The character algebra for module categories over Hopf algebras

Noelia Belén Bortolussi

Joint work with Martín Mombelli

CIEM - CONICET

Universidad Nacional de Córdoba, Argentina

Hamburg

August 20, 2020

Introduction

Shimizu's work:

[2016] $\rightsquigarrow$ *internal character theory* for finite tensor categories.

Introduction

Shimizu's work:

[2016] $\rightsquigarrow$ *internal character theory* for finite tensor categories.

For $\mathcal{C}$ = **finite tensor category over** $\Bbbk = \overline{\Bbbk}$

- adjoint algebra of $\mathcal{C}$: $\mathcal{A}_{\mathcal{C}} = \int_{X \in \mathcal{C}} X \otimes X^* \in \mathcal{Z}(\mathcal{C})$

  $\rightsquigarrow$ generalization of the adjoint representation of a Hopf algebra

Introduction

Shimizu's work:

[2016] $\rightsquigarrow$ *internal character theory* for finite tensor categories.

For $\mathcal{C}$ = **finite tensor category over** $\mathbb{k} = \overline{\mathbb{k}}$

- adjoint algebra of $\mathcal{C}$: $\mathcal{A}_{\mathcal{C}} = \int_{X \in \mathcal{C}} X \otimes X^* \in \mathcal{Z}(\mathcal{C})$

  $\rightsquigarrow$ generalization of the adjoint representation of a Hopf algebra
- space of central elements $CE(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{A}_{\mathcal{C}})$
- space of class functions $CF(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}_{\mathcal{C}}, \mathbf{1})$

Introduction

<u>**Shimizu's work:**</u>

[2016] $\rightsquigarrow$ *internal character theory* for finite tensor categories.

For $\mathcal{C}$ = **finite tensor category over** $\Bbbk = \overline{\Bbbk}$

- adjoint algebra of $\mathcal{C}$: $\mathcal{A}_{\mathcal{C}} = \int_{X \in \mathcal{C}} X \otimes X^* \in \mathcal{Z}(\mathcal{C})$

  $\rightsquigarrow$ generalization of the adjoint representation of a Hopf algebra
- space of central elements $CE(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{A}_{\mathcal{C}})$
- space of class functions $CF(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}_{\mathcal{C}}, \mathbf{1})$

For $\mathcal{C}$ =**pivotal finite tensor category over** $\Bbbk = \overline{\Bbbk}$

- internal character $\mathbf{ch}(X) \in CF(\mathcal{C})$ of $X \in \mathcal{C}$

Introduction

Main results

* The linear map $\mathbf{ch} : \Bbbk \otimes_{\mathbb{Z}} Gr(\mathcal{C}) \to CF(\mathcal{C})$, $\quad [X] \mapsto \mathbf{ch}(X)$, $\quad$ for $X \in \mathcal{C}$ is an injective algebra map.

Introduction

Main results

* The linear map $\mathbf{ch} : \Bbbk \otimes_{\mathbb{Z}} Gr(\mathcal{C}) \to CF(\mathcal{C})$, $\quad [X] \mapsto \mathbf{ch}(X)$, $\quad$ for $X \in \mathcal{C}$ is an injective algebra map.
* If $\mathcal{C}$ is unimodular, **ch** is an isomorphism $\Leftrightarrow$ $\mathcal{C}$ is semisimple.

[2018] $\rightsquigarrow$ generalization of $\mathcal{A}_{\mathcal{C}}$ and $CF(\mathcal{C})$.

Introduction

Main results

- The linear map $\mathbf{ch} : \Bbbk \otimes_{\mathbb{Z}} Gr(\mathcal{C}) \to CF(\mathcal{C})$, $[X] \mapsto \mathbf{ch}(X)$, for $X \in \mathcal{C}$ is an injective algebra map.
- If $\mathcal{C}$ is unimodular, **ch** is an isomorphism $\Leftrightarrow$ $\mathcal{C}$ is semisimple.

[2018] $\rightsquigarrow$ generalization of $\mathcal{A}_{\mathcal{C}}$ and $CF(\mathcal{C})$.

- $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$, $\mathcal{M}$ = indescomposable right exact left $\mathcal{C}$-module
- pivotal module category
- the *character theory* for pivotal module categories

Our work

Aim: Compute $\mathcal{A}_{\mathcal{D}}$, $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$ G-graded tensor category.

Our work

Aim: Compute $\mathcal{A}_{\mathcal{D}}$, $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$ G-graded tensor category.

We know that

- $\mathcal{C}_1$ is a tensor subcategory of $\mathcal{D}$
- $\mathcal{C}_g$ is an invertible $\mathcal{C}_1$-module

Our work

Aim: Compute $\mathcal{A}_{\mathcal{D}}$, $\mathcal{D} = \oplus_{g\in G}\mathcal{C}_g$ G-graded tensor category.

We know that

- $\mathcal{C}_1$ is a tensor subcategory of $\mathcal{D}$
- $\mathcal{C}_g$ is an invertible $\mathcal{C}_1$-module

IDEA: Relate $\mathcal{A}_{\mathcal{D}}$ with $\mathcal{A}_{\mathcal{C}_1}$ + DATA.

Our work

Aim: Compute $\mathcal{A}_{\mathcal{D}}$, $\mathcal{D} = \oplus_{g\in G}\mathcal{C}_g$ G-graded tensor category.

We know that

- $\mathcal{C}_1$ is a tensor subcategory of $\mathcal{D}$
- $\mathcal{C}_g$ is an invertible $\mathcal{C}_1$-module

IDEA: Relate $\mathcal{A}_{\mathcal{D}}$ with $\mathcal{A}_{\mathcal{C}_1}$ + DATA.

What we did? Let H be a finite dimensional Hopf algebra.

Compute $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$, $\mathcal{M} =$indescomposable exact Rep(H)-module.

Explicitly, for $H = \Bbbk G$ and $H = \Bbbk^G$.

Why? Rep$(\Bbbk^G)$ is a G-graded tensor category.

Contents of the talk

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

$\mathcal{C}$ = finite tensor category over $\mathbb{k} = \overline{\mathbb{k}}$

$\mathcal{M}$ = exact indecomposable left $\mathcal{C}$-module category

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

$\mathcal{C}$ = finite tensor category over $\Bbbk = \overline{\Bbbk}$

$\mathcal{M}$ = exact indecomposable left $\mathcal{C}$-module category

$Rex(\mathcal{M})$ is $\mathcal{C}$-bimodule category via

$$(X\overline{\otimes}F)(M) = X\overline{\otimes}F(M) \quad \text{and} \quad (F\overline{\otimes}X)(M) = F(X\overline{\otimes}M)$$

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

$\mathcal{C}$ = finite tensor category over $\Bbbk = \overline{\Bbbk}$

$\mathcal{M}$ = exact indecomposable left $\mathcal{C}$-module category

$Rex(\mathcal{M})$ is $\mathcal{C}$-bimodule category via

$$(X\overline{\otimes}F)(M) = X\overline{\otimes}F(M) \quad \text{and} \quad (F\overline{\otimes}X)(M) = F(X\overline{\otimes}M)$$

$\overline{\otimes}$ induces a $\mathcal{C}$-bimodule functor

Action functor: $\rho_{\mathcal{M}} : \mathcal{C} \to Rex(\mathcal{M}), \quad \rho_{\mathcal{M}}(X)(M) = X\overline{\otimes}M$

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

$\mathcal{C}$ = finite tensor category over $\Bbbk = \overline{\Bbbk}$

$\mathcal{M}$ = exact indecomposable left $\mathcal{C}$-module category

$Rex(\mathcal{M})$ is $\mathcal{C}$-bimodule category via

$$(X\overline{\otimes}F)(M) = X\overline{\otimes}F(M) \quad \text{and} \quad (F\overline{\otimes}X)(M) = F(X\overline{\otimes}M)$$

$\overline{\otimes}$ induces a $\mathcal{C}$-bimodule functor

Action functor: $\rho_{\mathcal{M}} : \mathcal{C} \to Rex(\mathcal{M}), \quad \rho_{\mathcal{M}}(X)(M) = X\overline{\otimes}M$

$\rho_{\mathcal{M}}$ has right adjoint

$$\rho^{ra}_{\mathcal{M}} : Rex(\mathcal{M}) \to \mathcal{C}, \quad \rho^{ra}_{\mathcal{M}}(F) = \int_{M\in\mathcal{M}} \underline{Hom}(M, F(M))$$

which is a $\mathcal{C}$-bimodule functor.

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

Consider the *relative center* 2-functor $\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}}\mathrm{Bimod} \to \mathrm{Cat}_{\Bbbk}$.

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

Consider the *relative center* 2-functor $\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}}\mathrm{Bimod} \to \mathrm{Cat}_{\Bbbk}$.

We have $\mathcal{Z}_{\mathcal{C}}(\rho^{ra}_{\mathcal{M}}) : \mathcal{Z}_{\mathcal{C}}(Rex(\mathcal{M})) \simeq \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

Consider the *relative center* 2-functor $\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}}\mathrm{Bimod} \to \mathrm{Cat}_{\Bbbk}$.

We have $\mathcal{Z}_{\mathcal{C}}(\rho^{ra}_{\mathcal{M}}) : \mathcal{Z}_{\mathcal{C}}(Rex(\mathcal{M})) \simeq \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.

Definition

The **adjoint algebra** of the module category $\mathcal{M}$ is

$$\mathcal{A}_{\mathcal{M}} := \mathcal{Z}_{\mathcal{C}}(\rho^{ra}_{\mathcal{M}})(Id_{\mathcal{M}}) \in \mathcal{Z}(\mathcal{C}).$$

$\mathcal{A}_{\mathcal{C}}$ is the adjoint algebra of the regular module category $\mathcal{C}$.

The definitions of $\mathcal{A}_{\mathcal{M}}$ and $CF(\mathcal{M})$

Consider the *relative center* 2-functor $\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}}\mathrm{Bimod} \to \mathrm{Cat}_{\Bbbk}$.

We have $\mathcal{Z}_{\mathcal{C}}(\rho^{ra}_{\mathcal{M}}) : \mathcal{Z}_{\mathcal{C}}(Rex(\mathcal{M})) \simeq \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.

Definition

The **adjoint algebra** of the module category $\mathcal{M}$ is

$$\mathcal{A}_{\mathcal{M}} := \mathcal{Z}_{\mathcal{C}}(\rho^{ra}_{\mathcal{M}})(Id_{\mathcal{M}}) \in \mathcal{Z}(\mathcal{C}).$$

$\mathcal{A}_{\mathcal{C}}$ is the adjoint algebra of the regular module category $\mathcal{C}$.

Definition

The **space of class functions** of $\mathcal{M}$ is $CF(\mathcal{M}) := Hom_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}_{\mathcal{M}}, \mathcal{A}_{\mathcal{C}})$.

The **adjoint algebra** of $\mathcal{M}$ is

$$\mathcal{A}_{\mathcal{M}} = \int_{M \in \mathcal{M}} \underline{Hom}(M, M) \in \mathcal{Z}(\mathcal{C})$$

with dinatural transformations $\pi_M^{\mathcal{M}} : \mathcal{A}_{\mathcal{M}} \xrightarrow{..} \underline{\mathrm{Hom}}(M, M)$.

The **adjoint algebra** of $\mathcal{M}$ is

$$\mathcal{A}_{\mathcal{M}} = \int_{M\in\mathcal{M}} \underline{Hom}(M,M) \in \mathcal{Z}(\mathcal{C})$$

with dinatural transformations $\pi^{\mathcal{M}}_M : \mathcal{A}_{\mathcal{M}} \xrightarrow{..} \underline{\mathrm{Hom}}(M,M)$.

The *half braiding* is the unique isomorphism $\sigma^{\mathcal{M}}_X : \mathcal{A}_{\mathcal{M}} \otimes X \to X \otimes \mathcal{A}_{\mathcal{M}}$

$$\begin{tikzcd}
\mathcal{A}_{\mathcal{M}} \otimes X \arrow[r, "\pi^{\mathcal{M}}_{X\overline{\otimes}M}\otimes \mathrm{id}_X"] \arrow[dd, "\sigma^{\mathcal{M}}_X"'] & \underline{\mathrm{Hom}}(X\overline{\otimes}M, X\overline{\otimes}M)\otimes X \arrow[d, "\mathfrak{b}_{X,M,X\overline{\otimes}M}"] \\
 & \underline{\mathrm{Hom}}(M, X\overline{\otimes}M) \arrow[d, "\mathfrak{a}_{X,M,M}"] \\
X \otimes \mathcal{A}_{\mathcal{M}} \arrow[r, "\mathrm{id}_X\otimes\pi^{\mathcal{M}}_M"] & X\otimes\underline{\mathrm{Hom}}(M,M).
\end{tikzcd}$$

The *product* and the *unit* of $\mathcal{A}_{\mathcal{M}}$ are the unique morphisms

$$m_{\mathcal{M}} : \mathcal{A}_{\mathcal{M}} \otimes \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{M}}, \quad u_{\mathcal{M}} : \mathbf{1} \to \mathcal{A}_{\mathcal{M}},$$

$$\begin{tikzcd}
\mathcal{A}_{\mathcal{M}} \otimes \mathcal{A}_{\mathcal{M}} \arrow[r, "\pi_M^{\mathcal{M}} \otimes \pi_M^{\mathcal{M}}"] \arrow[d, "m_{\mathcal{M}}"'] & \underline{\mathrm{Hom}}(M, M) \otimes \underline{\mathrm{Hom}}(M, M) \arrow[d, "\mathrm{comp}_M^{\mathcal{M}}"] \\
\mathcal{A}_{\mathcal{M}} \arrow[r, "\pi_M^{\mathcal{M}}"] & \underline{\mathrm{Hom}}(M, M)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\mathrm{coev}_{\mathbf{1},X}^{\mathcal{M}}"] \arrow[d, "u_{\mathcal{M}}"'] & \underline{\mathrm{Hom}}(M, M) \\
\mathcal{A}_{\mathcal{M}} \arrow[ur, "\pi_M^{\mathcal{M}}"'] &
\end{tikzcd}$$

$$\mathrm{coev}_{X,M}^{\mathcal{M}} : X \to \underline{\mathrm{Hom}}(M, X \overline{\otimes} M), \quad \mathrm{coev}_{X,M}^{\mathcal{M}} = \psi_{M, X \overline{\otimes} M}^{X}(\mathrm{id}_{X \overline{\otimes} M})$$

$$\psi_{M,N}^{X} : \mathrm{Hom}_{\mathcal{M}}(X \overline{\otimes} M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N))$$

The case $\mathcal{C} = \mathrm{Rep}(H)$

Let H be a finite dimensional Hopf algebra.

The case $\mathcal{C} = \mathrm{Rep}(H)$

Let H be a finite dimensional Hopf algebra.

Theorem

$$\{\textit{Exact indescomposable } \mathrm{Rep}(H) - \textit{modules}\}$$

$$\Updownarrow$$

$$\{{}_K\mathcal{M} : K = \textit{finite right } H\textit{-simple left } H\textit{-comodule algebra}\}$$

The case $\mathcal{C} = \mathrm{Rep}(H)$

Let H be a finite dimensional Hopf algebra.

Theorem

$$\{\textit{Exact indescomposable } \mathrm{Rep}(H) - \textit{modules}\}$$
$$\Updownarrow$$
$$\{{}_K\mathcal{M} : K = \textit{finite right } H\textit{-simple left } H\textit{-comodule algebra}\}$$

Action functor:

$$\rho_{{}_K\mathcal{M}} : \mathrm{Rep}(H) \to Rex({}_K\mathcal{M}) \simeq {}_K\mathcal{M}_K, \quad \rho_{{}_K\mathcal{M}}(X) = X \otimes_{\Bbbk} K$$

with K-actions $\quad s \cdot (x \otimes k) \cdot t = s_{(-1)} \cdot x \otimes s_{(0)} kt.$

We need $\quad \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}}) : \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \to \mathcal{Z}_{\mathrm{Rep}(H)}(\mathrm{Rep}(H))$

We need $\mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}}) : \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \to \mathcal{Z}_{\mathrm{Rep}(H)}(\mathrm{Rep}(H))$

Remark

- $\mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \simeq \mathrm{End}_{\mathrm{Rep}(H)}({}_K\mathcal{M}) \simeq {}^H_K\mathcal{M}_K$

We need $\quad \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}}) : \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \to \mathcal{Z}_{\mathrm{Rep}(H)}(\mathrm{Rep}(H))$

Remark

- $\mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \quad \simeq \quad \mathrm{End}_{\mathrm{Rep}(H)}({}_K\mathcal{M}) \quad \simeq \quad {}^H_K\mathcal{M}_K$
- $\mathcal{Z}_{\mathrm{Rep}(H)}(Rep(H)) = \mathcal{Z}(\mathrm{Rep}(H)) \quad \simeq \quad {}^H_H\mathcal{Y}D$

We need $\quad \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}}) : \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \to \mathcal{Z}_{\mathrm{Rep}(H)}(\mathrm{Rep}(H))$

Remark

- $\mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \quad \simeq \quad \mathrm{End}_{\mathrm{Rep}(H)}({}_K\mathcal{M}) \quad \simeq \quad {}^H_K\mathcal{M}_K$
- $\mathcal{Z}_{\mathrm{Rep}(H)}(Rep(H)) = \mathcal{Z}(\mathrm{Rep}(H)) \quad \simeq \quad {}^H_H\mathcal{Y}D$

For $P \in \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \simeq {}^H_K\mathcal{M}_K \mapsto \quad ?? \quad \in \mathcal{Z}_{\mathrm{Rep}(H)}(Rep(H)) \simeq {}^H_H\mathcal{Y}D$

We need $\quad \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}}) : \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \to \mathcal{Z}_{\mathrm{Rep}(H)}(\mathrm{Rep}(H))$

Remark

- $\mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \quad \simeq \quad \mathrm{End}_{\mathrm{Rep}(H)}({}_K\mathcal{M}) \quad \simeq \quad {}^H_K\mathcal{M}_K$
- $\mathcal{Z}_{\mathrm{Rep}(H)}(Rep(H)) = \mathcal{Z}(\mathrm{Rep}(H)) \quad \simeq \quad {}^H_H\mathcal{Y}D$

For $P \in \mathcal{Z}_{\mathrm{Rep}(H)}({}_K\mathcal{M}_K) \simeq {}^H_K\mathcal{M}_K \mapsto \quad ?? \quad \in \mathcal{Z}_{\mathrm{Rep}(H)}(Rep(H)) \simeq {}^H_H\mathcal{Y}D$

Definition

For each $P \in {}^H_K\mathcal{M}_K$, define the space

$$S^K(H,P) := \{\alpha \in Hom_K(H \otimes_{\Bbbk} K, P) : \alpha(h \otimes k) = \alpha(h \otimes 1) \cdot k\}.$$

Lemma

The space $S^K(H,P)$ is an object in ${}^H_H\mathcal{Y}D$.

Lemma

The space $S^K(H,P)$ is an object in ${}^H_H\mathcal{Y}D$.

- **<u>H-action:</u>** $(h\cdot\alpha)(x\otimes k)=\alpha(xh\otimes k), \quad x,h\in H, k\in K$
- **<u>H-coaction:</u>** $\lambda(\alpha)=\alpha^{-1}\otimes\alpha^{0}$, where for any $h\in H$, $k\in K$

$$\alpha^{-1}\otimes\alpha^{0}(h\otimes k)=\mathcal{S}(h_{(1)})\alpha(h_{(2)}\otimes 1)_{(-1)}h_{(3)}\otimes\alpha(h_{(2)}\otimes 1)_{(0)}\cdot k$$

Theorem

- $S^K(H,P) \simeq \int_{M \in {}_K\mathcal{M}} \underline{Hom}(M, P \otimes_K M)$ *as H-modules.*

  If $P = K$*, the isomorphism is an algebra map.*

Theorem

- *$S^K(H,P) \simeq \int_{M \in {}_K\mathcal{M}} \underline{Hom}(M, P \otimes_K M)$ as H-modules.*

  If $P = K$, the isomorphism is an algebra map.

- *For any $P \in {}^H_K\mathcal{M}_K$,*

$$S^K(H,P) \simeq \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}})(P) \quad \textit{in} \quad {}^H_H\mathcal{YD}.$$

Theorem

- $S^K(H,P) \simeq \int_{M \in {}_K\mathcal{M}} \underline{Hom}(M, P \otimes_K M)$ *as H-modules.*

  If $P = K$, *the isomorphism is an algebra map.*

- *For any* $P \in {}^H_K\mathcal{M}_K$,

$$S^K(H,P) \simeq \mathcal{Z}_{\mathrm{Rep}(H)}(\rho^{ra}_{{}_K\mathcal{M}})(P) \quad \textit{in} \quad {}^H_H\mathcal{YD}.$$

Corollary

There exists an isomorphism of algebras

$$S^K(H,K) \simeq \mathcal{A}_{{}_K\mathcal{M}} \quad \textit{in} \quad {}^H_H\mathcal{YD}.$$

Some explicit calculations

Example (Case $\mathbf{K} = H$)

$\mathcal{A}_{{}_H\mathcal{M}} \simeq S^H(H, H) \simeq H_{ad}$, underlying algebra H

H-**action** $\rightsquigarrow$ the adjoint action $h \triangleright x = h_{(1)} x S(h_{(2)})$, $h, x \in H$

H-**coaction** $\rightsquigarrow$ the coproduct

Some explicit calculations

Example (Case $\mathbf{K} = H$)

$\mathcal{A}_{_H\mathcal{M}} \simeq S^H(H,H) \simeq H_{ad}$, underlying algebra H

H-**action** $\rightsquigarrow$ the adjoint action $h \triangleright x = h_{(1)}xS(h_{(2)})$, $h, x \in H$

H-**coaction** $\rightsquigarrow$ the coproduct

Example (Case $\mathbf{K} = \Bbbk$)

$\mathcal{A}_{_\Bbbk\mathcal{M}} \simeq S^\Bbbk(H,\Bbbk) \simeq H^*_{ad}$, underlying algebra H^*

H-**action** $\rightsquigarrow (h \cdot f)(x) = f(xh)$

H-**coaction** $\rightsquigarrow \lambda(f) = f_{(-1)} \otimes f_{(0)}$, $< g, f_{(-1)} > f_{(0)} = \mathcal{S}(g_{(1)})fg_{(2)}$,

$h, x \in H$, $g \in H^*$.

Case H=$\Bbbk^G$, K= $\mathcal{K}(F,\psi,V)$

$F \subset G$ subgroup of a finite group G.

$\psi \in Z^2(F, \Bbbk^\times)$ a normalized 2-cocycle .

V a simple $\Bbbk_\psi F$-module

Case H=$\Bbbk^G$, K= $\mathcal{K}(F, \psi, V)$

$F \subset G$ subgroup of a finite group G.

$\psi \in Z^2(F, \Bbbk^\times)$ a normalized 2-cocycle .

V a simple $\Bbbk_\psi F$-module

$\mathcal{K}(F, \psi, V) = \mathrm{End}(V) \otimes_{\Bbbk_\psi F} \Bbbk G$ is a left $\Bbbk^G$-comodule algebra.

Case H=$\Bbbk^G$, K= $\mathcal{K}(F, \psi, V)$

$F \subset G$ subgroup of a finite group G.

$\psi \in Z^2(F, \Bbbk^\times)$ a normalized 2-cocycle .

V a simple $\Bbbk_\psi F$-module

$\mathcal{K}(F, \psi, V) = \mathrm{End}(V) \otimes_{\Bbbk_\psi F} \Bbbk G$ is a left $\Bbbk^G$-comodule algebra.

$S = \{representatives\ of\ \ right\ \ cosets\ F \backslash G : 1 \in S\} \subseteq G$

Case H=$\Bbbk^G$, K= $\mathcal{K}(F,\psi,V)$

$F \subset G$ subgroup of a finite group G.

$\psi \in Z^2(F,\Bbbk^\times)$ a normalized 2-cocycle .

V a simple $\Bbbk_\psi F$-module

$\mathcal{K}(F,\psi,V) = \mathrm{End}(V) \otimes_{\Bbbk_\psi F} \Bbbk G$ is a left $\Bbbk^G$-comodule algebra.

$S = \{\textit{representatives of right cosets } F\backslash G : 1 \in S\} \subseteq G$

Proposition (Description of $\mathcal{A}_{\mathcal{K}(F,\psi,V)}\mathcal{M}$)

The set $\{\alpha_{(f,s)} : (f,s) \in F \times S\}$ *is a basis for*

$$S^{\mathcal{K}(F,\psi,V)}(\Bbbk^G, \mathcal{K}(F,\psi,V)) \simeq \mathcal{A}_{\mathcal{K}(F,\psi,V)}\mathcal{M},$$

where $\quad \alpha_{(f,s)} = s^{-1}fs \otimes \overline{T_f \otimes s} \quad \in \quad \Bbbk G \otimes_{\Bbbk} \mathcal{K}(F,\psi,V).$

For any $(f, s) \in F \times S$, define

$$I(f,s) = \{(h,a) \in G \times G : ah^{-1}a = s^{-1}fs\}.$$

The $\Bbbk^G$-action and $\Bbbk^G$-coaction of $S^{\mathcal{K}(F,\psi,V)}(\Bbbk^G, \mathcal{K}(F,\psi,V))$ are

$\Bbbk^G$-**action** $\rightsquigarrow \lambda(\alpha_{(f,s)}) = \sum_{(h,a)\in I(f,s)} \delta_a \otimes h \otimes \overline{T_f \otimes sa}$

$$\Bbbk^G\text{-}\textbf{coaction} \rightsquigarrow \delta_g \cdot \alpha_{(f,s)} = \begin{cases} 0 & \text{if } g \neq s^{-1}fs \\ \alpha_{(f,s)} & \text{if } g = s^{-1}fs. \end{cases}$$

For any $(f, s) \in F \times S$, define

$$I(f, s) = \{(h, a) \in G \times G : ah^{-1}a = s^{-1}fs\}.$$

The $\Bbbk^G$-action and $\Bbbk^G$-coaction of $S^{\mathcal{K}(F,\psi,V)}(\Bbbk^G, \mathcal{K}(F, \psi, V))$ are

$\Bbbk^G$-**action** $\rightsquigarrow \lambda(\alpha_{(f,s)}) = \sum_{(h,a)\in I(f,s)} \delta_a \otimes h \otimes \overline{T_f \otimes sa}$

$$\Bbbk^G\text{-}\textbf{coaction} \rightsquigarrow \delta_g \cdot \alpha_{(f,s)} = \begin{cases} 0 & \text{if } g \neq s^{-1}fs \\ \alpha_{(f,s)} & \text{if } g = s^{-1}fs. \end{cases}$$

Proposition (Description of $CF({}_{\mathcal{K}(F,\psi,V)}\mathcal{M})$)

There exists a linear isomorphism $CF({}_{\mathcal{K}(F,\psi,V)}\mathcal{M}) \simeq \Bbbk^S$.

Bibliography

- K. Shimizu. *The monoidal center and the character algebra.* J. Pure Appl. Algebra 221, No. 9, 2338–2371 (2017).
- K. Shimizu, *Further results on the structure of (Co)ends in fintite tensor categories*, preprint arXiv:1801.02493. (2018)
- N. Bortolussi, M. Mombelli. *The character algebra for module categories over Hopf algebras*, preprint arXiv:1808.04810.

Gracias!

Danke schön!

Thank you!