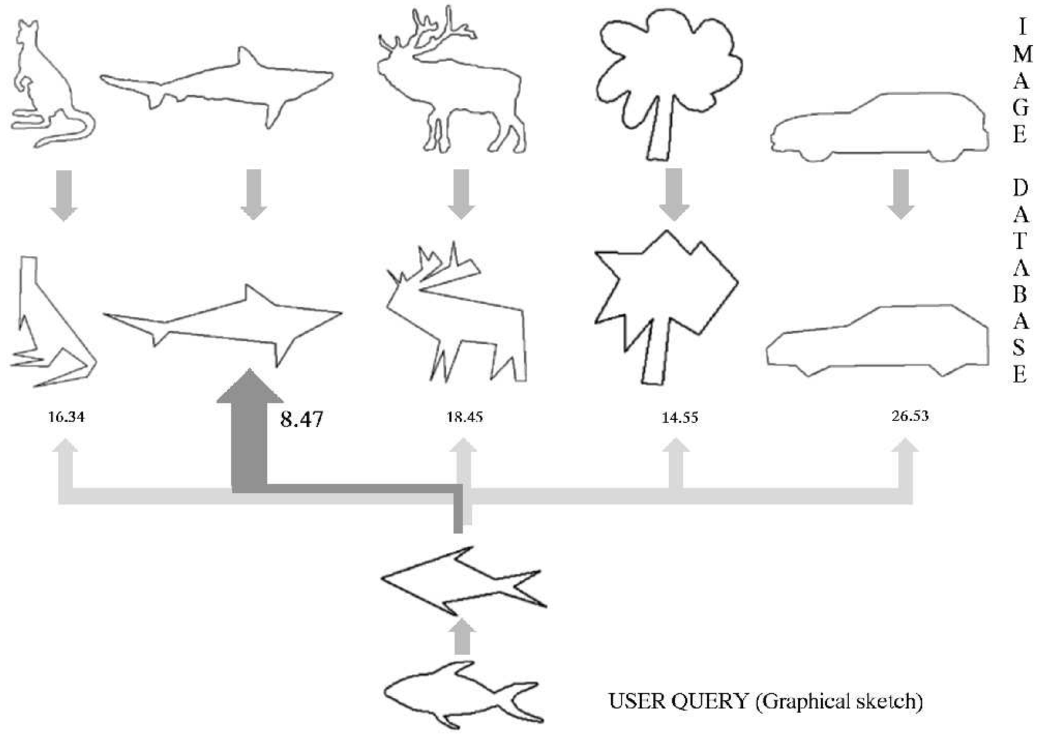

Figure 1. Retrieval of similar objects based on a similarity measure of object contours.

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Since contours of objects in digital images are distorted due to digitization noise and due to segmentation errors, it is necessary to neglect the distortions while at the same time preserving the perceptual appearance at the level sufficient for object recognition. To achieve this, the shape of objects is simplified by a novel discrete curve evolution method, before the similarity measure is applied. This approach allows us

- to reduce influence of noise and
- to simplify the shape by removing *irrelevant* shape features without changing *relevant* shape features.

The robustness of the discrete curve evolution method with respect to noisy deformations has been verified by numerous experiments (e.g., see Ref. 2 or online demos on our web site[3]). The continuity theorem (Theorem 1 in Section 2) gives a formal justification of this fact: if a polygon Q is close to a polygon P, e.g., Q is a distorted version of P, then the polygons obtained by their evolution remain close. Thus, continuity guarantees us the stability of the discrete curve evolution with respect to noise.

Moreover, the digital curve evolution allows us to find line segments in noisy images, due to the relevance order of the repeated process of linearization (see e.g., Figure 2). This fact follows from Theorems 1 and 2, which we prove in Section 2, since if polygon Q is close to polygon P, then first all vertices of Q are deleted that are not close to any vertex of P (Theorem 1), and then, whenever a vertex of P is deleted, then a vertex of Q that is close to it is deleted in the corresponding evolution step of Q (Theorem 2). Therefore, the linear parts of the original polygon will be recovered during the discrete curve evolution.

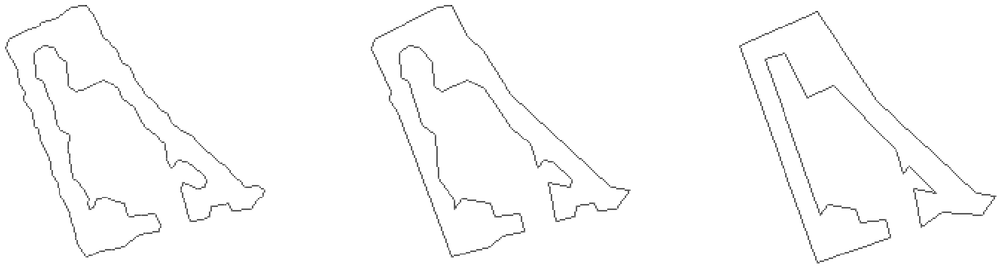

Figure 2. (a) $\rightarrow$ (b): noise elimination. (b) $\rightarrow$ (c): extraction of relevant line segments.

We assume that a closed polygon P is given. In particular, any boundary curve in a digital image can be regarded as a polygon without loss of information, with possibly a large number of vertices. We denote the set of vertices of P with $Vertices(P)$ and the set of edges with $Edges(P)$.

The discrete curve evolution produces a sequence of polygons $P = P^0, ..., P^m$ such that $|Vertices(P^m)| \leq 3$, where $| \, . \, |$ is the cardinality function. The process of the discrete curve evolution is very simple. The outline of the algorithm is the following.

For every evolution step $i = 0, ..., m-1$:

1. Each vertex v in P^i is assigned a relevance measure $K(v, P^i)$.
2. A polygon P^{i+1} is obtained after the vertices whose relevance measure is minimal are deleted from P^i.

The relevance measure $K(v, P^i)$ is defined below. A few stages of our curve evolution are illustrated in Figure 3.

The process of the discrete curve evolution is guaranteed to terminate, since in every evolution step, the number of vertices decreases by at least one. It is also obvious that this evolution converges to a convex polygon, since the evolution will reach a state where there are exactly three, two, one, or no vertices in P^m. Only when the set $Vertices(P^m)$ is empty, we obtain a degenerated polygon equal to the empty set, which is trivially convex. Thus, we obtain for every relevance measure K

PROPOSITION 1. *The discrete curve evolution converges to a convex polygon, i.e., there exists* $0 \leq i \leq m$ *such that* P^i *is convex, and if* $i < m$*, all polygons* $P^{i+1}, ..., P^m$ *are convex.* ■

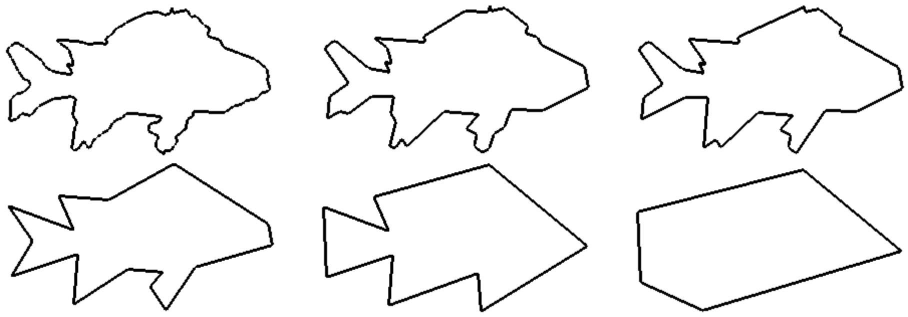

Figure 3. A few stages of our curve evolution. The first contour is a distorted version of the contour on www-site.[4]

Now we give a precise definition of the discrete curve evolution.

Definition: Let

$$K_{min}(P^i) = \min\{K(u, P^i) : u \in Vertices(P^i)\}$$

and

$$V_{min}(P^i) = \{u \in Vertices(P^i) : K(u, P^i) = K_{min}(P^i)\},$$

i.e., $V_{min}(P^i)$ contains the vertices whose relevance measure is minimal in P^i for $i = 0, ..., m-1$.

For a given polygon P and a relevance measure K, we call **discrete curve evolution** a process that produces a sequence of polygons $P = P^0, ..., P^m$, where $|Vertices(P^m)| \leq 3$, such that

$$Vertices(P^{i+1}) = Vertices(P^i) \setminus V_{min}(P^i).$$

An algorithmic definition of the discrete curve evolution is given in Ref. 5 (see also our www-site[3]).

The key property of this evolution is the order of the substitution. The substitution is done according to a relevance measure $K(v, P^i) = K(u, v, w)$, where u, w are neighbor vertices of vertex v in P^i. The relevance measure $K(v, u, w)$ is given by the formula

$$K(v, u, w) = \frac{\beta l_1 l_2}{l_1 + l_2}, \tag{1}$$

where β is the turn angle at vertex v in P^i, l_1 is the length of $\overline{vu}$, and l_2 is the length of $\overline{vw}$. (Both lengths are normalized with respect to the total length of polygon P^i.) The main property of this relevance measure is the following

- The higher the value of $K(v, u, w)$, the larger is the contribution to the shape of the polygon P^i of arc $\overline{vu} \cup \overline{vu}$.

A motivation for this measure and its properties are described in Ref. 5. Observe that the relevance measure is not a local property with respect to the polygon P, although its computation is local in P^i for every vertex v.

We prove in Theorem 1 that the discrete curve evolution is continuous: if polygon Q is close to polygon P, then the polygons obtained by their evolution are close. Moreover, we show that the evolution of Q will correspond to the evolution of P if Q approximates P (Theorem 2). Before we state these theorems, we need a few more definitions.

Definition: Let $p(P) = \min\{d(E, v) : E \in Edges(P) \text{ and } v \in Vertices(P) \text{ and } v \notin E\}$, i.e., $p(P)$ is the minimal distance from vertices to edges they do not belong. For example, in Figure 4 $p = p(P)$ is equal to the distance from vertex B to edge CD.

Definition: For every $E \in Edges(P)$ we call $Strip_H(E) = B(E, \frac{H}{2}) = \bigcup_{x \in E} B(x, \frac{H}{2})$ H-**strip** of P around E and we denote the set of all strips of P by $Strips_H(P) = \{Strip_H(E) : E \in Edges(P)\}$. Figure 4 shows the set of all strips of P.

Definition: Two strips $Strip_H(E)$ and $Strip_H(F)$ are **adjacent** if sides E and F are adjacent, i.e., intersect in a vertex of P.

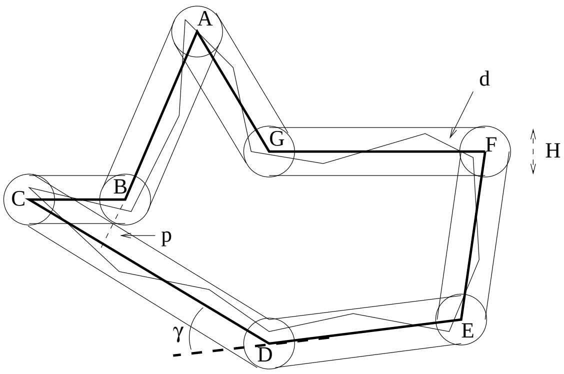

Figure 4.

Observe that if $H < p(P)$, then the intersection of two non-adjacent H-strips of P is empty.

Definition: For a closed polygon P and for every $v_0 \in Vertices(P)$, there exist exactly two linear orders $<_P$ on $Vertices(P) = \{v_0, v_1, ..., v_{k+1} = v_0\}$ such that $v_0 <_P v_1 <_P ... <_P v_{k+1} = v_0$ and $\overline{v_s v_{s+1}}$ is an edge of P, where $\overline{xy}$ is the line segment joining point x to y. We denote the set of all such linear orders on $Vertices(P)$ by $\mathcal{O}_P$. Clearly, $|\mathcal{O}_P| = 2|Vertices(P)|$.

Clearly, the set $Vertices(P)$ together with one of the linear orders $<_P$ uniquely determines $Edges(P)$. Any linear order $<_P \in \mathcal{O}_P$ can be extended to a linear order on all points of P, i.e., on $\bigcup Edges(P)$.

Definition: For any two points x, y in a H-strip S around an edge $\overline{uv}$ of P, we say that $x <_S y$ iff $\pi(x) <_P \pi(y)$, where $\pi : S \to \overline{uv}$ is the metric projection.

Definition: A polygon Q has **the same order** as polygon P if there exists a function $i : Vertices(Q) \to Strips(P)$ an order $<_P \in \mathcal{O}_P$ and an order $<_Q \in \mathcal{O}_Q$ such that

$q \in i(q)$ for all $q \in Vertices(Q)$,

$q_1 <_Q q_2$ iff $(i(q_1) <_P i(q_2))$ or $(i(q_1) = i(q_2)$ and $q_1 <_{i(q_1)} q_2$ in H-strip $i(q_1))$.

For example, see Figure 4. If $H < d$, where d is the smallest distance between two vertices of Q, then no two vertices of Q can be projected by π to the same point.

Definition: We say that $y \in Vertices(P)$ is a **neighbor (vertex)** of $x \in Vertices(P)$ if line segment $\overline{xy}$ is an edge of P. We also say in this case that x and y are **adjacent**.

Definition: A polygon Q is called an (H, d)-**approximation** of polygon P if $Q \subset B(P, \frac{H}{2})$, Q has the same order as polygon P, and the minimal length of edges of Q equal to d. (see e.g., Figure 4).

2. CONTINUITY OF THE DISCRETE CURVE EVOLUTION

Theorem 1 states that the discrete curve evolution is continuous. Intuitively, it says that if polygon Q is sufficiently close to a polygon P, then the polygons obtained by the evolution of Q will remain close to P. This will be the case until a stage k of the evolution of Q is reached at which exactly one vertex of Q^k is contained in $B(w, \epsilon)$ for every vertex w of P and all other vertices of Q are deleted.

THEOREM 1. **Continuity of the discrete curve evolution.** *Let P be a simple polygon. For every $0 < d$ and every $0 < \epsilon < \frac{d}{2}$, there exist $\delta > 0$ and $k \in \mathbf{Z}_+$ such that if polygon Q is (δ, d)-approximation of P, then*

$$\forall i \in \{0, 1, ..., k\} \quad Q^i \subset B(P, \epsilon)$$

and

$$\exists \text{ bijection } b : Vertices(Q^k) \to Vertices(P) \quad \forall v \in Vertices(Q^k)\ \forall w \in Vertices(P) \quad [v \in B(w, \epsilon) \Leftrightarrow w = b(v)].$$

Proof: Let $0 < d$ and $0 < \epsilon < \frac{d}{2}$ be given. Let H_0 be given by Theorem 3, below.

Let δ be determined by Proposition 2, below, for $\eta = \min\{\epsilon, \frac{H_0}{2}\}$.

Let polygon Q be (δ, d)-approximation of P. Then, by Proposition 2, Q is also a $(2\eta, d)$-approximation of P for which additionally holds that for every vertex v of P at least one vertex of Q is contained in $B(v, \eta)$.

Since $2\eta \leq H_0$, we obtain by Theorem 3 for $H = 2\eta$ that there exists an evolution stage k of Q such that

$$\forall i \in \{0, 1, ..., k\} \;\; Q^i \subset B(P, \eta)$$

and

$$\exists \text{ bijection } b : Vertices(Q^k) \to Vertices(P) \;\; \forall v \in Vertices(Q^k) \; \forall w \in Vertices(P) \;\; [v \in B(w, \eta) \Leftrightarrow w = b(v)].$$

Since $\eta \leq \epsilon$,

$$\forall i \in \{0, 1, ..., k\} \;\; Q^i \subset B(P, \epsilon),$$

and since at most one $v \in Vertices(Q^k)$ can be contained in $B(w, \epsilon)$ for $w \in Vertices(P)$, due to $\epsilon < \frac{d}{2}$, we obtain

$$\forall v \in Vertices(Q^k) \; \forall w \in Vertices(P) \;\; [v \in B(w, \epsilon) \Leftrightarrow w = b(v)],$$

which proves the theorem. ■

Theorem 2 says that if Q^k is a sufficiently close approximation of P and vertices of Q^k correspond to vertices of P, then the discrete curve evolution of Q^k follows the evolution of P.

THEOREM 2. *Let $P = P^0, ..., P^m$ be polygons obtained from a polygon P in the course of discrete curve evolution such that P^m is the first convex polygon and all minimal relevance measures are obtained for only one vertex, i.e., $|V_{min}(P^i)| = 1$ for $i = 0, ..., m-1$.*

There exists $0 < \xi$ such that if polygon Q^k be (ξ, d)-approximation of P and

$$\exists \textit{ bijection } b : Vertices(Q^k) \to Vertices(P) \;\; \forall v \in Vertices(Q^k) \; \forall w \in Vertices(P) \;\; [v \in B(w, \xi) \;\Leftrightarrow\; w = b(v)],$$

then for the evolution stages $i \in \{0, 1, ..., m-1\}$ of P we have:

$$v \in Vertices(Q^{k+i}) \setminus Vertices(Q^{k+i+1}) \;\Leftrightarrow\; b(v) \in Vertices(P^i) \setminus Vertices(P^{i+1}).$$

Proof: Let

$$C_i = \min\{K(u, P^i) - K_{min}(P^i) : u \in Vertices(P^i) \text{ and } K(u, P^i) \neq K_{min}(P^i)\}$$

and let

$$C_{min} < \min\{C_i : i = 0, ..., m-1\}.$$

If u, w are neighbors of vertex v in P^i, then the relevance measure is denoted by $K(v, P^i) = K(u, v, w)$. Since $K(u, v, w)$ is a continuous function of $l_1 = |vu|, l_2 = |v, w|$, and the turn angle β at v, it is also continuous with respect to the position of vertices:

$$\forall C > 0 \; \exists \epsilon_1 > 0 \; [u' \in B(u, \epsilon_1), v' \in B(v, \epsilon_1), w' \in B(w, \epsilon_1)] \;\Rightarrow\; |K(u, v, w) - K(u', v', w')| < C.$$

We take ξ to be the $\epsilon_1 > 0$ obtained for $\frac{C_{min}}{2}$ and for all triples of vertices v, u, w in P. Thus, we have

$$[u' \in B(u, \xi), v' \in B(v, \xi), w' \in B(w, \xi)] \;\Rightarrow\; |K(u, v, w) - K(u', v', w')| < \frac{C_{min}}{2}$$

and

$$\forall v \in Vertices(Q^k) \; \forall w \in Vertices(P) \;\; [v \in B(w, \xi) \;\Leftrightarrow\; w = b(v)].$$

Thus, for every there vertices v, u, w in Q^k we have

$$[u \in B(b(u), \xi), v \in B(b(v), \xi), w \in B(b(w), \xi)] \Rightarrow |K(u,v,w) - K(b(u), b(v), b(w))| < \frac{C_{min}}{2}.$$

Hence for $u_1, v_1, w_1, u_2, v_2, w_2$ vertices in Q^k

$$K(b(u_1), b(v_1), b(w_1)) - K(b(u_2), b(v_2), b(w_2)) > C_{min} \Rightarrow K(u_1, v_1, w_1) - K(u_2, v_2, w_2) > 0.$$

Consequently, $|V_{min}(Q^{k+i})| = 1$ for $i = 0, ..., m-1$.

Let $u \in V_{min}(P^i)$ and $z \in Vertices(P^i) \setminus V_{min}(P^i)$. Then $K(z, P^i) - K(u, P^i) > C_{min}$, and therefore, $K(b^{-1}(z), Q^{k+i}) - K(b^{-1}(u), Q^{k+i}) > 0$. Therefore, $u \in V_{min}(P^i)$ implies $b^{-1}(u) \in V_{min}(Q^{k+i})$, and consequently, for every $i \in \{0, 1, ..., m-1\}$:

$$v \in Vertices(P^i) \setminus Vertices(P^{i+1}) \text{ iff } b^{-1}(v) \in Vertices(Q^{k+i}) \setminus Vertices(Q^{k+i+1}).$$

■

THEOREM 3. *For every polygon P and every $0 < d$, there exist $0 < H_0 < \frac{2}{\sqrt{5}}d$ such that for every $H < H_0$ and for every (H,d)-approximation Q of P with the property that for every vertex v of P at least one vertex of Q is contained in $B(v, \frac{H}{2})$, there exists k such that:*

$$\forall i \in \{0, 1, ..., k\} \quad Q^i \text{ is } (H,d) - \text{approximation of } P$$

and

$$\exists \text{ bijection } b : Vertices(Q^k) \to Vertices(P) \ \ \forall v \in Vertices(Q^k) \ \forall w \in Vertices(P) \ \ [v \in B(w, \frac{H}{2}) \Leftrightarrow w = b(v)].$$

Proof: Let P be a polygon with the minimal turn angle γ and with $p = p(P)$ (see Figure 4).
Let Q be (H,d)-approximation of P for any $H < \frac{2}{\sqrt{5}}d$.

We divide vertices of Q into two types: We call every vertex q of Q that does not belong to $B(v, \frac{H}{2})$ for any vertex v of P **Type 1** vertex. We call every vertex q of Q that is contained in $B(v, \frac{H}{2})$ for some vertex v of P **Type 2** vertex.

We will show that there exists $H_0 < \frac{2}{\sqrt{5}}d$ (H_0 depends on γ, d) and an evolution stage k of Q such that for every $H < H_0$ all Type 2 vertices remain and all Type 1 vertices are deleted from Q^k.

Observe that at most one vertex q of Q can be contained in $B(v, \frac{H}{2})$, since $H < d$, where d is the minimal length of edges of Q. Hence, for (H,d)-approximation Q of P exactly one q of Q is contained in $B(v, \frac{H}{2})$ for every vertex v of P.

Step 1

We first show in Lemma 4 (below) that if x is a Type 1 vertex, then x and its direct neighbors $y, z \in Vertices(Q)$ are in the same H-strip S of P and x is between y, z with respect to $>_Q$, i.e., either $y >_Q x >_Q z$ or $z >_Q x >_Q y$. This implies that x is between y, z with respect to the order $>_S$ of points in H-strip S.

Then we show in Lemma 5 (below) that if three points x, y, z are in the same H-strip S of P and x is between y, z with respect to $>_S$, then the relevance measure $K(x,y,z)$ of $\overline{zx} \cup \overline{xy}$ is bounded by a function m_4 of H, d. This implies that $K(x,y,z) \le m_4(H,d)$ for every Type 1 vertex x of Q.

LEMMA 1. *Let $y \in Vertices(Q)$ and $i(y) = Strip_H(\overline{uv})$, where $u, v \in Vertices(P)$ and $v >_P u$ (see Figure 5). If $b \in B(v, \frac{H}{2}) \cap Vertices(Q)$ and y and b are neighbors in Q, than $b >_Q y$.*

Proof: Let $S = i(y)$ and $\overline{uv}$ be the side of P contained in S. Let S_v be the line segment of length H perpendicular to $\overline{uv}$ such that S_v is contained in S and $v \in S_v$ (see Figure 5).

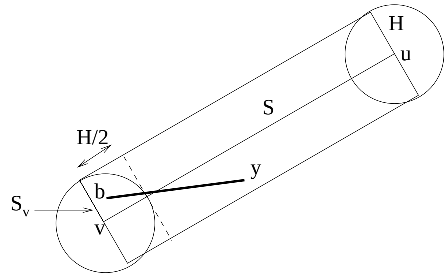

Figure 5.

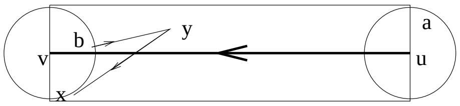

Figure 6.

Since $b \in B(v, \frac{H}{2})$, $d(b, S_v) \le \frac{H}{2}$. The diagonal of rectangle with side of length H and $\frac{H}{2}$ is equal to $\frac{\sqrt{5}}{2}H$. Since $H < \frac{2}{\sqrt{5}}d$, this diagonal is shorter than d. Therefore, if b is contained in such a rectangle no other vertex of Q can be contained in it. Since $i(y) = Strip_H(\overline{uv})$, we obtain $b >_Q y$. ■

LEMMA 2. *Let $x \in Vertices(Q)$ and $i(x) = Strip_H(\overline{uv})$, where $u, v \in Vertices(P)$ and $v > u$ (see Figure 6). If $a \in B(v, \frac{H}{2}) \cap Vertices(Q)$ and $b \in B(u, \frac{H}{2}) \cap Vertices(Q)$, than $b \ge_Q x \ge_Q a$.*

Proof: If $i(a) \neq i(x)$, then $i(x) >_P i(a)$ implies $x >_Q a$.

If $i(b) \neq i(x)$, then $i(b) >_P i(x)$ implies $b >_Q x$.

It remains to consider the case $i(a) = i(x) = i(b)$. We show that $b >_Q x$ by showing that the inverse order $x >_Q b$ leads to inconsistency. The proof of $x >_Q a$ is analog.

We assume that $x >_Q b$. Then x and b cannot be neighbor vertices of Q by Lemma 1. Thus, there exists $y \in Vertices(Q)$ such that y is neighbor of b and $x >_Q y >_Q b$ (see Figure 6). Yet, if $i(x) >_P i(y)$, then $b >_Q y$. If $i(x) = i(y)$, then $b >_Q y$ by Lemma 1. ■

LEMMA 3. *Let x be a Type 1 vertex of Q. If $y \in Vertices(Q)$ is a neighbor of x (i.e., $\overline{xy}$ is an edge of Q), then $i(x) = i(y)$ or $y \in B(v, \frac{H}{2}) \subseteq i(x)$ for some vertex v of P (see Figure 7(a)).*

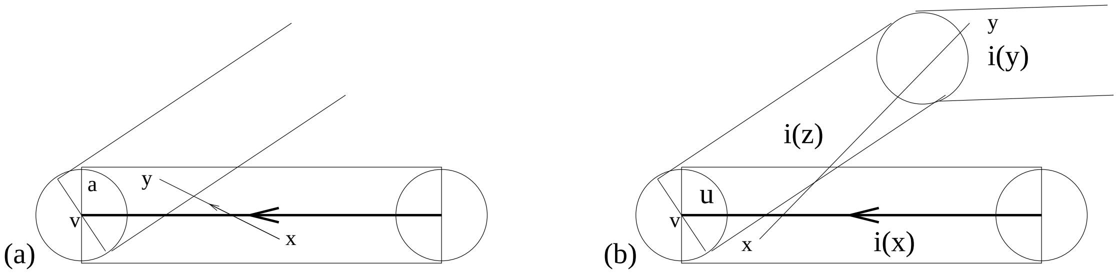

Figure 7.

Proof: We assume that $i(x) \neq i(y)$. Then $i(y) >_P i(x)$ or $i(x) >_P i(y)$, say $i(y) >_P i(x)$, which implies $y >_Q x$.

First we show that the intersection $i(x) \cap i(y)$ is not empty, i.e., $i(x)$ and $i(y)$ are neighbor strips. If this were not the case (see Figure 7(b)), then there exists z such that $i(z) \neq i(x)$, $i(z) \neq i(y)$, $i(z) \cap i(x) \neq \emptyset$, and $\overline{xy} \cap i(z)$ is nonempty. Since x is a Type 1 vertex of Q, there exists $u \in i(z) \cap i(x)$ (this intersection is equal to $B(v, \frac{H}{2})$ for some $v \in P$). If $i(u) = i(z)$, then $y >_Q u >_Q x$. If $i(u) = i(x)$, then $y >_Q u >_Q x$, by Lemma 2. Since $y >_Q u >_Q x$ contradicts the fact that $\overline{xy}$ is an edge of Q, we obtain that $i(x)$ and $i(y)$ are neighbor strips.

Therefore, there exists $v \in i(x) \cap i(y)$ a vertex of P and $a \in B(v, \frac{H}{2}) = i(x) \cap i(y)$ a vertex of Q (see Figure 7(a)). By Lemma 2, $a >_Q x$.

Since $H < p$, $B(v, \frac{H}{2})$ does not intersect any strip of P other than $i(x)$ and $i(y)$. Hence, $i(a) = i(x)$ or $i(a) = i(y)$.

If $i(a) = i(x)$, then $y >_Q a$, since $i(y) >_P i(x)$.

If $i(a) = i(y)$, then $y = a$ or $y >_Q a$, by Lemma 2.

If $y = a$, then $y = a \in B(v, \frac{H}{2})$.

If $y > a$, then $y > a > x$, which is inconstant with the fact that y is a direct neighbor of x. ■

LEMMA 4. *Let x be a Type 1 vertex of Q. If $y, z \in Vertices(Q)$ are different neighbors of x then $y, z \in i(x)$ and x is between y, z with respect to $<_Q$.*

Proof: Since $y, z \in Vertices(Q)$ are different neighbors of x, clearly, x is between y, z with respect to $<_Q$.

If $i(x) = i(y) = i(z)$, then $y, z \in i(x)$ is trivially true.

If $i(x) \neq i(y)$, then $y \in B(v, \frac{H}{2}) \subseteq i(x)$ for some vertex v of P, by Lemma 3. Consequently $y \in i(x)$. Similarly, $z \in i(x)$ if $i(x) \neq i(z)$. ■

LEMMA 5. *Let points x, y, z be in the same H-strip S of P and x be between y, z with respect to $>_S$. Then the relevance measure $K(x, y, z)$ of $\overline{zx} \cup \overline{xy}$ is bounded by a function m_4 of H, d:*

$$K(x, y, z) \leq m_4(H, d) = \frac{d\,H\,\left(\frac{\pi}{2} + \arcsin(\frac{H}{d})\right)}{d + H}.$$

Proof: Let S_y be the maximal line segment contained in S that contains points y and $\pi(y)$. Clearly, the length of S_y is H (see Figure 8(a)). S_z is similarly defined. Since x is between y, z with respect to $>_S$, x lies between S_y and S_z. Hence, the situation in Figure 8(b) is not possible.

Let L be the distance between S_y and S_z and let $R \subset S$ be the maximal rectangle contained in S with sides S_y and S_z. We call L the length of R and H the height of R.

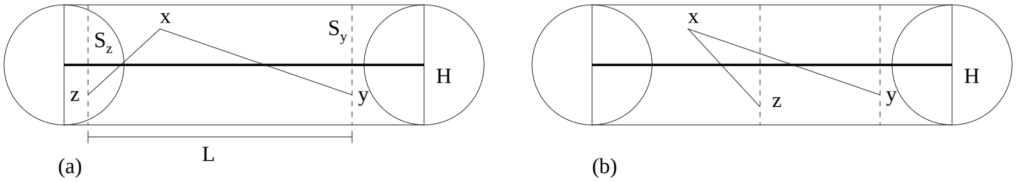

Figure 8.

Let y', z' be the endpoints of the side of R which has length L and which is further away form x than the other side of length L (Figure 9(a)). Since $|xz'| \geq |xz|$, $|xy'| \geq |xy|$, and turn angle $y'xz'$ is not smaller than turn angle yxz, we obtain $K(x, y', z') \geq K(x, y, z)$, where $|ab|$ is the length of the line segment $\overline{ab}$. Hence it is sufficient to compute the relevance measure for $y = y'$ and $z = z'$ (Figure 9(b)).

Let $\overline{M}$ be the side of length L of R that is different from side $y'z'$ and let x' be the metric projection of x on $\overline{M}$ (Figure 9(b)). Since $|x'z'| \geq |xz'|$, $|x'y| \geq |xy'|$ and the angle $y'x'z'$ is not smaller than $y'xz'$, we obtain $K(x', y', z') \geq K(x, y', z')$.

If x' is moving on side $\overline{M}$, then $K(x', y', z') \geq K(x, y, z)$ for $x, y, z \in R$. Since $K(x', y', z')$ is only a function of x', we will denote it by $m(x') = K(x', y', z')$. Since x' is moving on side $\overline{M}$, we can assume that $x' \in [0, L]$. Thus, we have $K(x, y, z) \leq m(x')$, where $x, y, z \in R$ and $x' \in [0, L]$.

The function $m(x)$ with $x \in (0, L)$ is given by the formula (see Figure 10(a))

$$m(x) = \frac{\sqrt{H^2 + (L - x)^2}\,\sqrt{H^2 + x^2}\,\left(\arctan(\frac{H}{L-x}) + \arctan(\frac{H}{x})\right)}{\sqrt{H^2 + (L - x)^2} + \sqrt{H^2 + x^2}} \tag{2}$$

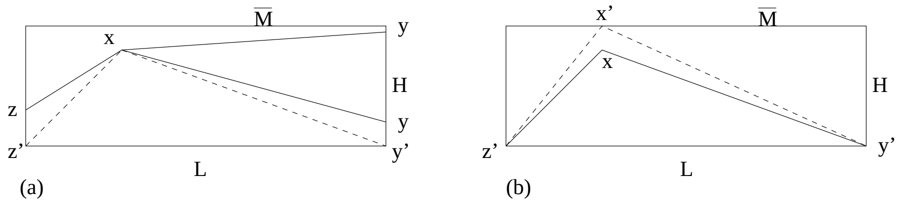

Figure 9.

where L and H are constant. It is easy to obtain by analyzing function m that m is symmetric with respect to $x = \frac{L}{2}$, m obtains its minimum at point $x = \frac{L}{2}$, and m is monotone decreasing for $x \in (0, \frac{L}{2}]$ and monotone increasing for $x \in [\frac{L}{2}, L)$.

Let x_0 be the point $x \in [0, L]$ for which the distance to z is equal to d (see Figure 10(b)). Since $H < d$, x_0 belongs to interval $(0, L)$. Since m is symmetric, m is defined for $x \in [x_0, L - x_0]$. Therefore, m reaches its maximum for $x = x_0$, and consequently, $K(x, y, z) \leq m(x_0)$.

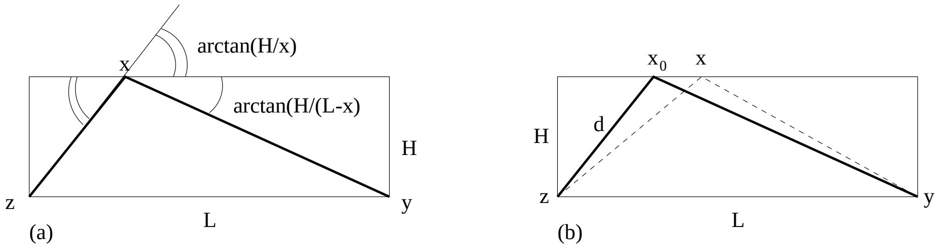

Figure 10.

Since the position of point x_0 is a function of H and d, we will treat $m(x_0)$ as a function of H, L, d, which we will denote by m_2. Consequently, we obtain $K(x, y, z) \leq m_2(H, L, d) = m(x_0)$ for $x, y, z \in [0, L] \times [0, H]$.

It remains to find an upper bound of the function $m_2(H, L, d)$ for $L \in (0, \infty)$. To achieve this, we find a more handy formula for $m_2(H, L, d) = m(x_0)$, i.e., we find $m_3(H, d, t) = m_2(H, L, d)$, where $t \in [0, \infty)$ and $t + H$ is the distance from x_0 to y (see Figure 11). The function m_3 is given by

$$m_3(H, d, t) = \frac{d\ (H + t)\ \left(\arcsin(\frac{H}{d}) + \arcsin(\frac{H}{H+t})\right)}{d + H + t}. \tag{3}$$

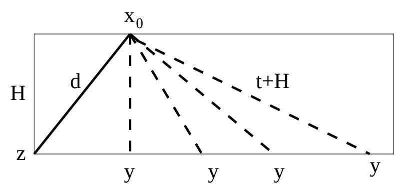

Figure 11.

We will show that

$$m_3(H, d, t) \leq m_4(H, d) = \frac{d\,H\ \left(\frac{\pi}{2} + \arcsin(\frac{H}{d})\right)}{d + H}, \tag{4}$$

where $m_4(H, d)$ is equal m_3 with $t = 0$.

When we multiply $m_4(H, d) - m_3(H, d, t) \geq 0$ by $(d + H)(d + H + t)$ and simplify it, we get

$$H\,\pi\ (d + H + t) - 2\,d\,t\ \arcsin(\frac{H}{d}) - 2\ (d + H)\ (H + t)\ \arcsin(\frac{H}{H + t}) \geq 0. \tag{5}$$

Since $\frac{\pi}{2}x \geq \arcsin x$, we obtain

$$(5) \geq H\pi(d+H) + H\pi t - H\pi t - H\pi(d+H) = 0 \tag{6}$$

■

It follows from Lemmas 4 and 5 that $K(x,y,z) \leq m_4(H,d)$ for every Type 1 vertex x of Q. Moreover, it holds $\lim_{H\to 0} m_4(H,d) = 0$ for every d.

Step 2

We show that the relevance measure of a Type 2 vertex is bounded from below. It follows from Lemma 6 that if x is a Type 2 vertex, then either the direct neighbors $y, z \in Vertices(Q)$ of x are in different strips or x is not between y, z with respect to $>_Q$.

LEMMA 6. *Let b be* **Type 2** *vertex, i.e., $b \in Vertices(Q)$ and $b \in B(v, \frac{H}{2})$ for some vertex v of P. Let $y, z \in Vertices(Q)$ be the direct neighbors of b. Then either $i(y) \neq i(z)$ or b is not between y, z with respect to $>_Q$.*

Proof: We assume that $i(y) = i(z)$. Then, by Lemma 1, either both $b >_Q y$ and $b >_Q z$ or both $b <_Q y$ and $b <_Q z$. Thus, b is not between y, z with respect to $>_Q$. ■

Let x be Type 2 vertex. Since $i(y) \subseteq i(y) \cup i(z)$, if $y, z \in i(y)$, then $y, z \in i(y) \cup i(z)$. Since minimum on a larger set can only be smaller, it is sufficient to consider the case in which $i(y) \neq i(y) \cup i(z)$. Therefore, we assume that $i(y) \neq i(z)$.

Since we seek an lower bound of K, and K is monotone increasing with respect to the length of line segments xy and xz, it is sufficient to consider the case in which $\overline{xy} = d$ and $\overline{xz} = d$. For every position of x (a Type 2 vertex), the turn angle yxz is the smallest if y, z lie on the outer boundary of $i(y) \cup i(z)$, see Figure 12(a). Since K is monotone increasing with respect to the turn angle, we assume that y, z lie on the outer boundary of $i(y) \cup i(z)$.

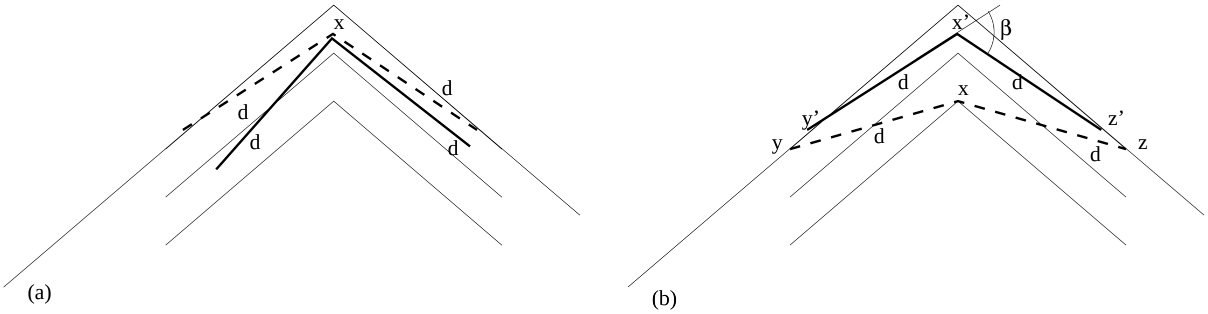

Figure 12.

Now we show that the smallest value of $K(x,y,z)$ is obtained if x lies at the inside corner of $i(y) \cup i(z)$, see Figure 12(b). Since $\overline{xy} = d$ and $\overline{xz} = d$, we have $K(x,y,z) = K(\beta)$, where β is the turn angle yxz, see Figure 12(b). Since $K(\beta)$ is a monotone decreasing function of the distance between y and z, it reaches the minimum when this distance is maximal, see Figure 13, which is the case if x lies in the inside corner of $i(y) \cup i(z)$.

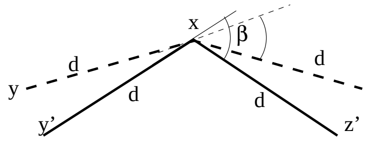

Figure 13.

For these positions of x, y, z, we have

$$K(x,y,z) = (\alpha - 2\arcsin\frac{H}{d})\frac{d}{2},$$

where α is the turn angle of $i(y) \cup i(z)$, see Figure 14. Since $\alpha \geq \gamma$, we obtain

$$K(x,y,z) \geq k(H,\gamma,d) = (\gamma - 2\arcsin\frac{H}{d})\frac{d}{2} \tag{7}$$

for any Type 2 vertex x and its neighbors y, z. Moreover, it holds $\lim_{H\to 0} k(H,\gamma,d) = \gamma\frac{d}{2}$ for every γ and d.

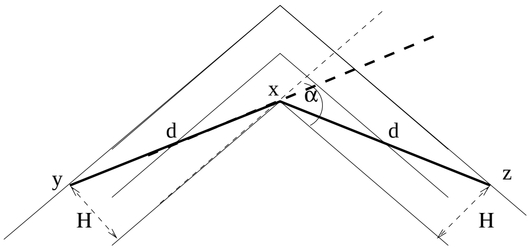

Figure 14.

Since $\lim_{H\to 0} m_4(H,d) = 0$, there exists $0 < H_0 < \frac{2}{\sqrt{5}}d$ such that, for every $0 < H < H_0$,

$$\frac{d\,H\,\left(\frac{\pi}{2} + \arcsin(\frac{H}{d})\right)}{d+H} = m_4(H,d) < k(H,\gamma,d) = (\gamma - 2\arcsin\frac{H}{d})\frac{d}{2}. \tag{8}$$

Therefore, we have proven that

$$K(x_1,y_1,z_1) \leq m_4(H,d) < k(H,\gamma,d) \leq K(x_2,y_2,z_2), \tag{9}$$

for every $0 < H < H_0$, every Type 1 vertex x_1, and every Type 2 vertex x_2. If the process of evolution is continued until the relevance measure $m_4(H,d)$ is obtained, then all Type 1 vertices are deleted and all Type 2 vertices remain, which proves the theorem. ∎

In the course of the proof of Theorem 3, we have proven the following two inequalities:

Corollary 1. *For every Type 1 vertex x_1 and its neighbors y_1, z_1 in polygon Q:*

$$K(x_1,y_1,z_1) \leq \frac{d\,H\,\left(\frac{\pi}{2} + \arcsin(\frac{H}{d})\right)}{d+H}. \tag{10}$$

For every Type 2 vertex x_2 and its neighbors y_2, z_2 in polygon Q:

$$(\gamma - 2\arcsin\frac{H}{d})\frac{d}{2} \leq K(x_2,y_2,z_2). \tag{11}$$

Proposition 2. *Let γ be a turn angle of polygon P for which expression $\sqrt{\frac{1}{\sin^2(\frac{\gamma}{2})} + \frac{1}{4\cos^2(\frac{\gamma}{2})}}$ is maximal. If*

$$\delta = \frac{\eta}{\sqrt{\frac{1}{\sin^2(\frac{\gamma}{2})} + \frac{1}{4\cos^2(\frac{\gamma}{2})}}}$$

and polygon Q is a (δ, d)-approximation of P, then Q is a $(2\eta, d)$-approximation of P for which additionally holds that for every vertex v of P at least one vertex of Q is contained in $B(v,\eta)$.

Proof: Let polygon Q be a (δ, d)-approximation of P. We want to find the smallest η such that for every vertex v of P at least one vertex of Q is contained in $B(v,\eta)$.

Let v be vertex of P with turn angle γ. We find such a η at vertex v.

Let e be the inner corner of the two δ-strips S_1 and S_2 that contain v, see Figure 15. Then the line ve divides the inner angle of $S_1 \cup S_2$ in two halves. Let ab be the line segment perpendicular to line ve such that a and b lie on the outside boundary of $S_1 \cup S_2$. Let η be the length of line segment vb.

Since $Q \subset B(P, \frac{\delta}{2})$, there must be a vertex of Q in $S_1 \cup S_2$ above the line ab. Therefore, there must be a vertex of Q in $B(v, \eta)$.

It remains to compute the length of η. Let point c be the perpendicular projection of point b on the inner side of S_2 in $S_1 \cup S_2$, see Figure 15.

Let u be neighbor vertex of v in P such that $\overline{uv} \subset S_1$. Let point f be the perpendicular projection of point e on $\overline{uv}$.

As can be seen in Figure 15, $\eta^2 = |\overline{ve}|^2 + |\overline{eb}|^2$. Since $\frac{\delta}{|\overline{eb}|} = \sin(\frac{\gamma}{2})$ in the triangle ebc and $\frac{\frac{\delta}{2}}{|\overline{ve}|} = \sin(90^o - \frac{\gamma}{2}) = \cos(\frac{\gamma}{2})$ in the triangle vef, we obtain

$$\eta = \delta \sqrt{\frac{1}{\sin^2(\frac{\gamma}{2})} + \frac{1}{4\cos^2(\frac{\gamma}{2})}}$$

■

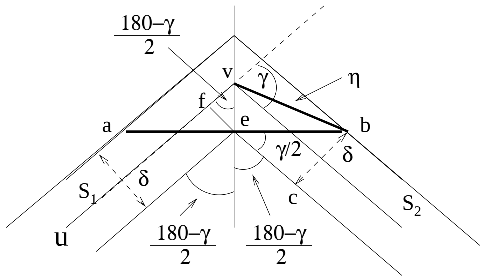

Figure 15.

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