Introductory Lecture
Vertex Algebras and their Representation Theory

Simon Lentner

University of Hamburg

21 August 2019

Content:

- Correlators and Conformal Field Theory
- Vertex Algebras and their representations
- The screening method

1) Correlators and Conformal Field Theory

Background and Motivation:

Consider a stochastical experiment:

- A space Ω of possible configurations ω.
- A probability function $P : \Omega \to \mathbb{R}$, maybe non-normalized
- Some observables (i.e. random variables) $\mathcal{O}_i : \Omega \to \mathbb{C}$.

Our main output are expectation values

$$\langle \mathcal{O}_i \rangle = \frac{\int_\Omega P(\omega)\, \mathcal{O}_i(\omega)\, \mathrm{d}\omega}{\int_\Omega P(\omega)\, \mathrm{d}\omega}$$

and more generally the n-correlators

$$\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = \frac{\int_\Omega P(\omega)\, \mathcal{O}_{i_1}(\omega) \cdots \mathcal{O}_{i_n}(\omega)\, \mathrm{d}\omega}{\int_\Omega P(\omega)\, \mathrm{d}\omega}$$

1) Correlators and Conformal Field Theory

In physics, Ω as set of "states of the world" (positions, speed,...)

- $P(\omega) = e^{-\frac{1}{T}\, L[\omega]}$ with L the total energy in the state ω and T the temperature of a **thermodynamical system**.

1) Correlators and Conformal Field Theory

In physics, Ω as set of "states of the world" (positions, speed,...)

- $P(\omega) = e^{-\frac{1}{T}\, L[\omega]}$ with L the total energy in the state ω and T the temperature of a **thermodynamical system**.
- $P(\omega) = e^{-\frac{\mathrm{i}}{\hbar}\, L[\omega]}$ with L the Langrangian of the system. Now P is a complex-valued amplitude of a **quantum system**.

In the classical limit $\hbar, T \to 0$ the system remains at minimal $L[\omega]$.

1) Correlators and Conformal Field Theory

Now we consider **quantum field theory** on a manifold Σ:

- Ω is e.g. the set of all functions $\phi_1, \ldots, \phi_n : \Sigma \to \mathbb{C}$ etc.
- The expectation value $\int_\Omega$ is the ill-defined path integral.
- $L[\omega]$ is a functional and finding minimal ω is a variational problem, leading to Euler-Langrange equations for $\phi_i^{\mathrm{classical}}$
- Typical observables $\mathcal{O}$ are the evaluations $\phi_i(z)$ and their derivatives at a fixed point $z \in \Sigma$.

1) Correlators and Conformal Field Theory

Now we consider **quantum field theory** on a manifold Σ:

- Ω is e.g. the set of all functions $\phi_1, \ldots, \phi_n : \Sigma \to \mathbb{C}$ etc.
- The expectation value $\int_\Omega$ is the ill-defined path integral.
- $L[\omega]$ is a functional and finding minimal ω is a variational problem, leading to Euler-Langrange equations for $\phi_i^{\mathrm{classical}}$
- Typical observables $\mathcal{O}$ are the evaluations $\phi_i(z)$ and their derivatives at a fixed point $z \in \Sigma$.

In **conformal field theory** the manifold is a surface Σ_g of genus g and correlators $\langle \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) \rangle$ are invariant under conformal maps.

We split the correlators into holomorphic and antiholomorphic **chiral correlators** $\langle \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) \rangle$, which are multivalued. They are function on the moduli space of complex structures of $\Sigma_{g,n}$.

1) Correlators and Conformal Field Theory

Example

A single free field on Σ with values in $\mathbb{C}$.

- Ω is the space of functions $\phi : \Sigma \to \mathbb{C}$
- $L[\phi] = \frac{1}{2} \int_\Sigma |\nabla \phi|^2 \, \mathrm{d}x\mathrm{d}y$ (no interaction, no external fields)
- The minima $\phi^{\text{classical}}$ are waves, solving $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$

1) Correlators and Conformal Field Theory

Example

A single free field on Σ with values in $\mathbb{C}$.

- Ω is the space of functions $\phi : \Sigma \to \mathbb{C}$
- $L[\phi] = \frac{1}{2} \int_\Sigma |\nabla \phi|^2 \, \mathrm{d}x\mathrm{d}y$ (no interaction, no external fields)
- The minima $\phi^{\text{classical}}$ are waves, solving $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$

Chiral correlators for $z = x + \mathrm{i}y$, for example of $\partial\phi(z)$ with itself:

$$\langle \partial\phi(z_1) \rangle = 0$$

$$\langle \partial\phi(z_2)\partial\phi(z_1) \rangle = \frac{1}{(z_2 - z_1)^2}$$

$$\langle \partial\phi(z_3)\partial\phi(z_2)\partial\phi(z_1) \rangle = 0$$

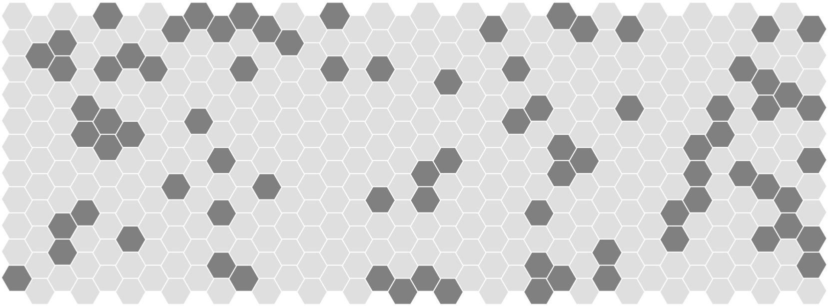
$$\langle \partial\phi(z_4)\partial\phi(z_3)\partial\phi(z_2)\partial\phi(z_1) \rangle = \frac{1}{(z_4-z_3)^2}\frac{1}{(z_2-z_1)^2} + \frac{1}{(z_4-z_2)^2}\frac{1}{(z_3-z_1)^2} + \frac{1}{(z_4-z_1)^2}\frac{1}{(z_3-z_2)^2}$$

Calculate?....**Axiomatize**!

1) Correlators and Conformal Field Theory
A statistical example: Critical Percolation

Take a rectangle with corners $z_1, z_2, z_3, z_4 \in \mathbb{C}$.
Fill it with a hexagonal lattice of mesh $\epsilon \to 0$.
Color each hexagon randomly with probability p.

With which probability $\mathbb{P}(p)$ exists a colored path left-to-right?

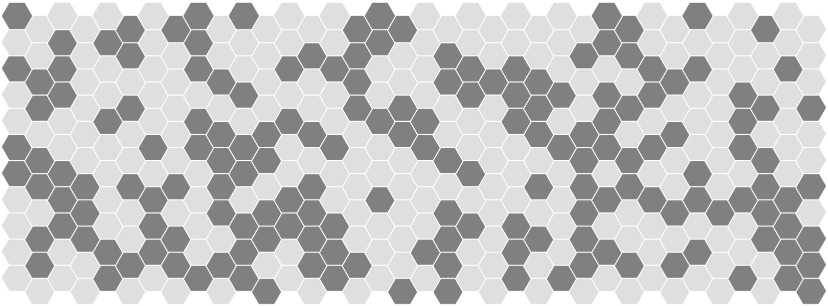


$p = 20\%$

1) Correlators and Conformal Field Theory
A statistical example: Critical Percolation

Take a rectangle with corners $z_1, z_2, z_3, z_4 \in \mathbb{C}$.
Fill it with a hexagonal lattice of mesh $\epsilon \to 0$.
Color each hexagon randomly with probability p.

With which probability $\mathbb{P}(p)$ exists a colored path left-to-right?

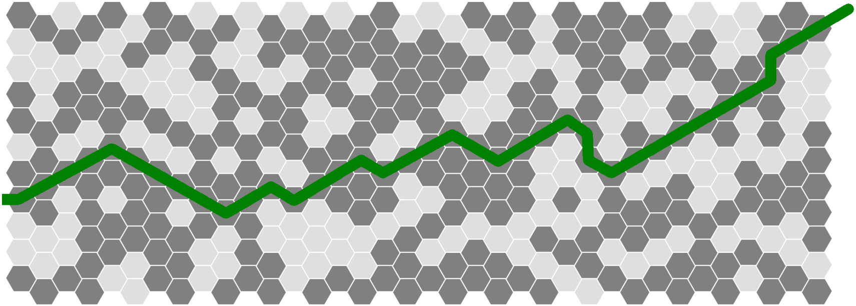


$p = 40\%$

1) Correlators and Conformal Field Theory
A statistical example: Critical Percolation

Take a rectangle with corners $z_1, z_2, z_3, z_4 \in \mathbb{C}$.
Fill it with a hexagonal lattice of mesh $\epsilon \to 0$.
Color each hexagon randomly with probability p.

With which probability $\mathbb{P}(p)$ exists a colored path left-to-right?

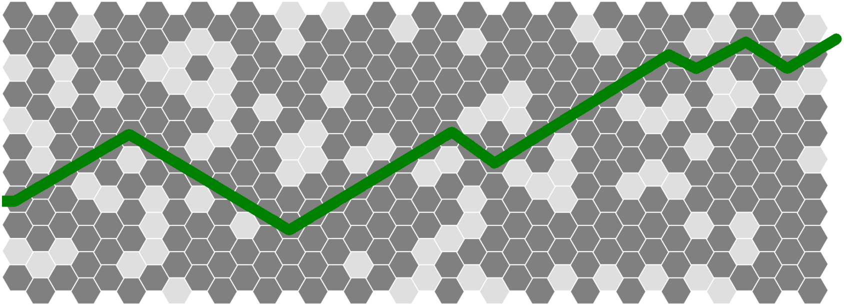


$p = 60\%$

1) Correlators and Conformal Field Theory
A statistical example: Critical Percolation

Take a rectangle with corners $z_1, z_2, z_3, z_4 \in \mathbb{C}$.
Fill it with a hexagonal lattice of mesh $\epsilon \to 0$.
Color each hexagon randomly with probability p.

With which probability $\mathbb{P}(p)$ exists a colored path left-to-right?



$p = 80\%$

Cardy (1992) conjectured from quantum field theory the formula

$$\mathbb{P}(p) \;=\; \begin{cases} 0 & \text{for } p < p_{crit} \\ \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)} z^{1/3} \cdot {}_2\mathrm{F}_1(z) & \text{for } p = p_{crit} \\ 1 & \text{for } p > p_{crit} \end{cases}$$

For $p = p_{crit}$ we have invariance under conformal transformations, in particular $\mathbb{P}(p)$ only depends on the crossratio $z := \frac{z_1 - z_2}{z_1 - z_3} \frac{z_3 - z_4}{z_2 - z_4}$.

Smirnov (2001) proved this for the hexagonal lattice, $p_{crit} := 50\%$.

This and other observables of critical percolation are manifestation of a conformal field theory $\mathcal{W}_{2,3}$ with logarithmic singularities.

Cardy (1992) conjectured from quantum field theory the formula

$$\mathbb{P}(p) = \begin{cases} 0 & \text{for } p < p_{crit} \\ \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)} z^{1/3} \cdot {}_2\mathrm{F}_1(z) & \text{for } p = p_{crit} \\ 1 & \text{for } p > p_{crit} \end{cases}$$

For $p = p_{crit}$ we have invariance under conformal transformations, in particular $\mathbb{P}(p)$ only depends on the crossratio $z := \frac{z_1 - z_2}{z_1 - z_3} \frac{z_3 - z_4}{z_2 - z_4}$.

Smirnov (2001) proved this for the hexagonal lattice, $p_{crit} := 50\%$.

This and other observables of critical percolation are manifestation of a conformal field theory $\mathcal{W}_{2,3}$ with logarithmic singularities.

Course of action

Axiomatize conformal field theory (e.g. Wighteman Axioms)
Axiomatize chiral conformal field theory (e.g. Vertex Algebra)
Study "only" their representation theory (modular tensor category)
$\Longrightarrow$ topological invariants DJW (1990) and modular form data.
Conversely reconstruct conformal field theories (screening method).

2) Vertex algebras

Definition

A *Vertex Operator Algebra (VOA)* is a collection of data:

- a graded vector space $\mathcal{V}$
- a vacuum vector 1
- a linear operator $\partial : \mathcal{V} \mapsto \mathcal{V}$
- a map called *vertex operator*

$$\mathrm{Y} : \ \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V}[[z, z^{-1}]]$$

- an action of the Virasoro algebra spanned by $\mathsf{L}_n, n \in \mathbb{Z}$,

fulfilling several compatibility axioms.

Definition

A *module* $\mathcal{M}$ of a VOA $\mathcal{V}$ is a vector space together with a map

$$Y_{\mathcal{M}}: \mathcal{V} \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}[[z, z^{-1}]]$$

and an action of the Virasoro algebra,
fulfilling similar compatibility axioms.

Definition

A *module* $\mathcal{M}$ of a VOA $\mathcal{V}$ is a vector space together with a map

$$Y_{\mathcal{M}} : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}[[z, z^{-1}]]$$

and an action of the Virasoro algebra,
fulfilling similar compatibility axioms.

Definition

An *intertwiner* between VOA modules $\mathcal{M}, \mathcal{N}, \mathcal{L}$ is a map

$$\mathrm{Y}_{\mathcal{M}, \mathcal{N}, \mathcal{L}} : \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to \mathcal{L}\{z\}[\log(z)]$$

where $\{z\}$ are power series with exponentials in $\mathbb{C}$,
fulfilling again similar compatibility axioms.

Theorem (Huang '08, Huang, Lepowsky, Zhang '11)

Let $\mathcal{V}$ be C_2-cofinite, then $\mathrm{Rep}(\mathcal{V})$ is a braided tensor category.

Idea of the proof.

The tensor product $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ is defined by having an intertwiner

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}} : \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)]$$

and being universal with respect to this property.

The braiding $c_{\mathcal{M},\mathcal{N}} : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{N} \to \mathcal{N} \otimes_{\mathcal{V}} \mathcal{M}$ is roughly defined by

$$c_{\mathcal{M},\mathcal{N}} \circ \mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}(z) = \mathrm{Y}_{\mathcal{N}\otimes\mathcal{M}}(-z)$$

where we analytically continue z to $-z$ counterclock-wise. □

Compare this to the tensor product $\otimes_{\mathcal{V}}$ over a commutative ring.

$\mathrm{Rep}(\mathcal{V})$ is a modular tensor category (conj. for nonsemisimple).

2) Vertex Algebras: The Heisenberg Algebra

Example

Take the vector space $\mathcal{V}_{\mathcal{H}} := \mathbb{C}[\partial\phi, \partial^2\phi, \cdots]$ with vertex operator

$$\mathrm{Y}(\partial\phi)\partial\phi = z^{-2} \cdot 1 + \sum_{k\geq 0} \frac{z^k}{k!} \partial\phi \, \partial^{1+k}\phi$$

- For every $a \in \mathbb{C}$ there is an irreducible module:

$$\mathcal{V}_a := \mathbb{C}[\partial\phi, \partial^2\phi, \cdots]e^{a\phi}, \qquad \mathrm{Y}_{\mathcal{V}_a}(\partial\phi)e^{a\phi} = az^{-1} \cdot e^{a\phi} + \cdots$$

- The tensor product product follows from some intertwiners

$$\mathcal{V}_a \otimes \mathcal{V}_b = \mathcal{V}_{a+b} \qquad \mathrm{Y}_{\mathcal{V}_a \otimes \mathcal{V}_b}(e^{a\phi})e^{b\phi} = z^{ab} \cdot e^{(a+b)\phi} + \cdots$$

- Hence the braiding is

$$c_{\mathcal{V}_a, \mathcal{V}_b} : \ \mathcal{V}_a \otimes \mathcal{V}_b \xrightarrow{e^{i\pi ab}} \mathcal{V}_b \otimes \mathcal{V}_a$$

2) Vertex Algebras: The Heisenberg Algebra

Example

The modular category of semisimple modules of $\mathcal{V}_{\mathcal{H}}$ is hence

$$\mathrm{Vect}_{\mathbb{C}}^{\sigma,\omega}, \quad \text{with} \quad \sigma(\lambda,\mu) = e^{\pi \mathrm{i}\, \lambda\mu}, \quad \omega = 1$$

The modular category of finite-length modules of $\mathcal{V}_{\mathcal{H}}$ is

$$\mathrm{Rep}(\mathbb{C}[X], \Delta, R) \quad \text{with} \quad \Delta(X) = 1 \otimes X + X \otimes 1, \quad R = \mathrm{e}^{\pi \mathrm{i}\, X \otimes X}$$

A typical 2-point correlation function on the sphere $\Sigma_{0,2}(z_1, z_2)$ is

$$\langle 1, \mathrm{Y}(\partial\phi, z_1)\mathrm{Y}(\partial\phi, z_2)1\rangle = \sum_{j\geq 0}(-1)^j \binom{-2}{j} z_1^{-2-j} z_2^j = (z_2 - z_1)^{-2}$$

The 0-point correlation functions on the torus $\Sigma_{1,0}(\tau)$, $q = e^{2\pi \mathrm{i}\tau}$

$$\mathrm{Tr}\langle 1, \mathrm{Y}(a_1, z_1)\mathrm{Y}(a_2, z_2)1\rangle = q^{-1/24} \sum_{n\geq 0} \dim(\mathcal{V}_{\mathcal{H}})_n = q^{\lambda^2/2}/\eta(q)$$

3) Vertex Algebras: The Lattice Vertex Algebra

Build a finite theory by simple-current-extension $Rep^{loc}(\Lambda)$ $\left(\mathrm{Vect}_{\mathbb{C}^n}^{\sigma,\omega}\right)$

Example

For an even integral lattice Λ, there is a Lattice VOA $\mathcal{V}_\Lambda$ with

$$\mathrm{Vect}_{\Lambda^*/\Lambda}^{\sigma,\omega} \quad \text{with} \quad \sigma(\lambda,\mu) := e^{\pi\mathrm{i}(\lambda,\mu)_\Lambda} \quad \omega(\lambda,\mu,\nu) \neq 0$$

Each module $\mathcal{V}_{\lambda+\Lambda}$ is a sum of Heisenberg modules $\displaystyle\bigoplus_{\lambda+\alpha\in\lambda+\Lambda} \mathcal{V}_{\lambda+\alpha}$

The 0-point correlation functions on the torus $\Sigma_{1,0}(\tau)$ are

$$\chi_{\lambda+\Lambda}(q) = \sum_{\lambda+\alpha\,\in\,\lambda+\Lambda} q^{|\lambda+\alpha|^2/2}/\eta(q)^{\mathrm{rank}} = \Theta_{\lambda+\Lambda}(z)/\eta(z)^{\mathrm{rank}}$$

These functions give a vector valued modular form

$$\chi_{\lambda+\Lambda}\left(\frac{az+b}{cz+d}\right) = \sum_{I} \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\lambda,\mu} \cdot \chi_{\mu+\Lambda}(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

for a mapping class group representation ρ, visible in MTC & TFT.

3) The screening method

Goal: Free field realization

Realize a complicated vertex algebra, such as $\mathcal{W}_{2,3}$
as a subalgebra of a Heisenberg VOA, Lattice VOA, or similar.

Dotsenko/Fadeev, Wakimoto, Feigin/Frenkel, Felder,....

Goal: Logarithmic conformal field theory

Construct one or all vertex algebras $\mathcal{V}$, such that $\mathrm{Rep}(\mathcal{V})$
is a given (nonsemisimple!) modular tensor category $\mathcal{C}$,
in particular the representation theory of a quantum group.

Feigin/Semikhatov/Tipunin/Gainutdinov, Adamovic/Milas,
Tsuchiya/Nagatomo/Wood, Creutzig/Gainutdinov/Runkel, L....

3) The screening method

Definition

Let $\mathcal{V}$ be a VOA and $\mathcal{M}$, $\mathcal{N}$ modules.
The tensor product $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ is defined by having an intertwiner

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}} : \ \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)].$$

Now, we fix $m \in \mathcal{M}$ and for all modules $\mathcal{N}$, we get a map

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}(m,z) : \ \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)].$$

Integrating around $z = 0$ defines the **screening operator**

$$\mathfrak{Z}_m : \mathcal{N} \to \overline{\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}} \quad m \in \mathcal{M}$$

E.g. the Heisenberg VOA has screening operators $\mathfrak{Z}_\lambda$ for all $\lambda \in \mathbb{C}^{rank}$

Local screening operators $\mathcal{M} = \mathcal{V}$ generate actions of Lie algebras, for example the Virasoro algebra, a semisimple Lie algebra $\mathfrak{g}$,...

3) The screening method

What are the relations of non-local screening operators in $\mathcal{V}$?

Theorem (L. 17, Huang-L. in progress)

If the involved singularities are not too severe in some sense, then the screening operators $\mathfrak{Z}_m$ fulfill the relations in the Nichols algebra $\mathcal{B}(\mathcal{M})$ in the braided tensor category $\mathrm{Rep}(\mathcal{V})$

Example (L. 17)

If $\lambda_1, \cdots, \lambda_n \in \mathbb{C}^n$ are small enough in some sense, then the screening operators $\mathfrak{Z}_{\lambda_1}, \cdots, \mathfrak{Z}_{\lambda_n}$ on the Heisenberg VOA fulfill the relations of the diagonal Nichols algebra with braiding

$$q_{ij} = e^{\pi \mathrm{i}(\lambda_i, \lambda_j)}$$

Conjecture

Does the VOA constructed as kernel of screening operators on $\mathcal{V}$ always have the modular tensor category ${}^{\mathcal{B}(\mathcal{M})}_{\mathcal{B}(\mathcal{M})}\mathcal{YD}(\mathrm{Rep}(\mathcal{V}))$...?

3) The screening method - some words on the proof

The theorem amounts to proving a statement in complex analysis that a relation in the Nichols algebra for $q_{ij} = e^{\pi \mathrm{i} m_{ij}}$ implies a linear relation between the generalized Selberg integrals

$$\mathrm{F}(m_{ij}, m_i) := \int \cdots \int_{[e^0, e^{2\pi \mathrm{i}}]^n} \prod_i z_i^{m_i} \prod_{i<j} (z_i - z_j)^{m_{ij}} \, \underset{\cdot}{z}_1 \ldots \underset{\cdot}{z}_n$$

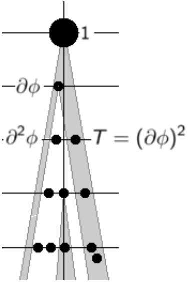
For example in degree 2 we have explicitly

$$\mathrm{F}(m_1, m_2, m_{12}) = \frac{e^{2\pi \mathrm{i} m_2} - 1}{2\pi \mathrm{i}} \frac{e^{2\pi \mathrm{i} m_1 + 2\pi \mathrm{i} m_{12}} - 1}{2\pi \mathrm{i}} \, \frac{1}{m_1 + m_2 + m_{12} + 2} \cdot$$
$$\cdot \left(\mathrm{B}(m_2 + 1, m_{12} + 1) + \frac{\sin \pi m_1}{\sin \pi (m_1 + m_{12})} \mathrm{B}(m_1 + 1, m_{12} + 1) \right)$$

We can see the Nichols algebra relation $x_1^2 = 0$ for m_{12} odd integer and the Nichols algebra relation $[x_1, x_2]_\pm = 0$ for $2m_{12}$ an integer outsides the poles at $m_{12} \in -\mathbb{N}$, where we get extensions.

Additional material

Structure of the Heisenberg algebra $\mathbb{C}[\partial\phi, \partial^2\phi, \ldots]$
as module over the Virasoro algebra [Feigin Fuks 1982]

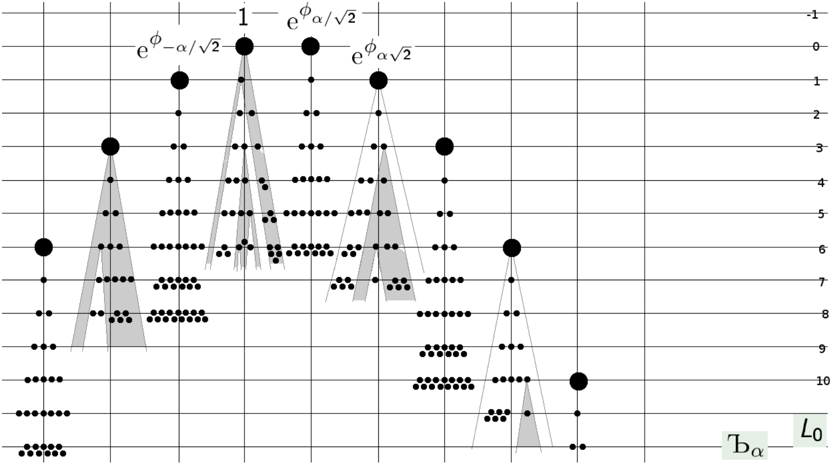


As Hilbert series we get the inverse eta function

$$\sum_{n\geq 0} \dim(V_n) q^n = \sum_{n\geq 0} p(n) q^n = \prod_{n\geq 0} (1 - q^n)^{-1}$$

Additional material

Structure of the Lattice Vertex algebra $\mathcal{V}_\Lambda,\ \Lambda = \sqrt{2}A_1$
and a second module of its four modules



Additional material

Actions of the screening operator $\mathfrak{Z}_{-\alpha/\sqrt{2}}$ with $(\mathfrak{Z}_{-\alpha/\sqrt{2}})^2 = 0$ and in grey kernel of the screening, the triplet algebra $\mathcal{W}_{2,1}$