On the classification of pre-Nichols algebras of diagonal type with finite Gelfand-Kirillov dimension

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Joint with Andruskiewitsch (arXiv:2002.11087) and Angiono-Campagnolo.

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Contents

Motivation

Hopf algebras: coradical and infinitesimal braiding

Fix field $\Bbbk = \overline{\Bbbk}$ with $\operatorname{char} \Bbbk = 0$. Let A be a Hopf algebra.

Definitions

- The coradical of A is $A_0 :=$ (direct) sum of simple subcoalgebras.
- Extends to coradical filtration $A = \bigcup_{n \geq 0} A_n$ — as coalgebra.

Hypothesis: the coradical is a Hopf subalgebra.

$\implies A = \bigcup_{n\geq 0} A_n$ is a Hopf algebra filtration.

$\rightsquigarrow$ Get associated graded Hopf algebra $\operatorname{gr} A$ endowed with $A_0 \hookrightarrow \operatorname{gr} A$ and $\operatorname{gr} A \twoheadrightarrow A_0$ composing to id.

$\rightsquigarrow$ Recover $\operatorname{gr} A \simeq R \# A_0$ with R graded Hopf algebra in ${}^{A_0}_{A_0}\mathcal{YD}$.

Definitions

$R = \bigoplus_{n \geq 0} R^n$ is the diagram and R^1 is the infinitesimal braiding of A.

Next: what kind of objects are these diagrams?

Generation in degree 1

- Fix a Hopf algebra H (with bijective antipode) and $V \in {}^H_H\mathcal{YD}$.

Definitions

A post-Nichols algebra of V is a corradically graded and connected Hopf algebra $\mathcal{E} = \bigoplus_{n\geq 0} \mathcal{E}^n$ in ${}^H_H\mathcal{YD}$ endowed with an iso $\mathcal{E}^1 \simeq V$.

A pre-Nichols algebra of V is a graded and connected Hopf algebra $\mathcal{B} = \bigoplus_{n\geq 0} \mathcal{B}^n$ in ${}^H_H\mathcal{YD}$ generated by $\mathcal{B}^1$ and endowed with $\mathcal{B}^1 \simeq V$.

- Restrict now to a group algebra $H = \Bbbk\Gamma$.

Generation in degree 1 conjecture, Andruskiewitsch-Schneider '00

Let $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$. Then the Nichols algebra itself is the unique finite dimensional post-Nichols algebra of V.

- Assume that Γ is abelian. $\rightsquigarrow$ Conjecture holds (Angiono).

Considering Gelfand-Kirillov dimension instead, the conjecture fails.

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Introducing the problem

Goal: classify finite **GK-dim** Hopf algebras with coradical $\Bbbk\Gamma$.

Following [AAH18] [1],

(A) classify (fin. dim.) $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $\operatorname{GK-dim} \mathcal{B}(V) < \infty$, then

(B) for such V, classify its finite **GK-dim** post-Nichols algebras,

which is related to

(C) classify finite **GK-dim** pre-Nichols algebras of V.

We focus on problem (C) for some families of diagonal braidings:

- quantum linear spaces,
- super type,
- Cartan type,
- standard type.

[1] [AAH18] N. Andruskiewitsch, I. Angiono, I. Heckenberger, *Liftings of Jordan and super Jordan planes*. Proc. Edinb. Math. Soc. **61** 661–672 (2018).

Background

Terminology

- Gelfand-Kirillov dim: let A be a $\Bbbk$-algebra generated by I (finite).

GK-dim A measures growth of $\{\text{words on } I \text{ of length} \leq k\}$ as $k \to \infty$.

Examples: $\cdot$ $\operatorname{GK-dim} A = 0$ iff $\dim A < \infty$,
$\cdot$ $\operatorname{GK-dim} \Bbbk[x_1, \dots, x_n] = n$, $\cdot$ $\operatorname{GK-dim} U(\mathfrak{g}) = \dim \mathfrak{g}$.

- Braidings of diagonal type: given $\mathbb{I} = \{1, \dots, \theta\}$ and $\mathfrak{q} \in M_\theta(\Bbbk^\times)$, consider $(V, c^{\mathfrak{q}})$ where

$$V \text{ has basis } (x_i)_{i \in \mathbb{I}}, \qquad c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad i, j \in \mathbb{I}.$$

- Dynkin diagram of such $\mathfrak{q}$: is the decorated graph with
  - vertices $\{1, \dots, \theta\}$, vertex i labelled by q_{ii};
  - edge between i and j iff $\widetilde{q}_{ij} := q_{ij} q_{ji} \neq 1$. Such edge is labelled $\widetilde{q}_{ij}$.
- PBW-type basis for $\mathcal{B}_{\mathfrak{q}} := \mathcal{B}(V, c^{\mathfrak{q}})$ is available (Kharchenkho).
- Roots of $\mathfrak{q}$: is the set of heights of the PBW generators.
- Root system of $\mathfrak{q}$: is the bundle of roots of all *reflections* of $\mathfrak{q}$.

Finite dim versus finite GK-dim

- All connected $\mathfrak{q}$'s such that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$ are known. More generally,

Theorem, [H09] [2]

Complete classification of connected $\mathfrak{q}$'s with finite root system.

- On the other hand, going back to problem (A) we have

Conjecture, [AAH] [3]

If $\operatorname{GK-dim} \mathcal{B}_{\mathfrak{q}} < \infty$, the root system of $\mathfrak{q}$ is finite.

Substantial progress has been made by Rosso and [AAH].

- However, regarding prob (C), for each connected $\mathfrak{q}$ with $\dim \mathcal{B}_{\mathfrak{q}} < \infty$ there is a finite GK-dim pre-Nichols algebra different from $\mathcal{B}_{\mathfrak{q}}$.

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[2] [H09] I. Heckenberger, *Classification of arithmetic root systems*. Adv. Math. 220, 59–124 (2009)

[3] [AAH] N. Andruskiewitsch, I. Angiono, I. Heckenberger, *On finite GK-dimensional Nichols algebras over abelian groups*. Mem. Amer. Math. Soc., to appear.

Eminent pre-Nichols algebras

The set $\mathfrak{Pre}(\mathfrak{q})$ of pre-Nichols algebras of $\mathfrak{q}$ is partially ordered:

$$\begin{tikzcd} & & & T(V) \arrow[dlll, two heads] \arrow[d, two heads] \arrow[drrr, two heads] & & & & \rightsquigarrow \text{ minimal} \\ \mathcal{B} \arrow[drrr, two heads] & \dots & \dots & \mathcal{B}' \arrow[d, two heads] & \dots & \dots & \mathcal{B}'' \arrow[dlll, two heads] & \\ & & & \mathcal{B}_{\mathfrak{q}} & & & & \rightsquigarrow \text{ maximal} \end{tikzcd}$$

$\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q}) :=$ subposet consisting of all elements with finite GK-dim.

Definition [Andruskiewitsch-S]

$\widehat{\mathcal{B}} \in \mathfrak{Pre}(\mathfrak{q})$ is eminent if it is the minimum of $\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q})$.

This means: GK-dim $\widehat{\mathcal{B}} < \infty$ and for each $\mathcal{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q})$ there exists a projection $\widehat{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ of braided Hopf algebras which is the identity on V.

Goal: decide if there exist (in such case find) eminent pre-Nichols algebras for each braiding of interest.

Quantum linear spaces

Terminology

- Recall: V has basis $(x_i)_{i\in\mathbb{I}}$ and braiding $c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$.
- Quantum linear space condition: $\quad q_{ij}q_{ji} = 1$ if $i \neq j \in \mathbb{I}$.
- Distinguished pre-Nichols: $\quad \widetilde{\mathcal{B}}_{\mathfrak{q}} := T(V)/\langle x_ix_j - q_{ij}x_jx_i \colon i < j\rangle$.

Satisfies GK-dim $= |\mathbb{I}|$.

Partition $\mathbb{I} = \mathbb{I}^{<3} \sqcup \mathbb{I}^3 \sqcup \mathbb{I}^{>3} \sqcup \mathbb{I}^\infty$, where

$$\mathbb{I}^\infty = \{i \in \mathbb{I} \colon q_{ii} \notin \mathbb{G}_\infty\}, \qquad \mathbb{I}^N = \{i \in \mathbb{I} \colon \operatorname{ord} q_{ii} = N\},\ N \geq 1,$$
$$\mathbb{I}^{>3} = \bigcup_{N>3} \mathbb{I}^N, \qquad \mathbb{I}^{<3} = \{i \in \mathbb{I} \colon q_{ii} = \pm 1\} = \mathbb{I}^1 \sqcup \mathbb{I}^2.$$

For $\star \in \{<3, 3, >3, \infty\}$ set $V^\star = \Bbbk\{x_i \colon i \in \mathbb{I}^\star\} \subset V$, so

$$V = V^{<3} \oplus V^3 \oplus V^{>3} \oplus V^\infty.$$

Example: super symmetric algebras

Super symmetric algebra condition: $V = V^1 \oplus V^2$, i.e., $q_{ii}^2 = 1$ for all i.

Lema [Andruskiewitsch-S]

If $\mathfrak{q}$ and $\mathfrak{p}$ have the same Dynkin diagram, there is a posets iso
$$\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q}) \simeq \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{p}) .$$

$\rightsquigarrow$ Instead of $(V, c^{\mathfrak{q}})$, consider $(V, c^{\mathfrak{p}})$ with $p_{ij} = \pm 1$.

- $\mathfrak{Pre}(\mathfrak{p})$ = super enveloping algebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{j\geq 1}\mathfrak{n}^j$ graded Lie super algebra generated by $\mathfrak{n}^1 \simeq V$.
- $\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{p})$ = super enveloping algebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{j\geq 1}\mathfrak{n}^j$ graded Lie super algebra generated by $\mathfrak{n}^1 \simeq V$ and $\dim \mathfrak{n} < \infty$.

Eminent pre-Nichols algebras might not exist.

Main result for quantum linear spaces

Theorem [Andruskiewitsch-S]

Recall: $V = V^{<3} \oplus V^3 \oplus V^{>3} \oplus V^\infty$.

(1) For $\star \in \{3, > 3, \infty\}$, the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V^\star)$ is eminent.

(2) Let $\mathcal{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$. For $\star \in \{\leq 3, > 3, \infty\}$, denote by $\mathcal{B}^\star$ the subalgebra of $\mathcal{B}$ generated by $V^\star$. Then

$$\mathcal{B} \simeq \mathcal{B}^{\leq 3} \underline{\otimes} \mathcal{B}^{>3} \underline{\otimes} \mathcal{B}^\infty.$$

(3) Assume V has basis $\{x_1, x_2\}$ with $x_1 \in V^3$, $x_2 \in V^1$. Then

$$\breve{\mathcal{B}}(V) = T(V)/\langle (\mathrm{ad}_c\, x_1)^4 x_2,\ (\mathrm{ad}_c\, x_2)^2 x_1 \rangle$$

is an eminent pre-Nichols algebra of V with $\mathrm{GK\text{-}dim} = 6$.

Moreover, $\breve{\mathcal{B}}(V)$ has a G_2-type PBW basis ...

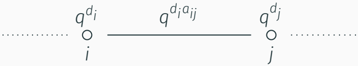
Non genuine quantum Serre relations

- Let $\mathsf{a} = (a_{ij})_{i,j\in\mathbb{I}}$ indecomposable symmetrizable GCM;

  $\mathsf{d} \in GL_\theta(\mathbb{Z})$ diagonal such that da is symmetric;

  $\mathfrak{g} = \mathfrak{g}(\mathsf{a})$ associated Kac-Moody algebra, $\mathfrak{g}(\mathsf{a}) = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$.

- Let $q \in \Bbbk^\times$ and consider the Dynkin diagram



- Fix $\mathfrak{q}$ with this Dynkin diagram

Beware: Cartan type, but associated Cartan matrix not necessarily a.

Define $\breve{\mathcal{B}}_{\mathfrak{q}} = T(V)/\langle (\mathrm{ad}_c\, x_i)^{1-a_{ij}} x_j \ i \neq j \in \mathbb{I}\rangle$.

Proposition [Andruskiewitsch-S]

GK-dim $\breve{\mathcal{B}}_{\mathfrak{q}} \geq \dim \mathfrak{g}^+$.

Cartan, super and standard types

Terminology

- Andruskiewitsch-Angiono organize Heckenbergers' list with a Lie-theoretic perspective:
    - Cartan,
    - super,
    - standard,
    - modular,
    - supermodular,
    - UFO.

- For each $\mathfrak{q}$ with $\dim \mathcal{B}_{\mathfrak{q}} < \infty$, Angiono [4] defines the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Some features:
    - if $\mathcal{B}_{\mathfrak{q}} = \mathfrak{u}_q^+(\mathfrak{g})$, then $\widetilde{\mathcal{B}}_{\mathfrak{q}} = \mathsf{U}_q^+(\mathfrak{g})$.
    - $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ inherits all Lusztig's isomorphisms from $\mathcal{B}_{\mathfrak{q}}$.
    - $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ has the same PBW generators of $\mathcal{B}_{\mathfrak{q}}$ ($\neq$ heights).
    - kernel of $\widetilde{\mathcal{B}}_{\mathfrak{q}} \twoheadrightarrow \mathcal{B}_{\mathfrak{q}}$ is generated by powers of root vectors.
    - $\operatorname{GK-dim} \widetilde{\mathcal{B}}_{\mathfrak{q}} < \infty$. It is asked: is $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ eminent?

Next: try to answer this for Cartan, super and standard types.
If the answer is negative, find eminent pre-Nichols algebras.

[4] [An] I. Angiono *Distinguished pre-Nichols algebras*. Transf. Groups **21** 1–33 (2016).

Cartan type A_2 at a third root of unity

Here $\mathcal{B}_{\mathbf{q}} = T(V)/\langle x_1^3, x_2^3, x_{12}^3, x_{112}, x_{221}\rangle$ and $\widetilde{\mathcal{B}}_{\mathbf{q}} = T(V)/\langle x_{112}, x_{221}\rangle$.

Define $\widehat{\mathcal{B}} = T(V)/\langle x_{1112}, x_{2112}, x_{1221}, x_{2221}\rangle$.

Let $\mathcal{Z}$ = subalgebra of $\widehat{\mathcal{B}}$ gen by $x_1^3,\ x_2^3,\ x_{12}^3,\ x_{112},\ x_{221}$.

Theorem [Andruskiewitsch-S]

(1) $\mathcal{Z}$ is a normal Hopf subalgebra. Moreover, $(\mathrm{ad}_c\,\widehat{\mathcal{B}})\mathcal{Z} = 0$.

(2) $\mathcal{Z}$ is a skew-polynomial algebra in five variables.

(3) There is an extension of braided Hopf algebras

$$\Bbbk \to \mathcal{Z} \hookrightarrow \widehat{\mathcal{B}} \twoheadrightarrow \mathcal{B}_{\mathbf{q}} \to \Bbbk.$$

(4) The pre-Nichols algebra $\widehat{\mathcal{B}}$ is eminent with $\mathrm{GK\text{-}dim}\,\widehat{\mathcal{B}} = 5$.

¡Gracias!