Crossed products of C^*-algebras for singular actions

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1. Setting - Crossed products

Start with a:-

Definition (C*-dynamical system)

• i.e. a triple $(\mathcal{A}, G, \alpha)$ consisting of a
C*-algebra $\mathcal{A}$, and a locally compact group G, and
• a strongly continuous action

$$\alpha : G \longrightarrow \mathrm{Aut}\,\mathcal{A}\,, \tag{1}$$

i.e. a group homomorphism such that $g \mapsto \alpha_g(A)$ is continuous for each $A \in \mathcal{A}$.

Such actions occur naturally, e.g. in studying time evolutions or symmetries of quantum systems.

Covariant representations

The natural class of representations of such a system respects the action:

Definition (Covariant representations)

A covariant representation of $(\mathcal{A}, G, \alpha)$ is a pair (π, U), where
• $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ and
• $U\colon G \to \mathrm{U}(\mathcal{H})$ is a continuous unitary representation satisfying

$$U(g)\pi(A)U(g)^* = \pi(\alpha_g(A)) \quad \text{for} \quad g \in G, a \in \mathcal{A}. \tag{2}$$

We write $\mathrm{Rep}(\alpha, \mathcal{H})$ for the set of covariant representations (π, U) of $(\mathcal{A}, G, \alpha)$ on $\mathcal{H}$.

It is a fundamental fact that the covariant representation theory of $(\mathcal{A}, G, \alpha)$ corresponds to the representation theory of a C*-algebra $\mathcal{C}$, hence can be analyzed with the usual C*-tools.

The crossed product

The correspondence between $\mathrm{Rep}(\alpha, \mathcal{H})$ and $\mathrm{Rep}(\mathcal{C}, \mathcal{H})$ takes the following form.

- There is a *-homomorphism $\eta_{\mathcal{A}} \colon \mathcal{A} \to M(\mathcal{C}) \equiv$ multiplier algebra of $\mathcal{A}$,
- a unitary homomorphism $\eta_G \colon G \to UM(\mathcal{C})$ such that

$$\eta_G(g)\eta_{\mathcal{A}}(A)\eta_G(g)^* = \eta_{\mathcal{A}}(\alpha_g(A)) \quad \text{for} \quad g \in G, A \in \mathcal{A}. \tag{3}$$

- Every representation $\rho \in \mathrm{Rep}(\mathcal{C}, \mathcal{H})$ has a unique extension $\widetilde{\rho} : M(\mathcal{C}) \to \mathcal{B}(\mathcal{H})$ such that the pair $(\widetilde{\rho} \circ \eta_{\mathcal{A}}, \widetilde{\rho} \circ \eta_G) \in \mathrm{Rep}(\alpha, \mathcal{H})$.

This bijective correspondence $\rho \leftrightarrow (\pi, U)$ preserves direct sums, subrepresentations and irreducibility

Crossed product

The function of this C*-algebra $\mathcal{C}$ is to carry the covariant representation theory of $(\mathcal{A}, G, \alpha)$.

It is called the crossed product of $(\mathcal{A}, G, \alpha)$, usually denoted by $\mathcal{A} \rtimes_\alpha G$, and it is constructed as a (C*-completion of a) skew convolution algebra of $L^1(G, \mathcal{A})$ with convolution product

$$(f * h)(t) = \int_G f(s)\, \alpha_s(h(s^{-1}t))\, ds\,.$$

In the case that $\mathcal{A} = \mathbb{C}$, this is just the usual group algebra $C^*(G) =: \mathcal{L}$ and η_G just becomes the usual $\eta \colon G \to UM(\mathcal{L})$ acting by left translations on $L^1(G, \mathcal{A})$.There is a bijection between continuous representations of G and nondegenerate representations of $\mathcal{L}$, given by
$U_{\mathcal{L}}(f) := \int_G f(s)\, U(s)\, ds$ for $f \in L^1(G)$.

A more useful characterization of $\mathcal{C} = \mathcal{A} \rtimes_\alpha G$ is as follows.

Crossed product

Definition (Crossed product - Raeburn)

Given a C*-dynamical system $(\mathcal{A}, G, \alpha)$, then the crossed product of $(\mathcal{A}, G, \alpha)$ is the unique C*-algebra $\mathcal{C}$ such that

• there are C*-algebra morphisms $\eta_{\mathcal{A}}\colon \mathcal{A} \to M(\mathcal{C})$, $\eta_{\mathcal{L}}\colon \mathcal{L} \to M(\mathcal{C})$ where
• $\eta_{\mathcal{L}}$ is non-degenerate i.e. $\mathrm{span}(\eta_{\mathcal{L}}(\mathcal{L})\mathcal{C})$ is dense in $\mathcal{C}$,
• The multiplier extension $\widetilde{\eta}_{\mathcal{L}}\colon M(\mathcal{L}) \to M(\mathcal{C})$ satisfies in $M(\mathcal{C})$ the relations

$$\widetilde{\eta}_{\mathcal{L}}(\eta(g))\eta_{\mathcal{A}}(A)\widetilde{\eta}_{\mathcal{L}}(\eta(g))^* = \eta_{\mathcal{A}}(\alpha_g(A)) \quad \text{for all} \quad A \in \mathcal{A}, \text{ and } g \in G.$$

• $\eta_{\mathcal{A}}(\mathcal{A})\eta_{\mathcal{L}}(\mathcal{L}) \subseteq \mathcal{C}$ and $\mathcal{C}$ is generated by this set as a C^*-algebra.
• For every covariant representation $(\pi, U) \in \mathrm{Rep}(\alpha, \mathcal{H})$ there exists a unique representation $\rho \in \mathrm{Rep}(\mathcal{C}, \mathcal{H})$ with

$$\rho(\eta_{\mathcal{A}}(A)\eta_{\mathcal{L}}(L)) = \pi(A)U_{\mathcal{L}}(L) \quad \text{for} \quad A \in \mathcal{A}, L \in \mathcal{L}.$$

2. Singular actions.

Above we assumed for $(\mathcal{A}, G, \alpha)$ that

- the map $\alpha : G \to \mathrm{Aut}\,\mathcal{A}$ is strongly continuous,
- the topological group G is locally compact, and
- we want to model the whole covariant representation theory for $\alpha : G \to \mathrm{Aut}\,\mathcal{A}$

Unfortunately many natural systems, both in physics and mathematics fail to satisfy these assumptions.

Failure in the first two cases, means the construction of a crossed product fails, and in the last case its representation theory is not the correct one we are interested in.

Example

On $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ define $(Qf)(x) = xf(x)$ and $Pf = if'$. Let $\mathcal{A} = C^*\{e^{itQ}, e^{itP} \mid t \in \mathbb{R}\} \subset \mathcal{B}(L^2(\mathbb{R}))$ and define $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{A}$ by $\alpha_t = \mathrm{Ad}\exp(itP^2)$. Then $\|e^{iQ} - \alpha_t(e^{iQ})\| = 2$ if $t \neq 0$.

3. Host algebras.

To construct a C*-algebra $\mathcal{C}$ which can play the role of the crossed product $\mathcal{A} \rtimes_\alpha G$ for such systems, we use Raeburn's approach.

As $C^*(G)$ will not exist if G is not locally compact, we generalize:

Definition (Host algebra)

A **host algebra** for a topological group G is a pair $(\mathcal{L}, \eta)$, where $\mathcal{L}$ is a C^*-algebra and $\eta\colon G \to UM(\mathcal{L})$ is a group homomorphism such that:

- For each non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{L}$, the representation $\widetilde{\pi} \circ \eta =: \eta^*(\pi)$ of G is continuous.
- For each complex Hilbert space $\mathcal{H}$, the map

$$\eta^*\colon \operatorname{Rep}(\mathcal{L}, \mathcal{H}) \to \operatorname{Rep}(G, \mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \circ \eta$$

  is injective.

We write $\operatorname{Rep}(G, \mathcal{H})_\eta$ for the range of η^*, and its elements are called $\mathcal{L}$-representations of G.

We call $(\mathcal{L}, \eta)$ a **full host algebra** if, in addition, we have:

- $\mathrm{Rep}(G, \mathcal{H})_\eta = \mathrm{Rep}(G, \mathcal{H})$ for each Hilbert space $\mathcal{H}$.

A full host algebra, carries precisely the continuous unitary representation theory of G, and if it is not full, it carries some subtheory of the continuous unitary representations of G. If we want to impose additional restrictions, e.g. a spectral condition, then we will specify a host algebra which is not full.

Host algebras need not exist, as there are topological groups with continuous unitary representations, but without irreducible ones, and η^* preserves irreducibility.

The existence of a host algebra for a fixed subclass of representations of G implies that this class of representations is "isomorphic" to the representation theory of a C^*-algebra.

If G is locally compact, then $\mathcal{L} = C^*(G)$ with the canonical map $\eta\colon G \to UM(C^*(G))$ is a full host algebra.

4. Crossed product hosts

Based on Raeburn's approach we define:

Definition (Crossed product hosts)

Let G be a topological group, let $(\mathcal{L}, \eta)$ be a host algebra for G and $(\mathcal{A}, G, \alpha)$ be a C^*-action (not necessarily cont.). A triple $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ is a **crossed product host** for $(\alpha, \mathcal{L})$ if

- $\eta_{\mathcal{A}} \colon \mathcal{A} \to M(\mathcal{C})$ and $\eta_{\mathcal{L}} \colon \mathcal{L} \to M(\mathcal{C})$ are morphisms of C^*-algebras.
- $\eta_{\mathcal{L}}$ is non-degenerate.
- We have in $M(\mathcal{C})$:

$$\widetilde{\eta}_{\mathcal{L}} \circ \eta(g)\, \eta_{\mathcal{A}}(A)\, \widetilde{\eta}_{\mathcal{L}} \circ \eta(g)^* = \eta_{\mathcal{A}}(\alpha_g(A)) \quad \text{for} \quad A \in \mathcal{A},\ g \in G$$

  where $\widetilde{\eta}_{\mathcal{L}} \colon M(\mathcal{L}) \to M(\mathcal{C})$ is the multiplier extension.
- $\eta_{\mathcal{A}}(\mathcal{A})\eta_{\mathcal{L}}(\mathcal{L}) \subseteq \mathcal{C}$ and $\mathcal{C}$ is generated by this set as a C^*-algebra.

A **full** crossed product host for $(\alpha, \mathcal{L})$ satisfies in addition:

- For every covariant representation (π, U) of $(\mathcal{A}, \alpha)$ on $\mathcal{H}$ for which U is an $\mathcal{L}$-representation of G, there exists a unique representation $\rho\colon \mathcal{C} \to \mathcal{B}(\mathcal{H})$ with

$$\rho(\eta_{\mathcal{A}}(A)\eta_{\mathcal{L}}(L)) = \pi(A)U_{\mathcal{L}}(L) \quad \text{for} \quad A \in \mathcal{A}, L \in \mathcal{L}.$$

Two crossed product hosts $(\mathcal{C}^{(i)}, \eta_{\mathcal{A}}^{(i)}, \eta_{\mathcal{L}}^{(i)})$, $i = 1, 2$, are **isomorphic** if there is an isomorphism $\Phi : \mathcal{C}^{(1)} \to \mathcal{C}^{(2)}$ such that $(\Phi(\mathcal{C}^{(1)}), \widetilde{\Phi} \circ \eta_{\mathcal{A}}^{(1)}, \widetilde{\Phi} \circ \eta_{\mathcal{L}}^{(1)}) = (\mathcal{C}^{(2)}, \eta_{\mathcal{A}}^{(2)}, \eta_{\mathcal{L}}^{(2)})$.

In the usual case, where $\alpha\colon G \to \mathrm{Aut}(\mathcal{A})$ is strongly continuous, and G is locally compact with $\mathcal{L} = C^*(G)$, then the crossed product algebra $\mathcal{A} \rtimes_\alpha G$ is a full crossed product host for $(\alpha, \mathcal{L})$.

However we have many examples beyond this.

In general a crossed product host need not exist.

This definition generalizes crossed products in four directions:

- The group G need not be locally compact,
- the action α need not be strongly continuous,
- the host algebra $\mathcal{L}$ does not have to coincide with $C^*(G)$ when G is locally compact,
- for a non-full crossed product host, we restrict to a subtheory of the covariant $\mathcal{L}$–representations:

Theorem

Let $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ be a crossed product host for $(\alpha, \mathcal{L})$, and define the homomorphism $\eta_G := \widetilde{\eta}_{\mathcal{L}} \circ \eta : G \to UM(\mathcal{C})$. Then for each Hilbert space $\mathcal{H}$ the map

$$\eta_\times^* : \operatorname{Rep}(\mathcal{C}, \mathcal{H}) \to \operatorname{Rep}(\alpha, \mathcal{H}), \quad \textit{given by} \quad \eta_\times^*(\rho) := \left(\widetilde{\rho} \circ \eta_{\mathcal{A}}, \widetilde{\rho} \circ \eta_G\right)$$

is injective, and its range
$\operatorname{Rep}(\alpha, \mathcal{H})_{\eta_\times} \subseteq \operatorname{Rep}_{\mathcal{L}}(\alpha, \mathcal{H}) \equiv$ *$\mathcal{L}$-representations of $(\mathcal{A}, G, \alpha)$, i.e. covariant representations (π, U) for which U is an $\mathcal{L}$-representation of G. If $\mathcal{C}$ is full, then we have equality:* $\operatorname{Rep}(\alpha, \mathcal{H})_{\eta_\times} = \operatorname{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$.

Theorem (Uniqueness Theorem)

Given a C^*-action $(\mathcal{A}, G, \alpha)$ and a host algebra $(\mathcal{L}, \eta)$ for G, let $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ and $(\mathcal{C}^\sharp, \eta^\sharp_{\mathcal{A}}, \eta^\sharp_{\mathcal{L}})$ be crossed product hosts for $(\alpha, \mathcal{L})$, such that $\mathrm{Rep}(\alpha, \mathcal{H})_{\eta_\times} = \mathrm{Rep}(\alpha, \mathcal{H})_{\eta^\sharp_\times}$ for any Hilbert space $\mathcal{H}$.

Then there exists a unique isomorphism $\varphi\colon \mathcal{C} \to \mathcal{C}^\sharp$ with $\widetilde{\varphi} \circ \eta_{\mathcal{A}} = \eta^\sharp_{\mathcal{A}}$ and $\widetilde{\varphi} \circ \eta_{\mathcal{L}} = \eta^\sharp_{\mathcal{L}}$.
In particular, full crossed product hosts for $(\alpha, \mathcal{L})$ are isomorphic.

Thus if two crossed product hosts carry the same covariant representation theory for $(\mathcal{A}, G, \alpha)$ and use the same host $\mathcal{L}$, they are isomorphic.

We now consider the more involved question of existence.

5. Existence of crossed product hosts

Given a C^*-action $(\mathcal{A}, G, \alpha)$ and a host algebra $(\mathcal{L}, \eta)$ for G,

observe that if we have a crossed product host $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ for $(\alpha, \mathcal{L})$, then for the universal representation $(\rho_u, \mathcal{H}_u)$ of $\mathcal{C}$, the corresponding covariant $\mathcal{L}$-representation $(\pi, U) = \big(\widetilde{\rho}_u \circ \eta_{\mathcal{A}}, \widetilde{\rho}_u \circ \eta_G\big) = \eta_\times^*(\rho_u)$ satisfies

$$\begin{aligned} \rho_u(\eta_{\mathcal{A}}(A)\eta_{\mathcal{L}}(L)) &= \pi(A)U_{\mathcal{L}}(L) \quad \text{for} \quad A \in \mathcal{A}, L \in \mathcal{L}, \\ \text{and} \quad \mathcal{C} \cong \rho_u(\mathcal{C}) &= C^*\big(\pi(\mathcal{A})U_{\mathcal{L}}(\mathcal{L})\big)\,. \end{aligned}$$

We can try to construct $\mathcal{C}$ in a similar way in any other covariant representation $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$ and ask when this is a crossed product host:

Theorem (Existence Theorem for crossed product hosts)

Given a C^*-action $(\mathcal{A}, G, \alpha)$ and a host algebra $(\mathcal{L}, \eta)$ for G, let $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$. Define

$$\mathcal{C} := C^*\big(\pi(\mathcal{A})U_{\mathcal{L}}(\mathcal{L})\big) \subset \mathcal{B}(\mathcal{H})\,.$$

Then $\pi(\mathcal{A}) \cup U_{\mathcal{L}}(\mathcal{L}) \subset M(\mathcal{C}) \subset \mathcal{B}(\mathcal{H})$, and we obtain morphisms

$$\eta_{\mathcal{A}} \colon \mathcal{A} \to M(\mathcal{C}) \quad \text{and} \quad \eta_{\mathcal{L}} \colon \mathcal{L} \to M(\mathcal{C})$$

determined by $\eta_{\mathcal{A}}(A)C := \pi(A)C$ and $\eta_{\mathcal{L}}(L)C := U_{\mathcal{L}}(L)C$ for $A \in \mathcal{A}$, $L \in \mathcal{L}$ and $C \in \mathcal{C}$.

Then the following are equivalent:

(i) *$(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ is a crossed product host.*

(ii) *$\pi(\mathcal{A})U_{\mathcal{L}}(\mathcal{L}) \subseteq U_{\mathcal{L}}(\mathcal{L})\mathcal{B}(\mathcal{H})$.*

(iii) *There exists an approximate identity $(E_j)_{j \in J}$ of $\mathcal{L}$ such that*

$$\|U_{\mathcal{L}}(E_j)\pi(A)U_{\mathcal{L}}(L) - \pi(A)U_{\mathcal{L}}(L)\| \to 0 \quad \text{for} \quad A \in \mathcal{A}, L \in \mathcal{L}.$$

This theorem shows how crossed product hosts can be constructed. It also isolates a distinguished class of representations:

Definition

A covariant $\mathcal{L}$-representation $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$ is called a **cross representation** for $(\alpha, \mathcal{L})$ if any of the equivalent conditions (i)-(iii) hold. Let $\mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$ denote the set of cross representations for $(\alpha, \mathcal{L})$ on $\mathcal{H}$.

We have examples beyond the usual case where $\mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$ is nonempty, and hence for which there exist crossed product hosts.

Example

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, $\mathcal{A} := \mathcal{B}(\mathcal{H})$, $G := \mathbb{R}$, $\mathcal{L} = C^*(\mathbb{R})$, H be a selfadjoint operator, $U_t := e^{itH}$ and $\alpha_t(A) := U_t A U_t^*$. Now α is strongly continuous iff H is bounded. Assume that H is unbounded, hence $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ is not strongly continuous. As U is strong operator continuous, $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$ where π is the identical representation $\pi(A) = A$ of $\mathcal{A}$. Then we have that $(\pi, U) \in \mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$ if and only if $(i\mathbf{1} - H)^{-1} \in \mathcal{K}(\mathcal{H})$.

In the case that $G = \mathbb{R}$ and $\mathcal{L} = C^*(\mathbb{R})$ we can rephrase the cross condition as follows.

A given covariant representation (π, U) is a cross representation iff for all $A \in \mathcal{A}$ we have

$$\lim_{t\to 0} \big(U_t - \mathbf{1}\big)\pi(B)U_{\mathcal{L}}(L) = 0 \text{ for all } L \in \mathcal{L} \text{ and } B \in \{A,\, A^*\}$$

$$\lim_{t\to\infty} \big(P[-t,t] - \mathbf{1}\big)\pi(B)U_{\mathcal{L}}(L) = 0 \text{ for all } L \in \mathcal{L} \text{ and } B \in \{A,\, A^*\}$$

where P is the spectral projector of the generator $H = H^*$ of $t \mapsto U_t = \exp(itH)$.

These conditions are easier to use, and it is e.g. obvious that if $U_{\mathcal{L}}(\mathcal{L}) \subseteq \mathcal{K}(\mathcal{H})$ then they are satisfied.

The class of cross representations for $(\alpha, \mathcal{L})$ satisfies a range of permanence properties, e.g. it is closed w.r.t.

- taking of subrepresentations,
- taking arbitrary multiples,
- restriction to α–invariant subalgebras of $\mathcal{A}$ and
- forming finite direct sums (but NOT infinite ones).

Theorem (Stability of cross representations w.r.t. bounded perturbations)

$\alpha\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ be a C^*-action, let $\mathcal{L} = C^*(\mathbb{R})$ and let $(\pi, U) \in \mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$ be a given cross representation.

Let $B = B^* \in \mathcal{B}(\mathcal{H})$ and define $U^B_t := \exp(it(H + B))$ where $U_t = \exp(itH)$ has generator $H = H^*$ (not necessarily bounded).

Assume that $\mathrm{Ad}\, U^B_t$ preserves $\pi(\mathcal{A})$, hence define the perturbed action $\alpha^B\colon \mathbb{R} \to \mathrm{Aut}(\pi(\mathcal{A}))$ by $\alpha^B_t := \mathrm{Ad}\, U^B_t$.

Then (π, U^B) is a cross representation of α^B.

Theorem (Existence Theorem for full crossed product hosts)

Let $(\mathcal{L}, \eta)$ be a host algebra for the topological group G and $\alpha\colon G \to \mathrm{Aut}(\mathcal{A})$ be a C^*-action. Then the following are equivalent:

(i) *There exists a full crossed product host $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ for $(\alpha, \mathcal{L})$.*

(ii) $\mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H}) = \mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$ *for all Hilbert spaces $\mathcal{H}$.*

If $\mathcal{C}$ is a full crossed product host, then all crossed product hosts are factor algebras of $\mathcal{C}$.

It is possible that there are crossed product hosts, but no full ones.

6. Special cases.

Theorem (Cross representations via compact operators)

Let $(\mathcal{A}, G, \alpha)$ be a C^*-action and $(\mathcal{L}, \eta)$ be a host algebra for G. If $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$ satisfies $\pi(\mathcal{A})U_{\mathcal{L}}(\mathcal{L}) \subseteq \mathcal{K}(\mathcal{H})$, then (π, U) is a cross representation for $(\alpha, \mathcal{L})$. This holds in particular if $U_{\mathcal{L}}(\mathcal{L}) \subseteq \mathcal{K}(\mathcal{H})$.

Example

Let $\mathcal{H} = L^2(\mathbb{R}^n)$, On $C_c^\infty(\mathbb{R}^n)$ define $(Q_j f)(\mathbf{x}) = x_j f(\mathbf{x})$ and $P_j f = i\partial f \big/ \partial x_j$. This defines a representation of the Heisenberg group $H_n(\mathbb{R})$ by $U(\mathbf{q}, \mathbf{p}, t) = \exp(iQ(\mathbf{q}))\exp(iP(\mathbf{p}))e^{it}$ where $Q(\mathbf{q}) = \sum_j^n q_j Q_j$ and $P(\mathbf{p}) = \sum_j^n p_j P_j$. Let $\mathcal{A} = \mathcal{B}(L^2(\mathbb{R}^n))$, and define $\alpha : H_n(\mathbb{R}) \to \mathrm{Aut}\,\mathcal{A}$ by $\alpha_g = \mathrm{Ad}\,U(g)$. Then α is not strongly continuous. However if we take $\mathcal{L} = C^*(H_n(\mathbb{R}))$, then $U_{\mathcal{L}}(\mathcal{L}) = \mathcal{K}(\mathcal{H})$ and hence the defining representation (π, U), $\pi(A) = A$, is a cross representation.

Theorem (Compact group actions)

Let $(U, \mathcal{H})$ be a continuous unitary representation of the compact group G. Then the following are equivalent:

(i) *For the identical representation π of $\mathcal{A} = \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, $\alpha_g(A) = U_g A U_g^*$ and $\mathcal{L} = C^*(G)$, the pair (π, U) is a cross representation of $(\alpha, \mathcal{L})$.*

(ii) *$\mathrm{Spec}(U)$ is finite or U is of finite multiplicity.*

If $(\mathcal{A}, G, \alpha)$ is a C^*-action where G is compact with host chosen as $\mathcal{L} = C^*(G)$, and if for a covariant representation $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$, U satisfies property (ii), then (π, U) is a cross representation.

The converse is not true, i.e. a cross representation (π, U) need not satisfy (ii), as can be seen by taking an infinite multiple of a cross representation for which $\mathrm{Spec}(U)$ is infinite.

We have numerous examples, many relevant for physics. For example

Example (The Fock representation for a bosonic quantum field.)

For a nonzero complex Hilbert space $\mathcal{H}$, the bosonic Fock space is

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}\,, \quad \otimes_s^n \mathcal{H} \equiv \text{symmetrized n-fold tensor product}$$

with a convention $\otimes_s^0 \mathcal{H} := \mathbb{C}$. The (dense) finite particle space is $\mathcal{F}_0(\mathcal{H}) := \mathrm{span}\{\otimes_s^n \mathcal{H} \mid n = 0, 1, \cdots\}$.
For $f \in \mathcal{H}$, define on $\mathcal{F}_0(\mathcal{H})$ a (closable) creation operator $a^*(f)$ by

$$a^*(f)\left(v_1 \otimes_s \cdots \otimes_s v_n\right) := \sqrt{n+1}\, f \otimes_s v_1 \otimes_s \cdots \otimes_s v_n,$$

and an essentially selfadjoint operator by $\varphi(f) := \big(a^*(f) + a(f)\big)/\sqrt{2}$ where $a(f)$ is the adjoint of $a^*(f)$.
Let $W(f) := \exp(i\overline{\varphi(f)})$, then the simple C*-algebra $\mathcal{A} := C^*\{W(f) \mid f \in \mathcal{H}\}$ is the Weyl algebra (in Fock representation π_F).

Example (continued)

Define an automorphic action $\alpha : U(\mathcal{H}) \to \mathrm{Aut}(\mathcal{A})$ by $\alpha_U(W(f)) := W(Uf)$.
Let H be selfadjoint, and consider the one-parameter group $t \to U_t = \exp(itH) \in \mathrm{U}(\mathcal{H})$.
Define the (strong operator continuous) unitary group $t \to \Gamma(U_t) \subset U\big(\mathcal{F}(\mathcal{H})\big)$ by

$$\Gamma(U_t)\big(v_1 \otimes_s \cdots \otimes_s v_n\big) := \big(U_t v_1 \otimes_s \cdots \otimes_s U_t v_n\big).$$

Then $\mathrm{Ad}\Gamma(U_t)$ preserves $\mathcal{A}$, and we have covariance $\alpha_{U_t}(A) = \Gamma(U_t)A\Gamma(U_t)^*$ for all $A \in \mathcal{A}$.
If $H \neq 0$ then $t \to \alpha_{U_t}$ in not strongly continuous.

We have proven that if $H \geq 0$, then $(\pi_F, \Gamma(U))$ is a cross representation for $(\alpha, C^*(\mathbb{R}))$.

7. Spectral conditions.

In physics we often have additional spectral conditions for the unitary implementers U_g, of a covariant pair (π, U), e.g. in the case that $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ gives time evolution, then the self-adjoint generator of $t \mapsto U_t = \exp(itH)$ must be non-negative. We will call such covariant representations positive.

To obtain such covariant representations we should choose the host algebra

$$\mathcal{L} = C^*_+(\mathbb{R}) := C^*\{f \in L^1(\mathbb{R}) \mid \mathrm{supp}(\widehat{f}) \subseteq [0, \infty)\} \cong C_0([0, \infty)).$$

We already have examples above of positive cross representations. We want to investigate the connection between the cross property and spectral conditions.

Positive covariant representations have a number of special features of which the Borchers-Arveson Theorem lists a few:

Theorem (Borchers-Arveson)

Let $(\mathcal{M}, \mathbb{R}, \alpha)$ *be a* W^**-dynamical system on a von Neumann algebra* $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$*. Then the following are equivalent:*

(i) *There is a positive strong operator continuous unitary one-parameter group* $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ *such that* $\alpha_t = \mathrm{Ad}\, U_t$ *on* $\mathcal{M}$*.*

(ii) *There is a positive strong operator continuous unitary one-parameter group* $U : \mathbb{R} \to \mathcal{M}$ *such that* $\alpha_t = \mathrm{Ad}\, U_t$ *on* $\mathcal{M}$*.*

If these conditions hold, then there is a unique implementing positive unitary group $U : \mathbb{R} \to \mathcal{M}$ *which is minimal in the sense that if* $\widetilde{U} : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ *is any other implementing positive unitary group, then* $t \mapsto \widetilde{U}_t U_t^*$ *is positive. (That is,* $\widetilde{H} \geq H$ *for the generators)*

Thus given a positive covariant representation, we can find another one (with the same π) for which U is inner, and there is a special minimal one amongst these.

Theorem

Let $(\mathcal{M}, \mathbb{R}, \alpha)$ be a W^*-dynamical system on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and let $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be a positive unitary one-parameter group such that $\alpha_t = \mathrm{Ad}\, U_t$ on $\mathcal{M}$.
Let $V : \mathbb{R} \to \mathcal{M}$ be the minimal positive one-parameter unitary group given by the Borchers–Arveson Theorem. Choose $\mathcal{L} = C^*_+(\mathbb{R})$, then:

(i) *If $(\mathcal{M}, V)$ is a cross representation then $(\mathcal{M}, U)$ is a cross representation.*

(ii) *Let $(\mathcal{A}, G, \alpha)$ be a C^*-action and let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation such that $\pi(\mathcal{A}) \subseteq \mathcal{M}$, and $\pi \circ \alpha_t = \mathrm{Ad}\, U_t \circ \pi = \mathrm{Ad}\, V_t \circ \pi$, with U, V, $\mathcal{M}$ as above. If (π, V) is a cross representation then (π, U) is a cross representation.*

Thus if the minimal positive covariant representation is cross, all other positive covariant representations are cross.

7. Understanding the cross condition.

Recall that for any covariant $\mathcal{L}$-representation (π, U) we can construct a precursor $\mathcal{C}$ for our crossed product host, but only for the cross representations is it an actual cross product host:-

Given a C^*-action $(\mathcal{A}, G, \alpha)$ and a host algebra $(\mathcal{L}, \eta)$ for G, let $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$. Define

$$\mathcal{C} := C^*\big(\pi(\mathcal{A})U_{\mathcal{L}}(\mathcal{L})\big)\,.$$

Then $\pi(\mathcal{A}) \cup U_{\mathcal{L}}(\mathcal{L}) \subset M(\mathcal{C}) \subset \mathcal{B}(\mathcal{H})$, and we obtain morphisms

$$\eta_{\mathcal{A}} \colon \mathcal{A} \to M(\mathcal{C}) \quad \text{and} \quad \eta_{\mathcal{L}} \colon \mathcal{L} \to M(\mathcal{C})$$

determined by $\eta_{\mathcal{A}}(A)C := \pi(A)C$ and $\eta_{\mathcal{L}}(L)C := U_{\mathcal{L}}(L)C$ for $A \in \mathcal{A}$, $L \in \mathcal{L}$ and $C \in \mathcal{C}$. We examine the structure of $\mathcal{C}$.

Fix a $(\pi, U) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$, $\mathcal{L}$, $\mathcal{C}$ and $\eta_{\mathcal{A}}$, $\eta_{\mathcal{L}}$ as above.

Definition

- A (nondegenerate) representation $\rho \in \mathrm{Rep}(\mathcal{C}, \mathcal{H})$ is an $\mathcal{L}$-representation if $\widetilde{\rho} \circ \eta_{\mathcal{L}} : \mathcal{L} \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation of $\mathcal{L}$. Write $\mathrm{Rep}_{\mathcal{L}}(\mathcal{C}, \mathcal{H})$ for the set of $\mathcal{L}$-representations of $\mathcal{C}$ on $\mathcal{H}$.
- Define $\mathcal{C}_{\mathcal{L}} := \eta_{\mathcal{L}}(\mathcal{L})\mathcal{C}\eta_{\mathcal{L}}(\mathcal{L}) \subset \mathcal{C}$.

- An $\mathcal{L}$-representation $\rho \in \mathrm{Rep}_{\mathcal{L}}(\mathcal{C}, \mathcal{H})$ determines a unitary $\mathcal{L}$-representation, $U^\rho : G \to \mathrm{U}(\mathcal{H})$ uniquely specified by $U^\rho(g) \cdot (\widetilde{\rho} \circ \eta_{\mathcal{L}})(L) = (\widetilde{\rho} \circ \eta_{\mathcal{L}})(\eta(g)L)$ for all $g \in G$, $L \in \mathcal{L}$.

Moreover we have covariance, i.e. $(\widetilde{\rho} \circ \eta_{\mathcal{A}},\ U^\rho) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H})$.

- Then $\mathcal{C}_{\mathcal{L}}$ is a (hereditary) C*-subalgebra with the property that an $\mathcal{L}$-representation ρ of $\mathcal{C}$ is uniquely determined by its restriction to $\eta_{\mathcal{L}}(\mathcal{L})\mathcal{C}\eta_{\mathcal{L}}(\mathcal{L})$ via the relation

$$\rho(C) = \operatorname*{s-lim}_i \operatorname*{s-lim}_j \rho(\eta_{\mathcal{L}}(E_i) C \eta_{\mathcal{L}}(E_j))$$

for any approximate identity (E_j) of $\mathcal{L}$.

Thus $\mathcal{C}_\mathcal{L}$ carries the information of the $\mathcal{L}$-representation ρ of $\mathcal{C}$, and can reproduce the unitaries of the covariant pairs. It can produce the other part π of the covariant pair for those $A \in \mathcal{A}$ which are in its relative multiplier:

Definition

For $\mathcal{C}_\mathcal{L}$ as above, let

$$\mathcal{A}_\mathcal{L} := \left\{ A \in \mathcal{A} \mid \eta_\mathcal{A}(A)\mathcal{C}_\mathcal{L} \subseteq \mathcal{C}_\mathcal{L} \quad \text{and} \quad \eta_\mathcal{A}(A^*)\mathcal{C}_\mathcal{L} \subseteq \mathcal{C}_\mathcal{L} \right\}.$$

Theorem

Given $(\mathcal{C}, \eta_\mathcal{A}, \eta_\mathcal{L})$ and $\mathcal{A}_\mathcal{L}$ as above;-

(i) $$\mathcal{A}_\mathcal{L} = \left\{ A \in \mathcal{A} \,\middle|\, \eta_\mathcal{A}(A)\eta_\mathcal{L}(\mathcal{L}) \subseteq \eta_\mathcal{L}(\mathcal{L})\mathcal{C} \;\; \text{and} \;\; \eta_\mathcal{A}(A^*)\eta_\mathcal{L}(\mathcal{L}) \subseteq \eta_\mathcal{L}(\mathcal{L})\mathcal{C} \right\},$$

(ii) *For any (hence all) approximate identities $\{E_j\}$ of $\mathcal{L}$ we have*

$$\mathcal{A}_\mathcal{L} = \left\{ A \in \mathcal{A} \,\middle|\, \big\|\big(\eta_\mathcal{L}(E_j) - \mathbf{1}\big)\eta_\mathcal{A}(B)\eta_\mathcal{L}(L)\big\| \to 0 \;\; \text{for } \; B = A \text{ and } A^*, \; \text{and for all } \; L \in \mathcal{L} \right\}.$$

We recognize in properties (i), (ii) the defining property for a cross representation. In fact we have:

Corollary

Given $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ *and* $\mathcal{A}_{\mathcal{L}}$ *as above;-*

(i) $\mathcal{A}_{\mathcal{L}} = \mathcal{A}$ *iff* $\mathcal{C}_{\mathcal{L}} = \mathcal{C}$ *iff* $(\pi, U) \in \mathrm{Rep}^{\times}_{\mathcal{L}}(\alpha, \mathcal{H})$.

(ii) *If* $(\mathcal{C}, \eta_{\mathcal{A}}, \eta_{\mathcal{L}})$ *is constructed from the universal covariant* $\mathcal{L}$*-representation of* $(\mathcal{A}, G, \alpha)$*, then a full crossed product host exists if and only if* $\mathcal{A}_{\mathcal{L}} = \mathcal{A}$.

The universal covariant $\mathcal{L}$-representation $(\pi_u, U_u) \in \mathrm{Rep}_{\mathcal{L}}(\alpha, \mathcal{H}_u)$ is the direct sum of all cyclic covariant $\mathcal{L}$-representations.

For the special case of a locally compact group G acting by a discontinuous action $\alpha : G \to \mathrm{Aut}\mathcal{A}$, with chosen host $\mathcal{L} = C^*(G)$ we have the following:-

Proposition

Let G be locally compact, $\mathcal{L} = C^*(G)$ and $(\mathcal{C}, \eta_\mathcal{A}, \eta_\mathcal{L})$ and $\mathcal{A}_\mathcal{L}$ as above;-

$$\mathcal{A}_\mathcal{L} = \Big\{ A \in \mathcal{A} \,\Big|\, \lim_{g \to \mathbf{1}} \eta_\mathcal{A}(\alpha_g(B))\eta_\mathcal{L}(L) = \eta_\mathcal{A}(B)\eta_\mathcal{L}(L) \quad \text{for all } L \in \mathcal{L} \text{ and } B \in \{A,\, A^*\} \Big\},$$

Thus we can specify the cross property in terms of a continuity property of the action.

Theorem

Let G be locally compact, $\mathcal{L} = C^*(G)$, and let $(\mathcal{C}, \eta_\mathcal{A}, \eta_\mathcal{L})$ be constructed from the universal covariant $\mathcal{L}$-representation of $(\mathcal{A}, G, \alpha)$, then the following are equivalent:

(i) *A full crossed product host exists.*

(ii) $\lim_{g\to\mathbf{1}} \eta_\mathcal{A}(\alpha_g(A))\eta_\mathcal{L}(L) = \eta_\mathcal{A}(A)\eta_\mathcal{L}(L)$ *for* $A \in \mathcal{A}$ *and* $L \in \mathcal{L}$.

(iii) *The conjugation action of G on $\mathcal{C}$ is strongly continuous.*

This implies that the maps $G \to M(\mathcal{C}), g \to \eta_\mathcal{A}(\alpha_g(A))$ are continuous w.r.t. the strict topology of $\mathcal{C}$ for all $A \in \mathcal{A}$.
If $\mathcal{A}$ is unital, then there is a converse, i.e. (i)-(iii) are equivalent to

(iv) *For every $A \in \mathcal{A}$, the map $G \to M(\mathcal{C}), g \mapsto \eta_\mathcal{A}(\alpha_g(A))$ is strictly continuous.*

Construction of a crossed product in the conventional sense, requires strong continuity of the action, i.e. $\lim_{g\to\mathbf{1}} \alpha_g(A) = A$ for all $A \in \mathcal{A}$.

For a full crossed product host, we simply replace this by continuity condition (ii), or by (iv) for unital $\mathcal{A}$.

For more information, consult:-

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Thank you!