Reflections of Yetter-Drinfeld modules over Nichols systems

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Let $\mathcal{C}$ be a braided strict monoidal category and let $\Bbbk$ be a field.

Definition

Let $\theta \in \mathbb{N}$, Γ be an abelian group such that $\mathbb{Z}^\theta \subset \Gamma$ and let R be a $\mathbb{N}_0^\theta$-graded connected Hopf algebra in $\mathcal{C}$. Let $V \in {}^R_R\mathcal{YD}(\mathcal{C})$ be a Γ-graded object, where there exists $n_0 \in \Gamma$ such that $V(n_0)$ generates V as an R-module. We say V is **well graded**, *if*

$$V(n_0) = V^{\mathrm{co}\,R} = \{v \in V \mid \rho_V^R(v) = 1 \otimes v\}.$$

Remark

We always have $V(n_0) \subset V^{\mathrm{co}\,R}$, since the R-coaction ρ_V^R is graded.

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \ldots, \theta\}$. Let $\alpha_1, \ldots, \alpha_\theta \in \mathbb{N}_0^\theta$ be the unit vectors. Let H be a Hopf algebra over some field $\Bbbk$ with bijective antipode and $\mathcal{C} := {}^H_H\mathcal{YD}$.

Definition

Let $Q \in \mathcal{C}$ be a Hopf algebra and $N_1, \ldots, N_\theta \in \mathcal{C}$ be finite-dimensional subobjects of Q in $\mathcal{C}$ and $N := (N_1, \ldots, N_\theta)$. The tuple $\mathcal{N} := (Q, N)$ is called a **pre-Nichols system**, *if*

1. *The algebra Q is generated by $N_1, \ldots, N_\theta$.*
2. *Q is an $\mathbb{N}_0^\theta$-graded Hopf algebra in $\mathcal{C}$ with $Q(\alpha_i) = N_i$ for all $i \in \mathbb{I}$.*

$\mathcal{N}$ is called a **Nichols system over** *$i \in \mathbb{I}$, if also*

3. *$\Bbbk[N_i]$ is strictly graded.*
4. *$(\operatorname{ad} \Bbbk[N_i])(N_j) \in {}^{\Bbbk[N_i]}_{\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})$ is well graded for all $j \in \mathbb{I} \setminus \{i\}$.*

We want to construct a reflection functor on ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$.

Proposition

Let $A \in \mathcal{C}$ and $R \in {}^{A}_{A}\mathcal{YD}(\mathcal{C})$ be Hopf algebras in the specific categories. Then the functor

$$
\begin{gathered}
{}^{R}_{R}\mathcal{YD}({}^{A}_{A}\mathcal{YD}(\mathcal{C})) \to {}^{R\#A}_{R\#A}\mathcal{YD}(\mathcal{C}), \\
\left(\left(V, \nu^{A}, \rho^{A}\right), \nu^{R}, \rho^{R}\right) \mapsto \left(V, \nu^{R}(\mathrm{id} \otimes \nu^{A}), (\mathrm{id} \otimes \rho^{A})\rho^{R}\right)
\end{gathered}
$$

and where morphisms are mapped onto oneself, is a braided strict monoidal isomorphism[1].

[1] Heckenberger and Schneider, "Hopf algebras and root systems", 2020, Proposition 3.8.7.

Definition

Let Γ be a partial ordered monoid with partial order denoted by $\leq$. Let R be a Γ-graded Hopf algebra in $\mathcal{C}$ and let V be a left R-module. We call V a **rational R-module**, *if for all $v \in V$ there exists $\gamma_0 \in \Gamma$, such that $R(\gamma)v = 0$ for all $\gamma \in \Gamma$, $\gamma_0 \leq \gamma$. Moreover we denote ${}^R_R\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ for the category of $\mathcal{YD}$-modules which are rational R-modules.*

Let A, B be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in the category $\mathcal{C} = {}^H_H\mathcal{YD}$.
There is a braided monoidal isomorphic functor[2]

$$(\Omega, \omega) : {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}},$$

such that

1. If $V \in {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ is $\mathbb{Z}$-graded, then $\Omega(V)$ is $\mathbb{Z}$-graded with $\Omega(V)(n) = V(-n)$ for all $n \in \mathbb{N}_0$.
2. If $V, V' \in {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ are $\mathbb{Z}$-graded and $f : V \to V'$ is a $\mathbb{Z}$-graded morphism, then $\Omega(f) = f$ as linear maps.

[2] Heckenberger and Schneider, "Hopf algebras and root systems", 2020, Proposition 12.3.2.

Let $\mathcal{N} = (Q, (N_1, \dots, N_\theta))$ be a pre-Nichols system and $i \in \mathbb{I}$ and $K_i := Q^{\mathrm{co}\,\Bbbk[N_i]} \in {}^{\Bbbk[N_i]}_{\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})$. Let (Ω_i, ω_i) be the braided monoidal isomorphism

$$(\Omega_i, \omega_i) : {}^{\Bbbk[N_i]}_{\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{\Bbbk[N_i^*]}_{\Bbbk[N_i^*]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}.$$

Definition

Assume $K_i \in {}^{\Bbbk[N_i]}_{\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$. *Let* $R_i(Q) := \Omega_i(K_i)\#\Bbbk[N_i^*]$ *and let*

$$(R_i, \omega_{R_i}) : {}^{Q}_{Q}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{R_i(Q)}_{R_i(Q)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$$

be the braided monoidal isomorphism defined as follows:

$$\begin{aligned} &{}^{Q}_{Q}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \xrightarrow{\cong} {}^{K_i\#\Bbbk[N_i]}_{K_i\#\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \xrightarrow{\cong} {}^{K_i}_{K_i}\mathcal{YD}({}^{\Bbbk[N_i]}_{\Bbbk[N_i]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}) \\ &\xrightarrow{(\Omega_i,\omega_i)} {}^{\Omega_i(K_i)}_{\Omega_i(K_i)}\mathcal{YD}({}^{\Bbbk[N_i^*]}_{\Bbbk[N_i^*]}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}) \xrightarrow{\cong} {}^{R_i(Q)}_{R_i(Q)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}. \end{aligned}$$

For $V \in {}^{Q}_{Q}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, $R_i(V)$ *is called the* i-**th reflection of** V.

Let $\mathcal{N} = (Q, N)$ be a pre-Nichols system and assume that N_j is irreducible in $\mathcal{C}$ for all $j \in \mathbb{I}$.
Let $0 \neq V \in {}^{Q}_{Q}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ be $\mathbb{Z}^\theta$-graded, such that there exists $n_0 \in \mathbb{Z}^\theta$, such that V is generated as a Q-module by $V(n_0)$, and assume $V(n_0)$ is irreducible in $\mathcal{C}$.

Definition

Let $k \in \mathbb{N}_0$, $i_1, \ldots, i_k \in \mathbb{I}$.

1. *We say $\mathcal{N}$* **admits the reflection sequence** *$(i_1, \ldots, i_k)$, if $k = 0$ or $\mathcal{N}$ is a Nichols-system over i_1, K_{i_1} is rational and $R_{i_1}(\mathcal{N})$ admits the reflection sequence $(i_2, \ldots, i_k)$.*
2. *Assume $\mathcal{N}$ admits the reflection sequence $(i_1, \ldots, i_k)$. We say V* **admits the reflection sequence** *$(i_1, \ldots, i_k)$, if $k = 0$ or $\Bbbk[N_{i_1}] \cdot V(n_0) \in {}^{\Bbbk[N_{i_1}]}_{\Bbbk[N_{i_1}]}\mathcal{YD}(\mathcal{C})$ is well graded and $R_{i_1}(V)$ admits the reflection sequence $(i_2, \ldots, i_k)$.*

This definition makes sure that the preconditions at the top of this slide are met after each reflection.

Let M be the convex hull of $\mathrm{Sup}(V) = \{n \in \mathbb{Z}^\theta \mid V(n) \neq 0\}$.

- The **interior** V° **of** V consists of all components $V(n)$, $n \in \mathrm{Sup}(V)$, such that n lies in the interior of M.
- The **boundary** δV **of** V consists of all components $V(n)$, $n \in \mathrm{Sup}(V)$, such that n lies on the boundary of M.

Proposition

Assume $\mathcal{N}$ and V admit all reflections. There exists a unique graded subobject $U \subset V^{\circ}$ of V in ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$, such that every proper graded subobject of V in ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$ is contained in U. Moreover V/U is irreducible in ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$.

Sketch of the proof:

- Show that if a graded subobject of V contains any component $V(n)$ on the boundary of V, then it must contain the entire edge of $\mathrm{Sup}(V)$, where n lies on, in particular it contains a vertex.
- Show that each vertex of $\mathrm{Sup}(V)$ is the generating compontent of some reflection of V, hence the only graded subobject of V containing a vertex is V.

Define the functor

$$\mathcal{I}_Q : {}^Q\mathcal{C} \to {}^Q_Q\mathcal{YD}(\mathcal{C}),$$

where a Q-comodule $U \in \mathcal{C}$ gets mapped to

$$\mathcal{I}_Q(U) = Q \otimes U \in {}^Q_Q\mathcal{YD}(\mathcal{C})$$

with Q-action $\mu_Q \otimes \mathrm{id}_U$ and coadjoint Q-coaction and where a Q-comodule morphism f gets mapped to $\mathcal{I}_Q(f) = \mathrm{id}_Q \otimes f$.

Remark

This construction is similar to Verma modules in the representation theory of Lie algebras.

Proposition

Assume that Q is finite dimensional and that $\mathcal{N}$ admits all reflections. Let $U \in {}^{Q}\mathcal{C}$ be an irreducible object in $\mathcal{C}$. Then the following are equivalent:

1. $\mathcal{I}_Q(U)$ *admits all reflections.*
2. $\mathcal{I}_Q(U)$ *is irreducible in* ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$.

Sketch of the proof:

1. Let $N \in \mathrm{Sup}(\mathcal{I}_Q(U)) = \mathrm{Sup}(Q)$ be such that there is no $n \in \mathbb{Z}^\theta$, $n > N$, such that $n \in \mathrm{Sup}(Q)$.
2. Show that each subobject of $\mathcal{I}_Q(U)$ contains an element in the componenent $\mathcal{I}_Q(U)(N)$, i.e. on the boundary of $\mathcal{I}_Q(U)$.
3. Use the previous proposition.

Assume $\mathcal{N}$ is of diagonal type, that is N_j is one-dimensional for all $j \in \mathbb{I}$. Let $0 \neq x_j \in N_j$ be a basis. Define $D \in (\Bbbk^\times)^{\theta\times\theta}$ as follows: For $j \in \mathbb{I}$ we have

$$c^{\mathcal{C}}_{N_j,N_j}(x_j \otimes x_j) = D_{jj} x_j \otimes x_j.$$

For $1 \leq j < k \leq \theta$ we have $D_{kj} = 1$ and

$$c^{\mathcal{C}}_{N_k,N_j} c^{\mathcal{C}}_{N_j,N_k}(x_j \otimes x_k) = D_{jk} x_j \otimes x_k.$$

Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ be the bicharacter that is given by $\chi(\alpha_j, \alpha_k) = D_{jk}$ for all $j, k \in \mathbb{I}$ and let $\lambda : \mathbb{Z}^\theta \to \Bbbk^\times$ be the character that is given by $\lambda(\alpha_j) = D_{jj}$ for all $j \in \mathbb{I}$.

For $v = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}^\theta$, $n_1, \ldots, n_\theta \in \mathbb{Z}$ we define the monomial $t_v := \prod_{i=1}^{\theta} t_i^{n_i}$. Let $\Delta_+^{\mathcal{N}}$ denote the positive roots of $\mathcal{N}$ and for $\gamma \in \mathbb{Z}^\theta$ let $m_\gamma := \min\{m \in \mathbb{N}_0 \mid (m+1)_{\chi(\gamma,\gamma)} = 0\}$.

Definition

If $\Delta_+^{\mathcal{N}}$ is finite and $m_\gamma \in \mathbb{N}_0$ for all $\gamma \in \Delta_+^{\mathcal{N}}$, then define the polynomial

$$P^{\mathcal{N}} := \prod_{\gamma \in \Delta_+^{\mathcal{N}}} \prod_{m=1}^{m_\gamma} t_\gamma - \lambda(\gamma)\,\chi(\gamma,\gamma)^{-m} \in \Bbbk[t_1, \ldots, t_\theta].$$

Remark

This construction is the same as the Shapovalov determinant for bicharacters of finite root systems[3].

[3] Heckenberger and Yamane, "Drinfel'd doubles and Shapovalov determinants", 2010, Section 7.

Let $U \in {}^{Q}\mathcal{C}$ be a one-dimensional object. Let $r_1, \ldots, r_\theta \in \mathbb{k}^\times$ be such that for $u \in U$, $u \neq 0$, $j \in \mathbb{I}$ we have

$$c^{\mathcal{C}}_{U,N_j} c^{\mathcal{C}}_{N_j,U}(x_j \otimes u) = r_j x_j \otimes u.$$

Theorem

Assume $\mathcal{N}$ admits all reflections and Q is finite-dimensional. The following are equivalent

1. $\mathcal{I}_Q(U)$ *is irreducible in* ${}^{Q}_{Q}\mathcal{YD}(\mathcal{C})$.
2. $P^{\mathcal{N}}(r_1, \ldots, r_\theta) \neq 0$.

Remark

Even more holds: We can determine precisely which reflection sequences $\mathcal{I}_Q(U)$ admits by looking at which factor of $P^{\mathcal{N}}$ vanishes for $(r_1, \ldots, r_\theta)$.