The adjoint algebra for 2-categories

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Based on:

N. Bortolussi and M. Mombelli, *The adjoint algebra for 2-categories*, Preprint arXiv:2005.05271

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Problem we want to attack:

G a finite group, $\mathcal{D} = \oplus_{g\in G}\mathcal{C}_g$ a G-graded tensor category

$$\big(\mathcal{D} \text{ is a } G\text{–extension of } \mathcal{C}_1\big)$$

- Compute Shimizu's adjoint algebra $Ad_{\mathcal{M}} \in \mathcal{Z}(\mathcal{D})$ for any left $\mathcal{D}$-module category $\mathcal{M}$.

- Might be useful to study nilpotent, solvable fusion categories (¿?)

$\mathcal{C}$ is nilpotent: $vect_{\Bbbk} \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n = \mathcal{C}$, $\mathcal{C}_i$ is a G_i-extension of $\mathcal{C}_{i-1}$.

Notation and Preliminaries

$\mathcal{C}$ a finite tensor category.

- ${}_{\mathcal{C}}\mathrm{Mod}$ is the 2-category of left $\mathcal{C}$-module categories.

- ${}_{\mathcal{C}}\mathrm{Mod}_{e}$ is the 2-category of exact indecomposable $\mathcal{C}$-module categories.

If $\mathcal{B}, \mathcal{B}'$ are 2-categories

- For any pseudofunctor $F : \mathcal{B} \to \mathcal{B}'$: $\mathit{PseudoNat}(F, F)$ is the monoidal category of pseudo-natural transformations;
- The *center* of $\mathcal{B}$ is $Z(\mathcal{B}) = \mathit{PseudoNat}(\mathrm{Id}_{\mathcal{B}}, \mathrm{Id}_{\mathcal{B}})$.

Objects in $Z(\mathcal{B})$ are pairs (V, σ):

$$V = \{V_A \in \mathcal{B}(A, A) \text{ 1-cells}, A \in \mathcal{B} \text{ a 0-cell } \},$$

$$\sigma = \{\sigma_X : V_B \circ X \to X \circ V_A\}, \text{ the half-braiding}$$

where, for any $X \in \mathcal{B}(A, B)$, σ_X is a natural isomorphism 2-cell such that

$$\sigma_{I_A} = \mathrm{id}_{V_A}, \quad \sigma_{X \circ Y} = (\mathrm{id}_X \circ \sigma_Y)(\sigma_X \circ \mathrm{id}_Y),$$

$\forall$ 0-cells $A, B, C \in \mathcal{B}$, and any pair of 1-cells $X \in \mathcal{B}(A, B)$, $Y \in \mathcal{B}(C, B)$.

- If $\mathcal{C}$ is a tensor category. There is a monoidal equivalence:

$$\mathcal{Z}({}_{\mathcal{C}}\mathrm{Mod}\,) \simeq_{\otimes} \mathcal{Z}(\mathcal{C})$$

$$((V_{\mathcal{M}}), \sigma) \longmapsto V_{\mathcal{C}}(\mathbf{1})$$

$V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}$-module functor.

- If $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$, $\Phi : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}_1}\mathrm{Mod}$ is the restriction 2-functor, then

$$PseudoNat(\Phi, \Phi) \simeq_{\otimes} \mathcal{Z}_{\mathcal{C}_1}(\mathcal{D})$$

Is the relative center.

- If $F : \mathcal{B} \to \mathcal{B}'$ is a 2-equivalence $\rightsquigarrow$ induces a monoidal equivalence $\widehat{F} : \mathcal{Z}(\mathcal{B}) \to \mathcal{Z}(\widetilde{\mathcal{B}})$

Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories. $\mathcal{N}$ be an invertible $(\mathcal{C}, \mathcal{D})$-bimodule category.

$$\theta^{\mathcal{N}} : {}_{\mathcal{C}}\mathrm{Mod} \to {}_{\mathcal{D}}\mathrm{Mod}$$

$$\theta^{\mathcal{N}}(\mathcal{M}) = \mathit{Fun}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$$

$$\begin{array}{ccc}
\mathcal{Z}(\mathcal{C}) & \xrightarrow{\quad\theta\quad} & \mathcal{Z}(\mathcal{D}) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathcal{Z}({}_{\mathcal{C}}\mathrm{Mod}) & \xrightarrow{\quad\widehat{\theta}^{\mathcal{N}}\quad} & \mathcal{Z}({}_{\mathcal{D}}\mathrm{Mod})
\end{array}$$

Identifying $\mathcal{D} = \mathrm{End}_{\mathcal{C}}(\mathcal{N})$

$$\theta(X)(N) = X \triangleright N$$

$X \in \mathcal{C}$, $N \in \mathcal{N}$.

Idea to solve the problem

Translate known constructions, problems to the realm of 2-categories

Idea to solve the problem

G a finite group, $\mathcal{B}$ a 2-category.

In [E. Bernaschini, C. Galindo and M. Mombelli, Group actions on 2-categories, Manuscripta Math. 159, No. 1-2, 81–115 (2019).]

We defined:

- An action of G on $\mathcal{B}$,
- The equivariantized 2-category $\mathcal{B}^G$
- There is a forgetful 2-functor $\Phi : \mathcal{B}^G \to \mathcal{B}$, such that
  1. G acts on $PseudoNat(\Phi)$, and
  2. $\mathcal{Z}(\mathcal{B}^G) \simeq_{\otimes} PseudoNat(\Phi)^G$.

Theorem

If $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$ *is a* G*-graded tensor category. Then*

- G *acts on* ${}_{\mathcal{C}_1}\mathrm{Mod}_{\,e}$
- *There exists a 2-equivalence* $({}_{\mathcal{C}_1}\mathrm{Mod}_{\,e})^G \simeq {}_{\mathcal{D}}\mathrm{Mod}_{\,e}$

Steps to solve the main problem:

- Define an anlogue of Shimizu's adjoint algebra for 2-categories.
- Prove that when applied to ${}_{\mathcal{C}}\mathrm{Mod}$ it coincides with Shimizu's version.
- How our adjoint algebra behaves under 2-equivalences.
- Apply all the above to $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$, ${}_{\mathcal{D}}\mathrm{Mod}$ and use $({}_{\mathcal{C}_1}\mathrm{Mod})^G \simeq {}_{\mathcal{D}}\mathrm{Mod}$

We did only the first 3 steps.

The main definition

Let $\mathcal{B}$ be a 2-category such that any 1-cell has right and left duals.

For any 0-cell $B \in \mathcal{B} \rightsquigarrow Ad_B \in \mathcal{Z}(\mathcal{B})$

$$(Ad_B)_A = \int_{X \in \mathcal{B}(A,B)} {}^*X \circ X \in \mathcal{B}(A,A)$$

For any $X \in \mathcal{B}(A, B)$ we shall denote by

$$\pi_X^{(A,B)} : \int_{X \in \mathcal{B}(A,B)} {}^*X \circ X \xrightarrow{\cdot\cdot} {}^*X \circ X$$

$$\begin{array}{ccc}
\left(\int_{X \in \mathcal{B}(A,B)} {}^*X \circ X\right) \circ Z & \xrightarrow{\pi_Y^{(A,B)} \circ \mathrm{id}_Z} & {}^*Y \circ Y \circ Z \\
\sigma_Z^B \big\downarrow & & \big\uparrow ev_Z \circ \mathrm{id}_{{}^*Y \circ Y \circ X} \\
Z \circ \left(\int_{X \in \mathcal{B}(C,B)} {}^*X \circ X\right) & \xrightarrow[\mathrm{id}_{Z} \circ \pi_{Y \circ Z}^{(C,B)}]{} & Z \circ {}^*Z \circ {}^*Y \circ Y \circ Z,
\end{array}$$

Results:

- For any 0-cell $B \in \mathcal{B}$, $Ad_B \in \mathcal{Z}(\mathcal{B})$ is an algebra.
- For equivalent 0-cells A, B the corresponding algebras Ad_A, Ad_B are isomorphic.

Theorem

$\mathcal{C}$ a tensor category. $\mathcal{M}$ an exact indecomp. $\mathcal{C}$-module category:

$$\Phi : \mathcal{Z}({}_{\mathcal{C}}\mathrm{Mod}\,) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$$

an algebra isomorphism $\Phi(Ad_{\mathcal{M}}) \simeq A_{\mathcal{M}}$(← *Shimizu's algebra*)

Results:

Theorem

If $F : \mathcal{B} \to \widetilde{\mathcal{B}}$ is a 2-equivalence, then

- *F induces a monoidal equivalence $\widehat{F} : \mathcal{Z}(\mathcal{B}) \to \mathcal{Z}(\widetilde{\mathcal{B}})$*
- *There is an algebra isomorphism $\widehat{F}(Ad_B) \simeq Ad_{F(B)}$.*

If $\mathcal{C}, \mathcal{D}$ are finite tensor categories. $\mathcal{N}$ is an invertible $(\mathcal{D}, \mathcal{C})$-bimodule category.

$$\theta^{\mathcal{N}} : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}}\mathrm{Mod}$$
$$\theta^{\mathcal{N}}(\mathcal{M}) = \mathit{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{M})$$

There is an isomorphism of algebras

$$\widehat{\theta}^{\mathcal{N}}(\mathit{Ad}_{\mathcal{M}}) \simeq \mathit{Ad}_{\mathit{Fun}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})}.$$

Here $\widehat{\theta}^{\mathcal{N}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ is the induced monoidal equivalence.