The double of the Jordan plane in odd characteristic

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Workshop: Hopf Algebras and Tensor Categories, Hamburg, August 19th 2020

Joint work with Nicolás Andruskiewitsch

The context

- Let $\mathbb{k}$ be an algebraic closed field of characteristic $p > 2$. Let $\Gamma \simeq \mathbb{Z}/p\mathbb{Z}$ be the cyclic group of p elements with generator g, written multiplicatively.

- The Jordan plane J is the algebra presented by generators x, y with relation $yx - xy + \frac{1}{2}x^2 = 0$.

- In characteristic 0 this is the Nichols algebra of the braided vector space $V = \mathbb{k}\{x, y\}$ with braiding:

$$c(x \otimes x) = x \otimes x, \qquad c(y \otimes x) = x \otimes y,$$
$$c(x \otimes y) = (x + y) \otimes x, \qquad c(y \otimes y) = (x + y) \otimes y.$$

• In odd characteristic, Cibils, Lauve and Witherspoon (2009) [CLW] showed that the Nichols algebra $\mathcal{B}(V)$ is the called restricted Jordan plane $\mathfrak{J} = J/(x^p, y^p)$.

• The braided vector space (V, c) can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with action $\rightharpoonup$ and coaction δ:

$$g \rightharpoonup x = x, \quad g \rightharpoonup y = y + x, \quad \delta(x) = g \otimes x, \quad \delta(y) = g \otimes y.$$

• We can define the finite dimensional Hopf algebra

$$H := \mathfrak{J} \# \Bbbk\Gamma$$

The plan.

- Present by generators and relations the Drinfeld double $D(H)$.

- Compute its simple modules.

- Define a Hopf algebra $\widetilde{D}$ of infinite dimension such that $\widetilde{D} \twoheadrightarrow D(H)$. This algebra will be the double of the Jordan plane.

- Understand various Frobenius-like maps and extensions of the algebra $\widetilde{D}$.

- In [AP] we also classify finite dimensional pre- and post-Nichols algebras of V.

The dual Hopf algebra H^*.

• Let be $W = \Bbbk\{u, v\}$ be the braided vector space given by

$$c(u \otimes u) = u \otimes u, \qquad c(u \otimes v) = v \otimes u,$$
$$c(v \otimes u) = u \otimes (v + u), \qquad c(v \otimes v) = v \otimes (v + u).$$

• $\mathcal{B}(W)$ is isomorphic to the Jordan plane as an algebra but with opposite comultiplication.

• $H^* \simeq \mathcal{B}(W) \# \Bbbk^\Gamma$.

• Also $\Bbbk^\Gamma \simeq \Bbbk[\zeta]/(\zeta^p - \zeta)$. This is the restricted enveloping algebra of Lie algebra of a one dimensional torus.

We have the following presentation for $(H^*)^{\mathrm{op}}$.

Lemma. $(H^*)^{\mathrm{op}}$ is the algebra presented by generators u, v and ζ with relations:

$$v^p = 0, \qquad u^p = 0, \qquad vu = uv - \frac{1}{2}u^2,$$
$$v\zeta = \zeta v + v, \; u\zeta = \zeta u + u, \; \zeta^p = \zeta.$$

Proposition. The Hopf algebra $D(H)$ is presented by generators u, v, ζ, g, x, y with the relations of H, $(H^*)^{\mathrm{op}}$ and

$$\begin{aligned} &\zeta y = y\zeta + y, && \zeta x = x\zeta + x, && vg = gv + gu, \\ &ug = gu, && vx = xv + (1 - g) + xu, && ux = xu, \\ &uy = yu + (1 - g) && vy = yv - g\zeta + yu, && g\zeta = \zeta g. \end{aligned}$$

$D(H)$ is an abelian extension of $\mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$.

Let $\mathbf{R}$ be the subalgebra of $D(H)$ generated by g, x, u. This is a normal commutative Hopf subalgebra.

Theorem. There exist a exact sequence of Hopf algebras:

$$\mathbf{R} \hookrightarrow D(H) \twoheadrightarrow \mathfrak{u}(\mathfrak{sl}_2(\Bbbk)).$$

As $\mathbf{R}$ is commutative and $\mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$ co-commutative, this is an abelian extension.

Remarks. • The algebra $\mathbf{R}$ is the Frobenius kernel of the algebraic group $\mathbf{G} = (\mathbf{G}_a \times \mathbf{G}_a) \rtimes \mathbf{G}_m$. Also $\mathbf{R}$ is local.

• $\operatorname{Irrep} D(H) \simeq \operatorname{Irrep} \mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$ as $D(H)\mathbf{R}^+$ is a nilpotent ideal. In particular there are p isomorphism classes of simple modules. There is an explicit description as quotients of Verma modules.

The double of the Jordan plane. We now define the Hopf algebra $\widetilde{D}$ as the algebra presented by generators u, v, ζ, g, x, y with the same relations as $D(H)$ but removing:

$$x^p = 0, \quad y^p = 0, \quad u^p = 0, \quad v^p = 0, \quad g^p = 1, \quad \zeta^p = \zeta.$$

$\widetilde{D}$ is a Hopf algebra with $\operatorname{GKdim} \widetilde{D} = 6$.

The Hopf subalgebra Z of $\widetilde{D}$ generated by x^p, y^p, u^p, v^p, g^p and $\zeta^{(p)} := \zeta^p - \zeta$ is central. We have the following result:

Theorem. There is an exact sequence of Hopf algebras:

$$Z \hookrightarrow \widetilde{D} \twoheadrightarrow D(H).$$

Remark. $Z \simeq \mathcal{O}(\mathbf{B})$ for certain soluble algebraic group $\mathbf{B}$ related with the 3×3 Heisenberg group.

Ring properties of $\widetilde{D}$.

• The algebra $\widetilde{D}$ admits an exhaustive ascending filtration $(\widetilde{D}_n)_{n\in\mathbb{N}_0}$ such that

$$\mathrm{gr}\,\widetilde{D} \simeq \Bbbk[T^{\pm}]\otimes\Bbbk[X_1,\dots,X_5].$$

• As a consequence $\widetilde{D}$ is a noetherian domain.

• $\widetilde{D}$ is a polynomial identity algebra as it is a free module over the central subalgebra Z.

A conmutative diagram. The previous maps can be summarized in the following diagram:

$$\begin{array}{ccccc}
\mathcal{O}(\mathbf{G}) & \hookrightarrow & \mathcal{O}(\mathbf{B}) & \twoheadrightarrow & \mathcal{O}(\mathbf{G}_a^3) \\
\text{Fr}\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(\mathbf{G}) & \hookrightarrow & \widetilde{D} & \twoheadrightarrow & U(\mathfrak{sl}_2(\mathbb{k})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{R} & \hookrightarrow & D(H) & \twoheadrightarrow & \mathfrak{u}(\mathfrak{sl}_2(\mathbb{k}))
\end{array}$$

All columns and rows are exact sequence of Hopf algebras.

The super Jordan plane

In [AAH] is introduced other indecomposable braided vector space of dimension two. Let $V_s = \Bbbk\{x_1, x_2\}$ with braiding

$$\begin{aligned} c(x_1 \otimes x_1) &= -x_1 \otimes x_1, & c(x_2 \otimes x_1) &= -x_1 \otimes x_2, \\ c(x_1 \otimes x_2) &= (-x_2 + x_1) \otimes x_1, & c(x_2 \otimes x_2) &= (-x_2 + x_1) \otimes x_2. \end{aligned}$$

• In characteristic 0 the Nichols algebra $\mathcal{B}(V_s)$ has GKdim 2. It is the called *super Jordan plane*.

• In characteristic $p > 2$, $\mathcal{B}(V_s)$ has finite dimension. Is the called *restricted super Jordan plane*.

In [AP2] we carried the same program for the super Jordan plane.

The double of the super Jordan plane The doubles of the super Jordan plane and its restricted version are essentially the bosonization of Hopf superalgebras $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ respectively. they fit in the following diagram:

$$\begin{array}{ccccc}
\mathcal{O}(\widetilde{\mathbf{G}}) & \hookrightarrow & \mathcal{O}(\widetilde{\mathbf{B}}) & \twoheadrightarrow & \mathcal{O}(\mathbf{G}_a^3) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(\mathfrak{G}) & \hookrightarrow & \widetilde{\mathcal{D}} & \twoheadrightarrow & U(\mathfrak{osp}(1|2)) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathbf{R}} & \hookrightarrow & \mathcal{D} & \twoheadrightarrow & \mathfrak{u}(\mathfrak{osp}(1|2))
\end{array}$$

This justifies the adjetive *super* given to the super Jordan plane.

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Finite dimensional pre-Nichols algebras

Theorem. Let $\mathcal{A}$ be a finite dimensional pre-Nichols algebra of V such that $J \twoheadrightarrow \mathcal{A}$. Then

$$\mathcal{A} \simeq \mathcal{G}(k, \ell, a) := J/(y^{p^k} - ax^{p^k}, x^{p^\ell}),$$

for some $k, \ell \in \mathbb{N}$, $a \in \Bbbk$.

Remark. The automorphism group of J acts over the set of pre-Nichols algebras $\mathcal{G}(k, \ell, a)$. Then we can see that

$$\mathcal{G}(k, \ell, a) \simeq \mathcal{G}(k, \ell, 0) := \mathcal{G}(k, \ell),$$

as braided Hopf algebras. This is **not** an isomorphism of pre-Nichols algebras.

The graded dual of the Jordan plane and post-Nichols algebras.

Given a graded locally finite Hopf algebra

$$A = \bigoplus_{n \in N_0} A_n, \qquad \text{dim}\, A_n < \infty, \forall n \in \mathbb{N}_0.$$

The graded dual $A^* := \bigoplus_{n \in N_0} (A_n)^*$ is also a Hopf algebra with the transpose structures.

We compute the graded dual of the Jordan plane. This will give us a post-Nichols algebra of W.

Theorem. The graded dual $\mathcal{E}$ of J is presented by generators $\mathrm{x}^{[n]}$, $\mathrm{y}^{[n]}$, $n \in \mathbb{N}$, and relations

$$\mathrm{x}^{[n]}\mathrm{x}^{[m]} = \binom{n+m}{n}\mathrm{x}^{[n+m]},$$

$$\mathrm{y}^{[n]}\mathrm{y}^{[m]} = \binom{n+m}{n}\mathrm{y}^{[n+m]},$$

$$\mathrm{x}^{[n]}\mathrm{y}^{[m]} = \sum_{k=0}^{m} \binom{n+k}{k}(-1)^k \frac{[-n]^{[k]}}{2^k}\mathrm{y}^{[m-k]}\mathrm{x}^{[n+k]},$$

$$\mathrm{x}^{[0]} = \mathrm{y}^{[0]} = 1.$$

Finite dimensional post-Nichols algebras

Let $\mathfrak{G}(k,\ell) = \mathcal{G}(k,\ell)^*$ be a post-Nichols algebra of W.

This is exactly the subalgebra of $\mathcal{E}$ generated by $\mathtt{x}^{[n]}$, $\mathtt{y}^{[m]}$, $n \in \mathbb{I}_{0,p^k-1}$, $m \in \mathbb{I}_{0,p^\ell-1}$. In particular $\mathcal{E} = \bigcup_{k,\ell\in\mathbb{N}} \mathfrak{G}(k,\ell)$.

We have then new examples of **coradically graded** pointed Hopf alegebras in positive characteristic p

$$\mathcal{H}_{k,\ell} = \mathfrak{G}(k,\ell)\#\Bbbk\Gamma.$$

Thanks.

References

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