Hopf-algebraic structures inspired by Kitaev models

Vincent Koppen

Universität Hamburg

Workshop: Hopf Algebras and Tensor Categories

21 August, 2020

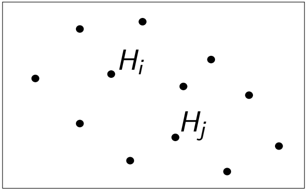
Background

2-dimensional gapped quantum many-body systems with a locally interacting Hamiltonian:

Background

2-dimensional gapped quantum many-body systems with a locally interacting Hamiltonian:

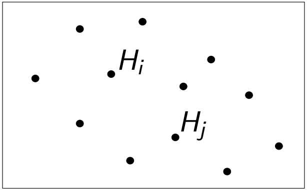
The Hilbert space $\mathcal{H} = \bigotimes_i H_i$ of such a system is composed of a set of finite-dimensional Hilbert spaces $(H_i)_i$, describing local degrees of freedom, associated to the vertices i of a hypergraph embedded in a surface.



Background

2-dimensional gapped quantum many-body systems with a locally interacting Hamiltonian:

The Hilbert space $\mathcal{H} = \bigotimes_i H_i$ of such a system is composed of a set of finite-dimensional Hilbert spaces $(H_i)_i$, describing local degrees of freedom, associated to the vertices i of a hypergraph embedded in a surface.

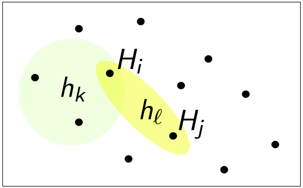


Example: $H_i = \mathbb{C}^2$ is a spin-$\frac{1}{2}$ system.

Background

2-dimensional gapped quantum many-body systems with a locally interacting Hamiltonian:

The Hilbert space $\mathcal{H} = \bigotimes_i H_i$ of such a system is composed of a set of finite-dimensional Hilbert spaces $(H_i)_i$, describing local degrees of freedom, associated to the vertices i of a hypergraph embedded in a surface.

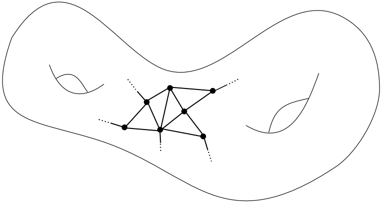


Example: $H_i = \mathbb{C}^2$ is a spin-$\frac{1}{2}$ system.

The Hamiltonian $h = \sum_k h_k$ is a sum of local interaction terms h_k that each involve only a bounded number of subsystems H_i.

Kitaev model

The Kitaev model is a family of such quantum many-body systems defined on any compact oriented surface Σ with cell decomposition.

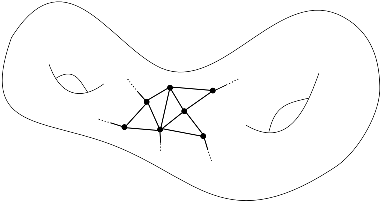


- Σ^2 : set of 2-cells
- Σ^1 : set of edges
- Σ^0 : set of vertices

The local degrees of freedom are associated to the edges

Kitaev model

The Kitaev model is a family of such quantum many-body systems defined on any compact oriented surface Σ with cell decomposition.



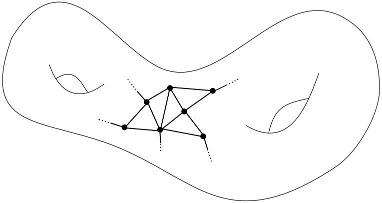
- Σ^2 : set of 2-cells
- Σ^1 : set of edges
- Σ^0 : set of vertices

The local degrees of freedom are associated to the edges and are given by a finite-dimensional semisimple complex Hopf C^*-algebra H. The Hilbert space is

$$\mathcal{H} = \bigotimes_{e \in \Sigma^1} H.$$

Kitaev model

The Kitaev model is a family of such quantum many-body systems defined on any compact oriented surface Σ with cell decomposition.



- Σ^2 : set of 2-cells
- Σ^1 : set of edges
- Σ^0 : set of vertices

The local degrees of freedom are associated to the edges and are given by a finite-dimensional semisimple complex Hopf C^*-algebra H. The Hilbert space is

$$\mathcal{H} = \bigotimes_{e \in \Sigma^1} H.$$

Example: The Kitaev model for $H = \mathbb{C}\mathbb{Z}_2 = \mathbb{C}^2$, considered on the torus, is known as the *toric code* (Kitaev, 1997).

Hopf algebras

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.
Every vector space we consider is finite-dimensional over $\mathbb{k}$.

Hopf algebras

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.
Every vector space we consider is finite-dimensional over $\mathbb{k}$.

Definition

A *Hopf algebra* is an algebra H together with algebra maps $\Delta : H \longrightarrow H \otimes H$ and $\varepsilon : H \longrightarrow \mathbb{k}$ turning H into a coalgebra such that there exists an antipode $S : H \longrightarrow H$.

Hopf algebras

Let $\Bbbk$ be an algebraically closed field of characteristic zero.
Every vector space we consider is finite-dimensional over $\Bbbk$.

Definition

A *Hopf algebra* is an algebra H together with algebra maps $\Delta : H \longrightarrow H \otimes H$ and $\varepsilon : H \longrightarrow \Bbbk$ turning H into a coalgebra such that there exists an antipode $S : H \longrightarrow H$.

Example

- The group algebra $\Bbbk G$ for a finite group G
- The dual Hopf algebra H^* for any Hopf algebra H

Hopf algebras

Let $\Bbbk$ be an algebraically closed field of characteristic zero.
Every vector space we consider is finite-dimensional over $\Bbbk$.

Definition

A *Hopf algebra* is an algebra H together with algebra maps $\Delta : H \longrightarrow H \otimes H$ and $\varepsilon : H \longrightarrow \Bbbk$ turning H into a coalgebra such that there exists an antipode $S : H \longrightarrow H$.

Example

- The group algebra $\Bbbk G$ for a finite group G
- The dual Hopf algebra H^* for any Hopf algebra H

Remark

The representation category H–mod *is a tensor category. Conversely, the Hopf algebra H can be reconstructed from the tensor category* H–mod *with the forgetful functor* $H\text{–mod} \to \mathrm{vect}(\Bbbk)$.

The Haar integral

Proposition

If H is a semisimple Hopf algebra, then there exists a unique Haar integral $\ell \in H$, i.e.:

- $x\ell = \varepsilon(x)\ell = \ell x$ *for all* $x \in H$,
- $\varepsilon(\ell) = 1$.

The Haar integral

Proposition

If H is a semisimple Hopf algebra, then there exists a unique Haar integral $\ell \in H$, i.e.:

- $x\ell = \varepsilon(x)\ell = \ell x$ *for all* $x \in H$,
- $\varepsilon(\ell) = 1$.

The defining properties of the Haar integral imply that it is an idempotent: $\ell^2 = \ell$.

The Haar integral

Proposition

If H is a semisimple Hopf algebra, then there exists a unique Haar integral $\ell \in H$, i.e.:

- $x\ell = \varepsilon(x)\ell = \ell x$ *for all* $x \in H$,
- $\varepsilon(\ell) = 1$.

The defining properties of the Haar integral imply that it is an idempotent: $\ell^2 = \ell$.

Example

Let G be a finite group.
Then the Haar integral of $\Bbbk G$ is $\frac{1}{|G|} \sum_{g \in G} g$.

Commuting-projector Hamiltonian

Using the given Hopf-algebraic structure, one constructs (Buerschaper-Mombelli-Christandl-Aguado 2010):

Commuting-projector Hamiltonian

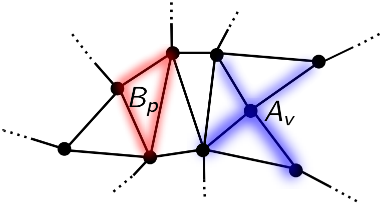
Using the given Hopf-algebraic structure, one constructs (Buerschaper-Mombelli-Christandl-Aguado 2010):

- a finite-dimensional vector space $\mathcal{H}_{\Sigma} = \bigotimes_{e \in \Sigma^1} H$,

Commuting-projector Hamiltonian

Using the given Hopf-algebraic structure, one constructs (Buerschaper-Mombelli-Christandl-Aguado 2010):

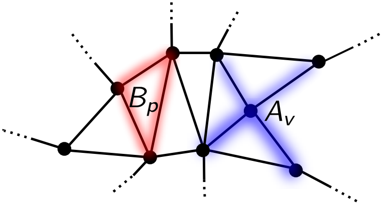
- a finite-dimensional vector space $\mathcal{H}_\Sigma = \bigotimes_{e \in \Sigma^1} H$,
- a family of local commuting projectors $(A_v)_{v \in \Sigma^0}$ and $(B_p)_{p \in \Sigma^2}$ on $\mathcal{H}_\Sigma$, using the Haar integrals of H and of H^*, respectively,



Commuting-projector Hamiltonian

Using the given Hopf-algebraic structure, one constructs (Buerschaper-Mombelli-Christandl-Aguado 2010):

- a finite-dimensional vector space $\mathcal{H}_\Sigma = \bigotimes_{e \in \Sigma^1} H$,
- a family of local commuting projectors $(A_v)_{v \in \Sigma^0}$ and $(B_p)_{p \in \Sigma^2}$ on $\mathcal{H}_\Sigma$, using the Haar integrals of H and of H^*, respectively,



giving rise to the Hamiltonian

$$h := \sum_{v \in \Sigma^0} (\mathrm{id} - A_v) + \sum_{p \in \Sigma^2} (\mathrm{id} - B_p) : \mathcal{H}_\Sigma \longrightarrow \mathcal{H}_\Sigma$$

Ground-state space of the Kitaev model

The ground-state space

$$\mathcal{H}_0 := \ker(h) = \mathrm{im}\Big(\prod_{v\in\Sigma^0} A_v \prod_{p\in\Sigma^2} B_p\Big)$$

is the state space of the three-dimensional topological field theory of Turaev-Viro type for the spherical fusion category H–mod. As such, it carries a mapping class group action.

Ground-state space of the Kitaev model

The ground-state space

$$\mathcal{H}_0 := \ker(h) = \mathrm{im}(\prod_{v\in\Sigma^0} A_v \prod_{p\in\Sigma^2} B_p)$$

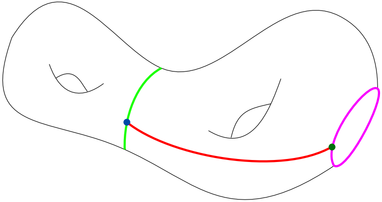
is the state space of the three-dimensional topological field theory of Turaev-Viro type for the spherical fusion category H–mod. As such, it carries a mapping class group action.

Remark

In applications to topological quantum computing, information is encoded in the subspace $\mathcal{H}_0 \subset \mathcal{H}$. Sufficiently local errors are dynamically corrected by the Hamiltonian. Quantum gates are implemented by the mapping class group action.

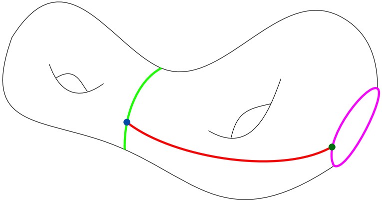
Defects in topological field theories

Topological field theories with defects (and boundaries) are defined on decorated stratified manifolds.

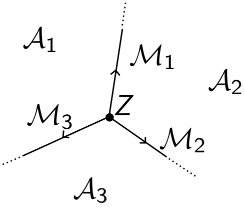


Defects in topological field theories

Topological field theories with defects (and boundaries) are defined on decorated stratified manifolds.



For theories of Turaev-Viro type, the defect label data are given in category-theoretic terms as follows (e.g. Kitaev-Kong 2011):



- $\mathcal{A}_i$: tensor category
- $\mathcal{M}_i$: bimodule category over the adjacent tensor categories
- Z : object in a cyclic relative Deligne product $\circlearrowleft\!\!\otimes_{\mathcal{A}_1}(\mathcal{M}_1 \boxtimes_{\mathcal{A}_2} \cdots \boxtimes_{\mathcal{A}_n} \mathcal{M}_n)$

Defects in Kitaev models

Based on *arXiv:2001.10578*

For the construction of a Kitaev model with defects, one first needs to realize the category-theoretic defect data in Hopf-algebraic terms:

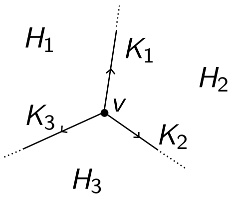
$$\text{tensor category } \mathcal{A} \rightsquigarrow \text{ Hopf algebra } H$$

$$\mathcal{A}_1\text{-}\mathcal{A}_2\text{-bimodule category } \mathcal{M} \rightsquigarrow H_1\text{-}H_2\text{-bicomodule algebra } K$$

For the construction of a Kitaev model with defects, one first needs to realize the category-theoretic defect data in Hopf-algebraic terms:

$$\text{tensor category } \mathcal{A} \rightsquigarrow \text{ Hopf algebra } H$$
$$\mathcal{A}_1\text{-}\mathcal{A}_2\text{-bimodule category } \mathcal{M} \rightsquigarrow H_1\text{-}H_2\text{-bicomodule algebra } K$$

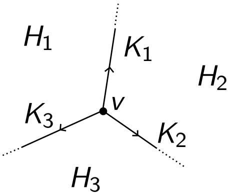
For a vertex



we define a $\Bbbk$-algebra $C_v := (H_1^* \otimes \cdots \otimes H_n^*) \gtrdot (K_1 \otimes \cdots \otimes K_n)$, which we call the *vertex algebra* for v.

Defects in Kitaev models

Consider a vertex



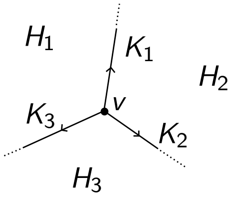
and recall that K_i–mod are bimodule categories.

Theorem

There is a canonical equivalence of categories
$C_v\text{–mod} \cong \circlearrowleft\!\!\otimes_{H_1\text{–mod}}(K_1\text{–mod} \boxtimes_{H_2\text{–mod}} \cdots \boxtimes_{H_n\text{–mod}} K_n\text{–mod}).$

Defects in Kitaev models

Consider a vertex



and recall that K_i–mod are bimodule categories.

Theorem

There is a canonical equivalence of categories
$C_v\text{–mod} \cong \circlearrowleft\!\!\otimes_{H_1\text{–mod}}(K_1\text{–mod} \boxtimes_{H_2\text{–mod}} \cdots \boxtimes_{H_n\text{–mod}} K_n\text{–mod})$.

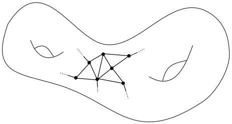
Remark

$\circlearrowleft\!\!\otimes_{\mathcal{A}_1}(\mathcal{M}_1 \boxtimes_{\mathcal{A}_2} \cdots \boxtimes_{\mathcal{A}_n} \mathcal{M}_n)$ *is the category of vertex labels for the state-sum construction of the modular functor by Fuchs-Schaumann-Schweigert (2019).*

Defects in Kitaev models

The input to our construction is:

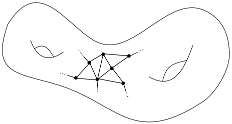
- A compact oriented surface Σ with cell decomposition $(\Sigma^0, \Sigma^1, \Sigma^2)$,



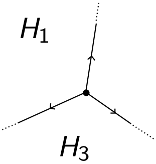
Defects in Kitaev models

The input to our construction is:

- A compact oriented surface Σ with cell decomposition (Σ^0, Σ^1, Σ^2),



- A decoration of the cells by:

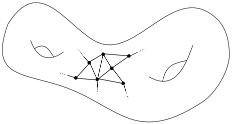


  - a semisimple Hopf algebra H_p for $p \in \Sigma^2$,

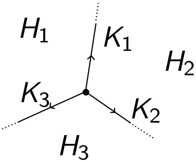
Defects in Kitaev models

The input to our construction is:

- A compact oriented surface Σ with cell decomposition (Σ^0, Σ^1, Σ^2),



- A decoration of the cells by:

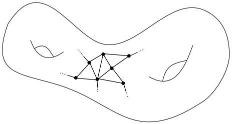


  - a semisimple Hopf algebra H_p for $p \in \Sigma^2$,
  - a semisimple bicomodule algebra K_e for $e \in \Sigma^1$ over the adjacent Hopf algebras,

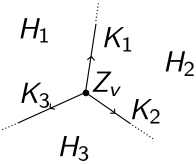
Defects in Kitaev models

The input to our construction is:

- A compact oriented surface Σ with cell decomposition (Σ^0, Σ^1, Σ^2),



- A decoration of the cells by:



  - a semisimple Hopf algebra H_p for $p \in \Sigma^2$,
  - a semisimple bicomodule algebra K_e for $e \in \Sigma^1$ over the adjacent Hopf algebras,
  - a representation Z_v of the vertex algebra $C_v = (H_1^* \otimes \cdots \otimes H_n^*) \circledgt (K_1 \otimes \cdots \otimes K_n)$ for every $v \in \Sigma^0$.

Defects in Kitaev models

Let Σ be a compact oriented surface with a decorated cell decomposition $(\Sigma^0, \Sigma^1, \Sigma^2)$.

Theorem

From these data, we construct a family of local commuting projectors $(A_v)_{v\in\Sigma^0}$ and $(B_p)_{p\in\Sigma^2}$ on the finite-dimensional vector space

$$\widetilde{\mathcal{H}}_{\Sigma} = \Big(\bigotimes_{e\in\Sigma^1} K_e^*\Big) \otimes \Big(\bigotimes_{v\in\Sigma^0} Z_v\Big),$$

giving rise to the Hamiltonian

$$h := \sum_{v\in\Sigma^0} (\mathrm{id} - A_v) + \sum_{p\in\Sigma^2} (\mathrm{id} - B_p)$$

on $\widetilde{\mathcal{H}}_{\Sigma}$.

Defects in Kitaev models

One crucial step in our construction of commuting projectors is the suitable generalization of the Haar integral to semisimple bicomodule algebras.

Defects in Kitaev models

One crucial step in our construction of commuting projectors is the suitable generalization of the Haar integral to semisimple bicomodule algebras. It is known that

Proposition

Any finite-dimensional semisimple $\Bbbk$-algebra A possesses a unique symmetric separability idempotent $\sum p^1 \otimes p^2 \in A \otimes A^{\mathrm{op}}$, i.e.:

- $\sum xp^1 \otimes p^2 = \sum p^1 \otimes p^2 x$ *for all* $x \in A$,
- $\sum p^1 p^2 = 1$,
- $\sum p^1 \otimes p^2 = \sum p^2 \otimes p^1$. *(symmetry)*

Defects in Kitaev models

One crucial step in our construction of commuting projectors is the suitable generalization of the Haar integral to semisimple bicomodule algebras. It is known that

Proposition

Any finite-dimensional semisimple $\mathbb{k}$-algebra A possesses a unique symmetric separability idempotent $\sum p^1 \otimes p^2 \in A \otimes A^{\mathrm{op}}$, i.e.:

- $\sum xp^1 \otimes p^2 = \sum p^1 \otimes p^2 x$ *for all* $x \in A$,
- $\sum p^1 p^2 = 1$,
- $\sum p^1 \otimes p^2 = \sum p^2 \otimes p^1$. *(symmetry)*

We furthermore need (and show) for our construction that

Lemma

If H is a Hopf algebra and A is an H-comodule algebra, then the symmetric separability idempotent $\sum p^1 \otimes p^2$ is contained in the H-coinvariant subspace $(A \otimes A^{\mathrm{op}})^{\mathrm{co}H} \subseteq A \otimes A^{\mathrm{op}}$.

Upshot:
We have constructed a quite explicit Hamiltonian model within a rather general framework for defects.

Upshot:
We have constructed a quite explicit Hamiltonian model within a rather general framework for defects.

Further work can be:
Show explicitly that the ground-state spaces of our Hamiltonians are naturally isomorphic to the vector spaces assigned to defect surfaces by the modular functor constructed by Fuchs-Schaumann-Schweigert 2019.

Idempotents for non-semisimple Hopf algebras

Based on *arXiv:1910.13161* jointly authored with Ehud Meir and Christoph Schweigert

Example

Let $H = \Bbbk G$ for a finite group G. The central orthogonal idempotents $(p_i)_i$ for the isotypic decomposition $\Bbbk G = \bigoplus_i S_i^{\dim(S_i)}$ of the regular representation are expressed by the *character-projector formula*:

$$p_i = \dim(S_i) \frac{1}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g$$

Example

Let $H = \Bbbk G$ for a finite group G. The central orthogonal idempotents $(p_i)_i$ for the isotypic decomposition $\Bbbk G = \bigoplus_i S_i^{\dim(S_i)}$ of the regular representation are expressed by the *character-projector formula*:

$$p_i = \dim(S_i)\frac{1}{|G|}\sum_{g\in G}\chi_i(g^{-1})g$$

For any *semisimple* Hopf algebra H, the orthogonal idempotents for the isotypic decomposition of the regular H-module are expressed by:

$$p_i := \dim(S_i)(S(\chi_i)\otimes \mathrm{id}_H)\Delta(\ell),$$

where $\ell \in H$ is the Haar integral, $(S_i)_i$ are the isomorphism classes of simple H-modules and $(\chi_i)_i$ their characters.

Idempotents for non-semisimple Hopf algebras

What about not necessarily semisimple Hopf algebras?

It is known that:

- If H non-semisimple, then the Haar integral does not exist. (Any integral $\ell \in H$, $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$, is nilpotent, since $\varepsilon(\ell) = 0$ due to Maschke's theorem.)

Idempotents for non-semisimple Hopf algebras

What about not necessarily semisimple Hopf algebras?

It is known that:

- If H non-semisimple, then the Haar integral does not exist. (Any integral $\ell \in H$, $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$, is nilpotent, since $\varepsilon(\ell) = 0$ due to Maschke's theorem.)
- However, if H semisimple, then $\ell = \frac{1}{\dim H}\chi_{H^*}$ due to Larson-Radford.

Idempotents for non-semisimple Hopf algebras

What about not necessarily semisimple Hopf algebras?

It is known that:

- If H non-semisimple, then the Haar integral does not exist. (Any integral $\ell \in H$, $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$, is nilpotent, since $\varepsilon(\ell) = 0$ due to Maschke's theorem.)
- However, if H semisimple, then $\ell = \frac{1}{\dim H}\chi_{H^*}$ due to Larson-Radford.
- If H is non-semisimple, then $p := \frac{1}{\dim H}\chi_{H^*}$ is still an idempotent.

Idempotents for non-semisimple Hopf algebras

What about not necessarily semisimple Hopf algebras?

It is known that:

- If H non-semisimple, then the Haar integral does not exist. (Any integral $\ell \in H$, $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$, is nilpotent, since $\varepsilon(\ell) = 0$ due to Maschke's theorem.)
- However, if H semisimple, then $\ell = \frac{1}{\dim H}\chi_{H^*}$ due to Larson-Radford.
- If H is non-semisimple, then $p := \frac{1}{\dim H}\chi_{H^*}$ is still an idempotent.

Our idea is hence to replace ℓ by p:

$$p_i := \dim(S_i)(S(\chi_i) \otimes \mathrm{id}_H)\Delta(p)$$

Question: Does this still give orthogonal idempotents?

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \mathbb{k}\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \mathbb{k}\langle g, x\rangle / (g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

- It has two simple (one-dimensional) modules: $\mathbb{k}_+$ and $\mathbb{k}_-$

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \Bbbk\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

- It has two simple (one-dimensional) modules: $\Bbbk_+$ and $\Bbbk_-$
- With our formula we indeed get orthogonal idempotents: $p_+ = \frac{1}{2}(1+g)$ and $p_- = \frac{1}{2}(1-g)$

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \mathbb{k}\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

- It has two simple (one-dimensional) modules: $\mathbb{k}_+$ and $\mathbb{k}_-$
- With our formula we indeed get orthogonal idempotents: $p_+ = \frac{1}{2}(1+g)$ and $p_- = \frac{1}{2}(1-g)$
- Furthermore, $H = Hp_+ \oplus Hp_-$ is a decomposition of the regular H-module, such that $Hp_\pm$ is a projective cover of $\mathbb{k}_\pm$

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \Bbbk\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

- It has two simple (one-dimensional) modules: $\Bbbk_+$ and $\Bbbk_-$
- With our formula we indeed get orthogonal idempotents: $p_+ = \frac{1}{2}(1+g)$ and $p_- = \frac{1}{2}(1-g)$
- Furthermore, $H = Hp_+ \oplus Hp_-$ is a decomposition of the regular H-module, such that $Hp_\pm$ is a projective cover of $\Bbbk_\pm$

H is called *basic* if all simple H-modules are one-dimensional.

Idempotents for non-semisimple Hopf algebras

Example

Let $H = \Bbbk\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be the four-dimensional Sweedler Hopf algebra.

- It has two simple (one-dimensional) modules: $\Bbbk_+$ and $\Bbbk_-$
- With our formula we indeed get orthogonal idempotents: $p_+ = \frac{1}{2}(1+g)$ and $p_- = \frac{1}{2}(1-g)$
- Furthermore, $H = Hp_+ \oplus Hp_-$ is a decomposition of the regular H-module, such that $Hp_\pm$ is a projective cover of $\Bbbk_\pm$

H is called *basic* if all simple H-modules are one-dimensional.

More generally: A Hopf algebra H is said to have the *Chevalley property* if the tensor product of any two semisimple H-modules is again semisimple.

Idempotents for non-semisimple Hopf algebras

This motivates:

Conjecture

Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the Chevalley property. Then we conjecture that the elements

$$p_i := \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)} \in H,$$

for $i \in I$, where I is the set of isomorphism classes of simple H-modules, define a set of orthogonal idempotents of H such that

$$H = \bigoplus_{i \in I} Hp_i$$

is an isotypic decomposition of the regular H-module, i.e. Hp_i is a projective cover of $S_i^{\oplus \dim(S_i)}$.

Idempotents for non-semisimple Hopf algebras

Let H be a Hopf algebra with the Chevalley property.

Idempotents for non-semisimple Hopf algebras

Let H be a Hopf algebra with the Chevalley property.

Theorem

Let $\chi : H \to \Bbbk$ be the character of a non-zero one-dimensional (hence, simple) H-module. Then $p_\chi = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p)) \in H$ is an idempotent such that $Hp_\chi \subseteq H$ is a χ-isotypic component of H.

Idempotents for non-semisimple Hopf algebras

Let H be a Hopf algebra with the Chevalley property.

Theorem

Let $\chi : H \to \Bbbk$ be the character of a non-zero one-dimensional (hence, simple) H-module. Then $p_\chi = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p)) \in H$ is an idempotent such that $Hp_\chi \subseteq H$ is a χ-isotypic component of H.

Let $\Lambda_0 \in H^*$ be the Haar integral of the maximal semisimple sub-Hopf-algebra $H_0^* = (H/J)^* \subseteq H^*$. Define the *Hecke algebra* $\mathcal{H}(H^*, H_0^*)$ as the subalgebra $\Lambda_0 H^* \Lambda_0 \subseteq H^*$.

Idempotents for non-semisimple Hopf algebras

Let H be a Hopf algebra with the Chevalley property.

Theorem

Let $\chi : H \to \Bbbk$ be the character of a non-zero one-dimensional (hence, simple) H-module. Then $p_\chi = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p)) \in H$ is an idempotent such that $Hp_\chi \subseteq H$ is a χ-isotypic component of H.

Let $\Lambda_0 \in H^*$ be the Haar integral of the maximal semisimple sub-Hopf-algebra $H_0^* = (H/J)^* \subseteq H^*$. Define the *Hecke algebra* $\mathcal{H}(H^*, H_0^*)$ as the subalgebra $\Lambda_0 H^* \Lambda_0 \subseteq H^*$.

Theorem

$$\sum_{i \in I} p_i = 1_H \quad \textit{(completeness relation)}$$

if and only if the Hecke algebra $\Lambda_0 H^* \Lambda_0$ has up to isomorphism only one simple module.

Idempotents for non-semisimple Hopf algebras

Let H_0^* be the maximal semisimple sub-Hopf-algebra of H^*.

Corollary

Let the Hopf algebra H be basic. Assume further that the associated Hecke algebra $\mathcal{H}(H^*, H_0^*)$ has, up to isomorphism, a unique simple $\mathcal{H}(H^*, H_0^*)$-module.
Then the conjecture holds for H, i.e.
$(p_i = \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)})_{i\in I}$ are orthogonal idempotents such that $H = \bigoplus_{i\in I} Hp_i$ is an isotypic decomposition for H.

Idempotents for non-semisimple Hopf algebras

Let H_0^* be the maximal semisimple sub-Hopf-algebra of H^*.

Corollary

Let the Hopf algebra H be basic. Assume further that the associated Hecke algebra $\mathcal{H}(H^*, H_0^*)$ has, up to isomorphism, a unique simple $\mathcal{H}(H^*, H_0^*)$-module.
Then the conjecture holds for H, i.e.
$(p_i = \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)})_{i\in I}$ are orthogonal idempotents such that $H = \bigoplus_{i\in I} Hp_i$ is an isotypic decomposition for H.

Remark

- *There are counter-examples to the conjecture for non-Chevalley Hopf algebras.*

Idempotents for non-semisimple Hopf algebras

Let H_0^* be the maximal semisimple sub-Hopf-algebra of H^*.

Corollary

Let the Hopf algebra H be basic. Assume further that the associated Hecke algebra $\mathcal{H}(H^*, H_0^*)$ has, up to isomorphism, a unique simple $\mathcal{H}(H^*, H_0^*)$-module.
Then the conjecture holds for H, i.e.
$(p_i = \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)})_{i \in I}$ are orthogonal idempotents such that $H = \bigoplus_{i \in I} Hp_i$ is an isotypic decomposition for H.

Remark

- *There are counter-examples to the conjecture for non-Chevalley Hopf algebras.*
- *We have computed an example of a 72-dimensional Hopf algebra, for which the conjecture holds, which is Chevalley but not basic. For this we used the computer algebra system Magma.*

Two natural open problems remain:

- Full proof the conjecture
- Application to the construction of a Kitaev model in the non-semisimple case

References

- M. Aguiar, *A note on strongly separable algebras*. Colloquium on Homology and Representation Theory (Vaquerías, 1998). Bol. Acad. Nac. Cienc. (Córdoba) 65 (2000), 51–60.
- O. Buerschaper, J. M. Mombelli, M. Christandl and M. Aguado, *A hierarchy of topological tensor network states*. J. Math. Phys. 54, 012201 (2013). arXiv:1007.5283 [cond-mat.str-el]
- J. Fuchs, G. Schaumann and C. Schweigert, *A modular functor from state sums for finite tensor categories and their bimodules*. arXiv:1911.06214 [math.QA]
- A. Kitaev, *Fault-tolerant quantum computation by anyons*. Annals Phys. 303 (2003) 2–30. arXiv:quant-ph/9707021
- A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls*. Comm. Math. Phys. 313 (2012), no. 2, 351–373. arXiv:1104.5047 [cond-mat.str-el]
- R. G. Larson, D. E. Radford, *Semisimple cosemisimple Hopf algebras*. Amer. J. Math. 110 (1988), no. 1, 187–195.