Some collaboration topics:

1. Derived TFT for nonsemisimple MTCs
2. More non-semisimple MTCs
3. New Borel-type subalgebras
Theorem (Lyubaschenko 1995)

Any modular tensor category $C$, possibly non-semisimple, produces a modular functor, assigning to a decorated surface $\Sigma_{g,n}$

$$Z : \Sigma_{g,n}^{X_1,\ldots,X_n} \mapsto \text{Hom}_C(X_1 \otimes \cdots \otimes X_n, L \otimes g)$$

with an action of the mapping class group $\Gamma_{g,n}$ of the surface and a compatible way of glueing surfaces.

Here $L \in C$ is the coend. For example $C = \text{Rep}(H)$ has $L = H_{\text{coadj}}^*$.

In conformal field theory, these are the spaces of chiral correlators, which are vector-valued modular forms under the action of $\Gamma_{g,n}$. 
Let $\mathcal{C}$ be a modular tensor category, possibly non-semisimple. We have an action of the mapping class group $\Gamma_{g,n}$ on the space

$$\mathcal{Z}^\bullet(\Sigma_{g,n}^{X_1,\ldots,X_n}) := \text{Ext}^\bullet_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L \otimes g)$$

Via the cup-product, we should consider as our new ring of scalars

$$\mathcal{Z}^\bullet(\Sigma_{0,0}) := \text{Ext}^\bullet_{\mathcal{C}}(1, 1)$$

E.g. the mapping class group $\text{SL}_2(\mathbb{Z})$ of the torus $\Sigma_{1,0}$ acts on Hochschild cohomology $\text{HH}^\bullet(H, H)$, as module over $\text{HH}^\bullet(H, \mathbb{K})$ for any finite-dimensional factorizable ribbon Hopf algebra $H$.

Schweigert, Woike give a homotopy coherent construction incl. glueing
Derived topological field theory for non-semisimple MTCs

Example

For $C = \frac{S_3 \times VD}{S_3}$ over a field $K$ of characteristic 3, the torus gives

$Z^0(\Sigma_{1,0}) = K[Z_3 P_1] \oplus K[Z_2 P_1] \oplus K$

$Z^i(\Sigma_{1,0}) = \begin{cases} 
   K[Z_3 P_1], & \text{for } 0 \neq i = 0, 3 \\
   K_{sgn}[Z_3 P_1], & \text{for } 0 \neq i = 1, 2 \end{cases}$

where we denote the standard representation of $SL_2(\mathbb{Z}_N)$ with for Dirichlet character $\chi : \mathbb{Z}_N^\times \to K^\times$ by $K\chi[Z_N P_1]$

Similar picture for representations of $u_q(sl_2)$ in characteristic 0, work in progress Gainutdinov, L., Schweigert.

What are the additional modular forms for the vertex algebra $W_{p,1}$?
Theorem (Gainutdinov-Lentner-Ohrmann 2019)

For a semisimple Lie algebra \( \mathfrak{g} \) and root of unity \( q \) (now even order) there exists a finite dimensional quasi-Hopf algebra \( \tilde{u}_q(\mathfrak{g}) \), such that \( \text{Rep}(\tilde{u}_q(\mathfrak{g})) \) is a modular tensor category

[EG10], [A12], [AGP12], [CGR17], [HLYY18], [N18]

Idea of Proof.

- Start with a semisimple modular tensor category \( \mathcal{C}_0 = \text{Vect}_A^{\sigma,\omega} \)
- The nonsemisimple modular tensor category is
  \[
  \mathcal{C} = \frac{\mathcal{B}(\mathcal{M})}{\mathcal{B}(\mathcal{M})\mathcal{YD}(\mathcal{C}_0)}
  \]
- For convenience, we work out generators and relations.

Should be the right categorical pendant to kernel of screening VOAs
Given a Lie algebra \( \mathfrak{g} \) of type ADE and a diagram automorphism \( \theta \), then we can construct a Nichols algebra over nonabelian groups

\[
\langle \theta \rangle \to G \to \Gamma
\]

by folding \( u_q(\mathfrak{g})^+ \) at \( q = i \) with abelian coradical \( \Gamma \).

Conversely all Hopf algebras with central grouplike \( \theta \) are foldings.
Theorem (Heckenberger-Vendramin 2014)

Classification of all finite-dimensional Nichols algebras of rank \( \geq 2 \) over finite nonabelian groups \( G \) (with full support).

In characteristic 0: Four exceptions in rank 2, 3 and the folded types above

Work in progress with G. Sanmarco, I. Angiono: Is the following true?

- All of the folded types are cocycle twists of the foldings.
- The liftings in the diagonal case, which are \( \theta \)-invariant, give all liftings of these families of nondiagonal Nichols algebras.

Further questions:

- Produces modular tensor categories, \( \langle \theta \rangle \)-graded extensions.
- Are they realized by kernel of screenings of orbifold models \( \mathcal{N}^\theta_\Lambda \)?
Borel subalgebras

[Lie]: Borel subalgebras of $\mathfrak{g}$ are the maximal solvable subalgebras.
[Levi]: Every Lie algebra is sum of semisimple and solvable subalgebra

Question (Heckenberger "Borel subalgebras")

*Find the maximal right-coideal subalgebras $C$ of $U_q(\mathfrak{g})$, i.e.*

$$\Delta(C) \subset C \otimes U_q(\mathfrak{g})$$

*which are basic algebras, i.e. all fin.-dim. irreps are 1-dimensional.*
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Example

$U_q(\mathfrak{sl}_2)$ has a family of non-homogeneous Borel subalgebras $B_\lambda$, isomorphic to a $q$-Weyl algebra, with algebra generators

$$\bar{E} := \bar{E} K^{-1} + \lambda K^{-1}, \quad \bar{F} := F + \lambda' K^{-1}, \quad \lambda \lambda' = -\frac{q}{(q-q^{-1})^2}$$

Different choices of $\lambda, \lambda'$ are connected by Hopf algebra autom., the closure $\lambda, \lambda' \to \infty$ are the lower/upper standard Borel algebras.
Only partial classification results in L., Vocke 2019, 2020:

- Structure of the graded algebra of a right-coideal subalgebras.
- Conjectural formula for all triangular basic right-coideal subalgebras.
- Proven non-basicness for all larger triangular right-coideal subalgebras.
- To prove basicness we need to construct the algebra: Smallest family of examples in type $A_n$.

Suffices to classify Borel subalgebras of $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$ and the triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$.
Triangular Borel subalgebras of $U_q(sl_4)$
It is interesting to study induced modules of 1-dim irreps!

\[ V(B, \chi) := U_q(g) \otimes_B \mathbb{C}_\chi \]

**Lemma**

*Any finite-dimensional irrep of $U_q(g)$ is a quotient of $V(B, \chi)$.***

**Example**

The induced $U_q(sl_2)$-module with respect to a Borel algebra $B_\lambda$

\[ V(B_\lambda, \chi) := U_q(g) \otimes_{B\lambda} \mathbb{C}_\chi \cong \mathbb{C}[K, K^{-1}]1_\chi \]

has a nontrivial submodule $V'$ iff for some $n \in \mathbb{N}_0$

\[ \chi(\bar{E}) = \epsilon q^n \cdot \lambda, \text{ equivalently } \chi(\bar{F}) = \epsilon q^{-n} \cdot \lambda' \]

It is cofinite of codimension $[V : V'] = n + 1$. 
Borel subalgebras

\[ V(B_\lambda, \chi) := U_q(g) \otimes_{B_\lambda} \mathbb{C}_\chi \cong \mathbb{C}[K, K^{-1}]1_\chi \] and the action is

\[ K = \begin{pmatrix} \cdots & 0 & \cdots \\ \vdots & 1 & 0 & \vdots \\ \vdots & 1 & 0 & \vdots \\ & & \cdots & \end{pmatrix} \]

\[ F = \begin{pmatrix} \cdots & q^{2(n-1)}f & -q^{-2n} \lambda' & \cdots \\ \vdots & q^{2n}f & -q^{-2(n+1)} \lambda' & \vdots \\ & & q^{2(n+1)}f & \vdots \\ & & & \cdots & \end{pmatrix} \]

\[ E = \begin{pmatrix} \cdots & -q^{-2(n-1)-2} \lambda & \cdots \\ & q^{-2(n-1)-2} e & -q^{-2n-2} \lambda \\ & & q^{-2n-2} e & -q^{-2(n+1)-2} \lambda \\ & & & \cdots & \end{pmatrix} \]