Differential Graded Modular Functors

Homotopy Coherent Mapping Class Group Actions and Excision for Hochschild Complexes of Modular Categories

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arXiv:2004.14343 [math.QA]

Workshop on 'Hopf Algebras and Tensor Categories' at the University of Hamburg
20th August 2020

Definition of a (differential graded) modular functor

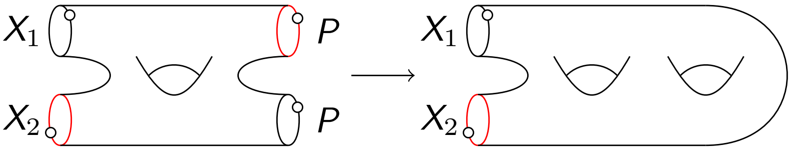
Let k be an algebraically closed field. For a k-linear category $\mathcal{C}$, we denote by $\mathcal{C}$-Surf^c the category

- whose objects are extended surfaces, i.e. compact oriented two-dimensional manifolds with boundary equipped with
  - a fixed point on every boundary component
  - and an orientation of the boundary (leading to *incoming* and *outgoing* boundary components)

  with a choice of a projective object of $\mathcal{C}$ for every boundary component
- and whose morphisms are generated by mapping classes (twisted by a cocycle coming from the framing anomaly) and sewings compatible with labels.

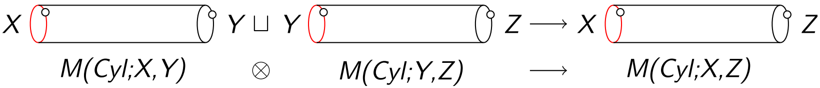
Disjoint union provides a symmetric monoidal structure.

Example of a sewing (incoming boundary in red):



Definition of a differential graded modular functor

For every symmetric monoidal functor $M : \mathcal{C}\text{-Surf}^c \longrightarrow \mathsf{Ch}_k$ the evaluation on cylinders provides a (non-unital) differential graded category, the *cylinder category* $\mathcal{Z}$, whose objects are the ones of $\operatorname{Proj}\mathcal{C}$:



Definition of a (differential graded) modular functor

If $s_P : (\Sigma, \underline{X}, P, P) \longrightarrow (\Sigma', \underline{X}')$ is a sewing that glues an incoming to an outgoing boundary, then evaluation of M on s provides a map

$$\int_{\mathbb{L}}^{P \in \mathcal{Z}} M(\Sigma; \underline{X}, P, P) \longrightarrow M(\Sigma'; \underline{X}') \ .$$

We say that M satisfies *excision* if this map in an equivalence.

Definition

A *differential graded modular functor* $M : \mathcal{C}\text{-}\mathsf{Surf}^c \longrightarrow \mathsf{Ch}_k$ for a linear category $\mathcal{C}$ is a symmetric monoidal functor whose cylinder category is equivalent to $\mathrm{Proj}\,\mathcal{C}$ and which satisfies excision.

Reminder on modular categories

Let $\mathcal{C}$ be a finite braided tensor category.

- A *ribbon structure* on $\mathcal{C}$ is a natural automorphism θ of $\mathrm{id}_{\mathcal{C}}$ which satisfies

$$\begin{aligned} \theta_{X \otimes Y} &= c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \ , \\ \theta_I &= \mathrm{id}_I \ , \\ \theta_{X^\vee} &= \theta_X^\vee \end{aligned}$$

  for all $X, Y \in \mathcal{C}$.
- We call the braiding of $\mathcal{C}$ *non-degenerate* if the only objects whose double braiding with all other objects is the identity are finite direct sums of the unit (see Shimizu and Brochier-Jordan-Safronov-Snyder for equivalent characterizations).
- A *modular category* is a finite ribbon category with non-degenerate braiding.
  (Semisimple version and non-semisimple version!)

Differential graded modular functor of a modular category

Semisimple modular categories are essentially equivalent to 3-2-1-dimensional topological field theories [Bartlett-Douglas-Schommer-Pries-Vicary 15].
This relies on $\mathcal{C}$ being semisimple! What can be saved in the non-semisimple case? A lot! Thanks to a construction due to [Lyubashenko 90s] one can build a modular functor.

Upshot

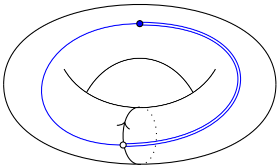
In the non-semisimple case, the modular functor is the shadow of a *differential graded* modular functor.

Theorem [Schweigert-W. 20]

Any modular category $\mathcal{C}$ gives rise in a canonical way to a differential graded modular functor $\mathfrak{F}_\mathcal{C} : \mathcal{C}\text{-}Surf^\mathsf{c} \longrightarrow Ch_k$.

The functor $\mathfrak{F}_{\mathcal{C}}$ is explicitly computable by choosing a marking (roughly, a cut system and a graph on the surface).

Example:



After choosing this marking there is a canonical equivalence

$$\int_{\mathbb{L}}^{X\in \mathrm{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \xrightarrow{\simeq} \mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2)\ ,$$

where the homotopy coend on the left hand side is actually the Hochschild complex

$$\dots \quad \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0,X_1\in \mathrm{Proj}\,\mathcal{C}} \mathcal{C}(X_1,X_0)\otimes \mathcal{C}(X_0,X_1) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0\in \mathrm{Proj}\,\mathcal{C}} \mathcal{C}(X_0,X_0)\ .$$

Differential graded modular functor of a modular category

As one proves via excision, for a closed surface Σ of genus $g+1$, $\mathfrak{F}_{\mathcal{C}}(\Sigma)$ is equivalent to the Hochschild chains of the $\mathrm{Proj}\,\mathcal{C}$-bimodule $M_g : (\mathrm{Proj}\,\mathcal{C})^{\mathrm{opp}} \otimes \mathrm{Proj}\,\mathcal{C} \longrightarrow \mathrm{Ch}_k$ defined by $M_g(X,Y) := \mathcal{C}(X, Y \otimes \mathbb{F}^{\otimes g})$, where $\mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^\vee$ is the canonical coend of $\mathcal{C}$.

Corollary [Schweigert-W. 20]

For any modular category $\mathcal{C}$, the Hochschild complex $CH(\mathcal{C}; M_g)$ comes with a canonical projective homotopy coherent action of the mapping class group of the closed surface of genus $g+1$.

In particular, the ordinary (categorical) Hochschild complex $\int_{\mathbb{L}}^{X \in \mathrm{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ carries a homotopy coherent projective $\mathrm{SL}(2,\mathbb{Z})$-action.

Differential graded modular functor of a modular category

Remark/Example

If $\mathcal{C}$ is given by finite-dimensional modules over a ribbon factorizable Hopf algebra A, $CH(\mathcal{C}; M_g)$ is equivalent to $CH\left(A; A \otimes (A^*_{\mathrm{coadj}})^{\otimes g}\right)$. For $A = D(G)$ for a finite group G, this complex is equivalent to $N_*(\mathrm{PBun}_G(\Sigma_{g+1}); k)$.

Remark

After the choice of a specific (!) marking,

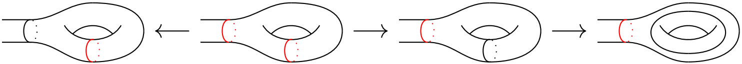
$$H_* \mathfrak{F}(\Sigma_g) \cong \left(\mathrm{Ext}^*(I, \mathbb{F}^{\otimes g})\right)^* \quad , \qquad (1)$$

and this isomorphism is compatible with the action established on the right hand side by Lentner-Mierach-Sommerhäuser-Schweigert. In particular, the Lyubashenko construction is recovered in zeroth homology.

Careful: (1) does not make sense for all markings and therefore is not compatible with sewing.

Proof, part I: The homotopy coherent Lego-Teichmüller game (anomaly-free case)

For a surface Σ, we define a *category* $\widehat{\mathrm{M}}(\Sigma)$ *of colored markings*. Objects are markings on Σ with a distinguished subset of colored cuts such that on each component of Σ, we have
$\#\text{colored cuts} + \#\text{boundary components} \geq 1$.
Morphisms (on the level of cuts) are generated by the F-move (cut removal), S-move (change to transversal cut) and the non-invertible uncolorings subjects to specific relations (morphisms for the graph part of the marking are the 'usual' ones).



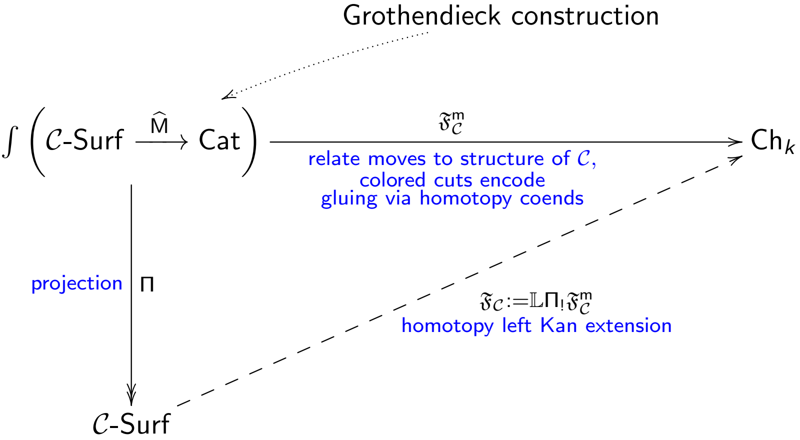
Proof, part I: The homotopy coherent Lego-Teichmüller game (anomaly-free case)

Theorem [Schweigert-W.]

For any surface Σ, the nerve of of the category $\widehat{\mathsf{M}}(\Sigma)$ of colored markings on Σ is a contractible space.

This builds on work of Grothendieck, Harer, Hatcher-Thurston, Moore-Seiberg, Bakalov-Kirillov.

Proof, part II: Homotopy left Kan extension (anomaly-free case)



Contractibility of $\widehat{\mathsf{M}}(\Sigma)$ for every surface Σ ensures computability of $\mathfrak{F}_{\mathcal{C}}$!

More results and directions

- Compute modular homologies $\mathfrak{F}_{\mathcal{C}}(\Sigma)//\mathrm{Map}(\Sigma)$ leading to a reasonable algebraic invariant of $\mathcal{C}$ (using fat graph computations with Müller and Wahl?).
- Investigate the E_2-structure on the differential graded Verlinde algebra, compute the Gerstenhaber bracket (with Schweigert).
- Computations for Drinfeld centers through string-net techniques (with Schweigert and Yang).
- Approaches using cyclic and modular operads (with Müller).
- Connection to factorization homology?