Pre-Nichols algebras of Cartan, super and standard type with finite Gelfand-Kirillov dimension

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Introduction

Following the same idea as in the previous talk, given a braided vector space V we are interested in finding all the pre-Nichols algebras of V with finite GK-dimension.

Now we will give a crucial definition of the talk:

Definition [ASa]

Let V be a braided vector space such that the Nichols algebra $\mathscr{B}_{\mathfrak{q}}$ satisfies that GKdim $\mathscr{B}_{\mathfrak{q}} < \infty$. All pre-Nichols algebra of V form a poset $\mathfrak{Pre}(V)$ with $T(V)$ minimal and $\mathscr{B}_{\mathfrak{q}}$ maximal. Denote $\mathfrak{Pre}_{\mathsf{GKd}}(V)$ the subposet of $\mathfrak{Pre}(V)$ with all finite GK-dimensional pre-Nichols algebras. We say that a pre-Nichols algebra is eminent if it is a minimum in $\mathfrak{Pre}_{\mathsf{GKd}}(V)$.

Introduction

Calculating eminent pre-Nichols algebras $\widehat{\mathscr{B}}_{\mathfrak{q}}$ reduces the problem of finding pre-Nichols algebras of V to finding quotients of $\widehat{\mathscr{B}}_{\mathfrak{q}}$.

In [AA] the braidings of diagonal type (of finite dimension) are grouped into the following types

1. Cartan type
2. Super type
3. Standard type
4. Modular type
5. Supermodular type
6. UFO.

Let's remember

Definition [An2]

The distinguished pre-Nichols algebra of $\mathscr{B}_{\mathfrak{q}}$ results from removing some kind of relations and adding quantum Serre relations in the given presentation. See [An1].

Introduction

The theorem that we prove is the following

Theorem [ASa][ACSa]

Let $\mathfrak{q}$ a matrix of Cartan, super or else standard type with connected Cartan matrix. Suppose further that $\mathfrak{q}$ is not of type:

- A_θ with $q = -1$.
- D_θ with $q = -1$.
- A_2 with $q \in \mathbb{G}'_3$.
- $\mathbf{A}_3(q|\{2\})$ with $q \in \mathbb{G}_\infty$.
- $\mathbf{A}_3(q|\{1,2,3\})$ with $q \in \mathbb{G}_\infty$.

then the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

We are currently working on the rest of the type of braidings.

Sketch of proof

To proof these results, we needed this conjecture:

Conjenture [AAH1, Conjeture 1.5]

The root system of a Nichols algebra of diagonal type with finite GKdim is finite.

The classification of all matrices with finite root system was provided in [H]. Also

Theorem [AAH2, Theorem 1.1]

If $\theta = 2$ or V is Cartan affine type, then the conjeture is true.

Moreover it holds in general for Cartan type and there are advances for $\theta = 3$. (Angiono-García Iglesias work in progress).

Sketch of proof

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Let $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ a braiding matrix with connected Dynkin diagram, $\theta \geq 2$, $V = (V, c)$ is a associated braided vector space with fixed basis $(x_i)_{i \in \mathbb{I}_\theta}$ and $\mathscr{B}$ is a pre-Nichols algebra of $\mathfrak{q}$ such that $\operatorname{GKdim} \mathscr{B} < \infty$.

Sketch of proof

Let $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ a braiding matrix with connected Dynkin diagram, $\theta \geq 2$, $V = (V, c)$ is a associated braided vector space with fixed basis $(x_i)_{i\in\mathbb{I}_\theta}$ and $\mathscr{B}$ is a pre-Nichols algebra of $\mathfrak{q}$ such that $\operatorname{GKdim}\mathscr{B} < \infty$. We determine conditions under which some defining relations from the presentation of Nichols algebras in [An1, Theorem 3.1] are annihilated in $\mathscr{B}$.

General sketch of proof

General sketch of proof

For a given relation x_u the strategy of the proof is the following:

(a) We suppose that $x_u \neq 0$, and either we check that $x_u \in \mathcal{P}(\mathscr{B})$ or we assume this fact.

(b) Let W be a braided subespace of $\mathcal{P}(\mathscr{B})$. Then $\operatorname{GKdim} \mathscr{B}(W) < \infty$. We compute the braiding matrix $\mathfrak{q}'$ of $V \oplus \Bbbk x_u$, then $\operatorname{GKdim} \mathscr{B}_{\mathfrak{q}'} < \infty$.

(c) On the other hand we prove that $\operatorname{GKdim} \mathscr{B}_{\mathfrak{q}'} = \infty$ using previous results, so we get a contradiction, so $x_u = 0$.

Some results

Here are some examples of the results obtained for some relations by the previous sketch

Lemma

Let $i \in \mathbb{I}_\theta$ *be a vertex of non-Cartan type such that* $q_{ii} \in \mathbb{G}'_N$. *Then* $x_i^N = 0$ *in* $\mathscr{B}$.

Lemma

Let $i, j \in \mathbb{I}_\theta$ *be such that* $\widetilde{q}_{ij} = 1$, $q_{ii}q_{jj} \neq 1$. *Then* $x_{ij} = 0$ *in* $\mathscr{B}$.

Lemma

Let $i, j \in \mathbb{I}_\theta$ *be such that* $q_{ii}q_{jj} = 1$, $\widetilde{q}_{ij} = 1$ *and either* $q_{ii}^2 \neq 1$ *or* $q_{jj}^2 \neq 1$. *Then* $x_{ij} = 0$ *in* $\mathscr{B}$.

Some Results

Lemma

Let $i, j \in \mathbb{I}_\theta$ be such that $q_{ii}q_{jj} = 1$, $\widetilde{q}_{ij} = 1$ and there exists $\ell \in I_\theta - \{i, j\}$ such that $\widetilde{q}_{i\ell}\widetilde{q}_{j\ell} \neq 1$. Then $x_{ij} = 0$ in $\mathscr{B}$.

Lemma

Let $i, j \in \mathbb{I}_\theta$ be such that $m_{ij} > 0$, $q_{ii}^{m_{ij}+1} \neq 1$ and one of the following hold:

(a) $q_{ii}^{m_{ij}+2} \neq 1$.

(b) $q_{ii}^{-m_{ij}(m_{ij}+1)} q_{jj}^2 \neq 1$.

Then $(\operatorname{ad}_c x_i)^{m_{ij}+1} x_j = 0$ in $\mathscr{B}$.

Lemma

Let $i, j, k \in \mathbb{I}_\theta$ be such that $q_{jj} = -1$, $\widetilde{q}_{ij} = \widetilde{q}_{jk}^{-1} \neq \pm 1$, $\widetilde{q}_{ik} = 1$. If either $q_{ii} = -1$ or $q_{kk} = -1$, then $[x_{ijk}, x_j]_c = 0$ in $\mathscr{B}$.

The theorem

This finishes the proof of our main result. Recall the theorem

Theorem [ASa][ACSa]

Let $\mathfrak{q}$ a matrix of Cartan, super or else standard type with connected Cartan matrix. Suppose further that $\mathfrak{q}$ is not of type:

- A_θ with $q = -1$.
- D_θ with $q = -1$.
- A_2 with $q \in \mathbb{G}'_3$.
- $\mathbf{A}_3(q|\{2\})$ with $q \in \mathbb{G}_\infty$.
- $\mathbf{A}_3(q|\{1,2,3\})$ with $q \in \mathbb{G}_\infty$.

then the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

Next we study the exceptional cases where the eminent pre-Nichols algebra is not the distinguished one.

The exceptions

Proposition [ASa]

Consider $\mathfrak{q}$ of type A_2 with $q \in \mathbb{G}_3'$. Let

$$\widehat{\mathscr{B}}_{\mathfrak{q}} = T(V)/\langle x_{1112}, x_{2112}, x_{2221}, x_{1221}\rangle.$$

Then $\widehat{\mathscr{B}}_{\mathfrak{q}}$ is the eminent pre-Nichols algebra of $\mathfrak{q}$. Moreover, $\operatorname{GKdim} \widehat{\mathscr{B}}_{\mathfrak{q}} = 5$.

The exceptions

Porposition [ACSa]

Consider $\mathfrak{q}$ of type $\mathbf{A}_3(q|\{2\})$. Let

$$\widehat{\mathscr{B}}_{\mathfrak{q}} = T(V)/\langle x_2^2, x_{13}, x_{112}, x_{332}\rangle.$$

Then $\widehat{\mathscr{B}}_{\mathfrak{q}}$ is the eminent pre-Nichols algebra of $\mathfrak{q}$.
Let $x_u = [x_{123}, x_2]_c$. The set

$$B = \left\{x_3^a x_{23}^b x_2^c x_u^d x_{123}^e x_{12}^f x_1^g : b, c, e, f \in \{0,1\}, \ a, d, g \in \mathbb{N}_0\right\}$$

is a basis of $\widehat{\mathscr{B}}_{\mathfrak{q}}$, so $\operatorname{GKdim} \widehat{\mathscr{B}}_{\mathfrak{q}} = 3$.

The exceptions

Proposition [ACSa]

Consider $\mathfrak{q}$ of type $\mathbf{A}_3(q|\{1,2,3\})$.*Let*

$$\widehat{\mathscr{B}}_{\mathfrak{q}} = T(V)/\langle x_1^2, x_2^2, x_3^2, x_{213}, [x_{123}, x_2]_c\rangle.$$

Then $\widehat{\mathscr{B}}_{\mathfrak{q}}$ is eminent pre-Nichols algebra of $\mathfrak{q}$. The set

$$B = \left\{x_3^a x_{23}^b x_2^c x_{13}^d x_{123}^e x_{12}^f x_1^g : a, c, e, g \in \{0,1\},\ b, d, f \in \mathbb{N}_0\right\}.$$

is basis of $\widehat{\mathscr{B}}_{\mathfrak{q}}$, so GKdim $\widehat{\mathscr{B}}_{\mathfrak{q}} = 3$.

Thanks!

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