The lifting method I

Agustín García Iglesias

FaMAF (Universidad Nacional de Córdoba)
CIEM (CONICET)

Workshop: Hopf Algebras and Tensor Categories
17.-21. August 2020
"Hamburg"

This talk is based on the articles

AAG **Liftings of Nichols algebras of diagonal type I. Cartan type A**
Int. Math. Res. Notices (2016),
with Nicolás Andruskiewitsch and Iván Angiono;

AG **Liftings of Nichols algebras of diagonal type II. All liftings are cocycle deformations**
Selecta Math. (2019),
with Iván Angiono.

Also,

- **Pointed Hopf algebras: a guided tour to the liftings**
  Rev. Colombiana Matem.,
  with Iván Angiono.

Main result

- We classify, up to isomorphism, all finite-dimensional pointed Hopf algebras with abelian group of group-like elements.
- We show that these are all cocycle deformations of an associated graded Hopf algebra.

The classification depends on a triple $(\mathfrak{q}, \Gamma, \boldsymbol{\lambda})$, where

- $\mathfrak{q} = (q_{ij}) \in \Bbbk^{\theta\times\theta}$ is a matrix.
- Γ is an abelian group.
- $\boldsymbol{\lambda} = (\lambda_r)_{r\in\mathcal{G}}$ is a family of scalars.

We provide an algorithm to construct a Hopf algebra $\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$, for each triple $(\mathfrak{q}, \Gamma, \boldsymbol{\lambda})$.

Let ζ be an Nth root of 1. Set

$$B_N = \mathbb{k}[x]/x^N, \qquad C_{mN} = \langle g \rangle \simeq \mathbb{Z}/mN\mathbb{Z}.$$

Consider C_{mN} acting on B_N via $g \cdot x = \zeta x$.
The (generalized) Taft algebra

$$\begin{aligned} T_{m,N} &= \mathbb{k}\langle x, g | gx = \zeta\, xg, g^{mN} = 1, x^N = 0 \rangle \\ &\simeq B_N \rtimes \mathbb{k}C_{mN}. \end{aligned}$$

Let ζ be an Nth root of 1. Set

$$B_N = \Bbbk[x]/x^N, \qquad C_{mN} = \langle g \rangle \simeq \mathbb{Z}/mN\mathbb{Z}.$$

Consider C_{mN} acting on B_N via $g \cdot x = \zeta x$.
The (generalized) Taft algebra

$$\begin{aligned} T_{m,N} &= \Bbbk\langle x, g | gx = \zeta\, xg, g^{mN} = 1, x^N = 0 \rangle \\ &\simeq B_N \# C_{mN}. \end{aligned}$$

For each $\lambda \in \Bbbk$, a deformation of $T_{m,N}$ is given by

$$T_{m,N}(\lambda) = \Bbbk\langle x, g | gx = \zeta\, xg, g^{mN} = 1, x^N = \lambda(1 - g^N) \rangle.$$

Invariants
the coradical

Let (A, m, Δ) be a Hopf algebra.

- A nonzero element $x \in A$ is group-like if
$$\Delta(x) = x \otimes x.$$
The set $G(A) = \{x \in A : x \text{ group-like}\}$ is a group *via* m.
- The coradical of A is $A_0 := \sum_{\substack{\text{simple} \\ \text{subcoalg.}}} C.$

Notice: $x \in G(A) \Rightarrow C_x := \Bbbk\{x\} \subset A$ is a simple subcoalgebra.

Definition. A Hopf algebra is called pointed if
$$A_0 = \Bbbk G(A).$$

The lifting method

in two slides

Let A be a Hopf algebra s.t. $H = A_0 \subseteq A$ is a Hopf subalgebra (vg. pointed).

There is a Hopf algebra filtration $A_0 \subset A_1 \subset \cdots \subset A$ so that

$$\operatorname{gr} A = A_0 \oplus \underbrace{A_1/A_0}_{:=V} \oplus A_2/A_1 \oplus \dots$$

is a graded Hopf algebra.

- $\operatorname{gr} A \simeq \mathcal{R}\# H$, $\mathcal{R} \in {}^H_H\mathcal{YD}$.
  - (V, c) is a braided vector space.
- $\Bbbk\langle V\rangle \leq \mathcal{R}$ is the Nichols algebra $\mathfrak{B}(V)$ of (V, c).

The lifting method

in two slides

Let A be a finite-dimensional pointed Hopf algebra; $A_0 = \Bbbk\Gamma$.

Program

1. Find all $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ s.t. $\dim \mathfrak{B}(V) < \infty$.
2. Find a presentation of such $\mathfrak{B}(V)$; for each V.
3. Check "$A_0 = \Bbbk\Gamma \implies \operatorname{gr} A = \mathfrak{B}(V)\#\Bbbk\Gamma$".
4. Find all deformations (aka liftings) of $\mathfrak{B}(V)\#\Bbbk\Gamma$.

Theorem Assume $A_0 = \Bbbk\Gamma$, abelian.

- [1] $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.
- [2] $\mathfrak{q} = (q_{ij}) \in \Bbbk^{\theta\times\theta}$ is a matrix, from a list $\mathcal{H}$.
- [3] $\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle\mathcal{G}_{\mathfrak{q}}\rangle$, for a minimal set of relations $\mathcal{G}_{\mathfrak{q}}$.
- [4] $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$, for some $\boldsymbol{\lambda} = (\lambda_r)_r \in \Bbbk^{\mathcal{G}}$.

---

[1] I.Angiono, J. Reine Angew. Math. 2013.

[2] I.Heckenberger, Adv. Math. 2009.

[3] I.Angiono, JEMS 2013.

[4] I.Angiono, AGI., Selecta M. 2018.

Insight

- *Any* $\mathfrak{q} = (q_{ij})_{1\leq i,j\leq \theta}$ defines a *braided vector space of diagonal type* $(V, c^{\mathfrak{q}})$, by setting $V = \Bbbk\{x_1, \dots, x_\theta\}$ and

$$c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \;\; i, j = 1, \dots, \theta.$$

- $\mathfrak{B}_{\mathfrak{q}}$ is the *Nichols algebra* associated to $(V, c^{\mathfrak{q}})$.
- $\dim \mathfrak{B}_{\mathfrak{q}} < \infty$ if and only if $\mathfrak{q} \in \mathcal{H}$.

Classification problem

- For each $\mathfrak{q} = (q_{ij}) \in \mathcal{H}$ and Γ (compatible), find all deformations of $\mathfrak{B}_{\mathfrak{q}} \# \Bbbk\Gamma$.
  That is, all A with $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}} \# \Bbbk\Gamma$.

Examples

Example 1.

- $\mathfrak{q} = (\zeta)$, $\zeta^N = 1$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk[x]/x^N$ (i.e. $\mathcal{G}_{\mathfrak{q}} = \{x^N\}$).
- $\Gamma = \mathbb{Z}/mN\mathbb{Z}$.

Set $\mathfrak{u}(\lambda) = \Bbbk[x]\#\Bbbk\Gamma/\langle x^N - \lambda(1 - g^N)\rangle$.

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then there is λ such that $A \simeq \mathfrak{u}(\lambda)$.

Examples

Example 1.

- $\mathfrak{q} = (\zeta)$, $\zeta^N = 1$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk[x]/x^N$ (i.e. $\mathcal{G}_{\mathfrak{q}} = \{x^N\}$).
- $\Gamma = \mathbb{Z}/mN\mathbb{Z}$.

Set $\mathfrak{u}(\lambda) = \Bbbk[x]\#\Bbbk\Gamma/\langle x^N - \lambda(1 - g^N)\rangle$.

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then there is λ such that $A \simeq \mathfrak{u}(\lambda)$.

Example 2.

- $\mathfrak{q} = \begin{pmatrix} \zeta & q_1 \\ q_2 & q \end{pmatrix}$, with $\zeta^3 = 1$, $q^{12} = 1$, $q_1 q_2 = q^{-1}$.
- $\Gamma = \mathbb{Z}/24\mathbb{Z}$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk\langle x_1, x_2\rangle/\langle\mathcal{G}_{\mathfrak{q}}\rangle$ (and $\dim = 72$), where $\mathcal{G}_{\mathfrak{q}}$ is

$$x_1^3, \qquad x_2^{12}, \qquad [x_2,[x_2,x_1]], \qquad [x_1,[x_1,x_2]]^4.$$

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then $A \simeq ??$.

Examples

Example 1.

- $\mathfrak{q} = (\zeta)$, $\zeta^N = 1$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk[x]/x^N$ (i.e. $\mathcal{G}_{\mathfrak{q}} = \{x^N\}$).
- $\Gamma = \mathbb{Z}/mN\mathbb{Z}$.

Set $\mathfrak{u}(\lambda) = \Bbbk[x]\#\Bbbk\Gamma/\langle x^N - \lambda(1 - g^N)\rangle$.

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then there is λ such that $A \simeq \mathfrak{u}(\lambda)$.

Example 2.

- $\mathfrak{q} = \begin{pmatrix} \zeta & q_1 \\ q_2 & q \end{pmatrix}$, with $\zeta^3 = 1$, $q^{12} = 1$, $q_1 q_2 = q^{-1}$.
- $\Gamma = \mathbb{Z}/24\mathbb{Z}$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk\langle x_1, x_2\rangle/\langle \mathcal{G}_{\mathfrak{q}}\rangle$, where $\mathcal{G}_{\mathfrak{q}}$ is

$$x_1^3, \qquad x_2^{12}, \qquad [x_2,[x_2,x_1]], \qquad [x_1,[x_1,x_2]]^4.$$

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$.

Examples

Example 1.

- $\mathfrak{q} = (\zeta)$, $\zeta^N = 1$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk[x]/x^N$ (i.e. $\mathcal{G}_{\mathfrak{q}} = \{x^N\}$).
- $\Gamma = \mathbb{Z}/mN\mathbb{Z}$.

Set $\mathfrak{u}(\lambda) = \Bbbk[x]\#\Bbbk\Gamma/\langle x^N - \lambda(1 - g^N)\rangle$.

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then there is λ such that $A \simeq \mathfrak{u}(\lambda)$.

Example 2.

- $\mathfrak{q} = \begin{pmatrix} \zeta & -q \\ \zeta & q \end{pmatrix}$, with $\zeta^3 = 1$, $q^{12} = 1$.
- $\Gamma = \mathbb{Z}/24\mathbb{Z}$.
- $\mathfrak{B}_{\mathfrak{q}} = \Bbbk\langle x_1, x_2\rangle/\langle \mathcal{G}_{\mathfrak{q}}\rangle$, where $\mathcal{G}_{\mathfrak{q}}$ is

$$x_1^3, \qquad x_2^{12}, \qquad [x_2, [x_2, x_1]], \qquad [x_1, [x_1, x_2]]^4.$$

- If $\operatorname{gr} A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) = \Bbbk\langle x_1, x_2\rangle\#\Bbbk\Gamma/\langle \tilde{\mathcal{G}}_{\mathfrak{q}}(\boldsymbol{\lambda})\rangle$.

~~Cocycle deformations~~Cleft objects

Warning: rather technical

Let (B, Δ) be a Hopf algebra.

- An algebra E is a (right) comodule algebra if it is a comodule with coaction $\rho \in \mathrm{Alg}(E, E \otimes B)$.

Example $E = B$, $\rho = \Delta$.

- A (right) cleft object is a right comodule algebra E provided with a c.i. comodule isomorphism $\gamma\colon B \xrightarrow{\simeq} E$ (the section).

[5] Schauenburg, Comm. Algebra (1996)

~~Cocycle deformations~~Cleft objects

Warning: rather technical

Let (B, Δ) be a Hopf algebra.

- An algebra E is a (right) comodule algebra if it is a comodule with coaction $\rho \in \mathrm{Alg}(E, E \otimes B)$.

Example $E = B$, $\rho = \Delta$.

- A (right) cleft object is a right comodule algebra E provided with a c.i. comodule isomorphism $\gamma \colon B \xrightarrow{\simeq} E$ (the section).

Let[5] $(E, \rho) \in \mathrm{Cleft}(B)$.

- There exists a Hopf algebra $A = L(E, B)$ such that E becomes a bicleft (L, B)-object.
- A is a cocycle deformation of B.
  - $A = B$ as coalgebras and $m_A = \sigma * m_B * \sigma^{-1}$.

[5] Schauenburg, Comm. Algebra (1996)

The plan

Given $B = \mathfrak{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, find all $E \in \mathrm{Cleft}(B)$ such that

$$\operatorname{gr} L(E, B) \simeq B.$$

- Check if $\operatorname{gr} A \simeq B$ implies $A \simeq L(E, B)$, some $E \in \mathrm{Cleft}(B)$.

The setting

Given $(\Gamma, \mathfrak{q})$, we define a set of deformation parameters

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\Gamma, \mathfrak{q}) \subset \Bbbk^{\mathcal{G}_{\mathfrak{q}}}.$$

For each $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, we construct

- a PBW deformation[6] $\mathcal{E}(\boldsymbol{\lambda})$ of $\mathfrak{B}_{\mathfrak{q}}$ in $\operatorname{Rep}\Gamma$;
- a Hopf algebra $\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$;

in such a way that

- $\mathcal{A}(\boldsymbol{\lambda}) = \mathcal{E}(\boldsymbol{\lambda})\#\Bbbk\Gamma$ is a right $\mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$-cleft object.
- $\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) \simeq L(\mathcal{A}(\boldsymbol{\lambda}), \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma)$.
- $\operatorname{gr}\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) \simeq \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.

---

[6] Heckenberger, Vendramin (2017).

The setting

Fix $\mathfrak{q} = (q_{ij})_{1 \le i,j \le \theta} \in \mathcal{H}$ and Γ *compatible* so that

- there are $g_1, \dots, g_\theta \in \Gamma$ and $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that

$$\chi_i(g_j) = q_{ji}, \qquad i, j = 1, \dots, \theta.$$

Let $V = \Bbbk\{x_1, \dots, x_\theta\}$ and $\mathcal{G} := \mathcal{G}_{\mathfrak{q}} \subset T(V)$ be such that

$$\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle \mathcal{G} \rangle,$$

- hence there are $(g_r)_{r \in \mathcal{G}} \in \Gamma$, $(\chi_r)_{r \in \mathcal{G}} \in \widehat{\Gamma}$.

The set of *deformation parameters* is

$$\mathbf{\Lambda} = \{\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} | \lambda_r = 0 \text{ if } \chi_r \neq \epsilon\} \subseteq \Bbbk^{\mathcal{G}}.$$

The algorithm

Setting:

- $\mathfrak{q}$ such that $\dim \mathfrak{B}_{\mathfrak{q}} < \infty$;
- $V, \mathcal{G}$ such that $\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle \mathcal{G} \rangle$;
- Γ *compatible* with $\mathfrak{q}$ $(V \ni x_i \rightsquigarrow (g_i, \chi_i) \rightsquigarrow (g_r, \chi_r))$,
- $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \Bbbk^{\mathcal{G}}$.

Preparation:

- Stratify $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_\ell$ so that for each $r \in \mathcal{G}_t$,

$$\Delta(\bar{r}) = \bar{r} \otimes 1 + 1 \otimes \bar{r} \quad \text{in} \quad \mathfrak{B}_{\mathfrak{q},t} := T(V)/\langle \cup_{i=0}^{t-1} \mathcal{G}_i \rangle.$$

We proceed step-wise $t \in \mathbb{I}_\ell := \{0, 1, \ldots, \ell\}$.

- Set $\mathcal{H}_t := \mathfrak{B}_{\mathfrak{q},t} \# \Bbbk\Gamma$.

Step t

Input

- A (braided) Hopf algebra $\mathfrak{B}_t := \mathfrak{B}_{\mathfrak{q},t}$ and $\mathcal{G}_t \subset \mathfrak{B}_t$ with
$$\Delta(r) = r \otimes 1 + 1 \otimes r, \qquad r \in \mathcal{G}_t.$$
- A Γ-algebra $\mathcal{E}_t$ with an algebra map $\rho\colon \mathcal{E}_t \to \mathcal{E}_t \otimes \mathfrak{B}_t$,
  - hence an algebra $\mathcal{A}_t := \mathcal{E}_t \# \Bbbk\Gamma$ and a section $\gamma_t\colon \mathfrak{B}_t \to \mathcal{E}_t$.
- A H. algebra $\mathfrak{u}_t$ with an algebra map $\delta\colon \mathcal{A}_t \to \mathfrak{u}_t \otimes \mathcal{A}_t$.

[7] Günther. Comm. Alg. (1999).

Step t

Input

- A (braided) Hopf algebra $\mathfrak{B}_t := \mathfrak{B}_{\mathfrak{q},t}$ and $\mathcal{G}_t \subset \mathfrak{B}_t$ with

$$\Delta(r) = r \otimes 1 + 1 \otimes r, \qquad r \in \mathcal{G}_t.$$

- A Γ-algebra $\mathcal{E}_t$ with an algebra map $\rho\colon \mathcal{E}_t \to \mathcal{E}_t \otimes \mathfrak{B}_t$,
  - hence an algebra $\mathcal{A}_t := \mathcal{E}_t \# \Bbbk\Gamma$ and a section $\gamma_t\colon \mathfrak{B}_t \to \mathcal{E}_t$.
- A H. algebra $\mathfrak{u}_t$ with an algebra map $\delta\colon \mathcal{A}_t \to \mathfrak{u}_t \otimes \mathcal{A}_t$.

Lemma[AG,[7]] $\mathcal{E}_{t+1} := \mathcal{E}_t / \langle \gamma_t(r) - \lambda_r : r \in \mathcal{G}_t \rangle \neq 0$ and

- $\mathcal{A}_{t+1} := \mathcal{E}_{t+1} \# \Bbbk\Gamma \in \mathrm{Cleft}(\mathcal{H}_{t+1}) \qquad (\rightsquigarrow \gamma_{t+1}).$

[7] Günther. Comm. Alg. (1999).

Step t

Input

- A (braided) Hopf algebra $\mathfrak{B}_t := \mathfrak{B}_{\mathfrak{q},t}$ and $\mathcal{G}_t \subset \mathfrak{B}_t$ with
$$\Delta(r) = r \otimes 1 + 1 \otimes r, \qquad r \in \mathcal{G}_t.$$
- A Γ-algebra $\mathcal{E}_t$ with an algebra map $\rho\colon \mathcal{E}_t \to \mathcal{E}_t \otimes \mathfrak{B}_t$,
  - hence an algebra $\mathcal{A}_t := \mathcal{E}_t \# \Bbbk\Gamma$ and a section $\gamma_t\colon \mathfrak{B}_t \to \mathcal{E}_t$.
- A H. algebra $\mathfrak{u}_t$ with an algebra map $\delta\colon \mathcal{A}_t \to \mathfrak{u}_t \otimes \mathcal{A}_t$.

Lemma[AG,[7]] $\mathcal{E}_{t+1} := \mathcal{E}_t / \langle \gamma_t(r) - \lambda_r : r \in \mathcal{G}_t \rangle \neq 0$ and

- $\mathcal{A}_{t+1} := \mathcal{E}_{t+1} \# \Bbbk\Gamma \in \mathrm{Cleft}(\mathcal{H}_{t+1}) \qquad (\rightsquigarrow \gamma_{t+1})$.

Proposition[AAG] If $\mathfrak{u}_{t+1} := \mathfrak{u}_t / \langle \tilde{r} - \lambda_r(1 - g_r) \rangle$, then

- $\mathfrak{u}_{t+1} \simeq L(\mathcal{A}_{t+1}, \mathcal{H}_{t+1})$ and
- $\operatorname{gr} \mathfrak{u}_{t+1} \simeq \mathfrak{B}_{t+1} \# \Bbbk\Gamma$.

where $\tilde{r} \in \mathfrak{u}_t$, $r \in \mathcal{G}_t$, is determined by
$$\tilde{r} \otimes 1 = \delta(\gamma_t(r)) - g_r \otimes \gamma_t(r).$$

[7] Günther. Comm. Alg. (1999).

Output

Let $\mathfrak{B}_0 = \mathcal{E}_0 = T(V)$, so $\mathfrak{u}_0 = T(V)\#\Bbbk\Gamma$ and $\gamma_0 = \mathrm{id}_{T(V)}$.

Theorem.[AAG] Fix $\boldsymbol{\lambda}$ in $\boldsymbol{\Lambda}$ and set

$$\mathcal{G}(\boldsymbol{\lambda}) = \{\gamma_t(r) - \lambda_r : r \in \mathcal{G}_t, t\} \subset T(V),$$
$$\widetilde{\mathcal{G}}(\boldsymbol{\lambda}) = \{\tilde{r} - \lambda_r(1 - g_r) : r \in \mathcal{G}_t, t\} \subset T(V)\#\Bbbk\Gamma.$$

Assume $\mathcal{E}(\boldsymbol{\lambda}) := T(V)/\langle\mathcal{G}(\boldsymbol{\lambda})\rangle \neq 0$. Then

- $\mathcal{A}(\boldsymbol{\lambda}) := \mathcal{E}(\boldsymbol{\lambda})\#\Bbbk\Gamma$ is a right cleft object for $\mathcal{H} = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.
- $\mathcal{L}(\boldsymbol{\lambda}) := L(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$ is a lifting of V.
- $\mathcal{L}(\boldsymbol{\lambda}) \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) := T(V)\#\Bbbk\Gamma/\langle\widetilde{\mathcal{G}}(\boldsymbol{\lambda})\rangle$.
- $\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$ is a cocycle deformation of $\mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.

Output

Let $\mathfrak{B}_0 = \mathcal{E}_0 = T(V)$, so $\mathfrak{u}_0 = T(V)\#\Bbbk\Gamma$ and $\gamma_0 = \mathrm{id}_{T(V)}$.

Theorem.[AAG] Fix $\boldsymbol{\lambda}$ in $\boldsymbol{\Lambda}$ and set

$$\mathcal{G}(\boldsymbol{\lambda}) = \{\gamma_t(r) - \lambda_r : r \in \mathcal{G}_t, t\} \subset T(V),$$
$$\widetilde{\mathcal{G}}(\boldsymbol{\lambda}) = \{\tilde{r} - \lambda_r(1 - g_r) : r \in \mathcal{G}_t, t\} \subset T(V)\#\Bbbk\Gamma.$$

Assume $\mathcal{E}(\boldsymbol{\lambda}) \coloneqq T(V)/\langle \mathcal{G}(\boldsymbol{\lambda})\rangle \neq 0$. Then

- $\mathcal{A}(\boldsymbol{\lambda}) \coloneqq \mathcal{E}(\boldsymbol{\lambda})\#\Bbbk\Gamma$ is a right cleft object for $\mathcal{H} = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.
- $\mathcal{L}(\boldsymbol{\lambda}) \coloneqq L(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$ is a lifting of V.
- $\mathcal{L}(\boldsymbol{\lambda}) \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) \coloneqq T(V)\#\Bbbk\Gamma/\langle \widetilde{\mathcal{G}}(\boldsymbol{\lambda})\rangle$.
- $\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$ is a cocycle deformation of $\mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.

Theorem.[AG] $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ iff $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$.

- If $\mathrm{gr}\, A = \mathfrak{B}_{\mathfrak{q}}\#\Bbbk\Gamma$, then $\exists\, \boldsymbol{\lambda} \in \boldsymbol{\Lambda} : A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$.

Classification

Theorem.[AAG,AG] Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$.

- isoclasses are parametrized by equivalence classes in $\boldsymbol{\Lambda}/\sim$.
- A is a cocycle deformation of $\operatorname{gr} A$.

[8] Andruskiewitsch, Angiono; Math. Zeitschrift, to appear

[9] Andruskiewitsch, Angiono, Pevtsova, Witherspoon; arXiv:2004.07149

Classification

Theorem.[AAG,AG] Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$.

- isoclasses are parametrized by equivalence classes in $\boldsymbol{\Lambda}/\sim$.
- A is a cocycle deformation of $\operatorname{gr} A$.

Consequences.

- Case A_0 is basic[8].
- Cohomology ring of A is finitely generated[9].

[8] Andruskiewitsch, Angiono; Math. Zeitschrift, to appear

[9] Andruskiewitsch, Angiono, Pevtsova, Witherspoon; arXiv:2004.07149

Pick $\zeta, q, q_1, q_2 \in \Bbbk$ with $\zeta^3 = 1$, $q^{12} = 1$, $q_1 q_2 = q^{-1}$.

- $\mathfrak{q} = \begin{pmatrix} \zeta & q_1 \\ q_2 & q \end{pmatrix}, \qquad \Rightarrow V = \Bbbk\{x_1, x_2\}.$
- Pick Γ (eg. $\mathbb{Z}/24\mathbb{Z}$).

We have that $\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle \mathcal{G} \rangle$, where $\mathcal{G} = \{r_1, r_2, r_3, r_4\}$ is

$$x_1^3, \qquad x_2^{12}, \qquad [x_2, [x_2, x_1]], \qquad (\diamondsuit)$$
$$[x_1, [x_1, x_2]]^4, \qquad (\clubsuit)$$

and $\mathcal{G}_0 = \{(\diamondsuit)\}$, $\mathcal{G}_1 = \{(\clubsuit)\}$ is an stratification.

Pick $\zeta, q, q_1, q_2 \in \Bbbk$ with $\zeta^3 = 1$, $q^{12} = 1$, $q_1 q_2 = q^{-1}$.

- $\mathfrak{q} = \begin{pmatrix} \zeta & q_1 \\ q_2 & q \end{pmatrix}$, $\qquad \Rightarrow V = \Bbbk\{x_1, x_2\}$.
- Pick Γ (eg. $\mathbb{Z}/24\mathbb{Z}$).

We have that $\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle\mathcal{G}\rangle$, where $\mathcal{G} = \{r_1, r_2, r_3, r_4\}$ is

$$x_1^3, \qquad x_2^{12}, \qquad [x_2, [x_2, x_1]], \qquad (\diamondsuit)$$

$$[x_1, [x_1, x_2]]^4, \qquad (\clubsuit)$$

and $\mathcal{G}_0 = \{(\diamondsuit)\}$, $\mathcal{G}_1 = \{(\clubsuit)\}$ is an stratification.

- Fix $\boldsymbol{\lambda} \in \boldsymbol{\Lambda} = \{(\lambda_{r_1}, \lambda_{r_2}, \lambda_{r_3}, \lambda_{r_4}) \in \Bbbk^4 : \lambda_i = 0 \text{ if } \chi_{r_i} \neq \epsilon\}$

$$= \{(\lambda_{r_1}, \lambda_{r_2}, 0, \lambda_{r_4}) \in \Bbbk^4 : \lambda_{r_i} \in \Bbbk\}$$

$$= \{(\lambda_1, \lambda_2, \lambda_3)\} \simeq \Bbbk^3.$$

- Fix $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \Bbbk^3$.

Step 0:
(*recall that* $\gamma_0 = \mathrm{id}_{T(V)}$, $\tilde{r} = r$, ...)

- $\mathfrak{B}_1 = T(V)$ modulo $(\diamondsuit)$, i.e.

$$x_1^3, \qquad x_2^{12}, \qquad x_{221}.$$

- $\mathcal{E}_1 = T(V)$ modulo

$$x_1^3 - \lambda_1, \qquad x_2^{12} - \lambda_2, \qquad x_{221}. \qquad (\diamondsuit')$$

- $\mathfrak{u}_1 = T(V)\#\Bbbk\Gamma$ modulo

$$x_1^3 - \lambda_1(1 - g_1^3), \qquad x_2^{12} - \lambda_2(1 - g_2^{12}), \qquad x_{221}. \qquad (\tilde{\diamondsuit})$$

Step 1 (and final):
$(\gamma_1 =?, \ \tilde{r} =?, \ \dots)$

- $\mathfrak{B}_2 = \mathfrak{B}_{\mathfrak{q}} = T(V)/\langle(\diamondsuit), r\rangle, \ r = x_{112}^4 \ (\clubsuit)$.

We look for $r' = \gamma_1(r)$, so $\rho(r') = r \otimes 1 + g_r \otimes r'$.
We compute

$$\rho(r) = r \otimes 1 + 1 \otimes r + c\lambda_1^2 \cdot \rho(x_2^2 x_{12}^2) - c\lambda_1^2 \, x_2^2 x_{12}^2 \otimes 1.$$

Hence $r' = r - c\lambda_1^2 \, x_2^2 x_{12}^2$.

- $\mathcal{E}_2 = \mathcal{E} = \mathcal{E}_1/\langle(\diamondsuit'), r' - \lambda_3\rangle. \ (\clubsuit')$

Now we compute $\delta(r')$ to find $\tilde{r}$. We get

$$\delta(r') - g_r \otimes r' \overset{!}{=} r' \otimes 1.$$

So $\tilde{r} = r - c\lambda_1^2 \, x_2^2 x_{12}^2$.

- $\mathfrak{u}_2 = \mathfrak{u} = \mathfrak{u}_1/\langle(\tilde{\diamondsuit}), \tilde{r} - \lambda_3(1 - g_{112}^4)\rangle. \ (\tilde{\clubsuit})$

We are done.

Pick $\zeta, q \in \mathbb{k}$ with $\zeta^3 = 1$, $q^{12} = 1$.

- $\mathfrak{q} = \begin{pmatrix} \zeta & -q \\ \zeta & q \end{pmatrix}$, $\qquad \Rightarrow V = \mathbb{k}\{x_1, x_2\}$.
- Set $\Gamma = \mathbb{Z}/24\mathbb{Z}$.
- $\mathfrak{B}_{\mathfrak{q}} = T(V)/\langle \mathcal{G} \rangle$, where $\mathcal{G}$ is

$$x_1^3, \qquad x_2^{12}, \qquad x_{221}, \qquad x_{112}^4.$$

Each $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{k}^3$ defines a Hopf algebra

$$\mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda}) = T(V)\#\mathbb{k}\Gamma/\langle \widetilde{\mathcal{G}}(\boldsymbol{\lambda}) \rangle,$$

where $\widetilde{\mathcal{G}}(\boldsymbol{\lambda})$ is

$$x_1^3 = \lambda_1(1 - g_1^3), \qquad x_2^{12} = \lambda_2(1 - g_2^{12}), \qquad x_{221} = 0,$$
$$x_{112}^4 = \lambda_3(1 - g_{112}^4) + c\lambda_1^2 x_2^2 x_{12}^2.$$

- If A satisfies $\operatorname{gr} A \simeq \mathfrak{B}_{\mathfrak{q}}\#\mathbb{k}\Gamma$, then $A \simeq \mathfrak{u}_{\mathfrak{q}}(\boldsymbol{\lambda})$.

Examples

Build your own!

Ready

- All $(\mathfrak{q}, \Gamma)$ with $(|\Gamma|, 210) = 1$. $\looparrowleft$[AS].
- Cartan: A_n, B_2, G_2. $\looparrowleft$[AAG,AG,GJ].
- Standard: G_2. $\looparrowleft$[J].
- Modular $\mathfrak{br}(2, a)$ $\looparrowleft$[GP].
- UFO: $\mathfrak{ufo}(7), \mathfrak{ufo}(8)$. $\looparrowleft$[H,GP].

Missing

- Other Cartan's and Standard type.
- Super and modular type.
- Other UFO's: $\mathfrak{ufo}(1), \dots, \mathfrak{ufo}(12)$.

Plus

- It also works for non-diagonal braidings!
  - non-abelian groups, non-pointed Hopf algebras, ...

- Set $G = \mathbb{S}_4$, $H = \Bbbk^G = \Bbbk\{\delta_t : t \in G\}$,
- $\mathcal{O} = \{(ij) : (ij) \text{ transposition in } G\}$.

Then $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$ is a braided vector space with

$$c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}.$$

Moreover, $\dim \mathfrak{B}(V) = 24^2$.

- $V \in {}^H_H\mathcal{YD}$ via:

$$\delta_t \cdot x_{(ij)} = \delta_{t,(ij)} x_{(ij)},$$
$$x_{(ij)} \mapsto \sum_{t \in G} \operatorname{sgn}(t) \delta_t \otimes x_{t^{-1}(ij)t}.$$

- Thus $\mathfrak{B}(V) \# \Bbbk^G$ is a graded Hopf algebra.

- Set $G = \mathbb{S}_4$, $H = \Bbbk^G = \Bbbk\{\delta_t : t \in G\}$,
- $\mathcal{O} = \{(ij) : (ij) \text{ transposition in } G\}$.

Then $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$ is a braided vector space with

$$c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}.$$

Moreover, $\dim \mathfrak{B}(V) = 24^2$.

- $V \in {}^H_H\mathcal{YD}$ via:

$$\delta_t \cdot x_{(ij)} = \delta_{t,(ij)} x_{(ij)},$$
$$x_{(ij)} \mapsto \sum_{t \in G} \operatorname{sgn}(t) \delta_t \otimes x_{t^{-1}(ij)t}.$$

- Thus $\mathfrak{B}(V) \# \Bbbk^G$ is a graded Hopf algebra. Deformations?

- Let $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$; $c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}$.
- [10] $\mathfrak{B}(V) = T(V)/\langle \mathcal{G} \rangle$, where $\mathcal{G}$ is

$$\begin{aligned} x_{(ij)}^2 &= 0; \\ x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} &= 0; \\ x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} &= 0, \end{aligned}$$

all different $(ij), (kl), (ik) \in \mathcal{O}$ with $(ij)(kl) = (kl)(ij)$.

[10] Milinski, Schneider Contemp. Math. (2000); Andruskiewitsch, Graña, Adv. Math. (2003).

- Let $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$; $c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}$.
- [11] $\mathcal{E}(\boldsymbol{\lambda}) = T(V)/\langle \mathcal{G}(\boldsymbol{\lambda}) \rangle$, where $\mathcal{G}(\boldsymbol{\lambda})$ is

$$
\begin{aligned}
x_{(ij)}^2 &= \lambda_{(ij)};\\
x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} &= 0;\\
x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} &= 0.
\end{aligned}
$$

are the PBW deformations of $\mathfrak{B}(V)$ (in H-mod).

[11] Heckenberger, Vendramin (2017); GI, Vay, JPPA (2018).

- Let $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$; $c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}$.
- [11] $\mathcal{E}(\boldsymbol{\lambda}) = T(V)/\langle \mathcal{G}(\boldsymbol{\lambda}) \rangle$, where $\mathcal{G}(\boldsymbol{\lambda})$ is

$$\begin{aligned} x_{(ij)}^2 &= \lambda_{(ij)}; \\ x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} &= 0; \\ x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} &= 0. \end{aligned}$$

  are the PBW deformations of $\mathfrak{B}(V)$ (in H-mod).
- $\mathcal{A}(\boldsymbol{\lambda}) = \mathcal{E}(\boldsymbol{\lambda})\#\Bbbk^{\mathbb{S}_4}$ is $\mathfrak{B}(V)\#\Bbbk^{\mathbb{S}_4}$-cleft.
- If $\mathcal{L}(\boldsymbol{\lambda}) := L(\mathcal{A}(\boldsymbol{\lambda}), \mathfrak{B}(V)\#\Bbbk^{\mathbb{S}_4})$, then $\operatorname{gr} \mathcal{L}(\boldsymbol{\lambda}) \simeq \mathfrak{B}(V)\#\Bbbk^{\mathbb{S}_4}$.

[11] Heckenberger, Vendramin (2017); GI, Vay, JPPA (2018).

- Let $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$; $c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}$.
- $\mathcal{L}(\boldsymbol{\lambda}) \simeq \mathfrak{u}(\boldsymbol{\lambda}) := T(V)\#\Bbbk^{\mathbb{S}_4}/\langle\widetilde{\mathcal{G}}(\boldsymbol{\lambda})\rangle$, where $\widetilde{\mathcal{G}}(\boldsymbol{\lambda})$ is

$$\begin{aligned} x_{(ij)}^2 &= \sum_{t \in \mathbb{S}_4} (\lambda_{(ij)} - \lambda_{t^{-1}(ij)t})\, \delta_t; \\ x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} &= 0; \\ x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} &= 0. \end{aligned}$$

These are (all) the deformations of $\mathfrak{B}(V)\#\Bbbk^{\mathbb{S}_4}$.

- Let $V = \Bbbk\{x_{(ij)} | (ij) \in \mathcal{O}\}$; $c(x_{(ij)} \otimes x_{(kl)}) = -x_{(ij)(kl)(ij)} \otimes x_{(ij)}$.
- $\mathcal{L}(\boldsymbol{\lambda}) \simeq \mathfrak{u}(\boldsymbol{\lambda}) := T(V)\#\Bbbk^{\mathbb{S}_4}/\langle\widetilde{\mathcal{G}}(\boldsymbol{\lambda})\rangle$, where $\widetilde{\mathcal{G}}(\boldsymbol{\lambda})$ is

$$\begin{aligned} x_{(ij)}^2 &= \sum_{t\in\mathbb{S}_4} (\lambda_{(ij)} - \lambda_{t^{-1}(ij)t})\,\delta_t; \\ x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} &= 0; \\ x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} &= 0. \end{aligned}$$

These are (all) the deformations of $\mathfrak{B}(V)\#\Bbbk^{\mathbb{S}_4}$.

Theorem.[GV] Classification of f.d. copointed Hopf algebras / $\Bbbk^{\mathbb{S}_4}$.

- All liftings are cocycle deformations.

The lifting method I

Agustín García Iglesias

FaMAF (Universidad Nacional de Córdoba)
CIEM (CONICET)

Workshop: Hopf Algebras and Tensor Categories
17.-21. August 2020
“Hamburg”