Exact results in AdS/CFT from localization

Part I

James Sparks

Mathematical Institute, Oxford

Based on work with Fernando Alday, Daniel Farquet, Martin Fluder, Carolina Gregory Jakob Lorenzen, Dario Martelli, and Paul Richmond

In the last few years there has been a flurry of interest in defining and studying supersymmetric gauge theories on curved backgrounds, preserving supersymmetry.

Some of the basic ideas and techniques go back to Witten's 1988 paper where he reformulated Donaldson's four-manifold invariants in terms of "topological quantum field theory."

Witten's theory is defined on any Riemannian four-manifold, but in the more recent developments, which work in a variety of dimensions, one requires certain specific types of geometric structure.

As in Witten's theory, for certain observables the path integral is exactly equal to its semi-classical approximation (a sort of fixed point theorem for supersymmetry in field space).

Essentially, one can take any supersymmetric gauge theory, coupled to matter, and formulate it on an appropriate class of background geometries.

This is interesting, because (a) certain observables may be computed exactly (non-perturbatively), and (b) these observables in general depend on the background geometry, leading to a richer structure than for the theory formulated in flat space.

In these talks I'll focus on the application of these ideas to the AdS/CFT duality, which is a conjecture relating strongly coupled gauge theories to semi-classical gravity.

Supersymmetric gauge theories are usually formulated in Minkowski spacetime, or in Wick-rotated Euclidean space.

To be concrete, consider a three-dimensional $\mathcal{N} = 2$ vector multiplet in Euclidean space $\mathbb{E}^3$.

This consists of a gauge field (connection) $\mathcal{A}$, two scalars σ, $\mathbf{D}$, and a fermion spinor field λ. All are valued in the Lie algebra $\mathfrak{g}$ of the gauge group $\mathbf{G}$.

These have associated supersymmetry transformations

$$\begin{aligned}
\delta\mathcal{A}_\mu &= -\tfrac{\mathrm{i}}{2}\lambda^\dagger\gamma_\mu\chi\ , \\
\delta\sigma &= -\tfrac{1}{2}\lambda^\dagger\chi\ , \\
\delta\mathsf{D} &= -\tfrac{\mathrm{i}}{2}(\mathsf{D}_\mu\lambda^\dagger)\gamma^\mu\chi + \tfrac{\mathrm{i}}{2}[\lambda^\dagger,\sigma]\chi\ , \\
\delta\lambda &= \left(-\tfrac{1}{2}\gamma^{\mu\nu}\mathcal{F}_{\mu\nu} - \mathsf{D} + \mathrm{i}\gamma^\mu\mathsf{D}_\mu\sigma\right)\chi\ , \\
\delta\lambda^\dagger &= 0\ .
\end{aligned}$$

Here χ is a constant spinor, γ_μ, $\mu = 1,2,3$, generate the Clifford algebra Cliff$(3,0)$, and $\mathsf{D}_\mu = \partial_\mu + \mathrm{i}[\mathcal{A}_\mu,\,\cdot\,]$ is the gauge covariant derivative.

Then one can check that the Yang-Mills action

$$S = \frac{1}{g_{YM}^2} \int \mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + D_\mu\sigma D^\mu\sigma + D^2 + i\lambda^\dagger\gamma^\mu D_\mu\lambda + i[\lambda^\dagger,\sigma]\lambda\right) ,$$

and Chern-Simons action

$$S = \frac{ik}{4\pi} \int \mathrm{Tr}\left(\epsilon^{\mu\nu\rho}(\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho + \tfrac{2i}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho) - \lambda^\dagger\lambda + 2D\sigma\right)$$

are invariant under δ. In doing so one uses $\partial_\mu\chi = 0$.

One can consider trying to define such theories on general Riemannian manifolds $(\mathbf{M}, \mathbf{g})$.

In doing so one first replaces $\partial_\mu \to \nabla_\mu$, where ∇_μ is the Levi-Civita connection.

But $\nabla_\mu \chi = 0$ would imply the metric is Ricci-flat, hence flat in dimension $\mathbf{d} = 3$.

There are two approaches: (a) Witten's topological twist, or (b) couple the theory to supergravity, and take a limit where $\mathbf{m}_{\mathsf{pl}} \to \infty$ ([Festuccia-Seiberg]).

For the topological twist one needs an R-symmetry – a global symmetry under which the Killing spinor χ is charged.

Witten considered $\mathcal{N} = 2$ gauge theory in four dimensions, which has an $\mathbf{SU(2)_R}$ symmetry. The spin group is Spin$(4) = \mathbf{SU(2)_+} \times \mathbf{SU(2)_-}$, and there is a chiral spinor χ transforming as $(\frac{1}{2}, \frac{1}{2})$ under $\mathbf{SU(2)_+} \times \mathbf{SU(2)_R}$.

Under the diagonal subgroup of $\mathbf{SU(2)_+} \times \mathbf{SU(2)_R}$ the spinor χ transforms as $\mathbf{1} \oplus \mathbf{0}$, resulting in a scalar satisfying $\partial_\mu \chi = 0$.

In more recent work, [Pestun] and [Kapustin-Willett-Yaakov] instead defined $\mathcal{N} = \mathbf{2}$ gauge theories on the round $\mathbf{S^4}$ and $\mathbf{S^3}$ (respectively), by appropriately modifying the Killing spinor equation for χ and supersymmetry transformations.

This, and subsequent generalizations, were reinterpreted by [Festuccia-Seiberg] in terms of minimally coupling the gauge theory to supergravity, and then taking a limit where the metric becomes non-dynamical.

There are typically other fields, in addition to the metric, in the gravity multiplet, which also become non-dynamical background fields. Setting the gravitino variation to zero leads to a Killing spinor equation for χ.

For example, in $\mathbf{d = 4}$ one can couple any supersymmetric gauge theory with a $\mathbf{U(1)}$ R-symmetry to new minimal supergravity.

The background fields are the metric, a $\mathbf{U(1)}$ gauge field $\mathbf{A}$, and a co-closed one-form $\mathbf{V}$, with Killing spinor equation

$$(\nabla_\mu - \mathrm{i}\mathbf{A}_\mu)\chi + \mathrm{i}\mathbf{V}_\mu\chi + \mathrm{i}\mathbf{V}^\nu\gamma_{\mu\nu}\chi \ = \ \mathbf{0}\ .$$

Here χ has positive chirality.

It turns out this is equivalent to $\mathbf{M_4}$ being equipped with an integrable complex structure $\mathbf{J}$, for which $\mathbf{g}$ is Hermitian. In particular $\mathbf{A}$ and $\mathbf{V}$ are fixed by this data [Dumitrescu-Festuccia-Seiberg].

At first sight this looks quite different to the topological twist, but they are not unrelated.

The spinor equation may be rewritten as

$$(\nabla^c_\mu - i A^c_\mu)\chi \;=\; 0\,,$$

where ∇^c is the Chern connection, preserving $\mathbf{g}$ and $\mathbf{J}$.

In particular this has $U(2) = U(1) \times_{\mathbb{Z}_2} SU(2)$ holonomy, and positive chirality spinors may be identified with $(\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes K^{1/2}$, where $K = \Lambda^{2,0}$ and A^c is the induced connection on $K^{-1/2}$. Thus χ is really a spin^c spinor, and effectively becomes a scalar with $\partial_\mu \chi = 0$.

A similar construction exists in $\mathbf{d = 3}$, with a similar Killing spinor equation

$$(\nabla_\mu - \mathrm{i}\mathbf{A}_\mu)\chi + \tfrac{\mathrm{i}}{2}\mathbf{h}\gamma_\mu\chi + \mathrm{i}\mathbf{V}_\mu\chi - \tfrac{\mathrm{i}}{2}\mathbf{V}^\nu\gamma_{\mu\nu}\chi \;=\; \mathbf{0}\,.$$

Notice the additional function $\mathbf{h}$.

It turns out this is equivalent to the three-manifold admitting an almost contact metric structure, together with an integrability condition [Closset-Dumitrescu-Festuccia-Komargodski].

Specifically $\mathbf{K}^\mu = \chi^\dagger\gamma^\mu\chi$ defines a nowhere zero vector field, and the corresponding foliation is transversely holomorphic.

Going back to our $\mathcal{N} = 2$ gauge multiplet in $\mathbf{d} = 3$, the supersymmetry transformations on such a background read

$$
\begin{aligned}
\delta \mathcal{A}_\mu &= -\tfrac{i}{2}\lambda^\dagger \gamma_\mu \chi \,, \\
\delta \sigma &= -\tfrac{1}{2}\lambda^\dagger \chi \,, \\
\delta D &= -\tfrac{i}{2} D_\mu(\lambda^\dagger \gamma^\mu \chi) + \tfrac{1}{2} h \lambda^\dagger \chi - \tfrac{1}{2} V_\mu \lambda^\dagger \gamma^\mu \chi + \tfrac{i}{2}[\lambda^\dagger, \sigma]\chi \,, \\
\delta \lambda &= \left[-\tfrac{1}{2}\gamma^{\mu\nu}\mathcal{F}_{\mu\nu} - (D - \sigma h) + i\gamma^\mu (D_\mu \sigma + i V_\mu \sigma)\right] \chi \,, \\
\delta \lambda^\dagger &= 0 \,.
\end{aligned}
$$

Here the background fields **A** and **V** also appear in the covariant derivatives, *e.g.* $D_\mu \chi \equiv (\nabla_\mu - iA_\mu)\chi + \frac{i}{2} V_\mu \chi$. The supersymmetric Yang-Mills action similarly receives minor modifications.

In quantum field theory we are interested in computing correlation functions

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\text{all fields}} e^{-S} \mathcal{O}_1 \cdots \mathcal{O}_n ,$$

where $\mathcal{O}_i$ are operators.

The theories I have described have the following *localization* property: if $\mathcal{O} = \mathcal{O}_{BPS}$ is a BPS operator, meaning $\delta\mathcal{O} = 0$ is invariant under supersymmetry, then

$$\langle \mathcal{O}_{BPS} \rangle \overset{\text{exactly}}{=} \int_{\delta-\text{invariant fields}} e^{-S} \mathcal{O}_{BPS} \cdot (\text{one-loop determinant}) .$$

Concretely, we have $\delta^2 = 0$ acting on any field.

We may then deform the original action by adding a δ-exact term

$$\mathbf{t}\,\delta \mathrm{Tr}\left[(\delta\lambda)^\dagger\lambda\right] \;,$$

where $\mathbf{t}$ is any real number. The expectation value of any δ-invariant operator is then independent of $\mathbf{t}$. The argument is due to Witten. For any operator $\mathcal{O}$ we have

$$\langle\, \mathcal{O}\,\rangle_{\mathbf{t}} \;=\; \int_{\text{fields}} \exp(\mathbf{t}\delta)\cdot \mathrm{e}^{-S}\mathcal{O} \;=\; \int_{\text{fields}} \mathrm{e}^{-S}\,(\mathcal{O} + \mathbf{t}\delta\mathcal{O})$$

is independent of $\mathbf{t}$, assuming the measure is δ-invariant, meaning $\langle\, \delta\mathcal{O}\,\rangle = 0$.

In particular we now have a term $\exp[-t(\delta\lambda)^\dagger\delta\lambda]$ in the integrand, which we may evaluate in the $t \to +\infty$ limit.

The dominant contribution then comes from field configurations with $\delta\lambda = 0$.

This is similar to the Duistermaat-Heckman theorem in symplectic geometry:

$$\int_M e^{-H+\omega} = \int_F \frac{e^{-H+\omega}}{\det(\text{normal})} ,$$

where H is a Hamiltonian function on (M, ω), and F is its critical set $\{dH = 0\}$.

One can argue that for the $\mathbf{d = 4}$ theories defined on a Hermitian manifold $\mathbf{(M_4, g, J)}$ the expectation values of δ-invariant operators depend on the background geometry only through the complex structure **J** (*cf.* [Johansen]).

Similarly, for $\mathbf{d = 3}$ such BPS observables depend only on the transversely holomorphic foliation.

These statements are established formally by showing that any deformation of the background preserving these structures is δ-exact.

However, one of the interesting features of these constructions is that one can calculate very explicitly!

In [Alday-Martelli-Richmond-JFS] we computed the localized partition function $\mathbf{Z} = \langle\, \mathbf{1}\, \rangle$ and BPS Wilson loop VEV $\langle\, \mathbf{W}\, \rangle$ for a general $\mathcal{N} = 2$ supersymmetric gauge theory coupled to matter, on $\mathbf{M_3} \cong \mathbf{S^3}$.

The background admits two Killing spinors, of opposite R-charge and related by charge conjugation. In particular

$$\mathsf{K} \;=\; \chi^\dagger \gamma^\mu \chi \partial_\mu \;=\; \partial_\psi \;.$$

is a Killing vector field.

The metric is locally

$$\mathrm{d}s_3^2 \;=\; \Omega(z,\bar{z})^2(\mathrm{d}\psi + a)^2 + c(z,\bar{z})^2\mathrm{d}z\mathrm{d}\bar{z} \; .$$

where z is a complex coordinate for the transversely holomorphic foliation by ∂_ψ.

Essentially the background is parametrized by an arbitrary choice of the functions $\Omega(z,\bar{z}), c(z,\bar{z})$, and local one-form $a = a(z,\bar{z})\mathrm{d}z + \text{c.c.}$, and imposing the Killing spinor equation then fixes everything else in terms of these.

If all the orbits of $\mathbf{K}$ close then $\mathbf{M_3}$ is the total space of a $\mathbf{U(1)}$ orbibundle over an orbifold Riemann surface Σ (a Seifert fibred 3-manifold).

On the other hand, if at least one orbit is open then $\mathbf{M_3}$ necessarily admits a $\mathbf{U(1)} \times \mathbf{U(1)}$ isometry, and we may write

$$\mathbf{K} \;=\; \partial_\psi \;=\; \mathbf{b_1}\partial_{\varphi_1} + \mathbf{b_2}\partial_{\varphi_2} \;,$$

where $\mathbf{b_1}, \mathbf{b_2} \neq \mathbf{0}$ can be thought of as parametrizing a choice of $\mathbf{K}$.

For the vector multiplet we find the localization equation $\delta\lambda = 0$ for $\mathbf{M_3} \cong \mathbf{S^3}$ implies

$$\mathcal{A} = 0 , \qquad \Omega\sigma = \sigma_0 = \text{constant} , \qquad \mathsf{D} = -\frac{\mathsf{h}}{\Omega}\sigma_0 .$$

A matter chiral multiplet consists of complex scalars ϕ and $\mathbf{F}$, together with a fermion spinor field ψ, in an arbitrary representation $\mathcal{R}$ of $\mathbf{G}$, plus superpotential.

These localize onto solutions of $\delta\psi = 0$, $\delta\psi^\dagger = 0$, which force everything $= 0$.

The classical action, evaluated on the localization locus, is given entirely by the Chern-Simons action:

$$\mathbf{S}_{\mathrm{CS}} = -\frac{\mathbf{ik}}{2\pi}\mathrm{Tr}(\sigma_0^2)\int_{M_3}\frac{\mathbf{h}}{\Omega^2}\sqrt{\det \mathbf{g}}\,\mathrm{d}^3\mathbf{x} = \frac{\mathbf{i}\pi\mathbf{k}}{|\mathbf{b}_1\mathbf{b}_2|}\mathrm{Tr}(\sigma_0^2)\ .$$

In the last step we have rewritten the integral in terms of an equivariant form on $\mathbb{R}^2 \oplus \mathbb{R}^2 \supset \mathbf{S}^3$, and used the Berline-Vergne fixed point formula (following a similar trick of [Martelli-JFS-Yau]).

Most of the work is in computing the one-loop determinants. Individual determinants cannot be computed in closed form for a general metric, but "most" eigenvalues pair and cancel by supersymmetry.

The final result for the partition function is

$$\mathbf{Z} \;=\; \langle\, 1\, \rangle \;=\; \frac{1}{|\mathbf{Weyl}|}\int_{\mathrm{Cartan}} \mathrm{d}\sigma_0\, \mathrm{e}^{-\frac{\mathrm{i}\pi k}{|b_1 b_2|}\mathrm{Tr}\,\sigma_0^2} \prod_{\alpha\in\Delta_+} 4\sinh\frac{\pi\sigma_0\alpha}{|b_1|}\sinh\frac{\pi\sigma_0\alpha}{|b_2|}$$

$$\cdot \prod_{\rho} s_\beta\left[\frac{\mathrm{i}(\beta+\beta^{-1})}{2}(1-R) - \frac{\rho(\sigma_0)}{\sqrt{|b_1 b_2|}}\right] .$$

Here we have defined $\beta = \sqrt{|b_1/b_2|}$, ρ denote weights in a weight space decomposition of the representation $\mathcal{R}$ for the matter fields, $\mathbf{R}$ is their R-charge, and $s_\beta(z)$ denotes the double sine function. This, and the sinh functions, arise from zeta function regularization.

It is also straightforward to insert BPS operators, for example the Wilson loop

$$\mathbf{W} \; = \; \mathrm{Tr}_{\mathcal{R}} \left[\mathcal{P} \exp \int_{\gamma} \mathrm{ds}(\mathrm{i}\mathcal{A}_{\mu}\dot{\mathrm{x}}^{\mu} + \sigma|\dot{\mathrm{x}}|) \right] \; ,$$

where $\mathbf{x}^{\mu}(\mathbf{s})$ parametrizes with worldline $\gamma =$ orbit of $\mathbf{K}$, is δ-invariant.

$\langle\, \mathbf{W} \,\rangle$ is then computed by inserting $\mathrm{Tr}_{\mathcal{R}} \mathrm{e}^{2\pi\ell\sigma_0}$ into the localized partition function, where $2\pi\ell =$ length of $\mathbf{K}$ orbit.

In particular we see that both the partition function and VEV of $\mathbf{W}$ depend on the background geometry only through $\mathbf{b_1}$, $\mathbf{b_2}$, parametrizing $\mathbf{K}$.

For comparison with AdS/CFT we should focus on field theories that in (conformally) flat space have an AdS gravity dual.

There are huge classes of these, described by Chern-Simons-quiver gauge theories, with $\mathbf{G} = \mathbf{U(N)^p}$, *e.g.* the maximally supersymmetric case is the ABJM theory, living on $\mathbf{N}$ M2-branes in flat space.

The gravity duals are M-theory backgrounds of the form $\mathrm{AdS}_4 \times \mathbf{Y_7}$, with $\mathbf{N}$ units of $*\mathbf{G_4}$ through the internal space $\mathbf{Y_7}$, and arise as *e.g.* near-horizon limits of $\mathbf{N}$ M2-branes at Calabi-Yau four-fold singularities [Martelli-JFS, many other authors].

The large **N** limit of the matrix model partition function was computed in [Martelli-Passias-JFS], using a saddle point method of [Herzog-Klebanov-Pufu-Tesileanu].

This involves the asymptotic expansion of the double sine function, and an ansatz for the saddle point eigenvalue distribution for σ_0.

The final results are extremely simple:

$$\begin{aligned} \log Z &= \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot \log Z_{\text{round } S^3} , \\ \log\langle W \rangle &= \tfrac{1}{2}\ell(|b_1| + |b_2|) \cdot \log\langle W \rangle_{\text{round } S^3} . \end{aligned}$$

In particular, the dependence on the background geometry factorizes from the dependence on the choice of gauge theory.

In AdS/CFT the three-dimensional background geometry $\mathbf{M_3} \cong \mathbf{S^3}$ arises as the conformal boundary of a four-manifold, in which gravity propagates.

The large $\mathbf{N}$ limit in the gauge theory is the same as the classical supergravity limit, and the conjecture relates gauge theory correlation functions to supergravity.

In the next talk I'll explain this relation, and then derive the same formulae in a purely classical computation in four-dimensional gravity. This amounts to a brute force proof of the conjecture for these particular observables.

I'll also present some recent results in $\mathbf{d = 5}$.