Nichols algebras and the cohomology of finite-dimensional Hopf algebras

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C I E M

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Joint with N. Andruskiewitsch, J. Pevtsova, S. Witherspoon.

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Let G be a finite group and $H = \mathbb{k}G$ its group algebra.

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- The result is meaningful when $\operatorname{char} \Bbbk > 0$.
- If $\operatorname{char} \Bbbk = 0$, then H is semisimple (Maschke theorem). Also, any cocommutative fd Hopf algebra is like this.

$p = \operatorname{char} \Bbbk > 0$: more cocommutative fd Hopf algebras.

Example

Let $\mathfrak{g}$ be a finite-dimensional *restricted* Lie algebra; i.e. it comes with a suitable map $x \mapsto x^{[p]}$.Let

$$H = \mathfrak{u}(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]} : x \in \mathfrak{g}\rangle$$

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Cocommutative fd. Hopf algebras $\longleftrightarrow$ finite group schemes, difficult to classify, some structure theorems.

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Let H be a cocommutative finite-dimensional Hopf algebra.

They also observe that the cohomology ring of a finite-dimensional *commutative* Hopf algebra is easily seen to be finite generated using the structure and add:

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> *We do not know whether it is reasonable to expect finite generation of the cohomology of an arbitrary finite-dimensional Hopf algebra.*

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- H **basic**: every simple H-module has dim 1.
- H **pointed**: every simple H-comodule has dim $1 \equiv H^*$ basic.

Here we have some Hopf algebras H that have **(fgc)**:

Ginzburg & Kumar 1993 char $\mathbb{k} = 0$, $H = \mathfrak{u}_q(\mathfrak{g})$, $\mathfrak{g}$ a simple Lie algebra, $q \in \mathbb{G}_\infty$ with restrictions on the order.
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Conjecture (Etingof & Ostrik 2005)

Finite generation of cohomology **(fgc)** *holds for finite tensor categories.*

Main result and scheme of the proof

From now on $\operatorname{char} \mathbb{k} = 0$.

Theorem (AAPW 2020)

Let H be a finite-dimensional pointed Hopf algebra such that $G(H)$ is abelian (but **no restrictions** *on $|G(H)|$). Then H has* **(fgc)**.

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Let H be a finite-dim. basic Hopf algebra such that $\mathrm{Hom}_{Alg}(H, \Bbbk)$ is abelian. Then H has **(fgc)**.

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Corollary (AAPW 2020)

*Let H be a finite-dim. Hopf algebra such that H fits into an extension $\Bbbk \to K \to H \to B \to \Bbbk$ with K semisimple and B pointed or basic as above. Then H has **(fgc)**.*

Approach I

Proposition (Negron & Plavnik 2018)

Let H a Hopf subalgebra of a fd Hopf algebra L. If L has **(fgc)**, *then so does H.*

$H \sim_{\mathrm{Mor}} H' :\Longleftrightarrow D(H) \simeq D(H')$ as quasitriangular Hopf algebras.
For example, let H be a finite-dimensional Hopf algebra.

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Corollary

$H \sim_{\mathrm{Mor}} H'$ & $D(H)$ satisfies **(fgc)**, *then so does H'.*

Approach II

Let H be a fd **pointed** Hopf algebra **with $G(H)$ abelian**.
Let $\mathrm{gr}\, H$ be the graded Hopf algebra arising from the coradical filtration of H: $H_0 = \Bbbk G(H) \subset H_1 \subset H_2 \cdots \subset H_n \dots$

Theorem (A. 2013)

$\mathrm{gr}\, H \simeq \mathcal{B}(V)\#\Bbbk G(H)$, where V is of diagonal type. **$\mathcal{B}(V)$ is the Nichols algebra of V.**

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Theorem (A. & García Iglesias 2019)

There exists a 2-cocycle $\sigma : H \otimes H \to \Bbbk$ such that $\operatorname{gr} H \simeq H_\sigma$.

Approach III

We now can present the different steps.

Step 1 If V is of diagonal type and $\dim \mathcal{B}(V) < \infty$, then $\mathcal{B}(V)$ has **(fgc)**. $\mathcal{B}(V)$ **is the Nichols algebra of** V.

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Step 3 If Γ is a finite abelian group, $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (hence of diagonal type), and $\dim \mathcal{B}(V \oplus \overline{V}) < \infty$, then the Drinfeld double $D(\mathcal{B}(V)\#\Bbbk\Gamma)$ has **(fgc)**.

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Step 4 H fd pointed Hopf algebra, $G(H)$ abelian.

$$H^* \sim_{\mathrm{Mor}} H \overset{\text{AGI '19}}{\sim}_{\mathrm{Mor}} \operatorname{gr} H \overset{\text{A '13}}{\simeq} \mathcal{B}(V)\#\Bbbk G(H).$$

By Step 1, $\mathcal{B}(V \oplus \overline{V})$ has **(fgc)**.
By Step 3, $D(\mathcal{B}(V)\#\Bbbk G(H))$ has **(fgc)**.
By **(Negron & Plavnik 2018)**, H and H^* have **(fgc)**.

Proof of Step 1: If V is of diagonal type and $\dim \mathcal{B}(V) < \infty$, then $\mathcal{B}(V)$ has (fgc)

- $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \implies$ **Nichols algebra** $\mathcal{B}(V) = \oplus_{n\in\mathbb{N}_0}\mathcal{B}^n(V)$: (graded) Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that

  $$\mathcal{B}^0(V) \simeq \Bbbk, \quad \mathcal{B}^1(V) \simeq V, \quad \mathcal{B}(V) = \Bbbk\langle V\rangle, \quad \operatorname{Prim}(\mathcal{B}(V)) = \mathcal{B}^1(V).$$

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- Kharchenko proved $\exists$ a **PBW-basis**: $\emptyset \neq S \subset \mathcal{B}(V)$ with a **total order** $<$, $h : S \mapsto \mathbb{N} \cup \{\infty\}$ (the *height*), such that

  $$B = \left\{s_1^{e_1} \dots s_t^{e_t} : t \in \mathbb{N}_0, s_1 > \cdots > s_t \in S, 0 < e_i < h(s_i)\right\}.$$

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  is a basis of $\mathcal{B}(V)$. Each $s \in S$ is a *PBW-generator* of $\mathcal{B}(V)$.
- We can choose the order $<$ to be convex: the monomial filtration is an algebra filtration $\implies$ the associated graded ring $\operatorname{gr} \mathcal{B}(V)$ is a quantum linear space.

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- The cohomology ring of $\operatorname{gr} \mathcal{B}(V)$ is well-known, but we provide a computation using the **Anick resolution** to relate it to permanent cycles in a suitable spectral sequence.

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- We reduce the verification of the statement to claims on root systems of Nichols algebras of diagonal type.
- We check these claims on each Nichols algebra of diagonal type in the list of Heckeberger, using the specific form of the PBW-basis as in the survey (A-Angiono).

Proof of Step 2: If V is of diagonal type, $V \in {}^K_K\mathcal{YD}$ with K semisimple and $\dim \mathcal{B}(V) < \infty$, then $\mathcal{B}(V)\#K$ has (fgc)

- (MPSW) H a Hopf algebra, R a bialgebra in ${}^H_H\mathcal{YD}$. Assume that either H or R is finite-dimensional. Then the (opposite of) the Hochschild cohomology $HH(R, \Bbbk)$ is a braided commutative graded algebra in ${}^H_H\mathcal{YD}$.

Putting together all these facts we get Step 2.

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- (Stefan & Vay 2016) K a ss Hopf algebra, R a fd K-module algebra. Then $\mathrm{H}(R\#K, \Bbbk) = \mathrm{H}(R, \Bbbk)^K$.

Putting together all these facts we get Step 2.

Proof of Step 3: If Γ is finite abelian, $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and $\dim \mathcal{B}(V) < \infty$, then $D(\mathcal{B}(V)\#\Bbbk\Gamma)$ has (fgc)

$V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is of diagonal type. By **(Beattie 2003)**

$$\operatorname{gr} D\left(\mathcal{B}(V)\#\Bbbk G(H)\right) \simeq \mathcal{B}(V \oplus \overline{V})\#\Bbbk\left(G(H) \times \widehat{G(H)}\right).$$

Then $\operatorname{gr} D\left(\mathcal{B}(V)\#\Bbbk G(H)\right)$ has **(fgc)** by Step 2.

Now the cohomology of $D\left(\mathcal{B}(V)\#\Bbbk G(H)\right)$ is controlled by the cohomology of $\operatorname{gr} D\left(\mathcal{B}(V)\#\Bbbk G(H)\right)$ by a classical spectral sequence argument (that requires however a detailed verification).

Muchas gracias

Danke schön

Thanks!