String Structures, Reductions and T-duality

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Outline

1. Topological T-duality and Courant algebroids

2. String structures and reduction of Courant algebroids

3. T-duality of string structures and heterotic Courant algebroids

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Electromagnetic duality

Maxwell's equations in vacuum (c = 1):

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Duality of order 4: $(\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E})$

The duality still holds if <u>both</u> electric and magnetic charges are included.

Phase space duality

Harmonic oscillator:

$$H=\frac{k}{2}x^2+\frac{1}{2m}p^2$$

Duality of order 4:

$$(x,p) \mapsto (p,-x)$$

 $(m,k) \mapsto (\frac{1}{k},\frac{1}{m})$

These are examples of *S*-duality.

T-duality: a toy example

Topological T-duality arose in the study of string theory compactifications.

Let V be a real *n*-dimensional vector space with basis $\{v_k\}_{k=1,...,n}$.

Let V^* be the dual space with basis $\{w_k\}_{k=1,...,n}$.

Fix a volume form on V and V^* .

The Fourier-Mukai transform is an isomorphism

$$\mathcal{FM}: \wedge^{\bullet} V^* \to \wedge^{\bullet} V, \qquad \mathcal{FM}(\phi)(v) = \int \phi(w) e^{\sum_{k=1}^{n} w_k \wedge v_k}$$

Geometric formulation of \mathcal{FM}

Let $\Lambda \subset V$ be a lattice and $\Lambda^* \subset V^*$ the dual lattice.

Define the torus $T^n = V/\Lambda$ and the dual torus $\hat{T}^n = V^*/\Lambda^*$.

Note that
$$\pi_1(T^n) = \Lambda = Irrep(\widehat{T^n})$$
 and $\pi_1(\widehat{T^n}) = \Lambda^* = Irrep(T^n)$.

In particular, $\hat{T}^n = \text{Hom}(\pi_1(T^n), S^1)$, so it parametrizes flat S^1 -bundles on T^n .

 $T^n \times \widehat{T}^n$ carries a universal S^1 -bundle \mathcal{P} called the Poincaré line bundle:

$$\mathcal{P}|_{T^n \times w} \cong \text{flat } S^1\text{-bundle on } T^n \text{ associated to } w$$

 $\mathcal{P}|_{v \times \widehat{T}^n} \cong \text{flat } S^1\text{-bundle on } \widehat{T^n} \text{ associated to } v$

Geometric formulation of $\mathcal{F}\mathcal{M}$

Now we have

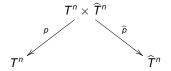
$$H^{\bullet}(T^{n},\mathbb{R}) = \wedge^{\bullet}V^{*}, \qquad H^{\bullet}(\widehat{T}^{n},\mathbb{R}) = \wedge^{\bullet}V$$

and

$$ch(\mathcal{P}) = e^{\sum_{k=1}^{n} w_k \wedge v_k} \in H^{\bullet}(T^n imes \widehat{T}^n, \mathbb{R})$$

The Fourier-Mukai transform is an isomorphism

$$\mathcal{FM} \colon H^ullet(\mathcal{T}^n,\mathbb{R}) o H^{ullet-n}(\widehat{\mathcal{T}}^n,\mathbb{R}), \qquad \mathcal{FM}(\phi) = \hat{p}_!ig(p^*(\phi) \wedge \textit{ch}(\mathcal{P})ig)$$



Topological T-duality

Idea: Replace T^n by a family of tori.

Possibilities include:

- $X = M \times T^n$
- $X \rightarrow M$ a principal T^n -bundle
- $X \rightarrow M$ an affine T^n -bundle
- X a Tⁿ-space (non-free action)
- singular fibrations (e.g. the Hitchin fibration, CY manifolds)

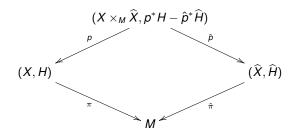
We shall consider the case when $X \rightarrow M$ is a principal torus bundle.

It turns out that an additional structure is needed on *X*, namely a bundle gerbe classified by its Dixmier-Douady class $[H] \in H^3(X, \mathbb{Z})$.

Topological T-duality

Theorem (Bouwknegt-Evslin-Mathai (2004), Bunke-Schick (2005))

There exists a commutative diagram



and

$$\mathcal{FM}: \ (\Omega^{\bullet}(X)^{T^{n}}, d_{H}) \to (\Omega^{\bullet-n}(\widehat{X})^{\widehat{T}^{n}}, d_{\widehat{H}}), \quad \mathcal{FM}(\omega) = \int_{T^{n}} e^{\mathcal{F}} \wedge \omega$$

is an isomorphism of the differential complexes, where $p^*H - \hat{p}^*\hat{H} = d\mathcal{F}$ and $\mathcal{F} = \langle p^*\theta \wedge \hat{p}^*\hat{\theta} \rangle$ for connections θ and $\hat{\theta}$ on X and \hat{X} respectively.

Remarks

> As a corollary, we have an isomorphism in twisted cohomology

$$H^{\bullet}(X,H)\cong H^{\bullet-n}(\widehat{X},\widehat{H})$$

This can be refined to an isomorphism in twisted K-theory,

$$K^{\bullet}(X,H)\cong K^{\bullet-n}(\widehat{X},\widehat{H})$$

- ► For circle bundles, the T-dual is unique up to isomorphism.
- ► For higher rank torus bundles, an additional condition on *H* is needed and the T-dual is not unique.

Example: Lens spaces L_p

Consider the action of \mathbb{Z}_p on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ given by $e^{\frac{2\pi i}{p}}(z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{p}} z_2)$

The quotient $L_{\rho}=S^3/\mathbb{Z}_{\rho}$ is an S^1 -bundle over S^2 with the Chern class

$$c_1(L_p) = p \in H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$$

Let $H = q \in H^3(L_p, \mathbb{Z}) \cong \mathbb{Z}$, then the T-dual pair is (L_q, p) .

In particular $L_0 = S^2 \times S^1$, so

$$(S^3,0) \iff (S^2 \times S^1,1)$$

Note that

$$egin{aligned} &\mathcal{K}^0(\mathcal{S}^3)=\mathcal{K}^1(\mathcal{S}^3)=\mathbb{Z}\ &\mathcal{K}^0(\mathcal{S}^2 imes\mathcal{S}^1,1)=\mathcal{K}^1(\mathcal{S}^2 imes\mathcal{S}^1,1)=\mathbb{Z} \end{aligned}$$

while

$$\mathcal{K}^0(\mathcal{S}^2 imes \mathcal{S}^1) = \mathcal{K}^1(\mathcal{S}^2 imes \mathcal{S}^1) = \mathbb{Z} \oplus \mathbb{Z}$$

Courant algebroids

A Courant algebroid on a smooth manifold X consists of a vector bundle $E \rightarrow X$ equipped with

- a bundle map $\rho \colon E \to TX$ called the anchor,
- a non-degenerate symmetric bilinear form $\langle , \rangle \colon E \otimes E \to \mathbb{R}$,
- an \mathbb{R} -bilinear operation $[,]: \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E),$

satisfying the following properties

- [a, [b, c]] = [[a, b], c] + [b, [a, c]]
- $[a,b] + [b,a] = d\langle a,b \rangle$
- $ho(a)\langle b,c
 angle=\langle [a,b],c
 angle+\langle b,[a,c]
 angle$
- $[a, fb] = f[a, b] + \rho(a)(f)b$
- $\rho[\mathbf{a}, \mathbf{b}] = [\rho(\mathbf{a}), \rho(\mathbf{b})]$

Exact Courant algebroids

A Courant algebroid *E* is **transitive** if the anchor ρ is surjective.

E is **exact** if it fits into an exact sequence

$$0 \to T^*X \to E \to TX \to 0$$

Exact Courant algebroids are classified by their Ševera class $H \in H^3(X, \mathbb{R})$.

An isotropic splitting $s: TX \rightarrow E$ fixes an isomorphism

 $E \cong TX \oplus T^*X$

where

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$$
$$[X + \xi, Y + \eta]_{H} = [X, Y] + \mathcal{L}_{X}\eta - i_{Y}d\xi + i_{Y}i_{X}H$$

with

$$H(X, Y, Z) = \langle [s(X), s(Y)], s(Z) \rangle.$$

Symmetries and generalised metric

Spin module $\Omega^{\bullet}(X)$: $(X + \xi) \cdot \omega = \iota_X \omega + \xi \wedge \omega$.

Abelian extension: $Aut(E) = Diff(X) \ltimes \Omega^2_{cl}(X)$

$$e^{B}(X+\xi)=X+\xi+\iota_{X}B$$

Extension class:

 $c(X,Y)=d\iota_X\iota_YH$

A generalised Riemannian metric is a self-adjoint orthogonal bundle map $G \in End(TX \oplus T^*X)$ for which $\langle Gv, v \rangle$ is positive definite.

 $G^2 = Id$ determines an orthogonal decomposition

 $TX \oplus T^*X = G_+ \oplus G_-$

where $G_{\pm} = \{X + B(X, \cdot) \pm g(X, \cdot) \mid X \in TX\}.$

Simple reduction

Consider a Lie group K acting freely on X.

Suppose the action lifts to a Courant algebroid E on X.

The simple reduction E/K is a vector bundle on X/K, which inherits the Courant algebroid structure on E.

E/K is <u>not</u> an exact Courant algebroid.

Buscher rules

Theorem (Cavalcanti-Gualtieri)

The map

$$\phi \colon (TX \oplus T^*X)/T^n \to (T\hat{X} \oplus T^*\hat{X})/\hat{T}^n$$
$$X + \xi \quad \mapsto \quad \hat{p}_*(\hat{X}) + p^*(\xi) - \mathcal{F}(\hat{X})$$

is an isomorphism of Courant algebroids.

The Buscher rules for (g, B) are given by

$$\widehat{G} = \phi(G)$$

Heterotic string theory

Conceived by the Princeton String Quartet in 1985.

Combines 26-dimensional bosonic left-moving strings with 10-dimensional right-moving superstrings.

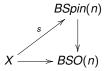
The theory includes a principal *G*-bundle $P \rightarrow X$ equipped with a connection.

The Green-Schwarz anomaly cancellation:

$$dH = \frac{1}{2}p_1(TX) - \frac{1}{2}p_1(P)$$

String structures

A spin structure on an oriented manifold X is a lift:

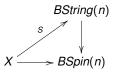


A string structure on a spin manifold X is a lift:

A string structure exists if and only if $\left[\frac{1}{2}p_1(S)\right] = 0$.

Equivalently, a string structure is $[H] \in H^3(P, \mathbb{Z})$, where $P \to X$ is the spin structure, such that the restriction of [H] to any fiber of P is the generator of $H^3(Spin(n), \mathbb{Z}) \cong \mathbb{Z}$.

String classes *H* are intimately related to extended actions and certain transitive Courant algebroids.



Heterotic Courant algebroids

Let *G* be a compact connected simple Lie group and $P \rightarrow X$ a principal *G*-bundle.

The Atiyah algebroid $\mathcal{A} := TP/G \rightarrow TX$ is a quadratic Lie algebroid,

$$\langle x, y \rangle = -k(x, y)$$

where *k* denotes the Killing form on g.

A transitive Courant algebroid \mathcal{H} is a heterotic Courant algebroid if

$$\mathcal{H}/T^*X\cong\mathcal{A}$$

is an isomorphism of quadratic Lie algebroids, where A is the Atiyah algebroid of some principal *G*-bundle *P*.

Classification of heterotic Courant algebroids

The obstruction for the Atiyah algebroid of *P* to arise from a transitive Courant algebroid \mathcal{H} is the first Pontryagin class $p_1(\mathcal{P}) \in H^4(X, \mathbb{R})$.

Theorem

Let $P \rightarrow X$ be a principal G-bundle and A a connection on P with curvature F. Let H^0 be a 3-form on X satisfying

 $dH^0+k(F,F)=0.$

Any heterotic Courant algebroid is isomorphic to one of the form

$$\mathcal{H} = TX \oplus \mathfrak{g}_P \oplus T^*X,$$

where

$$\langle (X, s, \xi), (Y, t, \eta) \rangle = \frac{1}{2} (i_X \eta + i_Y \xi) + \langle s, t \rangle$$

$$egin{aligned} & [X+s+\xi,Y+t+\eta]_{\mathcal{H}} = [X,Y] +
abla_X t -
abla_Y s - [s,t] - F(X,Y) \ & + \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H^0 \ & + 2 \langle t, i_X F
angle - 2 \langle s, i_Y F
angle + 2 \langle
abla s, t
angle, \end{aligned}$$

Extended action on Courant algebroids

Let *E* be an exact Courant algebroid on a *G*-manifold *X* and assume that the action lifts $G \rightarrow Aut(E)$.

If the infinitesimal action $\mathfrak{g} \to Der(E)$ on E is by inner derivations, we could consider a lift $\mathfrak{g} \to \Gamma(E)$.

A trivially extended action is a map $\alpha \colon \mathfrak{g} \to \Gamma(E)$ such that

- α is a homomorphism of Courant algebras,
- ▶ $\rho \circ \alpha = \psi$, where $\psi : \mathfrak{g} \to \Gamma(TX)$ denotes the infinitesimal *G*-action on *X*,
- the induced adjoint action of g on *E* integrates to a *G*-action on *E*.

Reduction by extended action

For an exact Courant algebroid $E \cong TX \oplus T^*X$ with a *G*-invariant Ševera class *H*, the extended action

$$\alpha \colon \mathfrak{g} \to \Gamma(E), \quad \mathbf{v} \mapsto \psi(\mathbf{v}) + \xi(\mathbf{v})$$

corresponds to solutions to $d_G(H + \xi) = c$, with the non-degenerate form $c(\cdot, \cdot) = -\langle \alpha(\cdot), \alpha(\cdot) \rangle \in \Omega^0(X, S^2\mathfrak{g}^*)^G$.

Two extended actions ξ , ξ' are equivalent if there exists an equivariant function $f: M \to \mathfrak{g}^*$ such that $\xi' = \xi + df$

Changing the invariant splitting of E corresponds to

$$H' + \xi' = H + \xi + d_G(B)$$

where $B \in \Omega^2(X)^G$ is the invariant 2-form relating the splittings.

The reduced Courant algebroid on X/G is defined by $E_{red} = Im(\alpha)^{\perp}/G$.

Heterotic Courant algebroids by reduction

Let $\sigma: P \to X$ be a *G*-bundle equipped with a *G*-invariant closed 3-form *H* on *P* and $E = TP \oplus T^*P$ with the *H*-twisted Dorfman bracket.

Since g comes with a natural pairing, it is natural to consider c = -k.

Proposition

Equivalence classes of solutions to $d_G(H + \xi) = -k$ are represented by pairs (H^0, A) satisfying

$$dH^0+k(F,F)=0.$$

The corresponding pair (H, ξ) is given by

$$H = \sigma^*(H^0) + CS_3(A), \qquad \xi = kA.$$

Hence, every heterotic Courant algebroid is obtained from an exact Courant algebroid via a trivially extended action.

Relation to string structures

The restriction of $H = \sigma^*(H^0) + CS_3(A)$ to any fibre of *P* is given by

$$\omega_3 = -\frac{1}{6}k(\omega, [\omega, \omega])$$

where $\omega \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan form.

A real string class is a class $H \in H^3(P, \mathbb{R})$ such that the restriction of H to any fibre of P coincides with ω_3 . Imposing integrality, (P, H) defines a string structure on X.

Let $\mathcal{EA}(P)$ and $\mathcal{SC}(P)$ denote the sets of equivalence classes of trivially extended actions and string classes on P respectively. The map

$$(H,\xi) \rightarrow [H]$$

is an isomorphism of $H^3(X, \mathbb{R})$ -torsors.

Consider a T^n -bundle $X \to M$ equipped with a string structure (P, H).

We assume that the T^n -action on X lifts to a T^n -action on P by principal bundle automorphisms, so we can view P as a principal $T^n \times G$ -bundle over M. Then $P_0 = P/T^n$ is a principal G-bundle over M.

Choose *H* to be a $T^n \times G$ -invariant representative for the string class.

Strategy

- Since P → P₀ is a principal Tⁿ-bundle, we can apply ordinary T-duality to the pair (P, H) to obtain a dual pair (P̂, Ĥ).
- ▶ The existence of a T-dual imposes the usual constraints on *H*.
- ► However, there is no guaranty that the *G*-action on P₀ lifts to an action on P̂ commuting with the T̂ⁿ-action.
- The restriction of *H* to the *G* × *Tⁿ*-fibres of *P* → *M* defines a class in *H*²(*G*, *H*¹(*Tⁿ*, ℤ)), which is the obstruction to *P* → *P*₀ being a pullback under σ₀: *P*₀ → *M* of a *Tⁿ*-bundle *X* → *M*.

T-duality commutes with reduction

Proposition

For commuting group actions, the simple reduction and reduction by extended action commute.

Theorem

The T-duality isomorphism

$$\phi \colon (TP \oplus T^*P)/T^n \to (T\widehat{P} \oplus T^*\widehat{P})/\widehat{T}^n$$

exchanges extended actions (H,ξ) and $(\widehat{H},\widehat{\xi})$, and we have the desired isomorphism

$$\mathcal{H}/T^n \cong ((TP \oplus T^*P)/T^n)_{red} \cong ((T\widehat{P} \oplus T^*\widehat{P})/\widehat{T}^n)_{red} \cong \widehat{\mathcal{H}}/\widehat{T}^n$$

The proof hinges on establishing the following identity,

$$\widehat{H} + \widehat{\xi} = H + \xi + d_G \langle \theta, \hat{\theta} \rangle$$

where $\widehat{A} - A = -\iota \langle \theta, \hat{\theta} \rangle$.

Remarks

- T-duality can be adapted to incorporate:
 - String structures
 - Trivially extended actions
 - Heterotic Courant algebroids
- > Heterotic Buscher rules are recovered via generalised metrics.
- The Pontryagin class $\frac{1}{2}p_1(TX)$ can be included.
- > The heterotic Einstein equations are preserved under T-duality.
- String structures allow for more flexibility in the possible changes in topology under T-duality.

Examples

Proposition

Let $c \in H^2(M, H^1(\hat{T}^n, \mathbb{Z}))$ and $\hat{c} \in H^2(M, H^1(T^n, \mathbb{Z}))$ be the Chern classes of $X \to M$ and $\hat{X} \to M$. Then the following holds in $H^4(M, \mathbb{R})$:

$$\langle \boldsymbol{c}, \hat{\boldsymbol{c}} \rangle = p_1(P_0).$$

Ordinary T-duality corresponds to $\langle c, \hat{c} \rangle = 0$.

- Higher dimensional Lens spaces.
- Homogeneous spaces $G \rightarrow G/H$.