PBW deformations: Examples and some theory

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Definition: PBW deformation

Let $A := \mathbb{K}\langle x_1, \ldots, x_n\rangle/(r_1, \ldots, r_m)$ be a graded algebra given by generators $x_1, \ldots, x_n$ and homogeneous relations $r_1, \ldots, r_m$. An algebra

$$D := \mathbb{K}\langle x_1, \ldots, x_n\rangle/(r_1 + t_1, \ldots, r_m + t_m)$$

with $\deg t_i < \deg r_i$ for all $1 \leq i \leq m$ is called a *deformation* of A. A deformation D of A is said to be a *PBW deformation*, if $\mathrm{gr}(D) \cong A$ as graded algebras. Here $\mathrm{gr}(D)$ denotes the associated graded algebra of D.

Natural questions

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- How can we parametrise these PBW-deformations?
- Properties?

Deciding whether a deformation is PBW or not

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- Computational approach: Gröbner bases
- Fact: A deformation with deformed relations $r_1 + t_1, \ldots, r_m + t_m$ is PBW iff the leading monomials of the Strong Gröbner basis of $\{r_1 + t_1, \ldots, r_m + t_m\}$ coincide with the leading monomials of the Strong Gröbner basis of $\{r_1, \ldots, r_m\}$.

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- Cons:
  - Algorithm might not terminate (non-commutative Gröbner bases!)
  - No structural argument.

Another method: Two birds with one stone

From now on: $\mathbb{K}$ algebraically closed with $\mathrm{char}(\mathbb{K}) = 0$.
Crucial lemma:

Lemma (Heckenberger, Vendramin (2018))

Let A be an $\mathbb{N}_0$-graded finite dimensional algebra over $\mathbb{K}$. Let $s, d \in \mathbb{N}$ and $(A(\lambda))_{\lambda \in \mathbb{K}^s}$ be an affine family of deformations of A such that $\dim A(\lambda) \leq d$ for all $\lambda \in \mathbb{K}^s$. Assume that $\dim A(\lambda) = d$ for all λ in a Zariski dense subset of $\mathbb{K}^s$. Then $A(\lambda)$ is a PBW deformation of A for all $\lambda \in \mathbb{K}^s$.

Example: Nichols algebra of type A_2

Let $N \in \mathbb{N}_{\geq 2}$, let q be a primitive root of unity of order N and let $q_{12} \in \mathbb{K}^{\times}$. Consider the $\mathbb{K}$-algebra A with generators x_1, x_2 and defining relations

$$x_1^N = x_2^N = x_{12}^N = 0, \qquad x_1 x_{12} - q q_{12} x_{12} x_1 = x_{12} x_2 - q q_{12} x_2 x_{12} = 0,$$

where $x_{12} = x_1 x_2 - q_{12} x_2 x_1$. It corresponds to the braiding matrix

$$Q = (q_{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix} q & q_{12} \\ q_{21} & q \end{pmatrix}$$

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$$Q = (q_{ij})_{1\leq i,j\leq 2} = \begin{pmatrix} q & q_{12} \\ q_{21} & q \end{pmatrix}$$

with $q_{12}q_{21} = q^{-1}$. Basis: $x_2^{n_2}x_{12}^{n_{12}}x_1^{n_1}$, $0 \leq n_1, n_{12}, n_2 < N$. Hence $\dim(A) = N^3$.

Example: Nichols algebra of type A_2

Like any other Nichols algebra of diagonal type of rank 2, A comes with an action of the free abelian group $\mathbb{Z}^2$ via

$$e_i \cdot x_j = q_{ij} x_j, \qquad 1 \leq i, j \leq 2.$$

We only look for deformations that respect this action. Assume that $q_{12}^N = 1$ and consider the deformation $A(\alpha_1, \alpha_2, \alpha_{12})$ with relations

$$x_1^N = \alpha_1, \qquad x_2^N = \alpha_2, \qquad x_{12}^N = \alpha_{12}$$
$$x_1 x_{12} - q q_{12} x_{12} x_1 = x_{12} x_2 - q q_{12} x_2 x_{12} = 0,$$

where $\alpha_1, \alpha_2, \alpha_{12} \in \mathbb{K}$.

Example: Nichols algebra of type A_2

One can show:

- $A(\alpha_1, \alpha_2, \alpha_{12})$ is semisimple iff

$$\alpha_{12}\big((-1)^N \alpha_{12} + q_{12}^{-\frac{N(N-1)}{2}} (1 - q^{-1})^N \alpha_1 \alpha_2\big) \neq 0.$$

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This means that the lemma can be applied and that $A(\alpha_1, \alpha_2, \alpha_{12})$ is a PBW deformation of A for every $\alpha_1, \alpha_2, \alpha_{12} \in \mathbb{K}$.

Observations

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- Is this true for every finite-dimensional Nichols-algebra (of diagonal type)?
- Counterexample to first question: Nichols-algebra of type $A(1,1)$.

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Definition: Ore extension

Let A be a $\mathbb{K}$-algebra, let σ be an automorphism of A and let D be a twisted derivation of A relative to σ, i.e. a linear map $D\colon A \to A$ such that:

$$D(ab) = D(a)b + \sigma(a)D(b)$$

for all $a, b \in A$.
Let $A_{\sigma,D}[x]$ be the vectorspace $A \otimes_{\mathbb{K}} \mathbb{K}[x]$ with multiplication rule

$$xa = \sigma(a)x + D(a).$$

In other words: $A_{\sigma,D}[x]$ consists of "polynomials in x with coefficients in A". The algebra $A_{\sigma,D}[x]$ is called an Ore extension.

Representation theory of iterated Ore extensions

Assume that A is a prime algebra (i.e. $aAb = 0$ implies $a = 0$ or $b = 0$ for every $a, b \in A$), let $x_1, \dots, x_n$ be a set of generators of A and let Z_0 be a central subalgebra of A. For every $i \in \{1, \dots, n\}$, let A^i be the subalgebra spanned by $x_1, \dots, x_i$ and let $Z_0^i = Z_0 \cap A^i$. Suppose that A^i is a finite module over Z_0^i and that for every $j < i \in \{1, \dots, n\}$

$$x_i x_j = b_{ij} x_j x_i + P_{ij} \quad \text{where } b_{ij} \in \mathbb{K}, P_{ij} \in A^{i-1}$$

and the formulas $\sigma_i(x_j) = b_{ij} x_j$ define an automorphism of A^{i-1} which is the identity on Z_0^{i-1}.

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and the formulas $\sigma_i(x_j) = b_{ij} x_j$ define an automorphism of A^{i-1} which is the identity on Z_0^{i-1}.
That means $A^i = A^{i-1}_{\sigma_i, D_i}[x_i]$ with $D_i(x_j) = P_{ij}$.

Representation theory of iterated Ore extensions

Let $\bar{A}$ be the iterated Ore extension with zero derivations, i.e. with relations $x_i x_j = b_{ij} x_j x_i$. This algebra is called the associated quasipolynomial algebra of A.

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Theorem (De Concini, Procesi)

Generically (i.e. on a Zariski-dense subset of the spectrum of A), every finite dimensional irreducible representation of A has the same dimension. This number equals $\deg(\bar{A})$.

How to compute $\deg(\bar{A})$

Let $H \in \mathbb{Z}^{n\times n}$ be a skew-symmetric matrix and let $\mathbb{K}_H[x_1, \ldots, x_n]$ be the algebra with defining relations $x_i x_j = q^{h_{ij}} x_j x_i$ for some $q \in \mathbb{K}^\times$.

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Suppose that q is a primitive m-th root of unity and view the matrix H as the matrix of a homomorphism $H\colon \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^n$. Let $h = \#\operatorname{im}(H)$.

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Theorem (De Concini, Procesi)

$\deg \mathbb{K}_H[x_1, \dots, x_n] = \sqrt{h}$.

Coming back to our previous example

Let us again consider the deformation $A(\alpha_1, \alpha_2, \alpha_{12})$ from the previous section, i.e. the $\mathbb{K}$-algebra with generators x_1, x_2 and defining relations

$$x_1^N = \alpha_1, \qquad x_2^N = \alpha_2, \qquad x_{12}^N = \alpha_{12} \tag{1}$$

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with the additional requirement that $q_{12}^N = 1$.
It is not not an iterated Ore extension, but if we omit the relations (1), it is. Since relations (2) force x_1^N, x_2^N, x_{12}^N to be central it suffices (Schur's Lemma) to look for finite dimensional irreducible representations of

$$B := \mathbb{K}\langle x_1, x_2 \rangle / (x_1 x_{12} - q q_{12} x_{12} x_1, x_{12} x_2 - q q_{12} x_2 x_{12}).$$

Coming back to our previous example

The relations of B as an iterated Ore extension with generators x_1, x_2, x_{12} read as

$$\begin{aligned} x_1 x_2 &= q_{12} x_2 x_1 + x_{12} \\ x_1 x_{12} &= q q_{12} x_{12} x_1 \\ x_{12} x_2 &= q q_{12} x_2 x_{12}. \end{aligned}$$

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Writing $q_{ij} = q^{l_{ij}}$, we obtain $H = \begin{pmatrix} 0 & 1+l_{12} & l_{12} \\ -1-l_{12} & 0 & 1+l_{12} \\ -l_{12} & -1-l_{12} & 0 \end{pmatrix}$ as the matrix for the homomorphism $H\colon \mathbb{Z}^3 \to (\mathbb{Z}/N\mathbb{Z})^3$

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Finally, we compute $\#\,\mathrm{im}(H) = N^2$, hence generically the dimension of every finite dimensional irreducible representation of $A(\alpha_1, \alpha_2, \alpha_{12})$ equals N.

Limitations and Outlook

Problems:

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- The trick from the A_2-example might not always work.

However, the theory can be applied to study the dimension of irreducible representations of coideal subalgebras of PBW deformations of certain Nichols algebras, e.g. of type A_n.

Thank You for your attention!

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