Algebras of non-local screenings and diagonal Nichols algebras

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Workshop: Hopf Algebras and Tensor Categories

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Outline of the project

- Let $v_1, \ldots, v_n \in \mathbb{C}^n$ spanning a non-integral lattice Λ.
- We study the **screening operators** $\mathfrak{Z}_{v_1}, \ldots, \mathfrak{Z}_{v_n}$.

Question: What are the algebra relations between them?

[Lent17]: Under a certain smallness condition on (v_i, v_j),
the screening operators generate the **Nichols algebra** $\mathcal{B}$
associated to the braiding $q_{ij} = e^{\mathrm{i}\pi(v_i, v_j)}$.

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Goal 1

[Heck06] contains a classification of diagonal braidings q_{ij} that lead to finite dimensional Nichols algebras.
Our goal is to find all realising lattices Λ for every braiding q_{ij}.

Goal 2

If the smallness condition on (v_i, v_j) fails, which algebra do we get instead of the Nichols algebra?

Motivation

Programme: construct examples of logarithmic / non-semisimple chiral conformal field theories / vertex algebras.

Conjecture (Waki86, FGST06, AM08, FT10)

The vertex algebra defined as

$$\mathcal{W} := \bigcap_i ker \mathfrak{Z}_{v_i}$$

has the following representation theory:

$$\mathcal{W}\text{-}Rep \;\simeq\; (Quantum\ Group_{\mathcal{B}})\text{-}Rep$$

i.e. $\mathcal{W}$ *is a logarithmic chiral conformal field theory.*

Example

For $\Lambda = \sqrt{2p}\Lambda_R(\mathfrak{sl}_2) \to$ the triplet algebra $\mathcal{W}_{p,1} = ker\mathfrak{Z}_{-\frac{\alpha}{\sqrt{p}}}$

FGST, NT: As abelian categories $\mathcal{W}_{p,1}$-Rep $\simeq$ $u_q(\mathfrak{sl}_2)$-Rep,
where q is a $2p$-th root of unity.

GR, CGR: Conjecturally $\mathcal{W}_{p,1}$-Rep $\simeq$ $\tilde{u}_q(\mathfrak{sl}_2)$-Rep
as modular tensor categories
where $\tilde{u}_q(\mathfrak{sl}_2)$ quasi Hopf algebra variant of $u_q(\mathfrak{sl}_2)$.

Example

For $\Lambda = \sqrt{2p}\Lambda_R(\mathfrak{g}) \to \mathcal{W} := \bigcap_i ker\mathfrak{Z}_{-\frac{\alpha_i}{\sqrt{p}}}$

Conjecturally: $\mathcal{W}$-Rep $\simeq$ $\tilde{u}_q(\mathfrak{g})$-Rep where q is $2p$-th root of unity.

GLO: constructed $\tilde{u}_q(\mathfrak{g})$.

Screening operators

Definition

Let $\mathcal{V}$ be a VOA and $\mathcal{M}$, $\mathcal{N}$ modules.
The tensor product $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ is defined by having an intertwiner

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}} : \ \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)].$$

Now, we fix $m \in \mathcal{M}$ and for all modules $\mathcal{N}$, we get a map

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}(m, z) : \ \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)].$$

Integrating around $z = 0$, we get

$$\mathfrak{Z}_m : \mathcal{N} \to \overline{\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}}$$

which we call *(non-local) screening operator* associated to $m \in \mathcal{M}$.

Example

We consider the n-dimensional *Heisenberg* VOA $\mathcal{V}_{\mathcal{H}}$.

- For every $a \in \mathbb{C}^n$ there is an irreducible module $\mathcal{V}_a$ generated by an element e^a
- The tensor product is $\mathcal{V}_a \otimes \mathcal{V}_b = \mathcal{V}_{a+b}$
- The braiding is $c_{\mathcal{V}_a,\mathcal{V}_b} : \mathcal{V}_a \otimes \mathcal{V}_b \stackrel{e^{\mathrm{i}\pi(a,b)}}{\longrightarrow} \mathcal{V}_b \otimes \mathcal{V}_a$
- For $v_1, \ldots, v_n \in \mathbb{C}^n$ we consider the screening operators $\mathfrak{Z}_{v_i}$ associated to the elements $e^{v_i} \in \mathcal{V}_{v_i}$:

$$\mathfrak{Z}_{v_i} : \mathcal{V}_a \longmapsto \overline{\mathcal{V}}_{a+v_i}$$

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In particular we will consider $v_1, \dots, v_n \in \mathbb{C}^n$ spanning a non-integral lattice Λ.

Nichols algebras

- Let (V, q_{ij}) be a vector space with diagonal braiding

$$c : x_i \otimes x_j \longmapsto q_{ij} \cdot x_j \otimes x_i$$

with $q_{ij} \in \mathbb{C}^\times$ and $\{x_1, \dots, x_n\}$ basis of V.

- We write (q_{ij}) as diagram, e.g. $\cdots \!-\!\overset{q_{11}}{\circ}\!\overset{q_{12}q_{21}}{\rule[0.5ex]{4em}{0.4pt}}\!\overset{q_{22}}{\circ}\!-\! \cdots$

- Let $\mathcal{B}(q_{ij})$ be the Nichols algebra of (V, q_{ij}).

Example ($\dim V = n$)

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra of rank n, with simple roots $\{\alpha_1, \dots, \alpha_n\}$ and root lattice Λ.
Let q be a primitive ℓ-th root of unity and $q_{ij} := q^{(\alpha_i, \alpha_j)}$ on V.
Then $\mathcal{B}(q_{ij}) = u_q(\mathfrak{g})^+$ is the positive part of the small quantum group.

Algebra of screenings

Theorem (Lent17)

Given a non-integral lattice Λ, a basis $\{v_1, \dots, v_n\}$, $m_{ij} := (v_i, v_j)$, consider elements e^{v_i} in modules $\mathcal{V}_{v_i}$ of the Heisenberg VOA $\mathcal{V}_{\mathcal{H}}$. The braiding is

$$e^{v_i} \otimes e^{v_j} \mapsto q_{ij}\; e^{v_j} \otimes e^{v_i}, \qquad q_{ij} := e^{\mathrm{i}\pi m_{ij}}$$

If the following condition on m_{ij} is satisfied:

$$\sum_{i<j} m_{ij} > -|J| + 1 \qquad \forall J \subseteq I,\ |J| \geq 2,\ i,j \in J, \quad (I \text{ index set})$$

the screening operators $\mathfrak{Z}_{v_1}, \dots, \mathfrak{Z}_{v_n}$ form the diagonal Nichols algebra $\mathcal{B}(q_{ij})$.

Example

If Λ is the rescaled root lattice of $\mathfrak{g}$, $m_{ij} = (\alpha_i, \alpha_j)r$, $r \in \mathbb{Q}$,
then the smallness condition holds if $0 \leq r < \frac{1}{(\alpha_i,\alpha_j)}$,
and the algebra of screenings is the Nichols algebra $u_q(\mathfrak{g})^+$.

Otherwise the algebra of screenings is an extension of the Nichols algebra.

Algebra of screenings

Theorem (F. - Lentner '19 -refinement)

In the setting of the previous theorem we have:

- *The truncation relation* $(\mathfrak{Z}_{v_i})^d = 0$ *holds if:*

$$m_{ii} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, d = \mathrm{ord}(q_{ii}).$$

- *The quantum Serre relation* $[\mathfrak{Z}_{v_i}, [\ldots [\mathfrak{Z}_{v_i}, \mathfrak{Z}_{v_j}]\ ]\ \ldots] = 0$ *holds if:*

$$m_{ii} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, -a_{ij} + 1$$

$$m_{ij} + k\frac{m_{ii}}{2} \notin -\mathbb{N} \qquad k = 0, \ldots, -a_{ij}$$

Proven by analytic continuation of the generalized Selberg integral.

Goal 1

Definition (Semik11, FL19)

Let Λ be a lattice, basis $\{v_1, \ldots, v_n\}$, $m_{ij} := (v_i, v_j)$.

We say that $(\Lambda,\ m_{ij})$ *realise* a given braiding q_{ij} iff

$$e^{\mathrm{i}\pi m_{ij}} = q_{ij} \qquad \begin{array}{ccc} q_{ii} & q_{ij}q_{ji} & q_{jj} \\ \circ & \text{———} & \circ \\ m_{ii} & 2m_{ij} & m_{jj} \end{array}$$

and m_{ij} is compatible with Nichols algebra reflections as follows:

$$2m_{ij} = a_{ij} m_{ii} \qquad \text{or} \qquad (1 - a_{ij}) m_{ii} = 2 \tag{1}$$

Moreover all the reflected matrices $\mathcal{R}^k(m_{ij})$ must fulfil (1) again.

Goal 1: find all lattices realising Nichols algebra braidings.

Example ($u_q(\mathfrak{sl}_3)^+$)

Consider the braiding $\overset{q^2}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$ where $q = e^{\mathrm{i}\pi r}$.

- If $q^2 \neq -1$ it is realised by rescaling by $r \in \mathbb{Q}$ a Lie algebra root lattice of type A_2, $\mathfrak{g} = \mathfrak{sl}_3$

$$\underset{2r}{\overset{q^2}{\circ}}\underset{-2r}{\overset{q^{-2}}{\text{———}}}\underset{2r}{\overset{q^2}{\circ}}$$

Example ($u_q(\mathfrak{sl}_3)^+$)

Consider the braiding $\overset{q^2}{\circ}\overset{q^{-2}}{\rule[0.5ex]{3em}{0.4pt}}\overset{q^2}{\circ}$ where $q = e^{\mathrm{i}\pi r}$.

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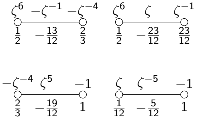
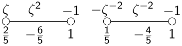
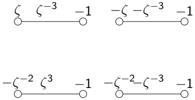
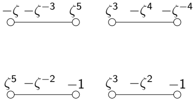
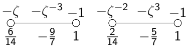
$$\underset{2r}{\overset{q^2}{\circ}}\underset{-2r}{\overset{q^{-2}}{\rule[0.5ex]{3em}{0.4pt}}}\underset{2r}{\overset{q^2}{\circ}}$$

- If $q^2 = -1$ it is *also* realised by rescaling by $r = \frac{p'}{2}$, $p' \in \mathbb{Z}$ odd a Lie superalgebra root lattice of type $A(1,0)$, $\mathfrak{g} = \mathfrak{sl}(2|1)$,

$$\underset{2r}{\overset{-1}{\circ}}\underset{-2r}{\overset{-1}{\rule[0.5ex]{3em}{0.4pt}}}\underset{1}{\overset{-1}{\circ}} \longrightarrow \underset{1}{\overset{-1}{\circ}}\underset{-2+2r}{\overset{-1}{\rule[0.5ex]{3em}{0.4pt}}}\underset{1}{\overset{-1}{\circ}}$$

Row	Braiding	Conditions	Solutions
$2'$	$\underset{r}{\overset{-1}{\circ}}\overset{-1}{\underset{-r}{-\!\!-\!\!-}}\underset{r}{\overset{-1}{\circ}}$ $\underset{r}{\overset{-1}{\circ}}\overset{-1}{\underset{-r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$ $\underset{1}{\overset{-1}{\circ}}\overset{-1}{\underset{-2+r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$		One solution according to A_2 (see $2''$). One solution according to $A(1,0)$ (see 3).
$2''$	$\underset{r}{\overset{q}{\circ}}\overset{q^{-1}}{\underset{-r}{-\!\!-\!\!-}}\underset{r}{\overset{q}{\circ}}$	$q \neq \pm 1$	Cartan, A_2
3	$\underset{r}{\overset{q}{\circ}}\overset{q^{-1}}{\underset{-r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$ $\underset{1}{\overset{-1}{\circ}}\overset{q}{\underset{-2+r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$	$q \neq \pm 1$	Super Lie, $A(1,0)$
$4'$	$\underset{r}{\overset{i}{\circ}}\overset{-1}{\underset{-2r}{-\!\!-\!\!-}}\underset{2r}{\overset{-1}{\circ}}$ $\underset{r}{\overset{i}{\circ}}\overset{-1}{\underset{-2r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$ $\underset{-r+1}{\overset{i}{\circ}}\overset{-1}{\underset{-2+2r}{-\!\!-\!\!-}}\underset{1}{\overset{-1}{\circ}}$		One solution according to B_2 (see $4'''$). One solution according to $B(1,1)$ (see 5).
$4''$	$\underset{r}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-2r}{-\!\!-\!\!-}}\underset{2r}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-2r}{-\!\!-\!\!-}}\underset{2r}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-\frac{8}{3}+2r}{-\!\!-\!\!-}}\underset{\frac{8}{3}-2r}{\overset{\zeta^{-1}}{\circ}}$	$\zeta \in \mathcal{R}_3$	One solution according to B_2 (see $4'''$). One solution according to 6.
$4'''$	$\underset{r}{\overset{q}{\circ}}\overset{q^{-2}}{\underset{-2r}{-\!\!-\!\!-}}\underset{2r}{\overset{q^2}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_3, \mathcal{R}_4$	Cartan, B_2

5	$\overset{q}{\underset{r}{\circ}}\overset{q^{-2}}{\underset{-2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\overset{-q^{-1}}{\underset{1-r}{\circ}}\overset{q^{2}}{\underset{-2+2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_4$	Super Lie, $B(1,1)$
6	$\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\overset{q}{\underset{r}{\circ}}$ $\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta^{-1}q}{\underset{-\frac{8}{3}+r}{\text{———}}}\overset{\zeta q^{-1}}{\underset{\frac{8}{3}-r}{\circ}}$	$\zeta \notin \mathcal{R}_3,\ q \neq 1, \zeta, \zeta^2$	
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_3$	No solution
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_{12}$	No solution
9	$\overset{-\zeta^{2}}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta}{\underset{-\frac{7}{6}}{\text{———}}}\overset{-\zeta^{2}}{\underset{\frac{2}{3}}{\circ}}$ $\overset{-\zeta^{2}}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta^{3}}{\underset{-\frac{3}{2}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\overset{-\zeta^{-1}}{\underset{\frac{1}{6}}{\circ}}\overset{-\zeta^{3}}{\underset{-\frac{1}{2}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$\zeta \in \mathcal{R}_{12}$	
10	$\overset{-\zeta}{\underset{\frac{5}{9}}{\circ}}\overset{\zeta^{-2}}{\underset{-\frac{10}{9}}{\text{———}}}\overset{\zeta^{3}}{\underset{\frac{2}{3}}{\circ}}$ $\overset{\zeta^{3}}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta^{-1}}{\underset{-\frac{14}{9}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\overset{-\zeta^{2}}{\underset{\frac{1}{9}}{\circ}}\overset{\zeta}{\underset{-\frac{4}{9}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$\zeta \in \mathcal{R}_9$	
11	$\overset{q}{\underset{r}{\circ}}\overset{q^{-3}}{\underset{-3r}{\text{———}}}\overset{q^{3}}{\underset{3r}{\circ}}$	$q \notin \mathcal{R}_3,\ q \neq \pm 1$	Cartan, G_2
12	$\overset{\zeta^{2}}{\underset{\frac{1}{2}}{\circ}}\overset{\zeta}{\underset{-\frac{7}{4}}{\text{———}}}\overset{\zeta^{-1}}{\underset{\frac{7}{4}}{\circ}}$ $\overset{\zeta^{2}}{\underset{\frac{1}{2}}{\circ}}\overset{-\zeta^{-1}}{\underset{-\frac{5}{4}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\overset{\zeta}{\underset{\frac{1}{4}}{\circ}}\overset{-\zeta}{\underset{-\frac{3}{4}}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$\zeta \in \mathcal{R}_8$	

13		$\zeta \in \mathcal{R}_{24}$	
14		$\zeta \in \mathcal{R}_5$	
15		$\zeta \in \mathcal{R}_{20}$	No solution
16		$\zeta \in \mathcal{R}_{15}$	No solution
17		$\zeta \in \mathcal{R}_7$	

Goal 2

- If the braiding is realised by rescaling a Lie algebra root lattice by $r \in \mathbb{Q}$, then one has:

Theorem

In the screening algebra:

- *For $r \geq 0$ all Nichols algebra relations hold (conjecturally also the non-simple truncation relations).*
  $\Longrightarrow$ *Small quantum group $u_q(\mathfrak{g})^+$*
- *For $r < 0$ all Nichols algebra relations hold except the truncation relations.*
  $\Longrightarrow$ *Conjecturally quantum group with infinite centre $U_q(\mathfrak{g})^+$*

- If the braiding is realised by rescaling a Lie superalgebra root lattice by $r', r'' \in \mathbb{Q}$, $\quad m_{ij} = \begin{cases} (\alpha_i, \alpha_j)_{\mathfrak{g}'} r' & \text{if } i \le f,\ j < f \\ 1 & \text{if } i = f = j \\ (\alpha_i, \alpha_j)_{\mathfrak{g}''} r'' & \text{if } i \ge f,\ j > f \end{cases}$

Theorem

In the screening algebra:
For $r', r'' \ge 0$ all Nichols algebra relations hold
(conjecturally also the non-simple truncation relations).

For $r', r'' < 0$ all Nichols algebra relations hold except the bosonic truncation relations.

For $r' > 0, r'' < 0$ or $r' < 0, r'' > 0$ the bosonic truncation relations on one side of the Dynkin diagram of the standard chamber fail.

- Otherwise: the algebra of screenings is always the Nichols algebra.

Thank you!

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