Nichols Algebras

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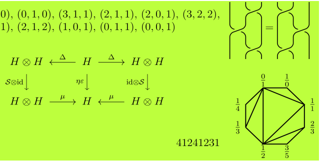

August 17, 2020

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- since $\sim$2000 autonomous research area - in MathSciNet, $\sim$120 articles with *Nichols* in title

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- The most powerful tools are available for categories of Yetter-Drinfeld modules (in particular over finite abelian groups).
- Problems with YD modules: there are many variations, and sometimes it is necessary to deal with more general settings.
- For the structure theory of Nichols algebras, the additional properties of the surrounding category are essential. It is difficult to get deep results on Nichols algebras in very general braided monoidal categories.

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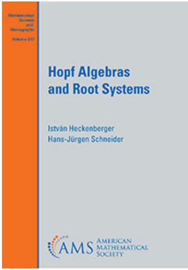

Further reading:
[HS] Heckenberger, Schneider: *Hopf Algebras and Root Systems*, Mathematical Surveys and Monographs 247, AMS, 2020.

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In many contexts, the flip map is not a morphism of the category and often alternatives are preferred.

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Yetter-Drinfeld modules over H form a braided monoidal category (morphisms: H-module H-comodule maps; notation: ${}^H_H\mathcal{YD}$)

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[HS, Thm. 3.10.4] If $\mathcal{S}_H$ iso and equalizer $(R, \iota : R \to A)$ of $((\mathrm{id}_A \otimes \pi)\Delta_A, \mathrm{id}_A \otimes \eta_H)$ exists ($R =$ right H-coinvariants) then R is a braided Hopf algebra (Hopf in ${}^H_H\mathcal{YD}(\mathcal{C})$) and $A \cong R\#H$.

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if $v, w \in V$, $vw = \mu(v \otimes w)$, then

$$\begin{aligned}\Delta(vw) =& \Delta(v)\Delta(w)\\ =& (1 \otimes v + v \otimes 1)(1 \otimes w + w \otimes 1)\\ =& 1 \otimes vw + v \otimes w + (1 \otimes v)(w \otimes 1) + vw \otimes 1,\end{aligned}$$

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Tensor Algebras as Braided Hopf Algebras

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If $c_{M,M} \neq -\mathrm{id}$, the ideal $(M^{\otimes 2}) = \sum_{k\geq 2} M^{\otimes k}$ of $T(M)$ generated by $M^{\otimes 2}$ is not a coideal of $T(M)$.

Nichols Algebra of a Yetter-Drinfeld Module

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$I(M) = \bigoplus_{k \geq 2} (I(M) \cap M^{\otimes k})$, hence $\mathcal{B}(M)$ is $\mathbb{N}_0$-graded:

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Only few (classes of) examples can be written down by generators and relations.

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- triangular type: partially classified, for those the defining relations are given
- group type: only few additional classes and exceptional examples of finite-dimensional Nichols algebras have been found; defining relations for those exceptional examples are known

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basis of $\mathcal{B}(V)$: $v_2^{n_1} v_{12}^{n_2} v_{112}^{n_3} v_{1112}^{n_4} v_{11112}^{n_5} v_1^{n_6}$, where $0 \leq n_1, n_5 < 2$, $0 \leq n_2, n_4 < 3$, $0 \leq n_3, n_6 < 18$.

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in [HS, 15.2], (essentially unique) root vector sequences are defined using right coideal subalgebras of $\mathcal{B}(V)$

Jordan Plane

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$\triangleright$	1	2	3	4
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$\dim(\mathcal{B}(V)) = 72.$

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non-abelian group type: look at appropriate subsets of the basis (subracks), use results on Nichols algebras of other type (like diagonal)

Skew Derivations

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the maps $\Delta_{n-1,1}$ give rise to "quantum differentials" (group type: skew derivations) by composing with $\mathrm{id} \otimes f$, $f \in V^*$.

Quantum Symmetrizer

for all $n \geq 1$,

$$\Delta_{1^n} : \mathcal{B}(V)(n) \to V^{\otimes n},$$
$$\Delta_{1^n} = (\Delta_{1,1} \otimes \mathrm{id}_{V^{\otimes n-2}}) \cdots (\Delta_{n-3,1} \otimes \mathrm{id}_{V^{\otimes 2}})(\Delta_{n-2,1} \otimes \mathrm{id}_V)\Delta_{n-1,1},$$

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is the **quantum symmetrizer**
it is injective and can be described in terms of reduced decompositions of elements of the symmetric group.

Coinvariants

if $V = U \oplus W$ in ${}^H_H\mathcal{YD}$ and $\pi : \mathcal{B}(V) \to \mathcal{B}(U)$, $\gamma : \mathcal{B}(U) \to \mathcal{B}(V)$ the Hopf algebra maps induced by projection to and inclusion of U, then the coinvariants $\mathcal{B}(V)^{\mathrm{co}\,\mathcal{B}(U)}$ of the Hopf algebra triple $(\mathcal{B}(V), \pi, \gamma)$ over $\mathcal{B}(U)$ are a Nichols algebra.

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this is used e.g. for triangular braidings

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take coinvariants $\mathcal{B}(V)^{\mathrm{co}\,\mathcal{B}(U)}$, transfer it to ${}^{\mathcal{B}(U)\#H}_{\mathcal{B}(U)\#H}\mathcal{YD}$ and then to another braided monoidal category ${}^{\mathcal{B}(U^*)\#H}_{\mathcal{B}(U^*)\#H}\mathcal{YD}$ using a braided monoidal equivalence.

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after bosonization with $\mathcal{B}(U^*)$, get new Nichols algebra in ${}^{H}_{H}\mathcal{YD}$.

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there is extensive literature on related topics