PI Flashlights

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Some collaboration topics:

1. Derived TFT for nonsemisimple MTCs
2. More non-semisimple MTCs
3. New Borel-type subalgebras

Derived Topological Field Theory for non-semisimple MTCs

Theorem (Lyubaschenko 1995)

Any modular tensor category $\mathcal{C}$, possibly non-semisimple produces a modular functor, assigning to a decorated surface $\Sigma_{g,n}$

$$\mathcal{Z}:\ \Sigma_{g,n}^{X_1,\ldots,X_n} \longmapsto \mathrm{Hom}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^{\otimes g})$$

with an action of the mapping class group $\Gamma_{g,n}$ of the surface and a compatible way of glueing surfaces.

Here $L \in \mathcal{C}$ is the coend. For example $\mathcal{C} = \mathrm{Rep}(H)$ has $L = H^*_{coadj}$.

In conformal field theory, these are the spaces of chiral correlators, which are vector-valued modular forms under the action of $\Gamma_{g,n}$.

Derived topological field theory for non-semisimple MTCs

Theorem (L., Mierach, Schweigert, Sommerhäuser 2020)

Let $\mathcal{C}$ be a modular tensor category, possibly non-semisimple. We have an action of the mapping class group $\Gamma_{g,n}$ on the space

$$\mathcal{Z}^\bullet(\Sigma_{g,n}^{X_1,\ldots,X_n}) \;:=\; \mathrm{Ext}^\bullet_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^{\otimes g})$$

Via the cup-product, we should consider as our new ring of scalars

$$\mathcal{Z}^\bullet(\Sigma_{0,0}) \;:=\; \mathrm{Ext}^\bullet_{\mathcal{C}}(1,1)$$

E.g. the mapping class group $\mathrm{SL}_2(\mathbb{Z})$ of the torus $\Sigma_{1,0}$ acts on Hochschild cohomology $\mathrm{HH}^\bullet(H,H)$, as module over $\mathrm{HH}^\bullet(H,\mathbb{K})$ for any finite-dimensional factorizable ribbon Hopf algebra H.

Schweigert, Woike give a homotopy coherent construction incl. glueing

Derived topological field theory for non-semisimple MTCs

Example

For $\mathcal{C} = {}^{\mathbb{S}_3}_{\mathbb{S}_3}\mathcal{YD}$ over a field $\mathbb{K}$ of characteristic 3, the torus gives

$$\mathcal{Z}^0(\Sigma_{1,0}) = \quad \mathbb{K}[\mathbb{Z}_3\mathbb{P}^1] \oplus \mathbb{K}[\mathbb{Z}_2\mathbb{P}^1] \oplus \mathbb{K}$$

$$\mathcal{Z}^i(\Sigma_{1,0}) = \begin{cases} \mathbb{K}[\mathbb{Z}_3\mathbb{P}^1], & \text{for } 0 \neq i = 0,3\ (4) \\ \mathbb{K}_{sgn}[\mathbb{Z}_3\mathbb{P}^1], & \text{for } 0 \neq i = 1,2\ (4) \end{cases}$$

where we denote the standard representation of $\mathrm{SL}_2(\mathbb{Z}_N)$ with for Dirichlet character $\chi : \mathbb{Z}_N^\times \to \mathbb{K}^\times$ by $\mathbb{K}_\chi[\mathbb{Z}_N\mathbb{P}^1]$

Similar picture for representations of $u_q(\mathfrak{sl}_2)$ in characteristic 0, work in progress Gainutdinov, L., Schweigert.

What are the additional modular forms for the vertex algebra $\mathcal{W}_{p,1}$?

More non-semisimple modular tensor categories I

Theorem (Gainutdinov-Lentner-Ohrmann 2019)

For a semisimple Lie algebra $\mathfrak{g}$ and root of unity q (now even order) there exists a finite dimensional quasi-Hopf algebra $\tilde{u}_q(\mathfrak{g})$, such that $\mathrm{Rep}(\tilde{u}_q(\mathfrak{g}))$ is a modular tensor category

[EG10], [A12], [AGP12], [CGR17], [HLYY18], [N18]

Idea of Proof.

- Start with a semisimple modular tensor category $\mathcal{C}_0 = \mathrm{Vect}_A^{\sigma,\omega}$
- The nonsemisimple modular tensor category is

$$\mathcal{C} = {}^{\mathfrak{B}(M)}_{\mathfrak{B}(M)}\mathcal{YD}(\mathcal{C}_0)$$

- For convenience, we work out generators and relations. $\square$

Should be the right categorical pendant to kernel of screening VOAs

More non-semisimple modular tensor categories II

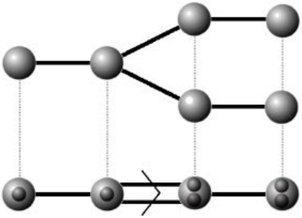
Theorem (L. 2013)

Given a Lie algebra $\mathfrak{g}$ of type ADE and a diagram automorphism θ, then we can construct a Nichols algebra over nonabelian groups

$$\langle\theta\rangle \to G \to \Gamma$$

by **folding** $u_q(\mathfrak{g})^+$ *at* $q = i$ *with abelian coradical* Γ.

Conversely all Hopf algebras with central grouplike θ are foldings.



More non-semisimple modular tensor categories II

Theorem (Heckenberger-Vendramin 2014)

Classification of all finite-dimensional Nichols algebras of rank ≥ 2 over finite nonabelian groups G (with full support).
In characteristic 0: Four exceptions in rank $2, 3$ and the folded types above

Work in progress with G. Sanmarco, I. Angiono: Is the following true?

- All of the folded types are cocycle twists of the foldings.
- The liftings in the diagonal case, which are θ-invariant, give all liftings of these families of nondiagonal Nichols algebras .

Further questions:

- Produces modular tensor categories, $\langle\theta\rangle$-graded extensions.
- Are they realized by kernel of screenings of orbifold models $\mathcal{V}_\Lambda^\theta$?

Borel subalgebras

[Lie]: Borel subalgebras of $\mathfrak{g}$ are the maximal solvable subalgebras.
[Levi]: Every Lie algebra is sum of semisimple and solvable subalgebra

Question (Heckenberger "Borel subalgebras")

Find the maximal right-coideal subalgebras C of $U_q(\mathfrak{g})$, i.e.

$$\Delta(C) \subset C \otimes U_q(\mathfrak{g})$$

which are **basic algebras**, *i.e. all fin.-dim. irreps are* 1*-dimensional.*

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Example

$U_q(\mathfrak{sl}_2)$ has a family of non-homogeneous Borel subalgebras B_λ, isomorphic to a q-Weyl algebra, with algebra generators

$$\bar{E} := EK^{-1} + \lambda K^{-1}, \qquad \bar{F} := F + \lambda' K^{-1}, \qquad \lambda\lambda' = -\frac{q}{(q-q^{-1})^2}$$

Different choices of λ, λ' are connected by Hopf algebra autom., the closure $\lambda, \lambda' \to \infty$ are the lower/upper standard Borel algebras.

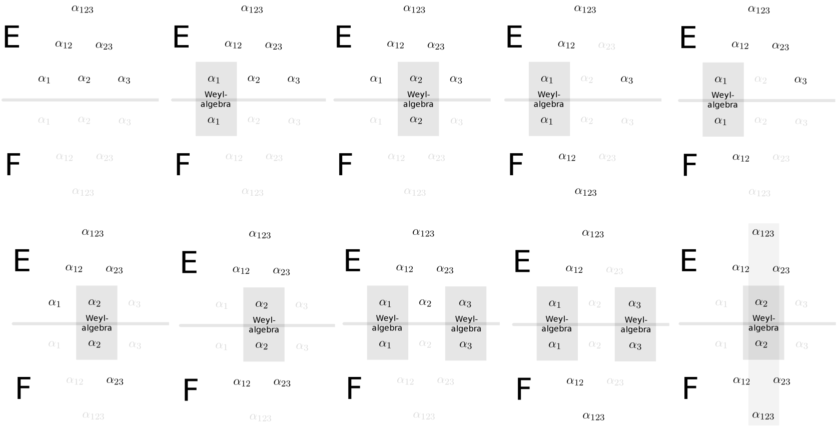
Borel subalgebras

Only partial classification results in L., Vocke 2019, 2020 :

- Structure of the graded algebra of a right-coideal subalgebras.
- Conjectural formula for all triangular basic right-coideal subalgebras.
- Proven non-basicness for all larger triangular right-coideal subalgebras.
- To prove basicness we need to construct the algebra: Smallest family of examples in type A_n.

Suffices to classify Borel subalgebras of $U_q(\mathfrak{sl}_2), U_q(\mathfrak{sl}_3)$ and the triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$

Triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$



Borel subalgebras

It is interesting to study induced modules of 1-dim irreps!

$$V(B,\chi) := U_q(\mathfrak{g}) \otimes_B \mathbb{C}_\chi$$

Lemma

Any finite-dimensional irrep of $U_q(\mathfrak{g})$ is a quotient of $V(B,\chi)$.

Example

The induced $U_q(\mathfrak{sl}_2)$-module with respect to a Borel algebra B_λ

$$V(B_\lambda,\chi) := U_q(\mathfrak{g}) \otimes_{B_\lambda} \mathbb{C}_\chi \cong \mathbb{C}[K,K^{-1}]1_\chi$$

has a nontrivial submodule V' iff for some $n \in \mathbb{N}_0$

$$\chi(\bar{E}) = \epsilon q^n \cdot \lambda, \text{ equivalently } \chi(\bar{F}) = \epsilon q^{-n} \cdot \lambda'$$

It is cofinite of codimension $[V : V'] = n+1$.

Borel subalgebras

$V(B_\lambda,\chi) := U_q(\mathfrak{g}) \otimes_{B_\lambda} \mathbb{C}_\chi \cong \mathbb{C}[K,K^{-1}]1_\chi$ and the action is

$$K = \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$F = \begin{pmatrix} \ddots & & & & \\ & q^{2(n-1)}f & -q^{-2n}\lambda' & & \\ & & q^{2n}f & -q^{-2(n+1)}\lambda' & \\ & & & q^{2(n+1)}f & \\ & & & & \ddots \end{pmatrix}$$

$$E = \begin{pmatrix} \ddots & & & & \\ & -q^{-2(n-1)-2}\lambda & & & \\ & q^{-2(n-1)-2}e & -q^{-2n-2}\lambda & & \\ & & q^{-2n-2}e & -q^{-2(n+1)-2}\lambda & \\ & & & & \ddots \end{pmatrix}$$