3d TQFTs from non-ssi modular cat.

(Ingo Runkel)

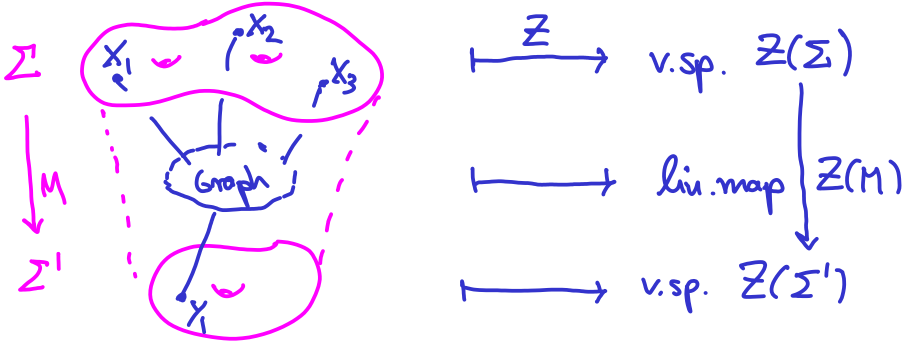
Outline:

- RT TQFT for ssi modular $\mathcal{C}$
- Lyubashenko inv. for non-ssi modular $\mathcal{C}$
- modified traces
- 3d TQFT for non-ssi modular $\mathcal{C}$

Reshetikhin–Turaev TQFT

Reshetikhin, Turaev '91, Turaev '94

$\mathcal{C}$: modular fusion category (= ssi modular tensor cat)



Constructed via surgery presentations of 3mf

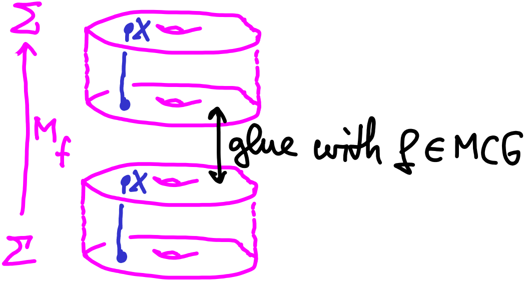
..., RT TQFT

In particular:

1. Closed 3mf + ribbon graph $\longmapsto$ number

$\rightsquigarrow$ invariants

2. Mapping cylinders



$\rightsquigarrow$ (projective) rep^n of surface MCGs

Lyubashenko invariants & MCG rep$^{\underline{n}}$

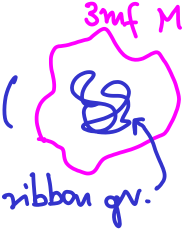
Lyubashenko '94

$\mathcal{C}$: non-ssi modular tensor cat.

Can still do ① and ② !

→ Key ingredient: coend $L = \int^{X \in \mathcal{C}} X^* \otimes X \quad \in \mathcal{C}$

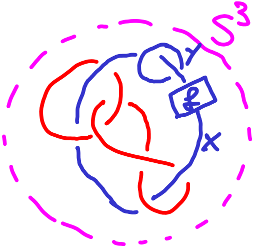
is a Hopf algebra in $\mathcal{C}$.

Lyu(

) :

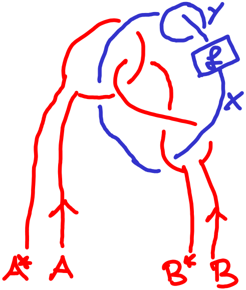
... Lyu.

1) surgery

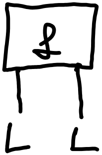


L : surgery rep$^{\underline{n}}$ of M

2) cut red ribbons $\longrightarrow$ dinatural xfer



$\xrightarrow[\text{univ. prop. of } L]{}$



3) compose with integral of L :

$$\mathrm{Lyu}(M) = (\text{const}) \cdot f \circ \Lambda^{\otimes \ell}$$

... Lyu.

Nice: eg $C = SF_N \leftarrow$ "symplectic fermion cat." $(N \in \mathbb{N})$

$\hat{=}$ rep$^{\underline{n}}$ of a fact. ribbon qHopf alg.

$$\mathrm{Lyu}(\text{lens space } L_{1,p}) = (\text{const})\, p^N$$

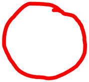
$(p \in \mathbb{N})$

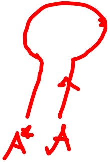
Lyu. inv. for one fixed C can distinguish all $L_{1,p}$ s.

(No RT-inv. for C ssi can do that.)

... Lyu.

Not nice: $\mathcal{C}$ non-ssi $\Longleftrightarrow$ $\mathrm{Lyu}(S^2 \times S^1) = 0$

1) $S^2 \times S^1 \simeq$  in S^3 (surgery)

2) 

$\rightsquigarrow$ counit $\varepsilon : L \to 1$ (cut surgery link)

3) Lem. $\mathcal{C}$ non-ssi $\Longleftrightarrow$ $\varepsilon \circ \Lambda = 0$ (compose w. integral)

So it is not part of a TQFT:

Lem $Z : \mathrm{Bord}_3 \longrightarrow \mathrm{Vect}$ a 3d TQFT (sym. mon. fun.)

Then

$$\dim Z(\Sigma) = Z(\Sigma \times S^1)$$

Would have $Z(S^2) = \{0\}$ $\Rightarrow$ $Z(\Sigma \xrightarrow{M} \Sigma') = 0$

for all M $\lightning$

(to cylinder = identity)

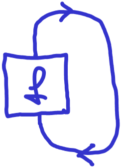
Modified trace

Geer, Kujawa, Patureau-Mirand '11

$\mathcal{C}$: finite tensor cat., spherical, non-ssi

- Categorical trace :

  Let $P \in \mathrm{Proj.}\,\mathcal{C}$, $f : P \to P$

  $\mathrm{tr}_P f =$  $= 0$

- Modified trace on $\mathrm{Proj}\,\mathcal{C}$ (Can define for more general tensor ideals)

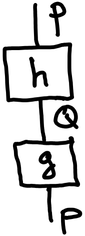
  $\{ t_P : \mathrm{End}(P) \longrightarrow k \mid P \in \mathrm{Proj}\,\mathcal{C} \}$

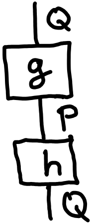
... modified trace

- Modified trace on Proj $\mathcal{C}$ (can define for more general tensor ideals)

$$\{ t_P : \mathrm{End}(P) \to k \mid P \in \mathrm{Proj}\,\mathcal{C} \}$$

1) cyclic

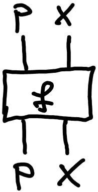
$t_P\Big($ 

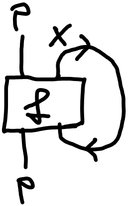
$\Big) = t_Q\Big($ 

$\Big)$

$P, Q \in \mathrm{Proj}\,\mathcal{C}$

2) partial trace

$t_{P\otimes X}\Big($ 

$\Big) = t_P\Big($ 

$\Big)$

$P \in \mathrm{Proj}\,\mathcal{C}$
$X \in \mathcal{C}$

... modified trace

Comments :

Beliakova, Blanchet, Gainutdinov '18
Berger, Gainutdinov, IR '18

- For $\mathcal{C} = \mathrm{Rep}\, H$ for H a fin.dim. pivotal (quasi-) Hopf alg :

  Can express mod. tr. in terms of (symmetrised) integrals

  $\rightarrow$ non-zero mod. tr. exists and unique up to scalars

Geer, Kujawa, Patureau-Mirand '11
Gainutdinov, IR '17

- For $\mathcal{C}$ modular tensor cat :

  non-zero mod. tr. exists and unique up to scalars

3d TQFT from non-ssi modular $\mathcal{C}$

De Renzi, Geer, Patureau-Mirand '17

- for $\mathcal{C} = \mathrm{Rep}\, H$, H fact. ribbon Hopf alg.

De Renzi, Gainutdinov, Geer, Patureau-Mirand, IR '19

- for $\mathcal{C}$ general (not nec. ssi) modular tens. cat

Want

$$Z : \overbrace{\mathrm{Bord}_3(\mathcal{C})}^{\text{anomaly}} \longrightarrow \mathrm{Vect}$$

$\mathcal{C}$-col. ribbon gr.

... but do not quite get this.

Instead:

... non-ssi TQFT

Omit some 3mf: Bord'_3

- same objects
- bordisms must have a projective ribbon in every connected component disjoint from incoming boundary

$\Sigma \sqcup -\Sigma \longrightarrow \emptyset$ ✓

$\emptyset \longrightarrow \Sigma \sqcup -\Sigma$ ✗

but can have

for P projective.

... non-ssi TQFT

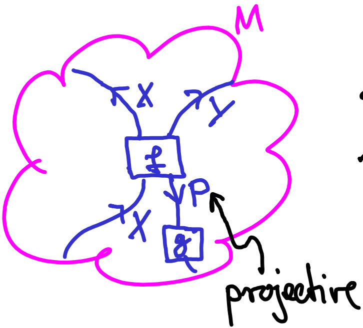
Get TQFT (sym. mon. fun.) $Z : \mathrm{Bord}_3' \longrightarrow \mathrm{Vect}$

- If C is ssi, $\mathbb{1}$ is proj., can add loop $\circlearrowleft^1$ to bordism $\rightsquigarrow$ recover Bord_3 and RT-TQFT

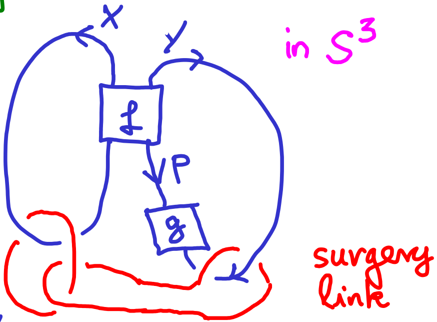
Suppose C is non-ssi.

- Empty closed 3mf $\emptyset \xrightarrow{M} \emptyset$ not allowed

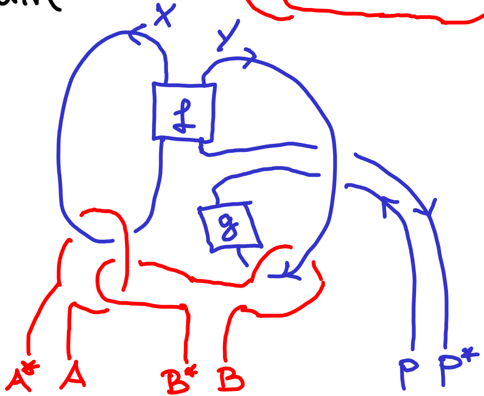
• For closed M with proj. ribbon :



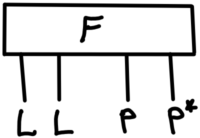
surgery $\longrightarrow$



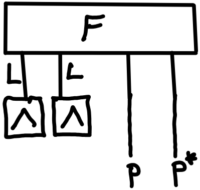
cut surgery link
+ proj. ribbon
$\longrightarrow$



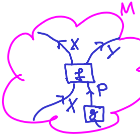
univ. property of coend

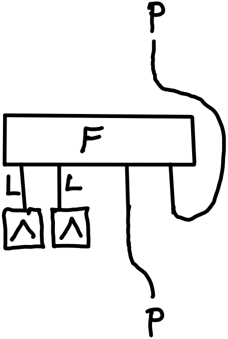


integral of L



modif. tr.

$L'\Big($ 

$\Big) = (\text{const})\ t_P\Big($ 

$\Big)$

Thank you!