Modular tensor categories and topological field theories

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Overview

1. Tensor categories and module categories
   - Some definitions
   - Categorical Morita equivalence
   - Radford's S^4 theorem and Nakayama functors
   - Relative Serre functors, pivotal module categories and Frobenius algebras
2. Topological field theories
   - Definition of topological field theories
   - From TFT to modular categories
   - Equivariant Frobenius-Schur indicators
   - Some lessons

Chapter 1

Tensor categories and module categories

Setting: finite tensor categories

Definition (Finite category)

Let k be a field. A k-linear category $\mathcal{C}$ is finite, if

1. $\mathcal{C}$ has finite-dimensional spaces of morphisms.
2. Every object of $\mathcal{C}$ has finite length.
3. $\mathcal{C}$ has enough projectives.
4. There are finitely many isomorphism classes of simple objects.

Remark

A linear category is finite, if and only if it is equivalent to the category A-mod of finite-dimensional A-modules over a finite-dimensional k-algebra.

Definition (Finite tensor category)

A finite tensor category is a finite rigid monoidal linear category.

In particular, the tensor product is exact in each argument. Examples: Finite-dimensional k-linear representations of a finite-dimensional k-Hopf algebra.

Drinfeld center

Braided categories naturally enter in the study of finite tensor categories:

Definition (Half-braiding, Drinfeld center)

Let $\mathcal{A}$ be a monoidal category.
A half-braiding for $V \in \mathcal{A}$ is a natural isomorphism

$$\sigma_V : V \otimes - \to - \otimes V$$

such that $\sigma_V(X \otimes Y) = (\mathrm{id}_X \otimes \sigma_V(Y)) \circ (\sigma_V(X) \otimes \mathrm{id}_Y)$ for all $X, Y \in \mathcal{C}$.
The Drinfeld center $\mathcal{Z}(\mathcal{A})$ has pairs (V, σ_V) as objects.

Remarks

1. $\mathcal{Z}(\mathcal{A})$ is a braided monoidal category.
2. The forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is exact.
   Left adjoint $L : c \mapsto \int^{x \in \mathcal{C}} x \otimes c \otimes {}^{\vee}x$
   Right adjoint $R : c \mapsto \int_{x \in \mathcal{C}} {}^{\vee}x \otimes c \otimes c$
3. $\mathcal{C}$ unimodular $\Leftrightarrow L \cong R$
   $\Leftrightarrow R(1) \in \mathcal{Z}(\mathcal{A})$ is a (commutative) Frobenius algebra (Shimizu 2017)

Module categories

Definition (Module categories)

Let $\mathcal{A}$ and $\mathcal{B}$ be linear monoidal categories.

1. A left $\mathcal{A}$-module category is a linear category $\mathcal{M}$ with a bilinear functor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{M}}) \xrightarrow{\sim} \otimes \circ (\mathrm{id}_{\mathcal{A}} \times \otimes) \qquad \lambda : \otimes \circ (\mathrm{id}_{\mathcal{A}} \times -) \xrightarrow{\sim} \mathrm{id}_{\mathcal{M}}$$

   satisfying obvious pentagon and triangle axioms. We write $a.m := a \otimes m$.
2. Right module categories are defined analogously.
3. An $\mathcal{A}$-$\mathcal{B}$ bimodule category is a linear category $\mathcal{D}$, with the structure of a left $\mathcal{A}$ and right $\mathcal{D}$-module category and a natural associator isomorphism $(a.d).b \cong c.(d.b)$.
4. Module functors, module natural transformations defined in obvious way.

Definition (Finite module categories)

Let $\mathcal{A}$ be a finite tensor category over k. A left $\mathcal{A}$-module category is finite, if the underlying category is a finite abelian category over k and the action is k-linear in each variable and right exact in the first variable.

Categorical Morita theory

Theorem (Schauenburg)

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ an indecomposable exact $\mathcal{C}$-module category. Then there is a braided monoidal equivalence $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}))$.

Definition

Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are categorically Morita equivalent, if there exists an exact $\mathcal{C}$-module category $\mathcal{M}$ and a tensor equivalence $\mathcal{D} \cong \mathcal{R}ex_{\mathcal{C}}(\mathcal{M})$.

Remarks

1. The bicategories of module categories of Morita equivalent finite tensor categories are equivalent.
2. $r : H \to A$ projection to a Hopf subalgebra, $r(H) \to B$ projection to a Hopf subalgebra B dual to A, then H-mod and $r(H)$-mod have equivalent categories of Yetter Drinfeld modules. (Barvels, Lentner, CS, 2015) and are even categorically Morita equivalent.

Classical objects like Hopf algebras deserve to be studied by their bicategory of module categories.

Equivariant generalizations of categorical Morita theory

Remarks

- For $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ a G-graded finite tensor category, denote by $\mathcal{Z}_G(\mathcal{C})$ the relative center with respect to the neutral component $\mathcal{C}_e$.
- $\mathcal{Z}_G(\mathcal{C})$ is a braided G-crossed tensor category.
- Its equivariantization (=orbifold) is the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Theorem (Jaklitsch, 2020)

Let $\mathcal{C}$ be a G-graded finite tensor category and $\mathcal{M}$ be an exact indecomposable G-graded $\mathcal{C}$-module category. Then there is an equivalence of braided G-crossed tensor categories $\mathcal{Z}_G(\mathcal{C}) \cong \mathcal{Z}_G(\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}))$.

Towards Radford's S^4 theorem: Eilenberg-Watts calculus

Classical result about finite categories:

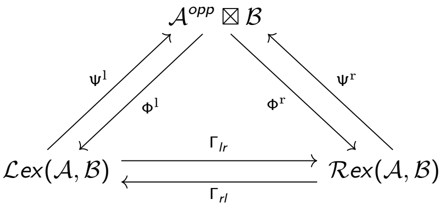
Proposition

Let A-mod and B-mod be finite tensor categories. Let

$$G : A\text{-mod} \to B\text{-mod}$$

be a right exact functor. Then $G \cong G({}_AA_A) \otimes_A -$.
The B-A-bimodule $G({}_AA_A)$ is a right A-module via the image of right multiplication $r_A : A \to A$ under $\mathrm{End}_A(A) \xrightarrow{G} \mathrm{End}_B(G(A))$.
A similar statement allows to express left exact functors in terms of bimodules.

Morita-invariant formulation: triangle of explicit adjoint equivalences, based on the Deligne product and (co)ends.



Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

For $\mathcal{A} = A\text{-mod}$:

$$N^r_{\mathcal{A}} = A^* \otimes_A - \cong \mathrm{Hom}_A(-,A)^* \quad \text{and} \quad N^l_{\mathcal{A}} = \mathrm{Hom}_A(A^*,-)\ .$$

For this reason, we call $N^r_{\mathcal{A}}$ and $N^l_{\mathcal{A}}$ Nakayama functors.

Proposition

1. *The Nakayama functors are adjoints,* $N^l_{\mathcal{A}} \dashv N^r_{\mathcal{A}}$.
2. $N^l_{\mathcal{A}}$ *equivalence* $\Leftrightarrow$ $N^r_{\mathcal{A}}$ *equivalence.* $\Leftrightarrow$ $\mathcal{A}$ *is selfinjective.*
3. $N^l_{\mathcal{A}} \cong \mathrm{id}_{\mathcal{A}}$ *and* $N^r_{\mathcal{A}} \cong \mathrm{id}_{\mathcal{A}}$ $\Leftrightarrow$ $\mathcal{A}$ *is symmetric Frobenius.*

Radford's S^4-theorem

For linear functors, we have

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite categories. Let $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$ such that F^{la} is left exact so that F^{lla} exists. Assume that F^{lla} is left exact as well.
Then there is a natural isomorphism

$$\varphi^l_F : \quad N^l_{\mathcal{B}} \circ F \cong F^{lla} \circ N^l_{\mathcal{A}}$$

that is coherent with respect to composition of functors.

Apply this to bimodule categories over finite tensor categories:

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite tensor categories and $\mathcal{M}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule.
Then the Nakayama functor has the structure of a twisted bimodule functor:

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Recovering Radford's S^4-theorem

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Observe

- The finite tensor category $\mathcal{A}$ is a bimodule over itself.
- $$N^l_{\mathcal{A}}(1) = \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,1)\otimes a = D_{\mathcal{A}}$$
  is the distinguished invertible object of $\mathcal{A}$.
- Compute
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(a\otimes 1) = a^{\vee\vee}\otimes N^l_{\mathcal{A}}(1) = a^{\vee\vee}\otimes D_{\mathcal{A}}$$
  and
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(1\otimes a) = N^l_{\mathcal{A}}(1)\otimes {}^{\vee\vee}a = D_{\mathcal{A}}\otimes {}^{\vee\vee}a$$
- We recover Radford's S^4-theorem in its categorical form $D_{\mathcal{A}}\otimes a\otimes D_{\mathcal{A}}^{-1} = a^{\vee\vee\vee\vee}$ [ENO, 2004]

Hence, the Radford's classical theorem can be seen as a statement about Nakayama functors as twisted module functors:

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Relative Serre functors

Definition (Fuchs, Schaumann, CS)

Let $\mathcal{M}$ be a $\mathcal{C}$-module. A right/left relative Serre functor is an endofunctor $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ / $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ of $\mathcal{M}$ together with a family

$$\begin{aligned} \underline{\mathrm{Hom}}(m,n)^{\vee} &\xrightarrow{\cong} \underline{\mathrm{Hom}}(n,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \\ {}^{\vee}\underline{\mathrm{Hom}}(m,n) &\xrightarrow{\cong} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(n),m) \end{aligned}$$

of isomorphisms natural in $m,n \in \mathcal{M}$.

- Relative Serre functors exist, iff $\mathcal{M}$ is an exact module category (i.e. $p.m$ is projective, if $p \in \mathcal{C}$ is projective).
- Serre functors are equivalences of categories.
- Serre functors are twisted module functors:

$$\phi_{c,m} : \ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \longrightarrow c^{\vee\vee}.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{c,m} : \ \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \longrightarrow {}^{\vee\vee}c.\,\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m)$$

Theorem

Let $\mathcal{M}$ be an exact $\mathcal{A}$-module. Then

$$N^{l}_{\mathcal{M}} \cong D_{\mathcal{A}}.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}} \quad \textit{and} \quad N^{r}_{\mathcal{M}} \cong D_{\mathcal{A}}^{-1}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

Pivotal module categories

Serre functors are twisted module functors:

$$\phi_{c,m} : \quad \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \longrightarrow c^{\vee\vee}.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{c,m} : \quad \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \longrightarrow {}^{\vee\vee}c.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)\,.$$

Definition (Schaumann 2015, Shimizu 2019)

A pivotal structure on an exact module category $\mathcal{M}$ over a pivotal finite tensor category $(\mathcal{C}, \pi)$ is an isomorphism of functors $\tilde{\pi} : \mathrm{id}_{\mathcal{M}} \rightarrow \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$
such that the following diagram commutes for all $c \in \mathcal{C}$ and $m \in \mathcal{M}$:

$$\begin{array}{ccc} c.m & \xrightarrow{\pi_c.\tilde{\pi}_m} & c^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \\ & {\scriptstyle \tilde{\pi}_{c.m}} \searrow \quad \nearrow {\scriptstyle \phi_{c,m}} & \\ & \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) & \end{array}$$

- For indecomposable exact module categories, the pivotal structure is unique up to scalar.
- The algebras $\underline{\mathrm{Hom}}(m, m) \in \mathcal{C}$ for m in a pivotal module category have the structure of symmetric Frobenius algebras.

Symmetric Frobenius algebras in the Drinfeld center

For CFT, we need symmetric Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.
Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-modules.
The functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a module category over $\mathcal{Z}(\mathcal{C})$:

$$(z.F)(m) := z.F(m)$$

with module functor structure given by half braiding:

$$(z.F)(c.m) = z.F(c.m) \cong (z \otimes c).F(m) \cong (c \otimes z).F(m) \cong c.(z.F)(m)$$

Theorem (Fuchs, CS 2020)

$\mathcal{C}$ a pivotal finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ exact $\mathcal{C}$-modules.

1. *The functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is an exact module category over $\mathcal{Z}(\mathcal{C})$ with relative Serre functor $N^r_{\mathcal{N}} \circ (D.-) \circ N^r_{\mathcal{M}}$.*
2. *If $\mathcal{C}$ is unimodular pivotal and $\mathcal{M}$ and $\mathcal{N}$ are pivotal $\mathcal{C}$-modules, then $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a pivotal $\mathcal{Z}(\mathcal{C})$-module category.*
3. *In particular, then $\underline{\mathrm{Nat}}(F,F)$ is a symmetric Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and $\underline{\mathrm{Nat}}(\mathrm{id}_{\mathcal{M}},\mathrm{id}_{\mathcal{M}})$ has a natural structure of a commutative symmetric Frobenius algebra.*

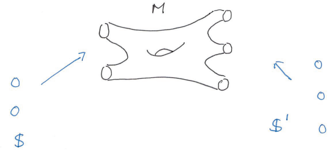
Chapter 2

Topological field theory

Warmup: Topological field theories in two dimensions

Definition (Cobordism category)

- Objects: closed oriented 1-manifolds.
- Morphisms: spans $\mathbb{S} \to M \leftarrow \mathbb{S}'$ with M oriented 2-manifold with boundary $\partial M \cong \overline{\mathbb{S}} \sqcup \mathbb{S}'$, up to diffeomorphism relative boundary.
- Monoidal product is disjoint union.



Definition (Two-dimensional topological field theory)

An (oriented) topological field theory is a symmetric monoidal functor $\mathrm{tft} : cob^{or}_{2,1} \to \mathrm{vect}$. (It is a *representation* of $cob^{or}_{2,1}$.)

Remark

Topological fact: $cob^{or}_{2,1}$ is the free symmetric monoidal category on a commutative Frobenius object.
Corollary: $\mathrm{tft}(\mathbb{S}^1)$ is a commutative Frobenius algebra. Any such Frobenius algebra gives a TFT.

Extended three-dimensional topological field theories

Knots and links are part of the picture:

Definition (2-vector spaces)

Denote by 2-vect the symmetric monoidal bicategory

- Objects: finitely semisimple k-linear abelian categories.
- 1-Morphisms: k-linear functors, 2-morphisms: k-linear natural transformations.
- The monoidal product is given by the Deligne product.

Definition (Cobordism bicategory)

- Objects: closed oriented 1-manifolds.
- 1-Morphisms: spans $\mathbb{S} \to M \leftarrow \mathbb{S}'$ with M oriented 2-manifold with boundary $\partial M \cong \overline{\mathbb{S}} \sqcup \mathbb{S}'$.
- 2-Morphisms: 3-manifolds with corners up to diffeomorphisms
- The monoidal product is given by disjoint union.

Definition (Extended topological field theory)

A 3-2-1 extended oriented topological field theory is a symmetric monoidal 2-functor $\mathrm{tft} : cob^{or}_{3,2,1} \to 2\text{-vect}$.

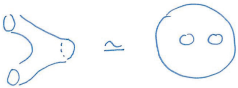
Evaluation of a 3-2-1 TFT

Definition (Extended topological field theory)

A 3-2-1 extended oriented topological field theory is a symmetric monoidal 2-functor $\mathrm{tft} : cob^{or}_{3,2,1} \to \text{2-vect}$.

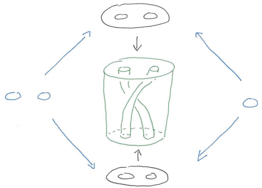
Examples:

1-morphism



gives tensor product
$\otimes : \mathrm{tft}(\mathbb{S}^1) \times \mathrm{tft}(\mathbb{S}^1) \to \mathrm{tft}(\mathbb{S}^1)$

2-morphism



gives braiding
$\otimes \Rightarrow \otimes^{opp}$

Modular tensor categories

Definition (Modular tensor category)

A modular tensor category $\mathcal{C}$ is a finite ribbon category such that the braiding is maximally non-degenerate. Various formulations exist and are equivalent [Sh]:

- Braided equivalence $\mathcal{C} \boxtimes \mathcal{C}^{rev} \cong \mathcal{Z}(\mathcal{C})$
- Coend $L := \int^{\mathcal{C}} U^{\vee} \otimes U$ has non-degenerate Hopf pairing $\omega_{\mathcal{C}}$
- Map $\mathrm{Hom}(1, L) \to \mathrm{Hom}(L, 1)$ induced by $\omega_{\mathcal{C}}$ is isomorphism.
- $\mathcal{C}$ has no transparent objects.

Remarks

- The Drinfeld center of a finite tensor category is a modular tensor category.
- The representation category of suitable vertex algebras or nets of observable algebras has naturally the structure of a modular tensor category:
  The chiral data of a (finite) conformal field theory are described by a modular tensor category.

Modular tensor categories

Remark

Topological fact [BDS-PV]:
$cob^{or}_{3,2,1}$ is free symmetric monoidal bicategory on a single anomaly-free modular object.
Corollary:
$\mathrm{tft}(\mathbb{S}^1)$ is a semisimple modular tensor category. Any such tensor category gives an extended TFT.

Remarks

- From a modular tensor category, one can construct a modular functor (Lyubashenko, $\sim$ 1995)
- Lyubashenko's modular functor is H^0 of a modular functor with values in chain complexes of vector spaces with homotopy coherent mapping class group action. (Woike, CS 2020, Lentner, Mierach, Sommerhäuser, CS 2020)
- This can be seen as a (substantial) generalization of the $\mathrm{SL}(2,\mathbb{Z})$ action on the center of a factorizable ribbon Hopf algebra (Sommerhäuser, Zhu,...)

Overview over constructions of 3d extended TFTs

	RT	TV
Method	Surgery	State sum / string nets
Input	Modular fusion category $\mathcal{C}$	Spherical fusion category $\mathcal{A}$
Special cases	Chern-Simons (including abelian Chern-Simons) (finite abelian group with quadratic form)	Kitaev's toric code Dijkgraaf-Witten theory finite group with 3-cocycle
$\mathrm{tft}(\mathbb{S}^1)$	$\mathcal{C}$	$\mathcal{Z}(\mathcal{A})$
Boundaries	only if $\mathcal{C}$ Witt-trivial	always
$\mathcal{C}_1$-$\mathcal{C}_2$-defects	if $\mathcal{C}_1, \mathcal{C}_2$ in same Witt class	always

State-sum TFT with boundaries

Remarks

- Framed modular functor from state sums for finite tensor categories and their bimodules (Fuchs, Schaumann, CS 2019)
- Boundary conditions: module categories, in particular $\mathcal{A}$ as a module category over itself.
- Different spherical fusion categories give same TV theory, e.g. G-rep and G-vect. Reflections of pivotal Hopf algebras are field theoretic dualities.

Flavours of TFTs

There are various flavours of TFTs, defined on different categories of cobordisms

Cobordism	Framed	Oriented
Input	fusion category	spherical fusion category
Boundary	exact module category	pivotal module category

Equivariant topological field theories:

$$G\text{-cob}^{3,2,1} \to 2\text{-vect}$$

	RT	TV
Input	G-graded spherical fusion category	G-modular tensor category

Theorem (Woike, CS)

For any group homomorphisms $G \to H$, there is a functor ("geometric orbifold construction") G-tft $\to$ H-tft that specializes to the orbifold construction (equivariantization) for $H = 1$.

Equivariant Frobenius Schur indicators and boundaries

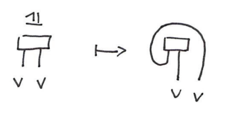
Recap

V a finite-dimensional irreducible $\mathbb{C}[G]$-module.

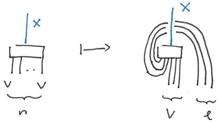
$$\nu_2(V) := \frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) \in \{0, \pm 1\}$$

$\nu = \pm 1 \Leftrightarrow$ non-deg. invariant bilinear form on V symmetric or antisymmetric.

$\nu_2(V)$ is the trace of the endomorphism on the one-dimensional vector space $\mathrm{Hom}(V \otimes V, 1)$:

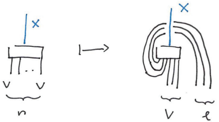


Generalization for pivotal categories: $V \in \mathcal{C}$ and $X \in \mathcal{Z}(\mathcal{C})$: [Kashina, Sommerhäuser, Zhu; Ng, Schauenburg]



Generalized Frobenius Schur indicator: $\nu_{V,X,(n,l)} := \mathrm{tr}\xi_{V,X,(n,l)}$.
Equivariance under $\mathrm{SL}(2,\mathbb{Z})$.

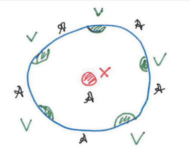
Application to the equivariant Frobenius-Schur indicators

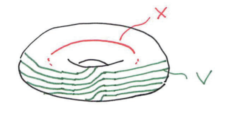


Generalized Frobenius Schur indicator:
$\nu_{V,X,(n,l)} := \mathrm{tr}\xi_{V,X,(n,l)}$.
Equivariance under $\mathrm{SL}(2,\mathbb{Z})$.

- Congruence subgroup conjecture for Drinfeld doubles of fusion categories
- FS indicators for big finite groups ($\sim 2 \cdot 10^{18}$ elements)

Theorem (Farnsteiner, 2020)





$\rightsquigarrow \mathrm{Hom}(V^{\otimes n}, X)$ *Solid torus with Wilson line* $\rightsquigarrow \nu_{V,X,(n,l)}$
$\mathrm{SL}(2,\mathbb{Z})$*-equivariance becomes geometric and follows from TFT axioms.*

Some lessons

Other representation-theoretic results can be understood in terms of TFT as well:

- TFT gives a homotopy action of $\mathrm{O}(3)$ on fusion categories. $\pi_1(\mathrm{o}(3))$ acts by autoequivalence $-^{\vee\vee}$ (Douglas, Schommer-Pries-Snyder)
- Relation between Brauer-Picard group of $\mathcal{A}$ and braided autoequivalences of $\mathcal{Z}(\mathcal{A})$ (Etingof, Nikshych, Ostrik, Meir)

Some lessons:

- Finite tensor categories should be studied at the same time as their bimodule categories and their Drinfeld centers.
- Various TFT constructions and the theory of finite tensor categories are deeply interwoven.
- 3d TFT (including boundaries and defects) is a natural organizing principle for mathematical structures related to finite tensor categories.