Exact results in AdS/CFT from localization

Part II

James Sparks

Mathematical Institute, Oxford

Based on work with Fernando Alday, Daniel Farquet, Martin Fluder, Carolina Gregory Jakob Lorenzen, Dario Martelli, and Paul Richmond

In the first part of this talk I described a computation of the partition function $\mathbf{Z} = \langle \mathbf{1} \rangle$ and BPS Wilson loop VEV $\langle \mathbf{W} \rangle$ for $\mathcal{N} = \mathbf{2}$ supersymmetric gauge theories on a class of three-manifolds $\mathbf{M_3} \cong \mathbf{S^3}$.

The background geometry has an almost contact structure, with Reeb vector field $\mathbf{K} = \mathbf{b_1}\partial_{\varphi_1} + \mathbf{b_2}\partial_{\varphi_2}$ in terms of the standard action of $\mathbf{U(1)^2}$ on $\mathbf{S^3} \subset \mathbb{R}^2 \oplus \mathbb{R}^2$.

For an appropriate class of $\mathbf{G} = \mathbf{U(N)^p}$ gauge theories, the large $\mathbf{N}$ limit of the partition function and Wilson loop may be computed analytically, leading to

$$\begin{aligned} \log \mathbf{Z} &= \frac{(|\mathbf{b_1}| + |\mathbf{b_2}|)^2}{4|\mathbf{b_1 b_2}|} \cdot \log \mathbf{Z}_{\text{round } \mathbf{S^3}} \, , \\ \log\langle \mathbf{W} \rangle &= \tfrac{1}{2}\ell(|\mathbf{b_1}| + |\mathbf{b_2}|) \cdot \log\langle \mathbf{W} \rangle_{\text{round } \mathbf{S^3}} \, . \end{aligned}$$

This class of gauge theories is expected to have a dual description in terms of four-dimensional supergravity.

In AdS/CFT the geometry $\mathbf{M_3}$ arises as the conformal boundary of a four-manifold $\mathbf{M_4}$ in which gravity propagates.

In the case at hand this is $\mathcal{N} = \mathbf{2}$ gauged supergravity in four dimensions – Einstein-Maxwell theory, with Abelian gauge field $\mathbf{A}$ and negative cosmological constant.

Near infinity ($\mathbf{r} = \infty$) the metric on $\mathbf{M_4}$ should take the form

$$\mathbf{ds^2_{M_4}} \;\simeq\; \frac{\mathbf{dr^2}}{\mathbf{r^2}} + \mathbf{r^2 ds^2_{M_3}} \ .$$

The equations of motion are

$$\begin{aligned} R_{\mu\nu} + 3g_{\mu\nu} &= 2(F_{\mu}{}^{\rho}F_{\nu\rho} - \tfrac{1}{4}F^2 g_{\mu\nu})\ , \\ \mathrm{d} *_4 F &= 0\ . \end{aligned}$$

A solution is supersymmetric if it admits a non-trivial solution to the Killing spinor equation

$$\left[\nabla_\mu - iA_\mu + \tfrac{1}{2}\Gamma_\mu + \tfrac{i}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right]\epsilon \ = \ 0\ .$$

Here Γ_μ generate Cliff$(4,0)$ in an orthonormal frame.

Any supersymmetric solution of this theory on M_4 uplifts to a supersymmetric solution of M-theory on $M_4 \times Y_7$ [Gauntlett-Varela]. A choice of internal space Y_7 determines the gauge theory on the conformal boundary $M_3 = \partial M_4$.

The AdS/CFT correspondence says that the large $\mathbf{N}$ gauge theory partition function $\mathbf{Z}$ should equal the supergravity partition function:

$$\log Z \;=\; -S_{SUGRA}\;.$$

More precisely, the right hand side is the least action solution to the Einstein equations, with fixed conformal boundary $\mathbf{M_3}$.

For $\mathbf{M_3} =$ round $\mathbf{S^3}$, this is the vacuum Euclidean AdS_4 (hyperbolic ball). Here the Maxwell $\mathbf{U(1)}$ gauge field $\mathbf{A = 0}$, and the metric is

$$ds^2_{EAdS_4} \;=\; \frac{dr^2}{r^2+1} + r^2 ds^2_{S^3_{round}}\;.$$

The on-shell action of any such solution to the Einstein equations is divergent, but it may be regularized:

$$\mathbf{S_{SUGRA}} = \mathbf{S_{Einstein-Maxwell}} + \mathbf{S_{Gibbons-Hawking}} + \mathbf{S_{counterterms}} .$$

Here

$$\begin{aligned}
\mathbf{S_{Einstein-Maxwell}} &= -\frac{1}{16\pi G_4}\int_{M_4}(R+6-F^2)\sqrt{\det g}\, d^4x \, , \\
\mathbf{S_{Gibbons-Hawking}} &= -\frac{1}{8\pi G_4}\int_{\partial M_4}\mathcal{K}\sqrt{\det\gamma}\, d^3x \, , \\
\mathbf{S_{counterterms}} &= \frac{1}{8\pi G_4}\int_{\partial M_4}(2+\tfrac{1}{2}R(\gamma))\sqrt{\det\gamma}\, d^3x \, ,
\end{aligned}$$

where $\mathbf{M_4}$ is cut off at some radius, γ_{ij} is the induced metric on $\partial \mathbf{M_4}$, $\mathcal{K}$ is the trace of the second fundamental form, and $\mathbf{R(\gamma)}$ denotes the Ricci scalar.

For Euclidean AdS_4 this gives

$$\mathbf{S_{SUGRA}} = \frac{\pi}{2\mathbf{G_4}} .$$

The four-dimensional Newton constant $\mathbf{G_4}$ is determined by the choice of internal space $\mathbf{Y_7}$, or equivalently choice of gauge theory on $\mathbf{M_3}$.

For example, when $\mathbf{Y_7}$ is a Sasaki-Einstein seven-manifold

$$\frac{\pi}{2\mathbf{G_4}} = \mathbf{N}^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}(\mathbf{Y_7})}} ,$$

and this formula has been shown to agree with the large $\mathbf{N}$ partition function on the round $\mathbf{S^3}$ for a variety of gauge theories in [Martelli-JFS], [Cheon-Kim-Kim], [Jafferis-Klebanov-Pufu-Safdi].

In this talk I want to focus on the dependence of the partition function on the choice of background geometry $\mathbf{M_3}$.

This is a Dirichlet filling problem: one should solve the Einstein equations with fixed conformal boundary data.

From gauge theory, we expect the least action solution to satisfy

$$\mathbf{S_{SUGRA}} = \frac{(|\mathbf{b_1}| + |\mathbf{b_2}|)^2}{4|\mathbf{b_1 b_2}|} \cdot \frac{\pi}{2\mathbf{G_4}} \ .$$

The local form of Euclidean supersymmetric solutions to Einstein-Maxwell theory was studied by [Dunajski-Gutowski-Sabra-Tod].

In particular, there is a class of *self-dual* solutions in which $*_4 \mathbf{F} = -\mathbf{F}$ is anti-self-dual, and the four-metric is Einstein with anti-self-dual Weyl tensor.

We also have a Killing vector

$$\mathbf{K} = \mathrm{i}\epsilon^\dagger \Gamma^\mu \Gamma_5 \epsilon \partial_\mu = \partial_\psi \ .$$

Self-dual Einstein metrics with a Killing vector have a rich geometric structure. They are (locally) conformal to a scalar-flat Kähler metric, with the metric determined entirely by a solution to the Toda equation:

$$\mathrm{d}s_4^2 \;=\; \frac{1}{y^2}\mathrm{d}s^2_{\mathrm{Kahler}} \;=\; \frac{1}{y^2}\Big[\mathcal{V}^{-1}(\mathrm{d}\psi+\phi)^2+\mathcal{V}(\mathrm{d}y^2+4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z})\Big]\,.$$

where $\mathcal{V}=1-\frac{1}{2}y\partial_y w$, the expression for $\mathrm{d}\phi$ is known (but complicated), and

$$\partial_z\partial_{\bar{z}}w+\partial_y^2\mathrm{e}^w \;=\; 0 \qquad \text{(Toda)}\,.$$

The conformal boundary is at $\mathbf{y = 0}$, and one can show that the structure induced on the conformal boundary is precisely the three-dimensional background geometry of [Closset-Dumitrescu-Festuccia-Komargodski].

In particular

$$\epsilon \;=\; \mathbf{y}^{-1/2}\left[\left(\mathbf{1}+\Gamma_0+\tfrac{1}{4}\mathbf{y}\mathbf{w}_{(1)}\Gamma_0\right)\begin{pmatrix}\chi\\ \mathbf{0}\end{pmatrix}+\mathcal{O}(\mathbf{y}^2)\right]\;,$$

where χ is a three-dimensional spinor satisfying the Killing spinor equation we saw last time, and we expand $\mathbf{w}(\mathbf{y},\mathbf{z},\bar{\mathbf{z}})=\mathbf{w}_{(0)}(\mathbf{z},\bar{\mathbf{z}})+\mathbf{y}\mathbf{w}_{(1)}(\mathbf{z},\bar{\mathbf{z}})+\mathcal{O}(\mathbf{y}^2)$.

Our strategy for constructing gravity duals to the boundary geometries on $\mathbf{M_3} \cong \mathbf{S^3}$ is to begin with an arbitrary $\mathbf{U(1)} \times \mathbf{U(1)}$-invariant self-dual Einstein metric on a four-ball $\mathbf{M_4} \cong \mathbf{B_4}$, which is asymptotically locally AdS with conformal boundary $\partial\mathbf{B_4} = [\mathbf{M_3}]$.

The space of such metrics is infinite-dimensional (a change of coordinates due to [Calderbank-Pedersen] maps the Toda equation to a *linear* eigenvalue equation on $\mathcal{H}^2 =$ hyperbolic upper half plane).

We then established a converse to the result in [Dunajski-Gutowski-Sabra-Tod]: any self-dual Einstein metric with a choice of Killing vector $\mathbf{K} = \partial_\psi$ determines a choice of conformal Kähler metric. Taking the Maxwell field $\mathbf{A}$ to have curvature $\mathbf{F} = \mathrm{d}\mathbf{A} = \frac{1}{2}$ Ricci-form of the conformal Kahler metric, the resulting background admits a Killing spinor ϵ (related to the canonical spin$^\mathbf{c}$ spinor for the conformal Kähler metric).

By construction, for each metric and each choice of Killing vector $\mathbf{K} = \mathbf{b_1}\partial_{\varphi_1} + \mathbf{b_2}\partial_{\varphi_2}$ we locally get a supersymmetric supergravity solution.

For fixed choice of self-dual Einstein metric, this leads to a one-parameter family of gauge fields $\mathbf{A}$ with anti-self-dual curvature $\mathbf{F} = \mathbf{dA}$, labelled by $\mathbf{b_1}/\mathbf{b_2}$, which are globally regular iff $\mathbf{b_1}/\mathbf{b_2} > \mathbf{0}$ or $\mathbf{b_1}/\mathbf{b_2} = -\mathbf{1}$.

One can then compute the regularized on-shell action $\mathbf{S_{SUGRA}}$ for any such solution.

Example: Euclidean AdS_4 has metric

$$\mathrm{d}s^2_{\mathrm{EAdS}_4} = \frac{\mathrm{d}r^2}{r^2+1} + r^2\left(\mathrm{d}\vartheta^2 + \cos^2\vartheta\,\mathrm{d}\varphi_1^2 + \sin^2\vartheta\,\mathrm{d}\varphi_2^2\right) .$$

Choosing the Killing vector $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$ the corresponding instanton **U(1)** gauge field is

$$A = \frac{\left(b_1 + b_2\sqrt{r^2+1}\right)\mathrm{d}\varphi_1 + \left(b_2 + b_1\sqrt{r^2+1}\right)\mathrm{d}\varphi_2}{2\sqrt{\left(b_2 + b_1\sqrt{r^2+1}\right)^2\cos^2\vartheta + \left(b_1 + b_2\sqrt{r^2+1}\right)^2\sin^2\vartheta}} .$$

Returning to the action for a general solution, the individual terms certainly depend on the detailed solution. For example

$$\frac{1}{16\pi G_4}\int_{B_4} F^2\sqrt{\det g}\,\mathrm{d}^4x \;=\; -\frac{\pi(|b_1+b_2|)^2}{8G_4|b_1 b_2|} + \frac{1}{256\pi G_4}\int_{M_3}\left(3w_{(1)}^3 + 4w_{(1)}w_{(2)}\right)\sqrt{\det g_3}\,\mathrm{d}^3x\;.$$

Here we have assumed the topology $M_3 \cong S^3$ and $M_4 \cong B_4$.

However, the final result is

$$\mathbf{S_{SUGRA}} = \frac{(|\mathbf{b_1}| + |\mathbf{b_2}|)^2}{4|\mathbf{b_1 b_2}|} \cdot \frac{\pi}{2\mathbf{G_4}} ,$$

agreeing with the field theory computation!

The Wilson loop in the fundamental representation maps to a supersymmetric M2-brane, wrapping a calibrated copy of the M-theory circle [Farquet-JFS], and with a minimal surface $\Sigma \subset \mathbf{B_4}$ with $\partial\Sigma = \gamma =$ orbit of Reeb vector $\mathbf{K}$.

$\log\langle \mathbf{W} \rangle_{\mathbf{gravity}}$ is identified with minus the regularized action of the M2-brane, and in [Farquet-JFS] we showed this reproduces the large $\mathbf{N}$ field theory result.

Although the two computations agree, it's not clear why.

For any self-dual Einstein background one can use the APS index theorem to write

$$\mathsf{S}_{\text{pure gravity}} \;=\; -\frac{3\pi}{4G_4}\eta(\partial M_4) + \frac{\pi}{4G_4}\left(\chi(M_4) + 3\sigma(M_4)\right) \;,$$

where $\eta(\partial M_4)$ is the APS eta invariant [Anderson].

[Operator $(-1)^p(*d - d*)$ acting on $\Omega^{2p}(\partial M_4)$ has eigenvalues λ_i, and define $\eta(s) = \sum_{\lambda_i \neq 0} \operatorname{sign} \lambda_i / |\lambda_i|^s$, $\eta = \eta(0)$.]

Including the instanton **A**, one can rewrite the whole action in terms of η and $\eta($Dirac coupled to **A**$)$.

We now change focus to $\mathbf{d = 5}$. [Imamura] has defined five-dimensional supersymmetric gauge theories on the $\mathbf{SU(3) \times U(1)}$-invariant squashed five-sphere background

$$\mathbf{ds_5^2 \ = \ \frac{1}{s^2}(d\tau + C)^2 + ds^2_{\mathbb{CP}^2}}$$

where $\frac{1}{2}\mathbf{dC} = \omega =$ Kähler form for the Fubini-Study metric on $\mathbb{CP}^2$. Here $\mathbf{s} =$ squashing parameter, with $\mathbf{s = 1}$ the round five-sphere.

There is also a background R-symmetry gauge field

$$\mathbf{A^R \ = \ \frac{1}{s^2}(1 + Q\sqrt{1-s^2})\sqrt{1-s^2}(d\tau + C) \ ,}$$

where $\mathbf{U(1)_R \subset SU(2)_R}$ and $\mathbf{Q = 1}$, $\mathbf{Q = -3}$ give rise to 3/4 BPS and 1/4 BPS solutions, respectively.

The *perturbative* partition function again localizes onto an integral over the constant mode $\boldsymbol{\sigma_0}$ of the scalar in the vector multiplet, and the final formula involves triple sine functions.

A particular class of five-dimensional gauge theories, with gauge group $\mathbf{USp(2N)}$ and arising from a D4-D8 system, is expected to have a large $\mathbf{N}$ description in terms of massive type IIA supergravity [Ferrara-Kehagias-Partouche-Zaffaroni], [Brandhuber-Oz].

In [Jafferis-Pufu] the large $\mathbf{N}$ limit of the partition function of these theories on the *round* sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_{\mathbf{6}} \times \mathbf{S}^{\mathbf{4}}$ supergravity solution.

In [Alday-Fluder-Gregory-Richmond-JFS] we computed the large **N** limit of the **USp(2N)** gauge theories on the squashed five-sphere, finding the free energy

$$\log Z = \frac{(|b_1| + |b_2| + |b_3|)^3}{27|b_1 b_2 b_3|} \cdot \log Z_{\mathrm{round}\, S^5} \ ,$$

where

$$\begin{cases} b_1 = b_2 = b_3 & 1/4 \text{ BPS} \\ b_1 = -1 - \sqrt{1-s^2}\ ,\ b_2 = b_3 = 1 - \sqrt{1-s^2} & 3/4 \text{ BPS} \end{cases}$$

There is again a supersymmetric Killing vector bilinear **K**, and embedding $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, this is $K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} + b_3 \partial_{\varphi_3}$.

We also computed the large $\mathbf{N}$ limit of BPS Wilson loops. If the worldline wraps the $\mathbf{S}^1_i \subset \mathbf{S}^5$ at the origin of two copies of $\mathbb{R}^2$, then we find

$$\log\langle W \rangle = \frac{|b_1| + |b_2| + |b_3|}{3|b_i|} \cdot \log\langle W \rangle_{\mathrm{round}\, S^5} .$$

We have reproduced these formulae from a dual supergravity computation.

We work in six-dimensional Romans $\mathbf{F(4)}$ gauged supergravity, which is a consistent truncation of massive IIA supergravity on $\mathbf{S}^4$ [Cvetic-Lu-Pope]. As well as the metric, there is a scalar $\mathbf{X}$, two-form potential $\mathbf{B}$, one-form potential $\mathbf{A}$, and an $\mathbf{SO(3)} \sim \mathbf{SU(2)}$ R-symmetry gauge field $\mathbf{A}_I$, $\mathbf{I = 1, 2, 3}$.

The one-form $\mathbf{A}$ is a Stueckelberg field, which may be set to $\mathbf{A} = \mathbf{0}$ by a gauge transformation. The $\mathbf{B}$-field then becomes massive, and the Euclidean action is

$$\begin{aligned} S_{\rm bulk} &= -\frac{1}{16\pi G_N}\int_{M_6}\Big[R * 1 - 4X^{-2}\mathrm{d}X\wedge *\mathrm{d}X \\ &\quad - \left(\tfrac{2}{9}X^{-6} - \tfrac{8}{3}X^{-2} - 2X^2\right) * 1 - \tfrac{1}{2}X^{-2}\left(\tfrac{4}{9}B\wedge *B + F_I\wedge *F_I\right) \\ &\quad - \tfrac{1}{2}X^4 H\wedge *H - \mathrm{i}B\wedge\left(\tfrac{2}{27}B\wedge B + \tfrac{1}{2}F_I\wedge F_I\right)\Big] \,. \end{aligned}$$

Notice the cubic Chern-Simons coupling for $\mathbf{B}$. Its curvature is $\mathbf{H} = \mathbf{dB}$.

A solution to the corresponding equations of motion is supersymmetric provided the Killing spinor equation and dilatino equation hold.

The squashed five-sphere background has $\mathbf{SU(3)} \times \mathbf{U(1)}$ symmetry, and one expects this to be preserved by the bulk filling. This leads to the ansatz

$$\begin{aligned} ds_6^2 &= \alpha^2(r)dr^2 + \gamma^2(r)(d\tau + C)^2 + \beta^2(r)ds^2_{\mathbb{CP}^2} \,, \\ B &= p(r)dr \wedge (d\tau + C) + \tfrac{1}{2}q(r)dC \,, \\ A_I &= f_I(r)(d\tau + C) \,, \end{aligned}$$

together with $\mathbf{X = X(r)}$.

We have constructed smooth, supersymmetric, asymptotically locally Euclidean AdS solutions with the topology $\mathbf{M_6 \cong B_6}$, with conformal boundary the squashed five-sphere backgrounds of [Imamura]. These may be given as expansions around the conformal boundary $\mathbf{r = \infty}$, and/or as expansions in the squashing parameter $\mathbf{s}$.

Reparametrization invariance allows us to set $\beta(r) = 3\sqrt{6r^2 - 1}/\sqrt{2}$ to its AdS_6 value, and an $SO(3)$ rotation sets $f_3(r) = f(r)$, $f_1(r) = f_2(r) = 0$.

For example, for the 3/4 BPS solution the first few terms in the expansion around $r = \infty$ are

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}} r + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \ldots, \\
\gamma(r) &= \frac{3\sqrt{3}}{s} r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \ldots, \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \ldots, \\
p(r) &= -\frac{i\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \ldots, \\
q(r) &= -\frac{3i\left(\sqrt{6}\sqrt{1-s^2}\right)}{s} r + \frac{\sqrt{\frac{2}{3}}i\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \ldots, \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \ldots .
\end{aligned}
$$

The parameter κ is uniquely determined by requiring this to extend to a smooth solution on the ball $\mathbf{M_6} \cong \mathbf{B_6}$. As an expansion in

$$\delta = \sqrt{-1 + s^{-1}}$$

this is

$$\frac{3\sqrt{3}}{4}\kappa = \delta + \frac{\sqrt{2}}{3}\delta^2 + \frac{113}{36}\delta^3 + \frac{25}{9\sqrt{2}}\delta^4 + \frac{1127}{288}\delta^5 + \frac{35}{9\sqrt{2}}\delta^6 + \ldots$$

Similar results hold in the 1/4 BPS case, except here we find a *two-parameter* family of solutions, leading to a new supersymmetric squashing of $\mathbf{S^5}$. In particular this includes a one-parameter subfamily of 1/2 BPS solutions.

As in four dimensions the regularized action is

$$\mathbf{S_{SUGRA}} \;=\; \mathbf{S_{bulk}} + \mathbf{S_{Gibbons-Hawking}} + \mathbf{S_{ct}} \;.$$

However, unlike in four dimensions the counterterms $\mathbf{S_{ct}}$ had not been computed.

This is a straightforward, but very long, computation. In particular the **B**-field is both massive and has a cubic Chern-Simons interaction, which leads to a much more complicated analysis than for more standard fields.

$$
\begin{aligned}
S_{ct} \quad = \quad & \frac{1}{8\pi G_N}\int_{\partial M_6}\Big\{\Big[\frac{4\sqrt{2}}{3} + \frac{1}{2\sqrt{2}}R(h) - \frac{1}{6\sqrt{2}}\|B\|_h^2 + \frac{3}{4\sqrt{2}}R(h)_{ij}R(h)^{ij} - \frac{15}{64\sqrt{2}}R(h)^2 - \frac{3}{4\sqrt{2}}\|F_I\|_h^2 \\
& + \frac{1}{12\sqrt{2}}\mathrm{Tr}_h B^4 + \frac{5}{8\sqrt{2}}\|d *_h B + \frac{i\sqrt{2}}{3}B\wedge B\|_h^2 - \frac{1}{4\sqrt{2}}\langle B, d\delta_h B + \frac{i\sqrt{2}}{3}d *_h B\wedge B\rangle_h - \frac{1}{\sqrt{2}}\|dB\|_h^2 \\
& + \frac{4\sqrt{2}}{3}(1-X)^2 - \frac{1}{\sqrt{2}}\langle \mathrm{Ric}(h)\circ B, B\rangle_h + \frac{9}{32\sqrt{2}}R(h)\|B\|_h^2 - \frac{13}{192\sqrt{2}}\|B\|_h^4\Big]\sqrt{\det h}\, d^5x \\
& - \frac{1}{4\sqrt{2}}B\wedge\Big[d *_h dB + \frac{\sqrt{2}i}{3}B\wedge\delta_h B - \frac{2}{9}B\wedge *_h(B\wedge B)\Big]\Big\} .
\end{aligned}
$$

Here $\mathrm{Ric}(h)_{ij} = R(h)_{ij}$ denotes the Ricci tensor of the boundary metric h_{ij}, with $R(h)$ the Ricci scalar. The inner product of two p-forms ν_1, ν_2 is defined by $\langle\nu_1,\nu_2\rangle_h\sqrt{\det h}\, d^5x = \nu_1\wedge *_h\nu_2$, which then also defines the square norm via $\|\nu\|_h^2 = \langle\nu,\nu\rangle_h$. The adjoint δ_h of d with respect to h_{ij} acting on the two-form B is $\delta_h B = *_h d *_h B$, and we have also defined $\mathrm{Tr}_h B^4 \equiv B_i{}^j B_j{}^k B_k{}^l B_l{}^i$. Finally, we have defined the p-form $(S\circ\nu)_{i_1\cdots i_p} \equiv S_{[i_1}{}^j\nu_{|j|i_2\cdots i_p]}$, where S_{ij} is any symmetric 2-tensor, and ν is any p-form.

Using this we may compute the holographic free energy. For example, for the 3/4 BPS solution we find

$$S_{\rm bulk} + S_{\rm Gibbons-Hawking} + S_{\rm ct} \;=\; -\frac{27\pi^2}{4G_N}\left(1 + \frac{8}{3}\delta^2 + \frac{16\sqrt{2}}{27}\delta^3 + \frac{68}{27}\delta^4 \right.$$
$$\left. + \frac{28\sqrt{2}}{27}\delta^5 + \frac{32}{27}\delta^6 + \ldots\right) \, .$$

This agrees with the field theory result!

The BPS Wilson loop maps to a fundamental string in type **IIA**, at the "pole" of the internal $\mathbf{S^4}$ [Assel-Estes-Yamazaki]. The renormalized string action is

$$\mathbf{S}_{\text{string}} = \int_{\Sigma} \left[\mathbf{X}^{-2} \sqrt{\det \gamma}\, \mathbf{d}^2\mathbf{x} + \mathbf{i}\mathbf{B} \right] - \frac{3}{\sqrt{2}} \text{length}(\partial \Sigma) \, ,$$

and also agrees with the large **N** field theory results.

There is clearly more to understand – some kind of geometric structure that explains the particularly simple forms for the BPS quantities being computed (in particular the factors $(|\mathbf{b}_1| + |\mathbf{b}_2|)^2/4|\mathbf{b}_1\mathbf{b}_2|$ and $(|\mathbf{b}_1| + |\mathbf{b}_2| + |\mathbf{b}_3|)^3/27|\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3|$ that appear in the partition functions in four and six dimensions, respectively).