## Categorical tori and their representations

A report on work in progress

#### Nora Ganter

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Crossed modules and categorical groups following Noohi

(Strict) categorical groups are (strict) monoidal groupoids  $G_1$  $s \downarrow \downarrow t$  $G_0$ with invertible objects (w.r.t. •).

A crossed module  $(G, A, \psi)$  encodes a strict categorical group

$$\begin{array}{c}
G \ltimes A \\
\rho r_1 & \downarrow & \rho r_1 \cdot \psi \\
G
\end{array}$$

group multiplication gives • and  $(g\psi(b), a) \circ (g, b) = (g, ab).$ 

#### **Crossed modules**

consist of a group G, a right Gmodule A and a homomorphism  $\psi: A \longrightarrow G$  with  $\psi(a^g) = g^{-1}\psi(a)g$  $\psi(a) \cdot b = a^{-1}ba.$ 

The crossed module of the categorical group  ${\mathcal G}$  above is

$$G = G_0$$
  

$$A = ker(s)$$
  

$$a^g = g^{-1} \bullet a \bullet g$$
  

$$\psi = t.$$

Example: the crossed module of a categorical torus

Two ingredients: A lattice  $\Lambda^{\vee}$  and a bilinear form J on  $\Lambda^{\vee}$ . From this, we form the crossed module

$$\Lambda^{\!\!\vee} imes U(1) \xrightarrow{\psi} \mathfrak{t} := \Lambda^{\!\!\vee} \otimes_{\mathbb{Z}} \mathbb{R}$$
 $(m, z) \longmapsto m,$ 

where the action of  $x \in \mathfrak{t}$  on  $\Lambda^{\vee} \times U(1)$  is given by

$$(m,z)^{x} = (m,z \cdot \exp(J(m,x))).$$

#### Categorical tori

The categorical torus  $\mathcal{T}$  is the strict monoidal category with

objects: t,

arrows:  $x \xrightarrow{z} x + m, \qquad x \in \mathfrak{t}, m \in \Lambda^{\lor}, z \in U(1),$ 

composition: the obvious one,

multiplication: addition on objects and on arrows

 $(x \xrightarrow{Z} x + m) \bullet (y \xrightarrow{W} y + n) = (x + y \xrightarrow{zw \exp(J(m, y))} x + y + m + n).$ 

# Classification

Schommer-Pries, Wagemann-Wockel, Carey-Johnson-Murray-Stevenson-Wang

Up to equivalence, the categorical torus  ${\mathcal T}$  only depends on the even symmetric bilinear form

I(m,n) = J(m,n) + J(n,m).

More precisely,

 $-I \in Bil_{ev}(\Lambda^{\vee},\mathbb{Z})^{S_2} = H^4(BT;\mathbb{Z}) \cong H^3_{gp}(T;U(1))$ 

classifies the equivalence class of the extension

 $pt/\!\!/ U(1) \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}.$ 

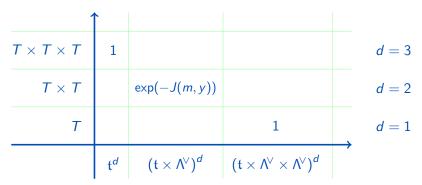
Examples:

- T<sub>max</sub> ⊂ G maximal torus of a simple and simply connected compact Lie group, Λ<sup>V</sup> coroot lattice, I<sub>bas</sub> basic bilinear form,
   (A = 1) an another Nieuroparties
- 2.  $(\Lambda_{Leech}, I)$  or another Niemeyer lattice.

Aussie-rules Lie group cohomology

 $H^3_{gp}(T; U(1)) = \check{H}^3(BT_{\bullet}; \underline{U(1)})$ 

and -1 corresponds to the Čech-simplicial 3-cocycle



where the non-trivial entry is short for

 $((x,m),(y,n)) \longmapsto \exp(-J(m,y)).$ 

Autoequivalences of the category of coherent sheaves

 $\widehat{T} = Hom(T, U(1))$ 

 $T_{\mathbb{C}} = \operatorname{spec} \mathbb{C}[\widehat{T}]$ 

 $\mathcal{C}ohT_{\mathbb{C}} \simeq \mathbb{C}[\widehat{T}] - \mathit{mod}^{\mathit{fin}}$ 

$$1Aut(CohT_{\mathbb{C}}) \simeq Bimod_{\mathbb{C}[\widehat{T}]}^{fin}$$
 [Deligne].

Inside  $1Aut(Coh(T_{\mathbb{C}}))$ , we have the full subcategory of direct image functors  $f_*$  of variety automorphisms f. This categorical group belongs to the crossed module

$$\mathbb{C}[\widehat{T}]^{\times} \xrightarrow{1} Aut_{var}(T_{\mathbb{C}}),$$

where f acts on  $\mathbb{C}[\widehat{T}]^{\times}$  by precomposition,  $\varphi \mapsto \varphi \circ f$ .

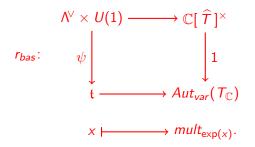
#### The basic representation of a categorical torus

The basic representation of  $\mathcal{T}$  is the strict monoidal functor

 $\varrho_{bas}: \mathcal{T} \longrightarrow 1Aut(\mathcal{C}oh(\mathcal{T}_{\mathbb{C}})).$ 

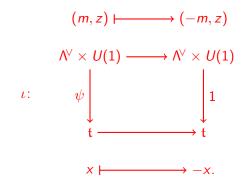
induced by the map of crossed modules

 $(m, z) \longmapsto z \cdot e^{2\pi i J(m, -)}$ 



### The involution $\iota$

The involution  $\iota$  of T, sending t to  $t^{-1}$  lifts to an involution of T, given by the map of crossed modules



This gives rise to an action of the group  $\{\pm 1\}$  by (strict monoidal) functors on the category  $\mathcal{T}$ .

### Extraspecial categorical 2-groups

The fixed points of  $\iota$  on T form the elementary abelian 2-group

 $T^{\{\pm 1\}} = T[2] \cong \Lambda^{\vee} / 2\Lambda^{\vee}.$ 

The categorical fixed points (or equivariant objects) of  $\iota$  on  ${\cal T}$  form an extension

$$pt/\!\!/ U(1) \longrightarrow \mathcal{T}^{\{\pm 1\}} \longrightarrow \widetilde{T[2]}$$

of the extraspecial 2-group  $\widetilde{T[2]}$  with Arf invariant

$$\phi(m) = \frac{1}{2}I(m,m) \mod 2\Lambda^{\vee}.$$

Example: In the example of the Leech lattice, T[2] is the subgroup of the Monster that is usually denoted  $2^{1+24}$ .

#### 1Automorphisms of the basic representation

Let  $\mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\}$  be the categorical group of the crossed module

$$\begin{array}{c} \Lambda^{\!\!\vee}\times\mathbb{C}^{\times} & \longrightarrow & \mathfrak{t}_{\mathbb{C}} \rtimes \{\pm 1\} \\ (m,z) & \longmapsto & (m,1), \end{array}$$

where -1 acts on everything by  $\iota$ .

Extend the basic representation to

 $\varrho_{bas}: \mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(Coh(T_{\mathbb{C}})),$ 

by setting  $r_{bas}(-1) := \iota$ . So,  $\varrho_{bas}(-1) := \iota_*$ .

Theorem: The 1automorphisms of this  $\rho_{bas}$  form the extraspecial categorical 2-group  $\mathcal{T}_{\mathbb{C}}^{\{\pm 1\}}$ .

### Normalizers

Let

$$\varrho: H \longrightarrow G = GL(V)$$

be a representation of a group H on some vector space. Then

$$Aut(\varrho) = C(\varrho) = \{g \in G \mid c_g \circ \varrho = \varrho\}$$

is the centralizer of (the image of)  $\varrho$  in G. Here  $c_g$  is conjugation by g.

Definition [Dror Farjoun, Segev]: The *injective normalizer* of  $\rho$  is the subgroup of  $Aut(H) \times G$  defined as

 $N(\varrho) = \{(f,g) \mid c_g \circ \varrho = \varrho \circ f\}.$ 

If  $\varrho$  is injective, this is the normalizer of its image.

# Towards the refined Monster? (In progress)

Theorem: The 1automorphisms of  $\mathcal{T}$  form an extension

$$pt/\!\!/\Lambda \longrightarrow 1Aut(\mathcal{T}) \longrightarrow O(\Lambda^{\!\vee}, I).$$

Here  $O(\Lambda^{\vee}, I)$  is the group of linear isometries of  $(\Lambda^{\vee}, I)$ .

Example: the Conway group

 $O(\Lambda_{Leech}^{\vee}, I) = Co_0.$ 

In spirit, the subgroup of the Monster, known as

$$2^{1+24}.Co_1 = \widetilde{T[2]} \rtimes (Co_0 / \{\pm id\})$$

wants to parametrize the isomorphism classes of some categorical variant of normalizer of  $\rho_{bas}$  :  $\mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(Coh(\mathcal{T}_{\mathbb{C}}))$ .